COMS30017
COMPUTATIONAL NEUROSCIENCE

LECTURE: INTRODUCTION TO DIFFERENTIAL EQUATIONS

PART-2

Dr. Rahul Gupta
xv20319@bristol.ac.uk

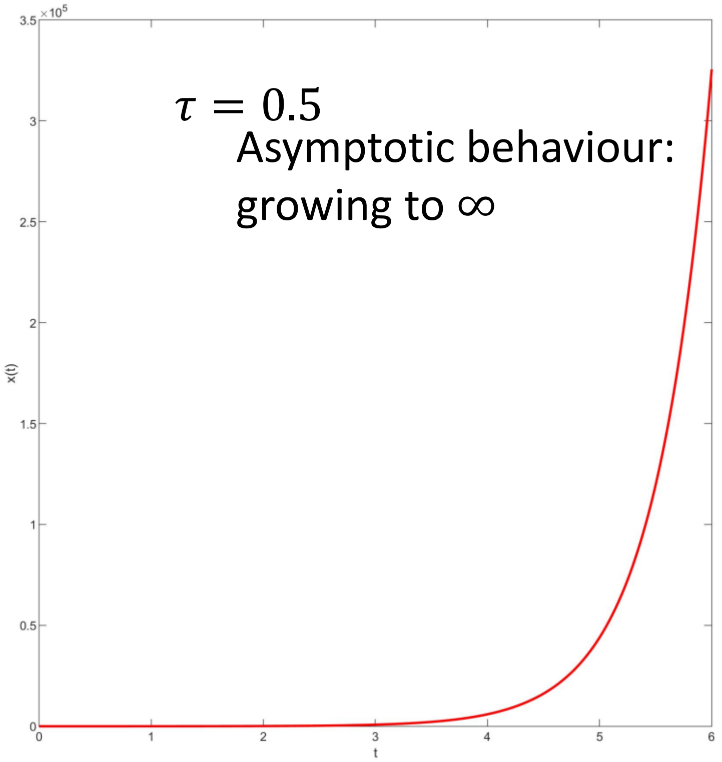
Let's try to solve 3 simple examples of the 1-D first-order linear ordinary differential equations

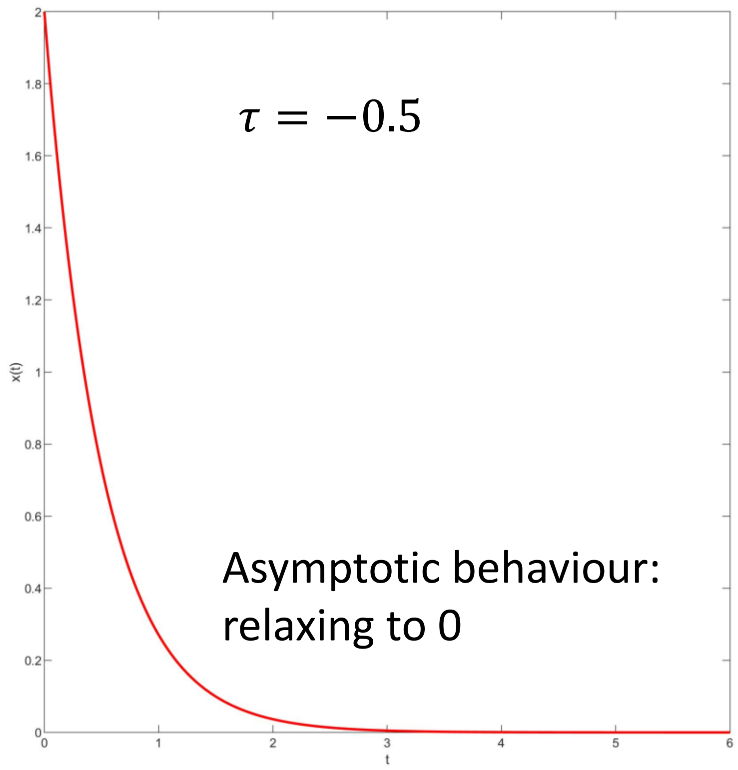
Example 1: $\tau \dfrac{dx}{dt} = x, \qquad given\ x(t = 0) = x_0$

Solution: $x(t) = x_0 e^{\frac{1}{\tau}t}$

If $\tau > 0$, $x(t)$ grows to infinity

If $\tau < 0$, $x(t)$ decays to zero and stay there





Example 2: $$\tau \frac{dx}{dt} = (x^* - x), \qquad given\ x(t=0) = x_0$$

$$\Rightarrow \quad \frac{dx}{dt} = -\frac{1}{\tau}x + \frac{1}{\tau}x^*, \qquad given\ x(t=0) = x_0$$

$$\Rightarrow \quad A(t) = \int_0^t a(s)ds = \int_0^t -\frac{1}{\tau}ds = -\frac{t}{\tau}$$

$$\Rightarrow \quad M(t) = \int_0^t e^{-A(s)}g(s)ds = \int_0^t e^{\frac{s}{\tau}}\frac{x^*}{\tau}ds = \frac{x^*}{\tau}\int_0^t e^{\frac{s}{\tau}}ds = x^*e^{\frac{t}{\tau}} - x^*$$

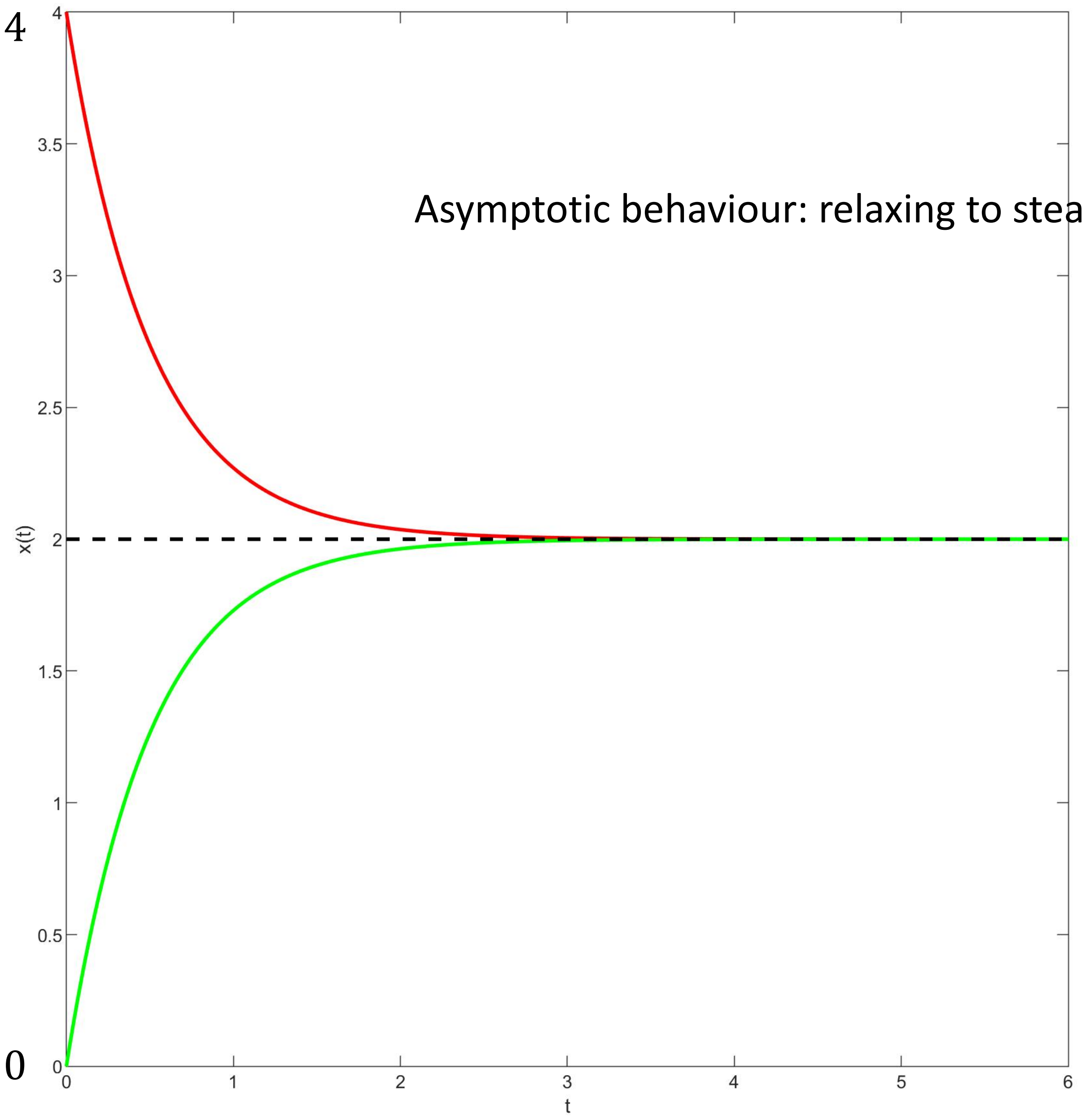
$$\Rightarrow \quad x(t) = x_0e^{-\frac{1}{\tau}t} + e^{-\frac{t}{\tau}}\left(x^*e^{\frac{t}{\tau}} - x^*\right)$$

Solution: $$x(t) = x^* + (x_0 - x^*)e^{-\frac{1}{\tau}t}$$

$x_0 = 4$

Asymptotic behaviour: relaxing to steady state x^*

$x^* = 2;\ \tau = 0.5;$



$x_0 = 0$

Example 3: $$\tau\frac{dx}{dt} = (x^* - x) + sin(\omega t)\ \ given\ x(t=0) = x_0$$

$$\Rightarrow \frac{dx}{dt} = -\frac{1}{\tau}x + \frac{1}{\tau}\left(x^* + sin(\omega t)\right)\ \ given\ x(t=0) = x_0$$

$$\Rightarrow A(t) = \int_0^t a(s)ds = -\frac{t}{\tau}$$

$$\Rightarrow M(t) = \int_0^t e^{-A(s)}g(s)ds = \int_0^t e^{\frac{s}{\tau}}\frac{1}{\tau}\left(x^* + sin(\omega s)\right)ds$$

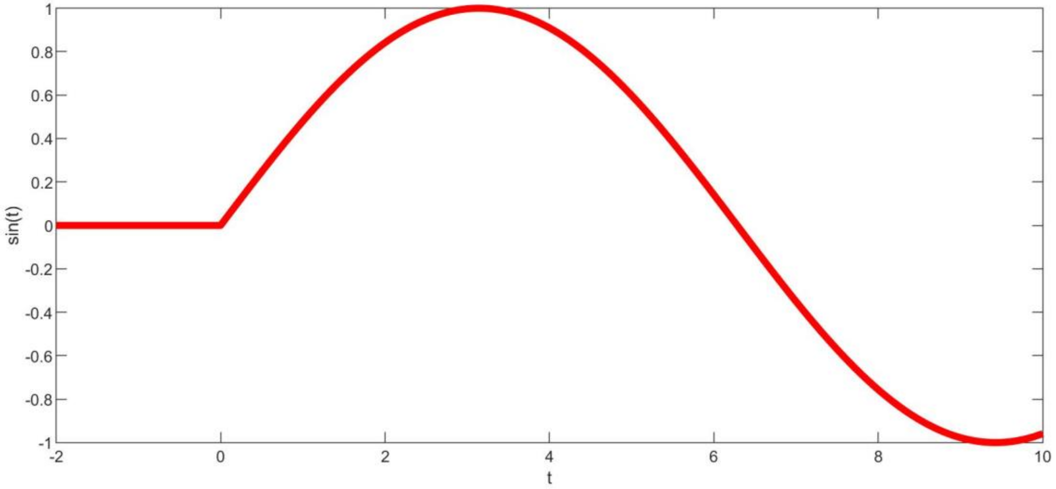
$$= \frac{x^*}{\tau}\int_0^t e^{\frac{s}{\tau}}ds + \int_0^t e^{\frac{s}{\tau}}\frac{1}{\tau}sin(\omega s)ds$$

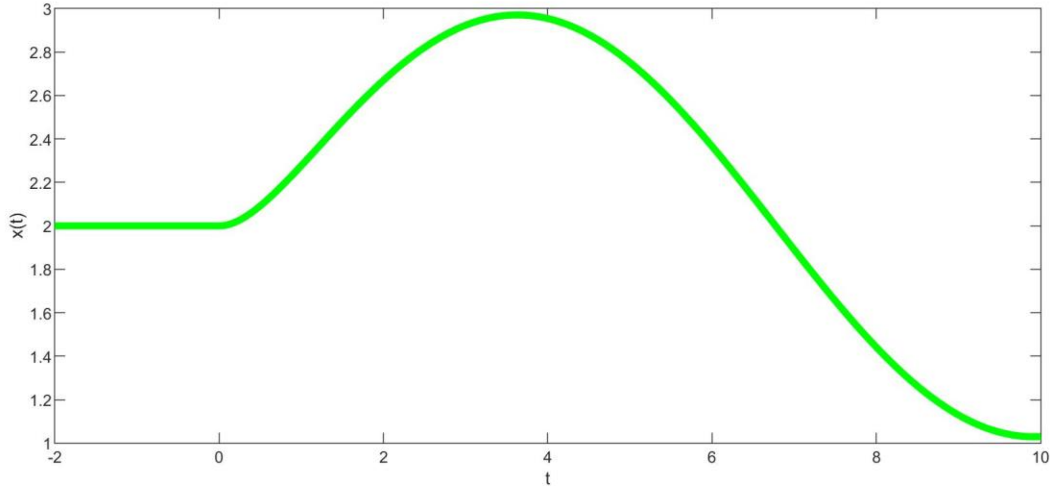
Solution: $$x(t) = x^* + (x_0 - x^*)e^{-\frac{1}{\tau}t} + \frac{\tau}{(1+\omega^2\tau^2)}\left(\left(\frac{1}{\tau}sin(\omega t) - \omega cos(\omega t)\right) + \omega e^{-\frac{t}{\tau}}\right)$$

If $\omega = 0$, $$x(t) = x^* + (x_0 - x^*)e^{-\frac{1}{\tau}t}$$

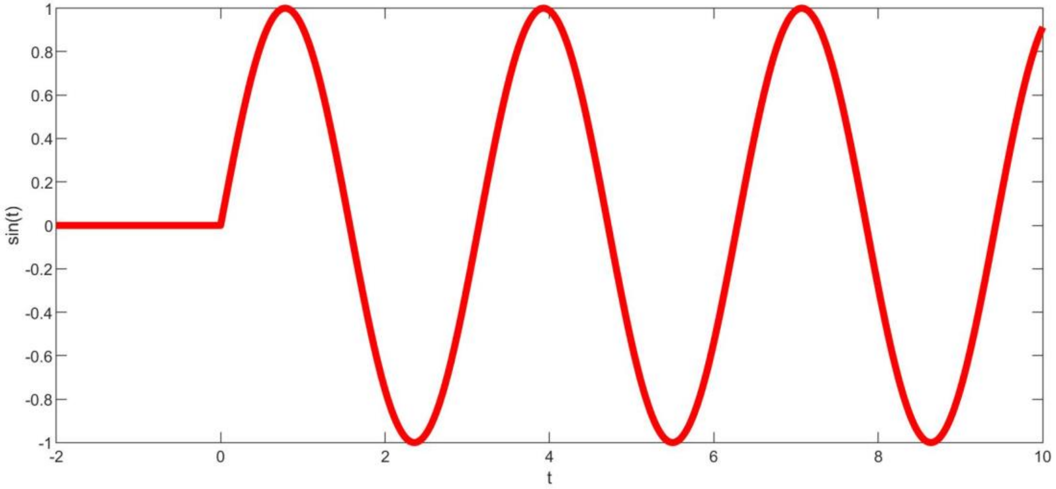
The phenomenon of LOW-PASS FILTERING in linear ODEs

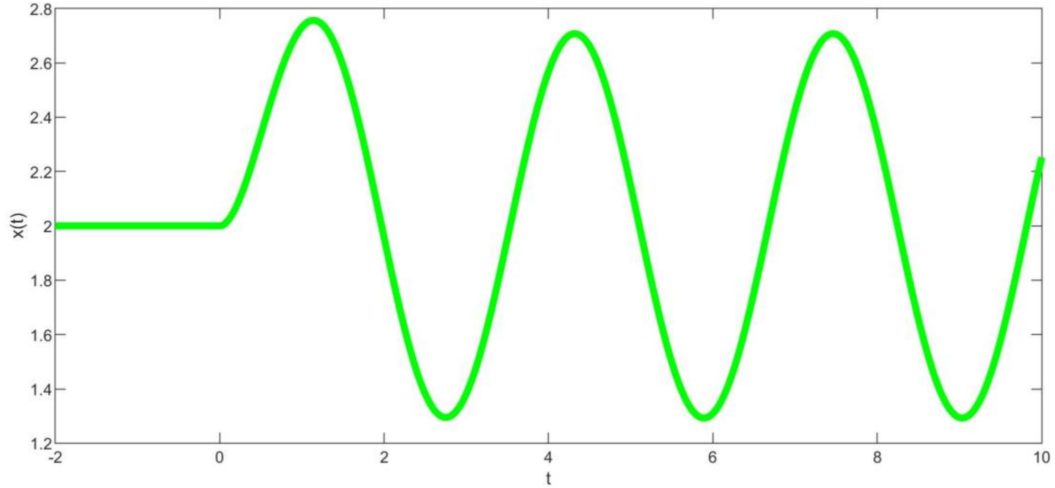
$x_0 = 2;\ x^* = 2;\ \tau = 0.5;\ \omega = 0.5$





$x_0 = 2;\ x^* = 2;\ \tau = 0.5;\ \omega = 2$



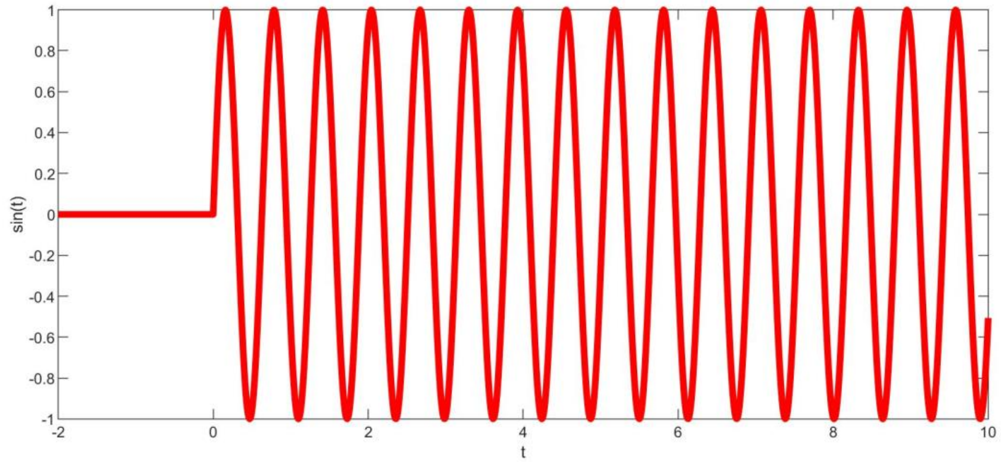


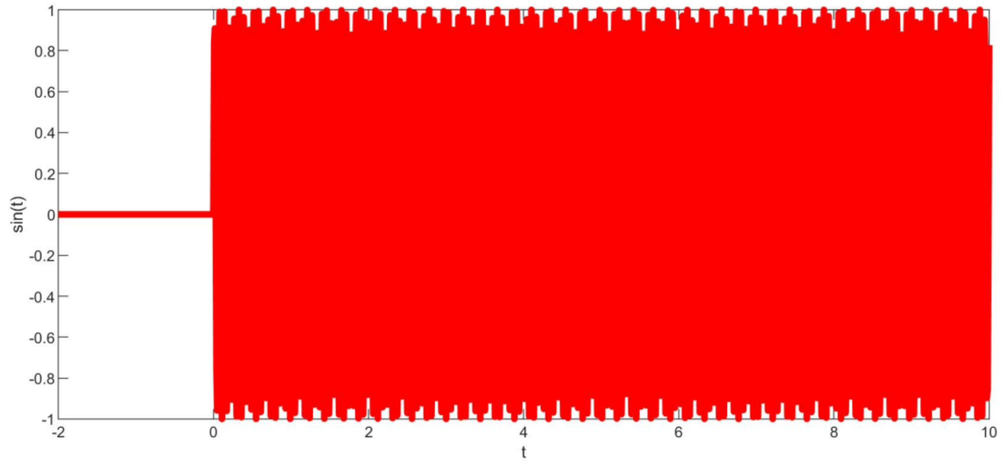
Do you observe the decreasing amplitude of $x(t)$ with increasing ω?

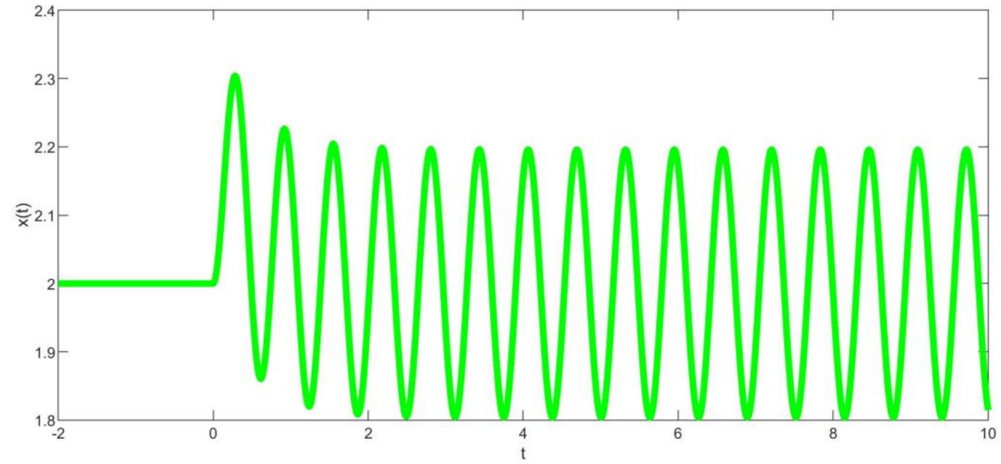
The phenomenon of LOW-PASS FILTERING in linear ODEs

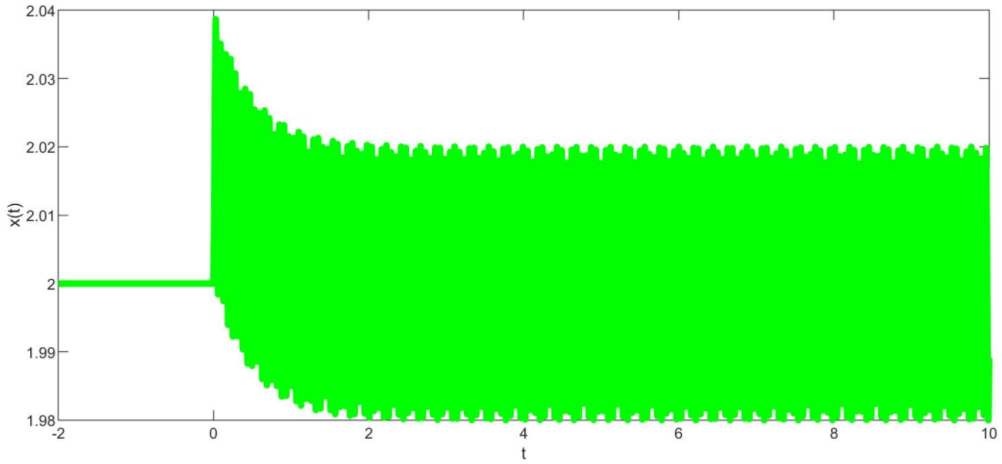
$x_0 = 2;\ x^* = 2;\ \tau = 0.5;\ \omega = 10$

$x_0 = 2;\ x^* = 2;\ \tau = 0.5;\ \omega = 100$









Do you observe the decreasing amplitude of $x(t)$ with increasing ω?

Intuition for ODE dynamics

- Linear differential equations do one of two things:

1. Converge to a steady-state value.
2. "Explode" to positive or negative ∞.

- Linear differential equations with a time varying input "try" to track the input signal.

- If the time constant(s) in the DEs are fast compared to the timescale of the input signal, the variables can track the input almost perfectly.

- In the converse case, if DE time constants are slow compared to the timescale of the input signal, then the variables will low-pass filter the input - they "average out" the high frequency components of the input.

Further Reading

Conor’s notes: https://github.com/coms30127/2019_20/notes/maths.pdf

Ordinary Differential Equations Revised Edition
by Morris Tenenbaum (Author), Harry Pollard (Author)