Guided waves in a plate

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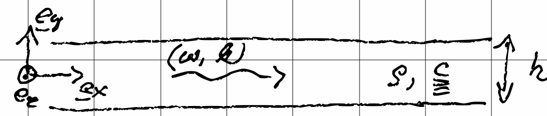

- harmonic particle displacements: $\underline{u}(x,y,z,t) = \underline{u}(x,y,z)e^{-i\omega t}$
- eq. of motion for $\underline{u}(x,y,z)$:

$$\begin{cases} \nabla \cdot \underline{\underline{T}} + \rho\omega^2 \underline{u} = \underline{0} & \text{balance of linear momentum} \\ \underline{\underline{T}} = \underline{\underline{\underline{\underline{c}}}} : \nabla \underline{u} & \text{material + kinematiks} \end{cases}$$

(1) $\nabla \cdot \underline{\underline{\underline{\underline{c}}}} : \nabla \underline{u} + \rho\omega^2 \underline{u} = \underline{0}$ Navier's eq.

- boundary conditions:

(2) $\underline{e}_y \cdot \underline{\underline{T}} = \underline{e}_y \cdot \underline{\underline{\underline{\underline{c}}}} : \nabla \underline{u} = \underline{0}$

- guided wave ansatz: $\underline{u}(x,y,z) = \underline{u}(y)e^{ikx}$

def. $\nabla := ik\underline{e}_x + \underline{e}_y \partial_y + 0\,\underline{e}_z$

$\Rightarrow \nabla \underline{u} = ik\,\underline{e}_x \underline{u} + \underline{e}_y \partial_y \underline{u}$

in (1):

$$(ik\underline{e}_x + \underline{e}_y\partial_y) \cdot \underline{\underline{\underline{\underline{c}}}} \cdot (ik\underline{e}_x + \underline{e}_y\partial_y) \cdot \underline{u} + \rho\omega^2 \underline{u} = \underline{0}$$

$$\Rightarrow \Big[(ik)^2 \underline{e}_x \cdot \underline{\underline{\underline{\underline{c}}}} \cdot \underline{e}_x + ik\,(\underline{e}_x \cdot \underline{\underline{\underline{\underline{c}}}} \cdot \underline{e}_y \partial_y + \underline{e}_y \cdot \underline{\underline{\underline{\underline{c}}}} \cdot \underline{e}_x \partial_y) + \underline{e}_y \cdot \underline{\underline{\underline{\underline{c}}}} \cdot \underline{e}_y \partial_y^2 \Big] \cdot \underline{u} + \rho\omega^2 \underline{u} = \underline{0}$$

$$\begin{cases} \Big[(ik)^2 \underline{\underline{c}}_{xx} + ik\,(\underline{\underline{c}}_{xy} + \underline{\underline{c}}_{yx})\partial_y + \underline{\underline{c}}_{yy}\partial_y^2 + \rho\omega^2 \underline{\underline{I}} \Big] \cdot \underline{u} = \underline{0} \\ \text{BC}: \Big[ik\,\underline{e}_y \cdot \underline{\underline{\underline{\underline{c}}}} \cdot \underline{e}_x + \underline{e}_y \cdot \underline{\underline{\underline{\underline{c}}}} \cdot \underline{e}_y \partial_y \Big] \cdot \underline{u} = 0 \end{cases}$$

$\leadsto$ guided wave eigenvalue problem (differential)

2. Discretization

$$\begin{cases} \left[(ik)^2\, \underline{\underline{c}}_{xx} + ik\,(\underline{\underline{c}}_{xy} + \underline{\underline{c}}_{yx})\,\partial_y + \underline{\underline{c}}_{yy}\,\partial_y^2 + \rho\omega^2 \underline{\underline{I}}\right]\cdot \underline{u} = \underline{0} \\ \text{BC}: \left[ik\, \underline{\underline{c}}_{yx} + \underline{\underline{c}}_{yy}\,\partial_y\right]\cdot \underline{u} = 0 \end{cases}$$

$u_x(y) \rightsquigarrow \bar{u}_x = [u_x(y_i)]_{N\times 1}$ at "collocation" points y_i

$$\bar{u} = \begin{bmatrix} \bar{u}_x \\ \bar{u}_y \\ \bar{u}_z \end{bmatrix}_{3N\times 1} \qquad \begin{aligned} \partial_y &\mapsto D_1 : N\times N \\ \partial_y^2 &\mapsto D_2 : N\times N \\ \text{scalar: } a &\mapsto a I_d : N\times N \end{aligned}$$

such that: $\left[\partial_y u_x(y)\big|_{y_i}\right] = D_1 \cdot \bar{u}_x$

e.g.: $\underline{\underline{c}}_{yy} \otimes D_2 = \begin{bmatrix} c_{11} D_2 & c_{12} D_2 & \boxed{c_{13} D_2} \\ c_{21} D_2 & \cdots & \\ \vdots & & \end{bmatrix}_{3N\times 3N}$

$\to$ $N\times N$ block

$\to$ Kronecker product

Discrete waveguide problem:

$$\left[(ik)^2 L_2 + ik\, L_1 + L_0 + \omega^2 M\right]\cdot u = 0$$

$$L_2 = \underline{\underline{c}}_{xx} \otimes I_d\,, \quad L_1 = (\underline{\underline{c}}_{xy} + \underline{\underline{c}}_{yx}) \otimes D_1$$

$$L_0 = \underline{\underline{c}}_{yy} \otimes D_2\,, \quad M = \rho\, \underline{\underline{I}} \otimes I_d$$

with BCs:

$$\left[(ik)\, B_1 + B_0\right]\cdot u = 0$$

$$B_1 = \underline{\underline{c}}_{yx} \otimes I_d\,, \quad B_0 = \underline{\underline{c}}_{yy} \otimes D_1$$

If $C_{ijkl} = C_{ijlk}$, $\underline{\underline{\underline{\underline{C}}}}$ operates only on the symmetric part of the arbitrary 2nd order tensor $\underline{\underline{Q}}$:

$$\delta_{ks} \quad , \quad \delta_{rl}$$

$$
\begin{aligned}
\underline{\underline{\underline{\underline{C}}}} : \underline{\underline{Q}} &= C_{ijkl}\, \underline{e}_i \underline{e}_j \underline{e}_k \underline{e}_l : Q_{rs}\, \underline{e}_r \underline{e}_s \\
&= C_{ijkl}\, Q_{lk}\, \underline{e}_i \underline{e}_j \\
&= C_{ijlk}\, Q_{lk}\, \underline{e}_i \underline{e}_j \qquad \text{(due to symmetry)} \\
&= C_{ijkl}\, Q_{kl}\, \underline{e}_i \underline{e}_j \\
&= C_{ijkl}\, \underline{e}_i \underline{e}_j \underline{e}_k \underline{e}_l : Q_{rs}\, \underline{e}_s \underline{e}_r \\
&= \underline{\underline{\underline{\underline{C}}}} : \underline{\underline{Q}}^T \\
&= \underline{\underline{\underline{\underline{C}}}} : \frac{1}{2}\left(\underline{\underline{Q}} + \underline{\underline{Q}}^T\right)
\end{aligned}
$$