Computing the Riemann Constant Vector

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Riemann Constant Vector

$$K_j(P) = \frac{1+\Omega_{jj}}{2} - \sum_{k\neq j}^{g} \oint_{a_k} \omega_k(Q) A_j(P,Q)\,\mathrm{d}Q$$

- Double integral: difficult to compute.

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Theorem

Let $P_0, P \in X$. Then

$$\boldsymbol{K}(P) = \boldsymbol{K}(P_0) + (g-1)\, \boldsymbol{A}(P_0, P).$$

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- Idea: most work to compute $\boldsymbol{K}(P_0)$ <u>once</u>.

Computing the RCV

Algorithm to compute $\boldsymbol{K}(P_0)$ inspired by two theorems:

Theorem

Let $\mathcal{C}$ be a divisor of degree $2g-2$. Then $\mathcal{C}$ is a canonical divisor if and only if

$$2\,\boldsymbol{K}(P) \equiv -\,\boldsymbol{A}(P, \mathcal{C}).$$

Computing the RCV

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Theorem

A vector $\boldsymbol{W} \in J(X)$ satisfies

$$\theta(\boldsymbol{W}, \Omega) = 0,$$

if and only if $\exists\, \mathcal{D} = P_1 + \cdots + P_{g-1}$ such that

$$\boldsymbol{W} = \boldsymbol{A}(P, \mathcal{D}) + \boldsymbol{K}(P).$$

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3. validate using

$$\theta\Big(\boldsymbol{A}(P_0, \mathcal{D}) + \boldsymbol{K}(P_0), \Omega\Big) = 0,$$

1. Computing a Canonical Divisor

Meromorphic differential:

$$\eta = \frac{p(x, y)\,\mathrm{d}x}{q(x, y)}$$

Valuation divisor:

$$(\eta)_{\mathsf{val}} = \sum_i p_i P_i - \sum_j q_j Q_j$$

where $P_i, Q_i \in X$ are poles and zeros, resp.

1. Computing a Canonical Divisor

Given $P \in X$,

$$P = \big(x_P(t), y_P(t)\big),$$

a <u>necessary</u> condition for $P \in (\eta)_{\mathsf{val}}$ is

$$p\big(x_P(t), y_P(t)\big)\Big|_{t=0} = 0, \qquad q\big(x_P(t), y_P(t)\big)\Big|_{t=0} = 0,$$
$$\text{or} \quad \frac{\mathrm{d}x_P}{\mathrm{d}t}(0) = x_P'(t)dt\Big|_{t=0} = 0.$$

Demo (Brief)

Localizing Differentials at Places

1. Computing a Canonical Divisor

Use the Abelian differentials of the 1st kind:

$$\omega_i = \frac{p_i(x, y)\,\mathrm{d}x}{\partial_y f(x, y)}$$