Computing the Riemann Constant Vector

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Kadomtsev–Petviashvili Equation

$u(x, y, t)$ = surface height of a 2D periodic shallow water wave.

$$\frac{3}{4} u_{yy} = \frac{\partial}{\partial x} \left(u_t - \frac{1}{4} \left(6 u u_x + u_{xxx} \right) \right)$$

Figure: Île de Ré, France

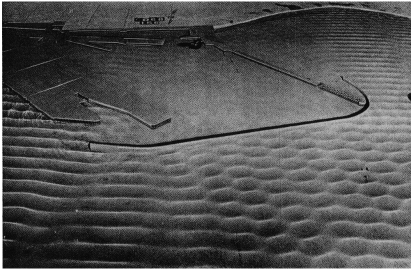

Figure: Model of San Diego Bay

Kadomtsev–Petviashvili Equation

"Finite-genus solutions:"

$$u = c + 2\partial_x^2 \log \theta\Big(\boldsymbol{U} x + \boldsymbol{V} y + \boldsymbol{W} t + \boldsymbol{A}\left(P^\infty, \mathcal{D}\right) - \boldsymbol{K}(P^\infty), \Omega \Big)$$

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- Riemann theta function $\theta : \mathbb{C}^g \times \mathfrak{h}_g \to \mathbb{C}$

$$\theta(z, \Omega) = \sum_{n \in \mathbb{Z}^g} e^{2\pi i\left(\frac{1}{2} n \cdot \Omega n + n \cdot z\right)}$$

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- Quantities come from compact, connected, Riemann surfaces built from algebraic curves.

Whirlwind Background - Curves

Given $f \in \mathbb{C}[x, y]$ construct an *algebraic curve*,

$$C = \{(\alpha, \beta) \in \mathbb{C}^{*2} \mid f(\alpha, \beta) = 0\}.$$

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Question

As $x \in \mathbb{C}^*_x$ varies on what surface is y single-valued?

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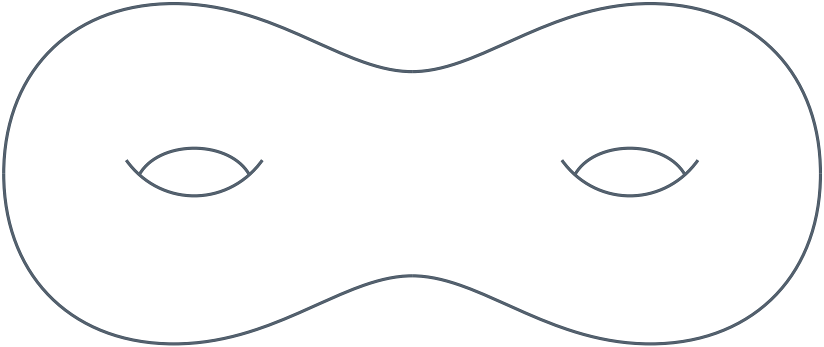
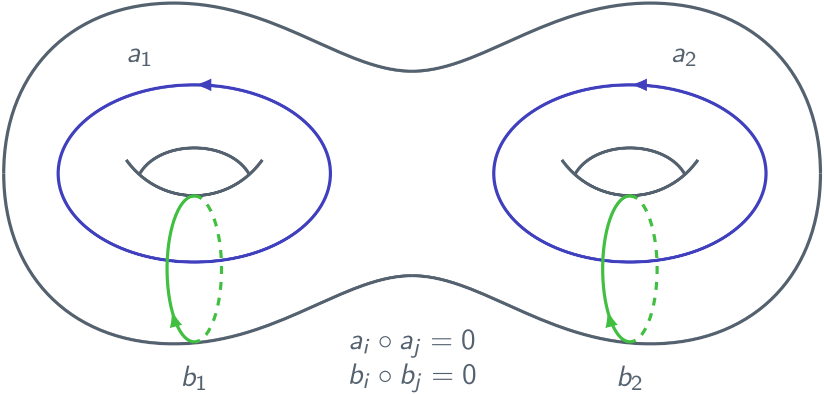
  - (optional board demo for undergraduates)

Whirlwind Background - Homology





$$a_i \circ a_j = 0$$
$$b_i \circ b_j = 0$$
$$a_i \circ b_j = \delta_{ij}$$
$$H_1(X, \mathbb{Z}) = \mathrm{span}\{a_1, \ldots, a_g, b_1, \ldots, b_g\}$$

Demo

Homology Basis

Question

What do we usually do with paths?

Whirlwind Background - One-Forms

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On C they look like

$$\nu = \nu(x, y)\,\mathrm{d}x$$

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Consider only $\omega \in \Omega^1_X$ *holomorphic* on X

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"Abelian differentials of the first kind"

$$\Gamma(X, \Omega^1_X)$$

Theorem

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- algorithm returns a non-normalized basis

$$\{\tilde{\omega}_1, \ldots, \tilde{\omega}_g\}$$

Demo

Abelian differentials of the first kind

Whirlwind Background - Period Matrix

Culmination of the theory: construct matrices

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- symmetric

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Fact: $\Omega = A^{-1}B$

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$$\tau = [I \ \ \Omega] \in \mathbb{C}^{g \times 2g}$$

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Jacobian of the Riemann surface

$$J(X) = \mathbb{C}^{g} / \Lambda$$

where

$$\Lambda = \mathbb{Z}^{g} \times \Omega \, \mathbb{Z}^{g}$$

Demo

Period matrix

A *place* $P \in X$ can be represented locally by a "Puiseux series"

$$P = \begin{cases} x_P(t) = \alpha + \lambda t^e, \\ y_P(t) = \sum_{k=0}^{\infty} \beta_k t^{n_k}, \end{cases}$$

Whirlwind Background - Places and Divisors

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$$P\Big|_{t=0} = (\alpha, \beta) \in C$$

- there can be distinct places with same projection on C.

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Valuation Divisors

Given $\nu \in \Omega^1_X$,

$$(\nu)_{\mathrm{val}} = \sum_{i=1}^{m} p_i P_i - \sum_{j=1}^{n} q_j Q_j$$

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$\mathcal{C} \in \mathrm{Div}(X)$ is *canonical* if $\mathcal{C}$ is a valuation divisor

Demo

Places and divisors

The Abel Map

Let $P \in X$ be a fixed place. The Abel Map

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Abel Map on Divisors

If $\mathcal{D} = \sum_i n_i P_i$ then

$$\boldsymbol{A}(P, \mathcal{D}) = \sum_i n_i \boldsymbol{A}(P, P_i)$$

Demo

The Abel Map

Riemann Constant Vector

The Riemann constant vector

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$$\boldsymbol{K} : X \to J(X)$$

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$$\boldsymbol{K}(P) = \big(K_1(P), \ldots, K_g(P)\big),$$

where

$$K_j(P) = \frac{1 + \Omega_{jj}}{2} - \sum_{k \neq j}^{g} \oint_{a_k} \omega_k(Q) A_j(P, Q) \, \mathrm{d}Q.$$

Riemann Constant Vector

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Theorem

Let $P_0, P \in X$. Then

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- Idea: most work from computing $\boldsymbol{K}(P_0)$ <u>once</u>.

Computing the RCV

Algorithm to compute $\boldsymbol{K}(P_0)$ inspired by two theorems:

Computing the RCV

Algorithm to compute $\boldsymbol{K}(P_0)$ inspired by two theorems:

Theorem

Let $\mathcal{C}$ be a divisor of degree $2g-2$. Then $\mathcal{C}$ is a canonical divisor if and only if

$$2\,\boldsymbol{K}(P_0) \equiv -\boldsymbol{A}(P_0, \mathcal{C}).$$

Computing the RCV

Algorithm to compute $\boldsymbol{K}(P_0)$ inspired by two theorems:

Theorem

A vector $\boldsymbol{W} \in J(X)$ satisfies

$$\theta(\boldsymbol{W}, \Omega) = 0,$$

if and only if $\exists\, \mathcal{D} = P_1 + \cdots + P_{g-1}$ such that

$$\boldsymbol{W} = \boldsymbol{A}(P_0, \mathcal{D}) + \boldsymbol{K}(P_0).$$

Combining the theorems:

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2. solve the equation

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1. Computing a Canonical Divisor

Meromorphic differential:

$$\nu = \frac{p(x, y)\,\mathrm{d}x}{q(x, y)}$$

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1. Computing a Canonical Divisor

Given $P \in X$,

$$P = \big(x_P(t), y_P(t)\big),$$

a necessary condition for $P \in (\nu)_{\mathrm{val}}$ is

$$\begin{aligned} p\big(x_P(t), y_P(t)\big)\Big|_{t=0} &= 0, \\ q\big(x_P(t), y_P(t)\big)\Big|_{t=0} &= 0, \quad \textit{or} \\ \frac{\mathrm{d}x_P}{\mathrm{d}t}(0) = x_P'(t)dt\Big|_{t=0} &= 0. \end{aligned}$$

Demo

Localizing Differentials at Places

Goal: find $\mathcal{P}$ containg the places in $(\nu)_{\mathsf{val}}$.

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Use "resultant sets":

- $R(f, p)(x) =$ resultant of $f(x, y)$ and $p(x, y)$ w.r.t. y,

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  have simultaneous solutions,
- **The point:** $P = (x_P(t), y_P(t))$ is a zero of $p(x,y)$ only if $x_P(0)$ is a root of $R(f,p)$,

1. Computing a Canonical Divisor

Given

$$\nu = \frac{p(x, y)\,\mathrm{d}x}{q(x, y)}$$

Define

$$\mathcal{X}_\nu^{(1)} = \{\alpha \in \mathbb{C} \mid R(f, p)(\alpha) = 0\}$$

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$$\mathcal{P} = \{P \in X \mid x_P(0) \in \mathcal{X}_\nu\}$$

Optimization

Use the Abelian differentials of the 1st kind:

$$\tilde{\omega}_i = \frac{p_i(x,y)\,\mathrm{d}x}{\partial_y f(x,y)}$$

- already computed the $\tilde{\omega}_i$'s,

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- no poles $\rightarrow$ terminate when deg reaches $2g - 2$,

Demo

Computing a Canonical Divisor

2. Finding the Half-Lattice Vector

Solve for $\boldsymbol{K}(P_0)$

$$2\,\boldsymbol{K}(P_0) \equiv -\,\boldsymbol{A}(P_0, \mathcal{C})$$

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Embed into $\mathbb{C}^g$

$$2\,\boldsymbol{K}(P_0) + \boldsymbol{A}(P_0, \mathcal{C}) = \boldsymbol{\lambda}$$

where

$$\boldsymbol{\lambda} \equiv \mathbf{0} \bmod \Lambda$$

Embed in $\mathbb{C}^g$

$\boldsymbol{\lambda}$ is one of the 2^{2g} lattice vectors in the fundamental region of Λ.

2. Finding the Half-Lattice Vector

Division by two is legal in $\mathbb{C}^g$

$$\boldsymbol{K}(P_0) = \boldsymbol{h} - \tfrac{1}{2}\,\boldsymbol{A}(P_0, \mathcal{C})$$

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Goal

Find an appropriate half-lattice vector $\boldsymbol{h}$.

Theorem

For any effective, degree $g-1$ divisor $\mathcal{D} = P_1 + \cdots + P_{g-1}$,

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\end{aligned}$$

The Point

It is necessary that

$$\theta\big(\boldsymbol{h}_j - \tfrac{1}{2}\,\boldsymbol{A}(P_0, \mathcal{C}), \Omega\big) = 0$$

for *at least one* of the 2^{2g} half lattice vectors $\boldsymbol{h}_j$.

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Basic Strategy

Evaluate above espression with each $\boldsymbol{h}_j$ to find a zero.

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for *at least one* of the 2^{2g} half lattice vectors $\boldsymbol{h}_j$.

Basic Strategy

Evaluate above espression with each $\boldsymbol{h}_j$ to find a zero.

Conjecture

This only works for *exactly one* $\boldsymbol{h}_j$.

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Numerical considerations:

- θ computation accurate to order ϵ (at best machine precision)

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- continue filtering by shifting by $\boldsymbol{A}(P_0,\mathcal{D})$

$$\left|\theta\left(\boldsymbol{h}_j - \tfrac{1}{2}\,\boldsymbol{A}(P_0,\mathcal{C}) + \boldsymbol{A}(P_0,\mathcal{D}),\Omega\right)\right| > \epsilon$$

(for any degree $g-1$ effective divisor $\mathcal{D}$)

Demo

Finding a half-lattice vector. Verifying results.

Solutions to KP:

$$u = c + 2\partial_x^2 \log\theta\Big(\boldsymbol{U}x + \boldsymbol{V}y + \boldsymbol{W}t + \boldsymbol{A}\left(P^\infty, \mathcal{D}\right) - \boldsymbol{K}(P^\infty), \Omega\Big),$$

Concluding Remarks and Next Steps

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Concluding Remarks and Next Steps

Solutions to KP:

$$u = c + 2\partial_x^2 \log \theta\left(\underbrace{\boldsymbol{U}x + \boldsymbol{V}y + \boldsymbol{W}t}_{\text{remaining work}} + \boldsymbol{A}\left(P^\infty, \mathcal{D}\right) - \underbrace{\boldsymbol{K}(P^\infty)}_{\text{now have this}}, \Omega\right),$$

Thank You!

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