Probabilistic Graphical Models

Homework 1

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Task 1: Urns

There are two urns containing colored balls. The first urn contains 50 red balls and 50 blue balls. The second urn contains 30 red balls and 70 blue balls. One of the two urns is randomly chosen (both urns have $1/2$ probability of being chosen) and then a ball is drawn at random from one of the two urns. If a red ball is drawn, what is the probability that it comes from the first urn?

Solution:

Let $U \in \{1, 2\}$ and $C \in \{r, b\}$ be the random variables for the urn and ball color, respectively. Then the probability in question is $P(U = 1 \,|\, C = r)$. To compute this probability, the probabilities $P(C = r \,|\, U = 1) = 1/2$, $P(U = 1) = 1/2$, and

$$\begin{aligned} P(C = R) &= P(C = r, U = 1) + P(C = r, U = 2) \\ &= P(C = r \,|\, U = 1)\, P(U = 1) + P(C = r \,|\, U = 2)\, P(U = 2) \\ &= \frac{1}{2} \cdot \frac{1}{2} + \frac{3}{10} \cdot \frac{1}{2} = \frac{2}{5} \end{aligned}$$

are needed. Using Bayes rule the probability can be computed straightforwardly:

$$P(U = 1 \,|\, C = r) = \frac{P(C = r \,|\, U = 1)\, P(U = 1)}{P(C = r)} = \frac{1/2 \cdot 1/2}{2/5} = \frac{1}{10}.$$

Thus, the probability that the first urn was chosen given that a red ball was drawn is 10 %.

Task 2: Recession Prediction

An economics consulting firm has created a model to predict recessions. The model predicts a recession with probability 80 % when a recession is indeed coming and with probability 10 % when no recession is coming. The unconditional probability of falling into a recession is 20 %. If the model predicts a recession, what is the probability that a recession will indeed come?

Solution:

This can also be computed using Bayes rule:

$$P(\text{Recession} \,|\, \text{Recession Predicted}) = \frac{P(\text{Recession Predicted} \,|\, \text{Recession})\, P(\text{Recession})}{P(\text{Recession Predicted})} = \frac{4/5 \cdot 1/5}{6/25} = \frac{2}{3}$$

The probability whether the model predicts a recession was again computed by marginalization,

$$\begin{aligned}P(\text{Recession Predicted}) &= P(\text{Recession Predicted}, \text{Recession}) + P(\text{Recession Predicted}, \neg\text{Recession}) \\ &= P(\text{Recession Predicted} \mid \text{Recession})\, P(\text{Recession}) \\ &\qquad + P(\text{Recession Predicted} \mid \neg\text{Recession})\, P(\neg\text{Recession}) \\ &= \frac{4}{5} \cdot \frac{1}{5} + \frac{1}{10}\frac{4}{5} = \frac{6}{25}.\end{aligned}$$

Hence, the probability for a recession given that the model predicts a recession is $2/3 \approx 67\,\%$.

Task 3: Sufficient Probabilities for Calculating A Conditional

Suppose we wish to calculate $P(X \mid Y, Z)$ and we have no conditional independence information. Which of the following sets of numbers are sufficient for calculation?

1. $P(Y, Z)$, $P(X)$, $P(Y \mid X)$, $P(Z \mid X)$
2. $P(Y, Z)$, $P(X)$, $P(Y, Z \mid X)$
3. $P(Y \mid X)$, $P(Z \mid X)$, $P(X)$

Solution:

By the relation $P(X \mid Y, Z) = P(X, Y, Z)/P(Y, Z) = P(Y, Z \mid X)\, P(X)/P(Y, Z)$, the second set of numbers are sufficient for calculating $P(X \mid Y, Z)$. This is again an application of Bayes rule but for two dependent random variables.

Task 4: Independence Properties

Prove or disprove (by providing a counterexample) each of the following properties of independence:

1. $X \perp (Y, W) \mid Z$ implies $X \perp Y \mid Z$
2. $X \perp Y \mid Z$ and $X \perp W \mid (Y, Z)$ implies $X \perp (W, Y) \mid Z$
3. $X \perp (Y, W) \mid Z$ and $Y \perp W \mid Z$ implies $(X, W) \perp Y \mid Z$

Solution:

Proof of First Statement: Assume $X \perp (Y, W) \mid Z$. Then $P(X, Y, W \mid Z) = P(X \mid Z)\, P(Y, W \mid Z)$ holds by definition. Integrating over W yields $P(X, Y \mid Z) = P(X \mid Z)\, P(Y \mid Z)$. Hence, $X \mid Z$ and $Y \mid Z$ are conditionally independent. $\square$

Proof of Second Statement: Using the chain rule of probability, the joint $P(X, W, Y \mid Z)$ can be decomposed as

$$P(X, W, Y \mid Z) \overset{\text{Chain Rule}}{=} P(X \mid Y, W, Z)\, P(Y, W \mid Z) \overset{\text{Premise 2}}{=} P(X \mid Y, Z)\, P(Y, W \mid Z) \overset{\text{Premise 1}}{=} P(X \mid Z)\, P(Y, W \mid Z),$$

where in the second and third step the premises $X \perp W \mid (Y, Z)$ and $X \perp Y \mid Z$ where used (in said order). Hence, $X \perp (W, Y) \mid Z$ holds. $\square$

Proof of Third Statement: By the chain rule of probability, exploiting the assumed conditional independencies and using the first statement proven earlier, it follows that

$$P(X, Y, W \mid Z) \overset{\text{Chain Rule}}{=} P(Y \mid X, W, Z)\, P(X, W \mid Z) \overset{\text{Premise 2}}{=} P(Y \mid X, Z)\, P(X, W \mid Z) \overset{\text{Premise 1}}{=} P(Y \mid Z)\, P(X, W \mid Z).$$

Thus, $(X, W) \perp Y \mid Z$ holds. $\square$