Guided waves in plates: weak form

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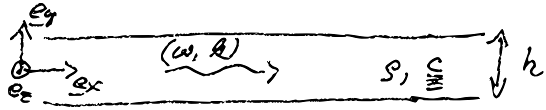

Guided wave problem:

$$\text{(S.1)}\quad \overbrace{\left[(ik)^2 \mathbf{C}_{xx} + ik\,\mathbf{C}_{xy}\partial_y + \left(ik\,\mathbf{C}_{yx}\partial_y + \mathbf{C}_{yy}\partial_y^2\right) + \rho\omega^2 \mathbf{I}\right]\cdot \mathbf{u}}^{\mathbf{r}(y)} = \mathbf{0}$$

$$\text{(S.2)}\quad \text{BC}: \mathbf{e}_y\cdot\mathbf{T} = \left[ik\,\mathbf{C}_{yx} + \mathbf{C}_{yy}\partial_y\right]\cdot\mathbf{u} = 0$$

⇝ Find $\mathbf{u}(y)$, such that (S.1) holds on $y \in (-h/2, h/2)$ and (S.2) is satisfied at $y = \pm h/2$.

⇝ "strong form"

<u>Derive the associated weak form:</u>

- introduce set of "trial solutions":

  $S := \{\mathbf{u} \mid \mathbf{u} \in H^1$ and $\mathbf{u}$ fulfills Dirichlet BC$\}$

  H^1: functions and its derivatives are square integrable.

- introduce a function space of "test functions":

  $V := \{\mathbf{v} \mid \mathbf{v} \in H^1$ and $\mathbf{v} = 0$ at Dirichlet boundary$\}$

- test (S.1) against $\mathbf{v} \in V$: $\int \mathbf{v}(y)\cdot\mathbf{r}(y)\,dy = 0$

$$\int_{-h/2}^{h/2} \mathbf{v}\cdot\left[(ik)^2 \mathbf{C}_{xx} + ik\,\mathbf{C}_{xy}\partial_y + \left(ik\,\mathbf{C}_{yx}\partial_y + \mathbf{C}_{yy}\partial_y^2\right) + \rho\omega^2 \mathbf{I}\right]\cdot \mathbf{u}\,dy = 0$$

$$\underset{(W)}{\Longrightarrow}\quad \int_{-h/2}^{h/2} \mathbf{v}\cdot\left[(ik)^2 \mathbf{C}_{xx} + ik\,\mathbf{C}_{xy}\partial_y\right]\cdot\mathbf{u}\,dy + \omega^2\int \mathbf{v}\cdot\rho\mathbf{I}\cdot\mathbf{u}\,dy$$

$$- \int_{-h/2}^{h/2} \partial_y\mathbf{v}\cdot\left[ik\,\mathbf{C}_{yx} + \mathbf{C}_{yy}\partial_y\right]\cdot\mathbf{u}\,dy + \left[\mathbf{v}\cdot\cancelto{0}{\left(ik\,\mathbf{C}_{yx} + \mathbf{C}_{yy}\partial_y\right)\cdot\mathbf{u}}\right]_{-h/2}^{h/2} = 0$$

⇝ Find $\mathbf{u}(y) \in S$, such that (W) holds <u>for all</u> $\mathbf{v} \in V$.

⇝ "weak form" (virtual work, variational form)

Discretization

Choose finite-dimensional $S^h \subset S$ and $V^h \subset V$:

- $\underline{v}^h = \sum_i \underline{v}_i \, P_i(y) \in V^h$
- $\underline{u}^h = \sum_j \underline{u}_j \, P_j(y) \in S^h$

$\Big\}$ Galerkin discretization: Use the same basis functions P_i in $\underline{v}^h$ and $\underline{u}^h$.

$\rightsquigarrow$ $P_i(y)$: interpolation / shape / basis functions

Insert into (W):

e.g.
$$\int \underline{v}_i \, P_i(y) \cdot \rho \, \underline{u}_j \, P_j(y) \, dy = \underline{v}_i \, \rho \underline{\underline{I}} \cdot \overbrace{\int P_i(y) P_j(y) \, dy}^{pp_{ij}} \cdot \underline{u}_j$$

$pp \stackrel{\text{def}}{=} [pp_{ij}]$: $\quad = v^T (pp \otimes \rho \underline{\underline{I}}) \cdot u = v^T M \cdot u$

$u = [\underline{u}_j]$, $v = [\underline{v}_i]$

$$\int \underline{v}_i \, P_i(y) \cdot \underline{\underline{c}}_{xy} \cdot \underline{u}_j \, P_j'(y) \, dy = \underline{v}_i \, \underline{\underline{c}}_{xy} \overbrace{\int P_i(y) P_j'(y) \, dy}^{ppd_{ij}} \cdot \underline{u}_j$$
$$= v^T (ppd \otimes \underline{\underline{c}}_{xy}) \cdot u$$

overall:

$pp_{ij} = \int P_i P_j \, dy \quad \rightsquigarrow \quad pp = [pp_{ij}]$

$pd_{ij} = \int P_i P_j' \, dy \quad \rightsquigarrow \quad pd = [pd_{ij}]$

$dd_{ij} = \int P_i' P_j' \, dy \quad \rightsquigarrow \quad dd = [dd_{ij}]$

$$v^T \Big[(ik)^2 \underbrace{c_{xx} \otimes pp}_{L_2} + ik \underbrace{\left(c_{xy} \otimes pd - c_{yx} \otimes pd^T \right)}_{L_1} + \underbrace{c_{yy} \otimes dd}_{L_0} + \omega^2 \underbrace{\rho I \otimes pp}_{M} \Big] \cdot u = 0$$

Note: commuting the Kronecker products is only a permutation in u and v.

- v is arbitrary !:

$$\Rightarrow \quad \left[(ik)^2 L_2 + ik L_1 + L_0 + \omega^2 M \right] \cdot u = 0$$