How to calculate derivates of complex valued functions and their partial component-wise derivates

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Calculating partial derivates $\frac{\partial x}{\partial \boldsymbol{z}}$ and $\frac{\partial y}{\partial \boldsymbol{z}}$ of complex valued function $f(\boldsymbol{z}) = x(\boldsymbol{z}) + i\, y(\boldsymbol{z})$ is non-trivial calculation because scaling factor is not 1 but $\frac{1}{2}$. I show two ways to calculate these derivates, first one uses Wirtinger calculus and other one Cauchy-Riemann equations. These results are well-cited in the literature but this paper works are reference/memore referesher.

Wirtinger calculus

You can solve for partial derivate of $\frac{\partial}{\partial \boldsymbol{z}}$ by solving the following linear equations. We derivate function using its component functions:

$$\frac{\partial}{\partial x} = \frac{\partial \boldsymbol{z}}{\partial x}\frac{\partial}{\partial \boldsymbol{z}} + \frac{\partial \bar{\boldsymbol{z}}}{\partial x}\frac{\partial}{\partial \bar{\boldsymbol{z}}} = \frac{\partial}{\partial \boldsymbol{z}} + \frac{\partial}{\partial \bar{\boldsymbol{z}}}$$

$$\frac{\partial}{\partial y} = \frac{\partial \boldsymbol{z}}{\partial y}\frac{\partial}{\partial \boldsymbol{z}} + \frac{\partial \bar{\boldsymbol{z}}}{\partial y}\frac{\partial}{\partial \bar{\boldsymbol{z}}} = i\frac{\partial}{\partial \boldsymbol{z}} - i\frac{\partial}{\partial \bar{\boldsymbol{z}}}$$

This gives linear equation which inverse we can solve and solve for $\frac{\partial}{\partial \boldsymbol{z}}$ (and $\frac{\partial}{\partial \bar{\boldsymbol{z}}}$),

$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} \frac{\partial}{\partial \boldsymbol{z}} \\ \frac{\partial}{\partial \bar{\boldsymbol{z}}} \end{pmatrix}$ and matrix inverse is

$\begin{pmatrix} \frac{\partial}{\partial \boldsymbol{z}} \\ \frac{\partial}{\partial \bar{\boldsymbol{z}}} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -i \\ 1 & +i \end{pmatrix}\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}$, which gives familiar Wirtinger formula $\frac{\partial}{\partial \boldsymbol{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$.

Cauchy-Riemann equations

Cauchy-Riemann equations are solved by assuming that derivates to complex numbered functions must be same if point of derivation is approached from any direction from the complex-plane. This means derivate must be also same if if approach the point from x-axis (real-value) and y-axis (imaginary-value).

Cauchy-Riemann equations $f(x, y) = u(x, y) + i\, v(x, y)$ are:

$u_x = v_y, v_x = -u_y$

If we plug these to our Wirtinger formula we get

$$\frac{\partial f}{\partial \boldsymbol{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) = \frac{1}{2}(u_x + i\, v_x - i\, u_y + v_y) = \frac{1}{2}(u_x + i\, v_x + i\, v_x + u_x) = u_x + i\, v_x = f'(\boldsymbol{z})$$

This is so because we get derivate along x-axis which must be correct if $f(\boldsymbol{z})$ is differentiable so derivation from any direction gives the correct result.

However, there these calculations are a bit formal and there is no intuitive solution which shows why we must scale derivates using $\frac{1}{2}$ and not just use straight-forward partial derivation calculation which gives wrong equation $\frac{\partial}{\partial \boldsymbol{z}} = \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$, which gives wrong scaling to the derivate.

The correct formula is $\frac{\partial f}{\partial \boldsymbol{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$.