Computing Solutions to the Kadomtsev–Petviashvili Equation

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- Bernard Deconinck
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Kadomtsev–Petviashvili Equation

$u(x, y, t)$ = surface height of a 2D shallow water wave,

$$\frac{3}{4} u_{yy} = \frac{\partial}{\partial x} \left(u_t - \frac{1}{4} \left(6 u u_x + u_{xxx} \right) \right) .$$

Only considering periodic solutions.

Figure: Île de Ré, France

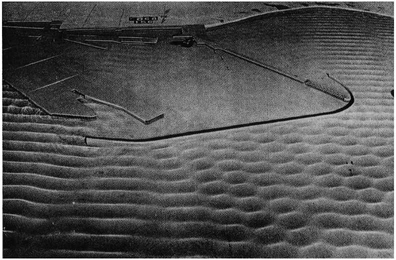

Figure: Model of San Diego Harbor

Finite Genus Solutions

$$u(x, y, t) = 2c + 2\partial_x^2 \log \theta\left(\boldsymbol{U}x + \boldsymbol{V}y + \boldsymbol{W}t + \boldsymbol{D}, \Omega\right)$$

Finite Genus Solutions

$$u(x, y, t) = 2c + 2\partial_x^2 \log \theta\left(\boldsymbol{U}x + \boldsymbol{V}y + \boldsymbol{W}t + \boldsymbol{D}, \Omega\right)$$

Riemann theta function,

$$\theta : \mathbb{C}^g \times \mathfrak{h}_g \to \mathbb{C}$$

where $\mathfrak{h}_g$ = space of *Riemann matrices*:

- $\Omega \in \mathbb{C}^{g\times g}$,
- $\Omega^T = \Omega$,
- $\text{Imag}(\Omega) > 0$.
- (g = "genus")

Finite Genus Solutions

$$u(x,y,t) = 2c + 2\partial_x^2 \log\theta\Big(\boldsymbol{U}x + \boldsymbol{V}y + \boldsymbol{W}t + \boldsymbol{D}, \Omega\Big)$$

$\boldsymbol{U}, \boldsymbol{V}, \boldsymbol{W}, \boldsymbol{D} \in \mathbb{C}^g$, $\Omega \in \mathfrak{h}_g$ derived from compact Riemann surface

- **For this talk**: *compact Riemann surface* = complex solutions to plane algebraic curve

$$C = \big\{(\lambda,\mu) \in \mathbb{C}^* \;\big|\; f(\lambda,\mu) = 0,\; f \in \mathbb{C}[\lambda,\mu]\big\}.$$

Finite Genus Solutions

$$u(x,y,t) = 2c + 2\partial_x^2 \log\theta\Big(\boldsymbol{U}x + \boldsymbol{V}y + \boldsymbol{W}t + \boldsymbol{D}, \Omega\Big)$$

Goal

Given a Riemann surface X and a *divisor* $\mathcal{D}$ on X compute the corresponding solution to the KP equation.

- *Note: solutions not necessarily real or bounded.*

Journey to the Initial Value Problem

$$
\begin{tikzcd}
u(x, y, 0) \arrow[rr, "\text{Deconinck}"] \arrow[d, dashed] & & X, \mathcal{D} \arrow[d, "\begin{array}{l}\text{A) Krichever, Deconinck, van Hoeij} \\ \text{B) Frauendiener, Klein}\end{array}"] \arrow[dll, dashed, "\text{Bobenko}"'] \\
u(x, y, t) & & \boldsymbol{U}, \boldsymbol{V}, \boldsymbol{W}, \boldsymbol{D}, \Omega \arrow[ll, "\text{Krichever}"] \arrow[ll, dashed, bend left, "\text{Dubrovin, Flickinger, Segur}"]
\end{tikzcd}
$$

Journey to the Initial Value Problem

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
u(x,y,0) \arrow[r, "\text{Deconinck}"] \arrow[d, dashed] & X, \mathcal{D} \arrow[d, "\begin{array}{l}\text{A) Krichever, Deconinck, van Hoeij}\\ \text{B) Frauendiener, Klein}\end{array}"] \arrow[dl, dashed, "\text{Bobenko}"'] \\
u(x,y,t) & \boldsymbol{U}, \boldsymbol{V}, \boldsymbol{W}, \boldsymbol{D}, \Omega \arrow[l, thick, "\text{Krichever}"] \arrow[l, dashed, bend left, "\text{Dubrovin, Flickinger, Segur}"]
\end{tikzcd}
$$

Krichever

The finite-genus formulae are indeed solutions to KP. (Where parameters are obtained from a Riemann surface.)

Journey to the Initial Value Problem

$$
\begin{tikzcd}
u(x,y,0) \arrow[r, "\text{Deconinck}"] \arrow[d, dashed] & X, \mathcal{D} \arrow[dl, dashed, "\text{Bobenko}"'] \arrow[d, thick, "{\begin{array}{l}\textbf{A) Krichever, Deconinck, van Hoeij} \\ \text{B) Frauendiener, Klein}\end{array}}"] \\
u(x,y,t) & \boldsymbol{U}, \boldsymbol{V}, \boldsymbol{W}, \boldsymbol{D}, \Omega \arrow[l, "\text{Krichever}"'] \arrow[l, dashed, bend left, "\text{Dubrovin, Flickinger, Segur}"]
\end{tikzcd}
$$

Krichever, Deconinck, van Hoeij

Krichever's inverse procedure + Deconinck / van Hoeij's computational framework used to obtain necessary parameters.

Journey to the Initial Value Problem

$$\begin{tikzcd}[column sep=huge, row sep=huge]
u(x,y,0) \arrow[r, "\text{Deconinck}"] \arrow[d, dashed] & X, \mathcal{D} \arrow[dl, dashed, "\text{Bobenko}"'] \arrow[d, Rightarrow, "\substack{\text{A) Krichever, Deconinck, van Hoeij} \\ \textbf{B) Frauendiener, Klein}}"] \\
u(x,y,t) & \boldsymbol{U}, \boldsymbol{V}, \boldsymbol{W}, \boldsymbol{D}, \Omega \arrow[l, "\text{Krichever}"] \arrow[l, dashed, bend left, "\text{Dubrovin, Flickinger, Segur}"]
\end{tikzcd}$$

Frauendiener, Klein

Alt. approach: use "Real Riemann surfaces".

Journey to the Initial Value Problem

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
u(x,y,0) \arrow[r, "\text{Deconinck}"] \arrow[d, dashed] & X, \mathcal{D} \arrow[dl, dashed, "\text{Bobenko}"'] \arrow[d, "\substack{\text{A) Krichever, Deconinck, van Hoeij} \\ \text{B) Frauendiener, Klein}}"] \\
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\end{tikzcd}
$$

Deconinck

Begin with initial condition $u(x, y, 0)$ of the form,

$$u(x, y, 0) = 2c + 2\partial_x^2 \log \theta\Big(\boldsymbol{u}x + \boldsymbol{v}y + \boldsymbol{d}, \Omega\Big).$$

Journey to the Initial Value Problem

$$
\begin{tikzcd}
u(x,y,0) \arrow[r, "\text{Deconinck}"] \arrow[d, dashed] & X, \mathcal{D} \arrow[dl, dashed, "\text{Bobenko}"'] \arrow[d, "\substack{\text{A) Krichever, Deconinck, van Hoeij} \\ \text{B) Frauendiener, Klein}}"] \\
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\end{tikzcd}
$$

Dubrovin, Flickinger, Segur

Alt. approach: algebraic conditions on parameters.

- *Only works in $g = 2, 3$ case.*

Journey to the Initial Value Problem

$$\begin{tikzcd}[column sep=huge, row sep=huge]
u(x,y,0) \arrow[r, "\text{Deconinck}"] \arrow[d, dashed] & X, \mathcal{D} \arrow[d, "\begin{array}{l}\text{A) Krichever, Deconinck, van Hoeij} \\ \text{B) Frauendiener, Klein}\end{array}"] \arrow[dl, dashed, very thick, "\text{Bobenko}"] \\
u(x,y,t) & \boldsymbol{U}, \boldsymbol{V}, \boldsymbol{W}, \boldsymbol{D}, \Omega \arrow[l, "\text{Krichever}"] \arrow[l, bend left, dashed, "\text{Dubrovin, Flickinger, Segur}"]
\end{tikzcd}$$

Bobenko

Alt. approach: use "Schottky Uniformization" techniques.

Journey to the Initial Value Problem

$$\begin{tikzcd}
u(x,y,0) \arrow[r, "\text{Deconinck}"] \arrow[d, dashed] & X, \mathcal{D} \arrow[d, "\begin{array}{l}\text{A) Krichever, Deconinck, van Hoeij} \\ \text{B) Frauendiener, Klein}\end{array}"] \arrow[dl, dashed, "\text{Bobenko}"'] \\
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\end{tikzcd}$$

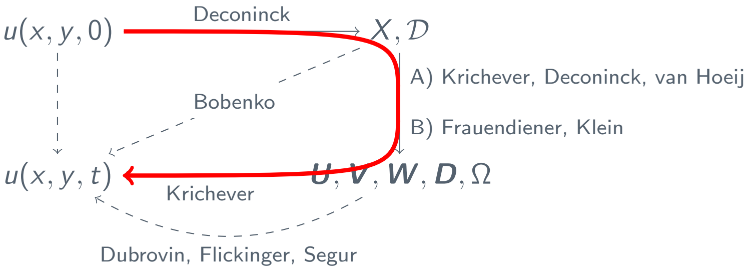
Goal

Realizing the map

$$X, \mathcal{D} \quad \longrightarrow \quad u(x,y,t)$$

in Sage using the software package “Abelfunctions”.

Journey to the Initial Value Problem



Future Goal

Solving the initial value problem

$$u(x, y, 0) \quad \longrightarrow \quad u(x, y, t)$$

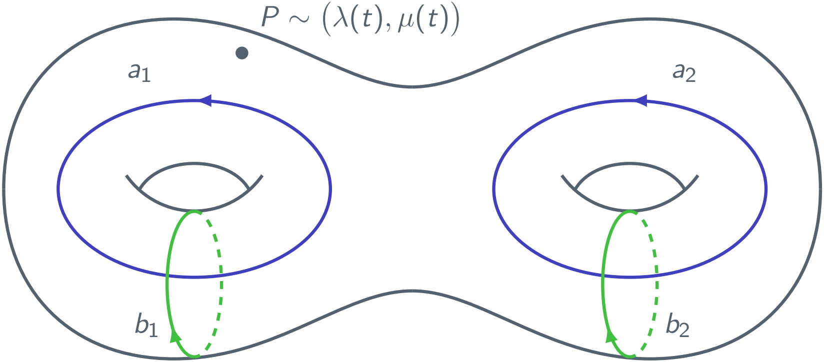
in Sage using the software package “Abelfunctions”.

abelfunctions

A Sage library for computing with Abelian functions, Riemann surfaces, and complex algebraic curves.

https://github.com/cswiercz/abelfunctions
https://www.cswiercz.info/abelfunctions

Riemann Surfaces in One Picture



$$\omega_1 = h_1(\lambda, \mu)\, \mathrm{d}\lambda \quad \cdots \quad \omega_g = h_g(\lambda, \mu)\, \mathrm{d}\lambda$$

Demonstration

$$X, \mathcal{D} \quad \longrightarrow \quad u(x, y, t)$$

Steps:

Demonstration

$$X, \mathcal{D} \quad \longrightarrow \quad u(x, y, t)$$

Steps:

- Given $f \in \mathbb{C}[x, y]$ compute X,

Demonstration

$$X, \mathcal{D} \quad \longrightarrow \quad u(x, y, t)$$

Steps:

- Given $f \in \mathbb{C}[x, y]$ compute X,
- compute period matrix Ω,

Demonstration

$$X, \mathcal{D} \quad \longrightarrow \quad u(x, y, t)$$

Steps:

- Given $f \in \mathbb{C}[x, y]$ compute X,
- compute period matrix Ω,
- compute frequencies $\boldsymbol{U}, \boldsymbol{V}, \boldsymbol{W}$;

Demonstration

$$X, \mathcal{D} \quad \longrightarrow \quad u(x, y, t)$$

Steps:

- Given $f \in \mathbb{C}[x, y]$ compute X,
- compute period matrix Ω,
- compute frequencies $\boldsymbol{U}, \boldsymbol{V}, \boldsymbol{W}$;
- given some $\mathcal{D} \in \mathrm{Div}(X)$ compute $\boldsymbol{D} = A(P_\infty, \mathcal{D}) - K(P_\infty)$,

Demonstration

$$X, \mathcal{D} \quad \longrightarrow \quad u(x, y, t)$$

Steps:

- Given $f \in \mathbb{C}[x, y]$ compute X,
- compute period matrix Ω,
- compute frequencies $\boldsymbol{U}, \boldsymbol{V}, \boldsymbol{W}$;
- given some $\mathcal{D} \in \mathrm{Div}(X)$ compute $\boldsymbol{D} = A(P_\infty, \mathcal{D}) - K(P_\infty)$,
- evaluate finite genus solution formula.

Demo

Thank You

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