Investments Class
Problem set 3

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March 10, 2020

Exercise 1. (Efficient portfolios)

- The return on the portfolio is linear in the returns of the risky assets and the risk-free return:

$$
\begin{aligned}
\mu_P &= \mu^T\omega + (1 - \mathbb{1}^T\omega)R_f \\
\mu_P - R_f &= \mu^T\omega + -\mathbb{1}^T\omega R_f \\
\mu_P - R_f &= (\mu - R_f\mathbb{1})^T\omega
\end{aligned}
$$

  where $\mathbb{1}$ is the $N \times 1$ vector of ones. The maximization is then formulated in terms of the Lagrangian with no constraints:

$$\mathcal{L} = (R_f + (\mu - \mathbb{1}R_f)^T\omega) - \gamma\frac{1}{2}\omega^T\Sigma\omega.$$

  The FOC condition with respect to ω reads as:

$$\frac{\partial\mathcal{L}}{\partial\omega} = \mu - \mathbb{1}R_f - \gamma\Sigma\omega = 0$$

  and solving for ω yields the following optimal ω^*:

$$\omega^* = \gamma^{-1}\Sigma^{-1}(\mu - R_f\mathbb{1})$$

  The ω^* is optimal since the Hessian $\frac{\partial^2\mathcal{L}}{\partial\omega\partial\omega^\top} = -\Sigma$ is negative semi-definite.

$$
\begin{aligned}
cov[R, R_p] &= cov\left[R, R_f + (\omega^*)^T(R - R_f\mathbb{1})\right] \\
&= cov\left[R, (\omega^*)^T R\right] a \\
&= cov[R - \mu, R - \mu]\,\omega^* \\
&= var[R, R]\,\omega^* \\
&= \Sigma\omega^* \\
&= \Sigma\gamma^{-1}\Sigma^{-1}(\mu - R_f\mathbb{1}) \\
&= \gamma^{-1}(\mu - R_f\mathbb{1}) \\
&\Downarrow \\
\gamma cov[R, R_p] &= \mu - R_f\mathbb{1} \quad (\star)
\end{aligned}
$$

- Using the definition of μ_P and the previously proven identity, it follows that

$$\begin{aligned}
\mu_P = \mathbb{E}[R_p] &= \mathbb{E}\left[R_f + \omega^T(R - R_f\mathbb{1})\right] \\
&= R_f + \omega^T(\mu - R_f\mathbb{1}) \\
&= \omega^T\mu + (1 - \omega^T\mathbb{1})R_f \\
&= \omega^T(\mu - \mathbb{1}R_f) + R_f \\
&\overset{\star}{=} \omega^T\gamma cov[R, R_p] + R_f \\
&= \gamma cov\left[\omega^T R, R_p\right] + R_f \\
&= \gamma cov[R_p, R_p] + R_f \\
&= \gamma\sigma_P^2 + R_f \\
&\Downarrow \\
\mu_P - R_f &= \gamma\sigma_P^2 \\
&\Downarrow \\
\gamma &= \frac{\mu_P - R_f}{\sigma_P^2} \\
&\Downarrow \\
(\mu - R_f\mathbb{1}) &\overset{\star}{=} \gamma cov[R, R_p] = \frac{cov[R, R_p]}{\sigma_P^2}(\mu_P - R_f) = \beta_P(\mu_P - R_f) \quad (\star\star)
\end{aligned}$$

  Where $\beta_P = (\beta_{1,P}, \ldots, \beta_{N,P})^T$.

- A linear regression model is

$$Y = X_1\beta_1 + \cdots + X_N\beta_N + \epsilon$$

  with the following assumptions:

  A1. **Linearity** between Y and X_i.

  A2. **Full rank** of X.

  A3. **Exogeneity**: $\mathbb{E}[\epsilon \mid X_1, \cdots, X] = 0$.

  A4. **Homoskedasticity and nonautocorrelation**: $cov(\epsilon_i, \epsilon_j | X_i, X_j) = 0 \quad \forall i \neq j$ and $var(\epsilon_i \mid X_1, \ldots, X_N) = \sigma^2$.

  A5. **Normality of the errors**: $\epsilon \mid X_1, \ldots, X_N \sim N(0, \sigma^2)$.

  The above assumptions are satisfied when regressing the excess return of an asset on the excess return of the market portfolio:

$$\begin{aligned}
R_i &= R_f + \beta_i(R_P - R_f) + \epsilon_i \\
R_i - R_f &= \beta_i(R_P - R_f) + \epsilon_i
\end{aligned}$$

  The exogeneity assumption is verified by taking the expectation of the model:

$$
\begin{aligned}
\mathbb{E}\left[R_i - R_f\right] &= \beta_i \mathbb{E}\left[R_P - R_f\right] + \mathbb{E}\left[\epsilon_i\right] \\
\mu_i - R_f &= \beta_i(\mu_P - R_f) + \mathbb{E}\left[\epsilon_i\right] \\
0 &\overset{\star\star}{=} \mathbb{E}\left[\epsilon_i\right]
\end{aligned}
$$

To check the homoskedasticity assumption, take the covariance of each side of the model with the stochastic return on the market portfolio:

$$
\begin{aligned}
cov(R_i - R_f, R_P) &= \beta_i cov(R_p - R_f, R_P) + cov(\epsilon_i, R_p) \\
cov(R_i, R_P) &= \beta_i cov(R_P, R_P) + cov(\epsilon_i, R_p) \\
cov(R_i, R_P) &= \frac{cov(R_i, R_P)}{\sigma_P^2} cov(R_P, R_P) + cov(\epsilon_i, R_p) \\
cov(\epsilon_i, R_p) &= 0
\end{aligned}
$$

The previously found expression for $\beta_{i,P}$ is consistent:

$$
\beta_i = \frac{cov(R_i - R_f, R_P - R_f)}{var(R_P - R_f)} = \frac{cov(R_i, R_P)}{\sigma_P^2} = \beta_{i,P}
$$

When regressing on R_i alone and not on the excess above R_f, the intersect is still R_f as it follows from the previously found identity and the fact that $\mathbb{E}\left[\epsilon_i\right] = 0$.

- **Proposition [Sharpe-Ratios].** All the mean-variance efficient portfolios have the same Sharpe ratio where we define its Sharpe ratio as $SR_p = \frac{\mu_P - R_f}{\sigma_P}$:

  Proof. To show that all the portfolios have the same Sharpe ratio we will suppose that there exists another portfolio P' and show that $SR_P = SR_{P'}$. In the previous subtask we found that

  $$(\mu - R_f \mathbb{1}) = \beta_P(\mu_P - R_f)$$

  multiplying on the left by the asset weights of the alternative mean-variance efficient portfolio $\omega_{P'}$ we get that

  $$
  (\omega_{P'})^T(\mu - R_f\mathbb{1}) = (\omega_{P'})^T \frac{cov(R, R_P)}{\sigma_P^2}(\mu_P - R_f) \tag{1}
  $$

  $$
  (\mu_{P'} - R_f) = \frac{cov(R_{P'}, R_P)}{\sigma_P^2}(\mu_P - R_f) \tag{2}
  $$

  All the mean-variance efficient portfolios are a linear combination of the tangent portfolio R_{tan} and the risk-free assets R_f. Let η and η' be the weights of the tangent portfolio in the portfolios P and P':

  $$
  \begin{aligned}
  R_P &= \eta \times R_{tan} + (1 - \eta)R_f \\
  R_{P'} &= \eta' \times R_{tan} + (1 - \eta')R_f
  \end{aligned}
  $$

Then the covariance between the two portfolios is

$$cov(R_P, R_{P'}) = \eta\eta' cov(R_{tan}, R_{tan}) = \eta\eta' \tag{3}$$

while the variances are:

$$\sigma_P^2 = var(R_P) = \eta^2 \tag{4}$$
$$\sigma_{P'}^2 = var(R_{P'}) = \eta'^2 \tag{5}$$

Plugging Equation 4 and Equation 3 in Equation 2 yields:

$$\begin{aligned}
(\mu_{P'} - R_f) &= \frac{\eta\eta'}{\eta^2}(\mu_P - R_f) \\
\frac{(\mu_{P'} - R_f)}{\eta'} &= \frac{(\mu_P - R_f)}{\eta} \\
\frac{(\mu_{P'} - R_f)}{\sigma_{P'}} &= \frac{(\mu_P - R_f)}{\sigma_P} \\
SR_P &= SR_{P'}
\end{aligned}$$

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Exercise 2. (Portfolio math) First we show that a convex combination of two minimum variance frontier portfolios w_a and w_b is a minimum variance frontier portfolio.

w is a minimum variance frontier portfolio if it solves the optimization problem and thus can be written as:

$$w = \lambda\Sigma^{-1}\mathbb{1} + \gamma\Sigma^{-1}\mu$$

with $\lambda = \frac{C-\mu_p B}{\Delta}$ and $\gamma = \frac{\mu_p A - B}{\Delta}$.

Subsequently consider two arbitrary minimum variance porfolios w_a and w_b with returns μ_a, μ_b respectively and write $w = \alpha w_a + (1-\alpha)w_b$. Then

$$\begin{aligned}
w &= \alpha(\lambda_a\Sigma^{-1}\mathbb{1} + \gamma_a\Sigma^{-1}\mu) + (1-\alpha)(\lambda_b\Sigma^{-1}\mathbb{1} + \gamma_b\Sigma^{-1}\mu) \\
&= (\alpha\lambda_a + (1-\alpha)\lambda_b)\Sigma^{-1}\mathbb{1} + (\alpha\gamma_a + (1-\alpha)\gamma_b)\Sigma^{-1}\mu
\end{aligned}$$

But A, B, C, D do not depend on the portfolio but only on the available assets so they do not change between w_a and w_b. So we obtain:

$$w = \frac{C - (\alpha\mu_a + (1-\alpha)\mu_b)B}{\Delta}\Sigma^{-1}\mathbb{1} + \frac{(\alpha\mu_p + (1-\alpha)\mu_b)A - B}{\Delta}\Sigma^{-1}\mu$$

The return of w equals $\alpha\mu_a + (1-\alpha)\mu_b$ and its variance is $\alpha^2 w_a^\top \Sigma w_a + (1-\alpha)^2 w_b^\top \Sigma w_b + 2\alpha(1-\alpha)w_a^\top \Sigma w_b$

Finally, w is of the form of a minimum variance frontier portfolio solving the following optimization problem:

$$\min_w \frac{1}{2} w^\top \Sigma w \text{ such that } w^\top \mu = \alpha\mu_a + (1-\alpha)\mu_b \text{ and } w^\top \mathbb{1} = 1$$

Conversely, if we have the minimum variance portfolio w and want to write it as a combination of two different portfolios w_a and w_b (with return μ_a and μ_b), we can define $\alpha = \frac{\mu_p - \mu_b}{\mu_a - \mu_b}, 1 - \alpha = \frac{\mu_a - \mu_p}{\mu_a - \mu_b}$. Thus,

$$\begin{aligned} \alpha w_a + (1-\alpha) w_b &= \frac{C - (\alpha\mu_a + (1-\alpha)\mu_b)B}{\Delta}\Sigma^{-1}\mathbb{1} + \frac{(\alpha\mu_p + (1-\alpha)\mu_b)A - B}{\Delta}\Sigma^{-1}\mu \\ &= \frac{C - (\mu_p)B}{\Delta}\Sigma^{-1}\mathbb{1} + \frac{(\mu_p)A - B}{\Delta}\Sigma^{-1}\mu = w \end{aligned}$$

because $\frac{\mu_p - \mu_b}{\mu_a - \mu_b}\mu_a + \frac{\mu_a - \mu_p}{\mu_a - \mu_b}\mu_b = \frac{\mu_p(\mu_a - \mu_b) - \mu_a\mu_b + \mu_a\mu_b}{\mu_a - \mu_b} = \mu_p$
This proves that any minimum variance frontier portfolio can be replicated by a convex combination of two minimum variance frontier portfolios.

For the second part of the exercise, we follow the hint. The expected return of the convex combination of the two portfolios is: $\mathbb{E}[Y] = wR + (1-w)R_{\text{min}}$. In addition, the variance of this portfolio is given by:

$$Var(Y) = w^2 Var(R) + (1-w)^2 Var(R_{\text{min}}) + 2w(1-w)Cov(R, R_{\text{min}})$$

Since the global minimum-variance portfolio has the minimal variance among all portfolios, we can see that $Var(Y) \geqslant Var(R_{\text{min}})$ and this value is achieved when $w = 0$.

To find the minimum algebraically, we can differentiate the variance with respect to w:

$$\frac{\partial Var(Y)}{\partial w} = 2wVar(R) - 2(1-w)Var(R_{\text{min}}) + 2(1-2w)Cov(R, R_{\text{min}})$$

and equate it with 0:

$$w(2Var(R) + 2Var(R_{\text{min}}) - 4Cov(R, R_{\text{min}})) = 2Var(R_{\text{min}}) - 2Cov(R, R_{\text{min}})$$

that leads to:

$$\hat{w} = \frac{Var(R_{\text{min}}) - Cov(R, R_{\text{min}})}{Var(R) + Var(R_{\text{min}}) - Cov(R, R_{\text{min}})}$$

But to be the unique minimum, $\hat{w}$ needs to be 0 as we saw previously.

Thus, $Var(R_{\text{min}}) = Cov(R, R_{\text{min}})$.

Exercise 3. After downloading the data, we compute the mean return, standard deviation, and correlation matrix for returns over the entire sample period.

	Mean of returns
vwretd	0.008877
b2ret	0.004718
tmytm	0.003626

	Standard deviation of returns
vwretd	0.043508
b2ret	0.008161
tmytm	0.002559

Correlation matrix of returns			
	vwretd	b2ret	tmytm
vwretd	1.000000	0.092795	-0.021777
b2ret	0.092795	1.000000	0.240501
tmytm	-0.021777	0.240501	1.000000

The Tangency portfolio can be computed with the following formula:

$$w_{tan} = \frac{\Sigma^{-1}(\mu - R_0\mathbf{1})}{B - AR_0}$$

Where

$$A = \mathbf{1'}\Sigma^{-1}\mathbf{1}$$

And

$$B = \mathbf{1'}\Sigma^{-1}\mu$$

We obtain weights of 14% in the stock and 86% in the bond. The mean, standard deviation and Sharpe ratio of the Tangency portfolio are the following

	Metrics of tangency portfolio
mean	0.005309
std	0.009761
Sharpe ratio	0.172410

We do the same for a portfolio that invests 60% in stocks and 40% in bonds (the 60/40 portfolio):

	Metrics of 60/40 portfolio
mean	0.007213
std	0.026607
Sharpe ratio	0.134831

Where we have computed the mean of a portfolio as follows

$$\mu_p = w'\mu$$

The portfolio standard deviation

$$\sigma_p = \sqrt{w'\Sigma w}$$

And finally the Sharpe ratio

$$SR = \frac{\mu - \bar{R}^f}{\sigma}$$

We then compute the unlevered risk-parity (RP) portfolio following the AFP paper

$$w_{i,unleverd} = \frac{1}{\sum_i \hat{\sigma}_i^{-1}} \hat{\sigma}_i^{-1}$$

We find that approximately 15% should be invested in stocks and 85% in bonds. The metrics of this portfolio are the following

	Metrics of RP-unlevered portfolio
mean	0.005358
std	0.010057
Sharpe ratio	0.172252

We perform similarly for the levered RP portfolio where the weights are computed as follows

$$w_{levered} = \frac{\sigma_{60/40}}{\sigma_{RP-unlevered}} w_{RP-unlevered}$$

Since the weights don't add up to 1, we conclude that we have to go short on T-bills by

$$1 - \sum_i w_{i,levered}$$

We find that we need to invest 41% in stocks, 224% in bonds and -165% in T-Bills. The metrics of this portfolio are the following

	Metrics of RP-levered portfolio
mean	0.008209
std	0.026607
Sharpe ratio	0.172252

We plot the RP-levered and RP-unlevered portfolios on the efficient frontier along with the Tangency and 60/40 portfolio:

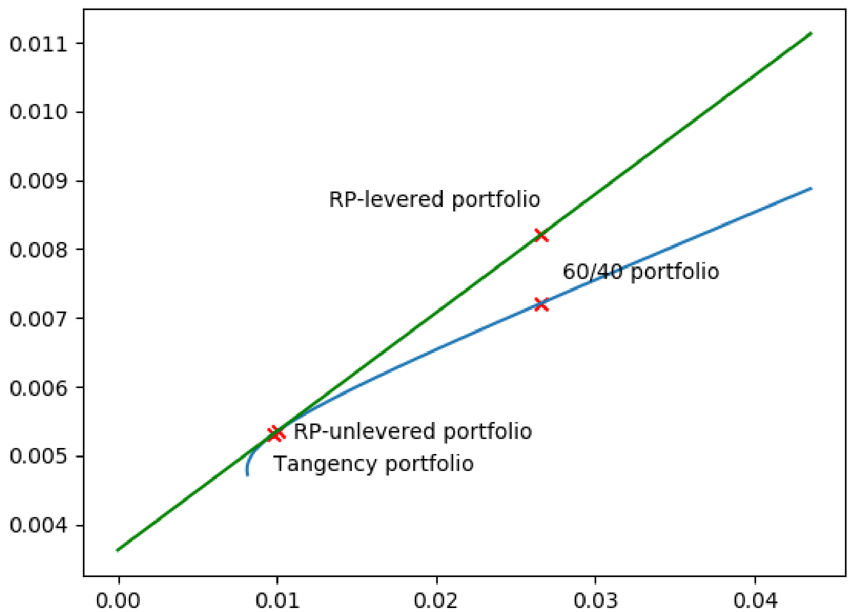

Figure 1: RP-levered, RP-unlevered, Tangency, and 60/40 portfolio with efficient frontier

What explains the difference between the RP and RP-unlevered portfolio performance?

We now follow the same method but now by using the three-year monthly excess returns instead of the full sample. We obtain a mean of returns of the RP-unlevered portfolio using a rolling-window of 0.005280 and a mean of returns of the RP-levered portfolio using a rolling-window of 0.007845. These metrics are almost identical (slightly lower) than their full-window counterparts. **Why?**

We now consider an investor who has a mean-variance utility U

$$U = \mu_p - \frac{a}{2}\sigma_p^2$$

Using the full-sample estimates of the means and covariance matrix stocks and bonds, the optimal portfolio can be computed as follows:

$$w_0 = \frac{1}{a}\Sigma^{-1}(\mu - R_0\mathbf{1})$$

We end up with 42% in stocks, 253% in bonds and −195% in T-Bills. The metrics are the following

	Metrics of optimal portfolio
mean	0.008580
std	0.028735
Sharpe ratio	0.172410