COMS30017
COMPUTATIONAL NEUROSCIENCE

LECTURE: INTRODUCTION TO DIFFERENTIAL EQUATIONS

Dr. Rahul Gupta
xv20319@bristol.ac.uk

Intended learning outcomes

- What are differential equations and how are they useful tools for modelling real- world systems?
- Types of ordinary differential equations (ODEs): Dimensionality, Order and Linearity.
- How to solve 1-D first-order linear ODEs?
- Get an intuition for the typical dynamics in linear ODEs: response to external inputs and steady-state behaviours.

What are DIFFERENTIAL EQUATIONS????

- Differential equations (DEs) are by far the most common formalism for modelling real-word systems.

- Ubiquitous in physics, chemistry, engineering, geoscience, biology.

- DYNAMICS of a SYSTEM obviously implies looking at the pattern or profile of changes in a variable or some variables of interest associated with the system over a course of time – DYNAMICAL SYSTEMS

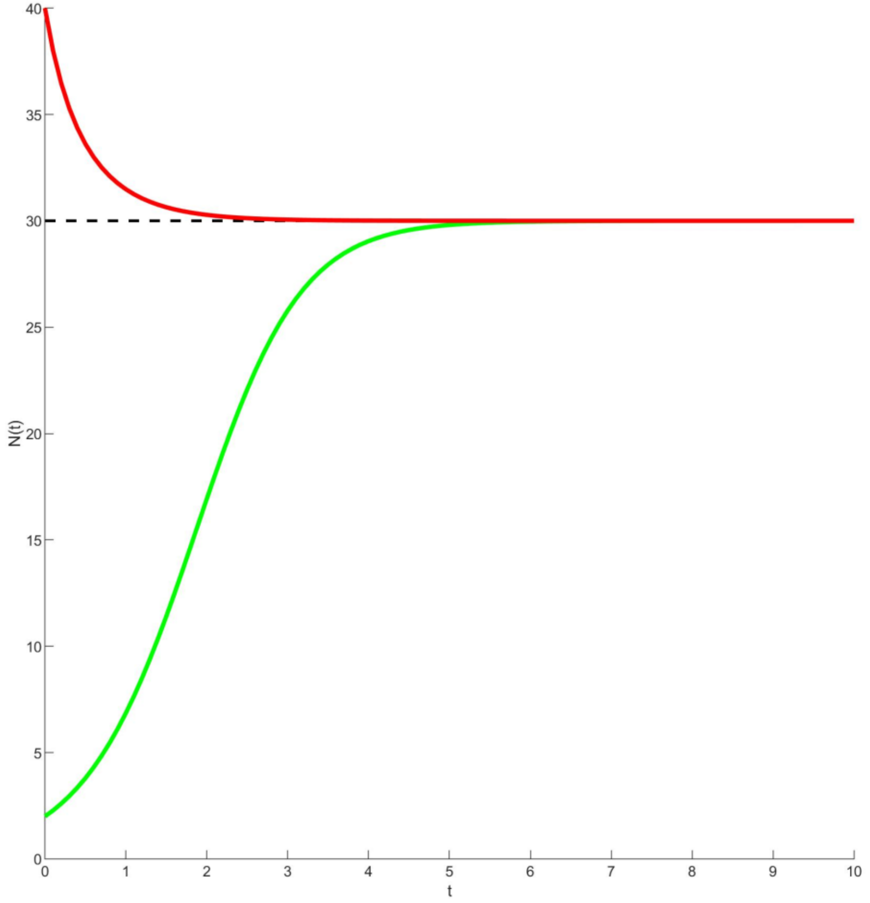
- FORCES or FACTORS which keep constantly acting on and impacting the variable(s) collectively constitute the source of continuous change in the variable.

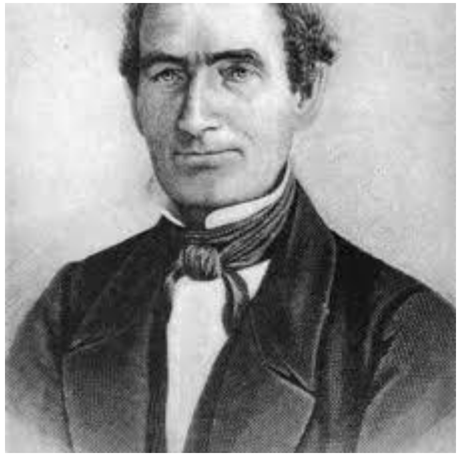
- DIFFERENTIAL EQUATIONS are the systematic means to represent these continuous and instantaneous changes in the variable in response to the forces or factors acting and driving those changes.

Verhulst Model of Population Dynamics (1845, 1847)

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$$

r=1.5; Maximum reproduction rate
K= 30; Carrying capacity of a local region
N(t)= The rabbit population in the region





Pierre Verhuslt

*wikipedia

The instantaneous change in the variable x at time, t

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) = f$$

The driving force, which is the source of change at every time t that the variable x experiences in itself.

Dimensionality of differential equation: The number of variables involved in the differential equations defines the dimensionality of the differential equation.

1-D Differential Equation

$$\frac{dx}{dt} = f(x,t)$$

2-D Differential Equation

$$\frac{dx_1}{dt} = f_1(x_1, x_2, t), \ \frac{dx_2}{dt} = f_2(x_1, x_2, t)$$

n-D Differential Equation

$$\frac{dx_1}{dt} = f_1(x_1, x_2, \dots, x_n t), \ \frac{dx_2}{dt} = f_2(x_1, x_2, \dots, x_n t), \dots$$

$$\frac{dx_n}{dt} = f_n(x_1, x_2, \dots, x_n t)$$

ORDINARY DIFFERENTIAL EQUATION

$$\frac{dx}{dt} \qquad \frac{d^2x}{dt^2} \qquad \frac{d^nx}{dt^n}$$

1st Order 2nd Order nth Order

PARTIAL DIFFERENTIAL EQUATION

$$\frac{\partial x}{\partial t} \qquad \frac{\partial^2 x}{\partial t^2}$$

Order of differential equation: The highest order derivative of the variable in the differential equation defines the order of the differential equation

Let's decompose f

f

Component that strictly depends on the value of the variable itself $f(x, t)$

An external feed, which doesn't care about what the value of the variable is. $g(t)$

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) = f \quad \Longrightarrow \quad \frac{\mathrm{d}x}{\mathrm{d}t}(t) = f(x, t) + g(t)$$

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) = \boxed{f(x,t)} + \boxed{g(t)}$$

LINEAR DIFFERENTIAL EQUATION

$ax(t), a(t)x(t)$

NON-LINEAR DIFFERENTIAL EQUATION

$a(t)x(t) + b(t)x(t)^n, n \neq 0$

$sin(x), e^x, tanh(x)$

If absent:

AUTONOMOUS OR HOMOGENOUS DIFFERENTIAL EQUATION

If present:

NON-AUTONOMOUS OR INHOMOGENEOUS DIFFERENTIAL EQUATION

Can be linear or non-linear

We are going to study only 1-D First-Order Linear Ordinary Differential Equations

The Canonical Form of

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) = a(t)x(t) + g(t)$$

Example 1:

$$4\frac{dx}{dt}(t) + 2x(t) = 0 \Rightarrow \frac{dx}{dt}(t) = -\frac{1}{2}x(t)$$

$$\tau\frac{dx}{dt}(t) + tx(t) = \cos(t) \Rightarrow \frac{dx}{dt}(t) = -\frac{t}{\tau}x(t) + \frac{1}{\tau}\cos(t)$$

$$\frac{1}{sin^2(t)}\frac{dx}{dt} + t^2x(t) = 1 + e^{-2t} \Rightarrow \frac{dx}{dt} = -sin^2(t)t^2x(t) + sin^2(t)(1 + e^{-2t})$$

What does solving the ODE mean?

- It is to find the explicit form of the unknown variable $x(t)$, which when differentiated with respect to t returns the given differential equation.

- In other words, $x(t)$ is the integral of the differential equation over a time interval $[t_0, t]$, where t_0 is the point of start. Generally, $t_0 = 0$

- Then you need somewhere to start, i.e. you need to be given or mentioned already what $x(t)$ is at t_0.

- This is why ODEs are also called **INITIAL VALUE PROBLEM**.

Analytical Solution of 1-D first-order linear ODEs

Given

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) = a(t)x(t) + g(t), \qquad x(0) = x_0$$

Step 1: Compute $A(t) = \int_0^t a(s)ds$

Step 2: Compute $M(t) = \int_0^t e^{-A(s)} g(s)ds$

Step 3: Compute $x(t) = x_0 e^{A(t)} + e^{A(t)} M(t)$ ⟸ Analytical Solution

Case 1: Homogeneous or Autonomous Differential Equations $g(t) = 0$

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) = a(t)x(t) + g(t), \qquad x(0) = x_0$$

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) = a(t)x(t), \qquad x(0) = x_0$$

Step 1: Compute $$A(t) = \int_0^t a(s)ds$$

Step 2: Compute $$M(t) = \int_0^t e^{-A(s)}.0.ds = 0$$

Step 3: Compute $$x(t) = x_0 e^{A(t)} + e^{A(t)} M(t) = x_0 e^{A(t)}$$

Case 1 Special: $a(t) = a$, Constant Coefficient

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) = ax(t), \qquad x(0) = x_0$$

$$A(t) = \int_0^t a(s)ds = \int_0^t ads = at$$

$$x(t) = x_0 e^{A(t)} = x_0 e^{at}$$