Description of project task

TANA21/22: Beräkningsmatematik — *Andrew R. Winters* — *Autumn 2019*

The problem to be solved in the project will be the numerical solution of a two-point boundary value problem governed by a second-order, linear ordinary differential equation (ODE)

$$\begin{aligned} y''(x) &= p(x)y'(x) + q(x)y(x) + r(x) \quad \text{where} \quad x \in (a,b), \\ y(a) &= \alpha, \\ y(b) &= \beta. \end{aligned}$$

Physical applications of such a two-point boundary value problem are numerous such as the load and deformation of a length of a beam or the heating of a wire as electricity is run through it. This problem uses an ODE model for a function in the interior of the interval where the function is fixed with *boundary conditions* by two constant values at either side of the interval.

Below we divide the end goal to solve this two-point boundary value problem into mini-projects. This structure will help organize the numerical approximation into manageable pieces that will also assist in debugging the code. In a sense, the earlier mini-projects act as a computational "toolbox" that can be applied to solve the larger problem. Broadly, the mini-projects are implementing and testing:

1. A routine to solve a symmetric positive definite linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$.
2. An interpolation routine to approximate a function $f(x)$.
3. A routine to solve the two-point boundary value problem given above.

It is important to note:

- TANA21 students: Implement and solve mini-projects 1 and 3.
- TANA22 and 9AMA73 students: Implement and solve mini-projects 1, 2 and 3.

Linear solver for symmetric positive definite matrix

Write a linear solver routine for a linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$ where the matrix $\mathbf{A}$ is symmetric positive definite. The implementation should be done in MATLAB. The linear solver will come in two versions:

Version Eins: Compute the Cholesky factorization of the matrix $\mathbf{A}$. Solution of the system can then be done with forward/backward substitution as discussed in the lecture.

Version Zwei: Use a variant of the Gauss-Seidel iterative method known as *Successive Over-relaxation (SOR)* to solve the system. This method takes the form

$$x_i^{(k+1)} = (1-\omega)x_i^{(k)} + \frac{\omega}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)}\right)$$

and is equivalent to Gauss-Seidel when $\omega = 1$.

For either version, the implementation should be verified to solve the linear system $\mathbf{Ax} = \mathbf{b}$. A specific task for each version are:

Version Eins: Vary the size n of the matrix system and computationally estimate the amount of work for the Cholesky factorization. Use the `tic` and `toc` MATLAB commands to measure the computation time.

Version Zwei: For over-relaxation the parameter $\omega \in (1, 2)$ where $\omega = 1$ corresponds to standard Gauss-Seidel. Vary the value of ω and observe its effect on the number of iterations required by SOR to achieve a prescribed solution tolerance of $\texttt{tol} = 10^{-9}$.

Interpolation

Write a MATLAB routine that creates a polynomial interpolant for a smooth function $f(x)$. The two versions are:

Version Eins: A global interpolating polynomial in Newton form. Create the necessary coefficients using a Newton divided difference table.

Version Zwei: A piecewise interpolating polynomial using linear splines.

Specific tasks for either interpolation procedure:

Version Eins: Experiment using uniform interpolation nodes and Chebyshev interpolation nodes to observe/remove Runge phenomena of the global interpolation.

Version Zwei: Verify the convergence order of h^2 for a linear spline where h is the largest sub-interval size.

Two-point boundary value problem

Use central difference approximations of any derivatives to numerically approximate the solution of the ODE on the given interval with the prescribed boundary conditions:

$$y'' = -\frac{2}{x}y' + \frac{2}{x^2}y + \frac{\sin(\ln(x))}{x^2}, \quad 1 \leq x \leq 2 \quad \text{with} \quad y(1) = 1 \;\text{ and }\; y(2) = 2$$

The ODE has the exact solution

$$y = c_1 x + \frac{c_2}{x^2} - \frac{3}{10}\sin(\ln(x)) - \frac{1}{10}\cos(\ln(x)),$$

with

$$c_1 = \frac{11}{10} - c_2, \quad c_2 = \frac{1}{70}\left[8 - 12\sin(\ln(2)) - 4\cos(\ln(2))\right].$$

This exact solution should be used to verify the solution accuracy of the approximate solution.

Specific tasks for this procedure are:

All students: Discuss the discretization and assembly of the discrete matrix representation of this ODE. To solve the resulting linear system you will use the linear solver from mini-project 1.

All students: Verify the second order accuracy of the ODE discretization with increasing sets of nodes $n = 10, 20, 40$ and 80.

TANA22/9AMA73 only: Use the interpolation routines from mini-project 2 to approximate the functions

$$p(x) = -\frac{2}{x} \quad q(x) = \frac{2}{x^2}, \quad r(x) = \frac{\sin(\ln(x))}{x^2}$$

on $n = 6, 12$ Chebyshev nodes. How does the resolution of the polynomial interpolant affect the quality of the approximate solution to the two-point boundary value problem?