EPFL

Fin404 Derivatives

Master in Financial Engineering

Spring 2019

Problem set 2. Due date: March 6

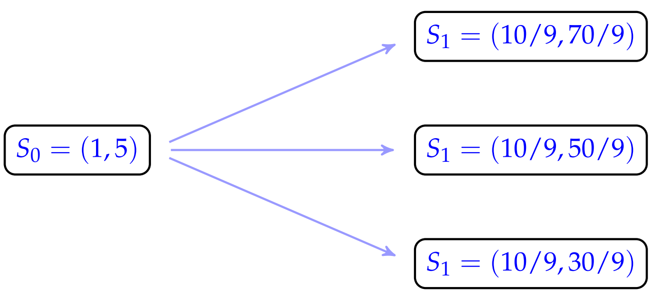
Basic questions

Question 1. Show that a trading strategy π is self-financing if and only if its value process satisfies the difference equation

$$\Delta X_t^{\pi} = \sum_{k=0}^{n} \pi_{kt} \Delta S_{k,t}, \qquad t \geq 1.$$

How do you interpret this equation? Why is this characterization of self-financing strategies **<u>less useful</u>** than the one we established in class?

Question 2. Consider the one period model represented as



Show that this single period model is free of arbitrage opportunities. Suppose that, in addition to the stock and the riskless asset, you can also trade in a European call option with strike $k = 50/9$ and initial price $c_0 = 0.9$. Determine whether the extended model is arbitrage free. If not construct a strategy that generates riskless profits.

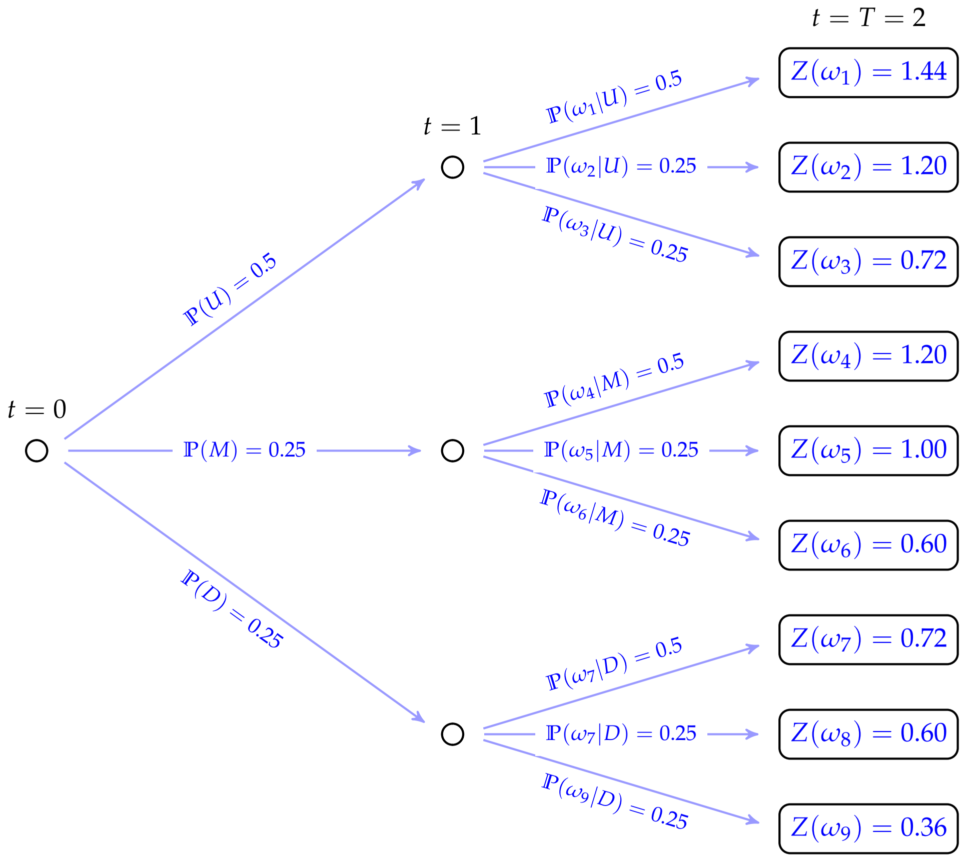


Figure 1: Event tree for Question 3.

Question 3. Let $U = \{\omega_1, \omega_2, \omega_3\}$, $M = \{\omega_4, \omega_5, \omega_6\}$, and $D = \{\omega_7, \omega_8, \omega_9\}$. Assume that the tree is represented by Figure 1 and consider the measure defined by

$$\mathbb{Q}(\{\omega_k\}) = Z(\omega_k)\mathbb{P}(\{\omega_k\}), \qquad k = 1, \ldots, 9.$$

Show that $\mathbb{Q}$ is a probability measure that is equivalent to $\mathbb{P}$ and compute the probabilities $\mathbb{Q}(U)$, $\mathbb{Q}(D)$, $\mathbb{Q}(\omega_1|U)$, $\mathbb{Q}(\omega_3|U)$, $\mathbb{Q}(\omega_4|M)$, $\mathbb{Q}(\omega_6|M)$, $\mathbb{Q}(\omega_7|D)$ and $\mathbb{Q}(\omega_9|D)$ required to label the branches of the tree.

	Strike	Barrier	Maturity	Price
SQ Stock	–	–	–	100.0
Zero coupon bond	–	–	1 Year	
Zero coupon bond	–	–	2 Year	0.97
Forward price SQ	–	–	1 Year	101.01
Forward price SQ	–	–	2 Year	
European Call on SQ	100	–	1 Year	
American Call on SQ	100	–	1 Year	
European Call on SQ	100	–	2 Year	22.0
European Put on SQ	100	–	1 Year	14.0
European Put on SQ	100	–	2 Year	22.0
Up and out Call on SQ	100	140	1 Year	
Up and in Call on SQ	100	140	1 Year	1.0
Down and out Put on SQ	90	95	2 Year	

Table 1: Price table for Exercise 1

Problems

Turn in each of the **underlined exercises**.

Exercise 1. Fill in the blanks in Table 1 under the assumption that SQ *does not pay dividends over the next year*. Make sure to **justify all your calculations**.

Hint. An up and in (resp. out) option gets activated (resp. deactivated) if the underlying asset price exceeds the specified barrier at least once during the life of the option. Similarly, a down and out option gets deactivated is the underlying asset price drops below the specified barrier at least once during the life of the option.

Exercise 2. Consider the one period model represented as

$$\begin{tikzcd} & \boxed{S_{11}(\omega_1) = 210} \\ \boxed{S_{10} = 100} \arrow[ur] \arrow[r] \arrow[dr] & \boxed{S_{11}(\omega_2) = 105} \\ & \boxed{S_{11}(\omega_3) = 52.5} \end{tikzcd}$$

and assume that the price of the riskless asset is given by $S_{00} = 1$ at time zero and by $S_{01} = 1.05$ in all states at time one.

a) Show that this single period model is free of arbitrage opportunities.

b) Suppose that in addition to the stock and the riskless asset you can also trade in a European call option with strike $k = 200$ and initial price $C_0 = 1$. Determine whether the extended market is arbitrage free. If not construct a strategy that allows to generate riskless profits.

c) Suppose that in addition to the stock and the riskless asset you can also trade in a European call option with strike $k = 209$ and initial price $C_0 = 0.32$. Determine whether the extended market is arbitrage free. If not construct a strategy that allows to generate riskless profits.

d) Consider a derivative that costs p_0 and which will pay you

$$p_0 + 1_{\{S_{11}=105\}}1050.$$

Compute the set of prices p_0 such that the market remains arbitrage free after the introduction of this derivative.

Exercise 3. A **numéraire** is a *strictly positive* process N_t such that $N_t = X_t^\pi$ for some self financing strategy π with $N_0 = X_0^\pi = 1$. In this exercise we consider a fixed numéraire within the multiperiod model of the lecture and denote by

$$S_t^N = S_t/N_t = (S_{0t}/N_t, S_{1t}/N_t, \dots, S_{nt}/N_t)^\top$$

the vector of securities prices expressed in units of the numéraire. Prove that a trading strategy π is self-financing if and only if its $N-$denominated value process $X_t^{\pi,N} = X_t^\pi/N_t$ satisfies the difference equation

$$\Delta X_t^{\pi,N} = \sum_{k=0}^{n} \pi_{kt} \Delta S_{kt}^N, \qquad t = 1, 2, \dots, T.$$

Remark. Note that, as long as it is strictly positive, the price of any asset or portfolio of assets can be used as a numéraire. In the lecture we used the price of the riskless asset but any other choice, such as the price of a stock or a zero-coupon bond, works equally well.

Exercise 4. Consider the same model as in the lecture and assume that the interest rate is constantly equal to r. Let x, α and β be strictly positive constants and consider the *self-financing* strategy defined by

$$\pi_{kt} = \mathbf{1}_{\{k=1\}}(\alpha - \beta X_{t-1})/S_{1,t-1}, \qquad (k,t) \in \{1,2,...,n\} \times \{1,2,\ldots,T\}$$

and the initial value $X_0 = x$.

a) Show that the riskless part of the portfolio satisfies $\pi_{0t}S_{0t-1} = aX_{t-1} + b$ for some constants (a,b) to be determined.

b) Use the self-financing property to show that the value of the trading strategy evolves according to a difference equation of the form

$$X_t = f(R_t) + g(R_t)X_{t-1} \tag{1}$$

where the $\mathcal{F}_t-$measurable random variable

$$R_t = (e^{-r\Delta}S_{1,t}/S_{1,t-1}) - 1$$

denotes the excess return on the first risky asset over $(t-1,t]$ and $f,g : \mathbb{R} \to \mathbb{R}$ are functions to be determined.

c) Derive an explicit solution to (1) by showing that

$$X_t = \Phi(x; R_1, R_2, \ldots, R_t)$$

for some function $\Phi : \mathbb{R} \times \mathbb{R}^t \to \mathbb{R}$ to be determined. Use this solution to express π_{0t} as a function of x and the stock returns $(R_1, R_2, \ldots, R_{t-1})$.

Exercise 5. **(Midterm Spring 2012)** Fill in the blanks in table 2 below under the assumption that markets are dynamically complete and that the stock of IBM does not pay any dividends *over the next month*.

Exercise 6. Consider the model of the lecture, let $\eta \in \mathbb{R}^{n+1}$ be such that $\sum_{k=0}^{n} \eta_k = 1$ and define a trading strategy by letting

$$\pi_{kt} = \eta_k(X^{\pi}_{t-1}/S_{k,t-1}), \qquad (k,t) \in \{0,1,\ldots,n\} \times \{1,\ldots,T\}$$

Security	Strike	Maturity	Price
IBM Stock	–	–	100
Zero coupon bond	–	1M	0.99
Zero coupon bond	–	3M	
Forward price IBM stock	–	1M	
Forward price IBM stock	–	3M	
European Call on IBM	100	1M	
European Call on IBM	90	3M	8.0
European Call on IBM	110	3M	0.5
European Call on IBM	115	3M	0.25
European Put on IBM	100	1M	1.0
European Put on IBM	90	3M	0.5
European Put on IBM	110	3M	
European Put on IBM	115	3M	16.5

Table 2: Price table for Exercise 5

and requiring that $X_0^\pi = 1$.

a) Explain why π is referred to as a *constant proportions* trading strategy and show that it is self-financing.

b) Show that the value process solves

$$X_t^\pi = X_{t-1}^\pi \left(1 + \sum_{k=0}^{n} \eta_k R_{kt}\right), \qquad t = 1, 2, \ldots, T$$

where the random variable

$$R_{kt} = \Delta S_{kt} / S_{k,t-1} = (S_{kt}/S_{k,t-1}) - 1$$

represents the return on security k over the period $(t-1, t]$.

c) Show that the discounted value process satisfies

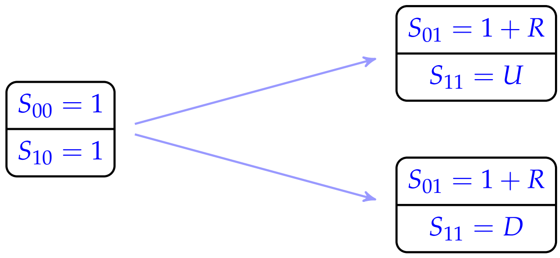
$$\hat{X}_t^\pi = \hat{X}_{t-1}^\pi \left(1 + \sum_{k=1}^{n} \eta_k \hat{R}_{kt}\right), \qquad t = 1, 2, , \ldots, T,$$

where

$$\hat{R}_{kt} = \Delta \hat{S}_{kt}/S_{k,t-1} = (\hat{S}_{kt}/\hat{S}_{k,t-1}) - 1$$

Solve this equation by providing an expression for the discounted value process that only depends on η and $(\hat{R}_{ks})_{s=1,k=0}^{t,n}$.

Exercise 7. Consider the one period model represented as

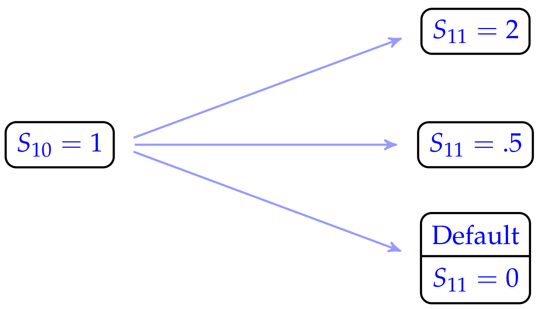


for some constants $R > 0$ and $D < U$. Assume that trading $|\pi|$ units of the risky asset at time zero entails a transaction cost of $\varepsilon|\pi|S_{10}$ for some constant $\varepsilon \in [0, 1)$.

a) Define an arbitrage opportunity in this model with transaction cost.

b) Show that the market is arbitrage free if and only if $D < \alpha S_{10}$ and $U > \beta S_{10}$ for some α, $\beta > 0$ to be determined.

Exercise 8. Let S_{1t} denote the price of one share of SwissQuote stock, consider the one period model represented as



and assume that the market price of the riskless asset is equal to 1 at time zero and to $S_{01} = 1$ at time one (i.e. $r = 0$).

a) Show that this single period model is free of arbitrage opportunities.

b) Assume that in addition to the riskless asset and the stock you can also trade a zero coupon bond with maturity $T = 1$ and initial price P_0 that pays 1 at date T in all states where SwissQuote is alive and zero otherwise. Compute the maximal value of P_0 such that the market remains arbitrage free after the introduction of the zero coupon bond.

Exercise 9. Consider a one period model with two stocks. Assume that the interest rate is equal to zero and that the price of stock $k \in \{1, 2\}$ satisfies

$$S_{k1} = \xi_k S_{k0}$$

where the random variables (ξ_1, ξ_2) are iid with

$$\xi_k \in \left\{ \frac{1}{U_k}, U_k \right\}$$

for some arbitrary constants $U_1, U_2 \neq 1$.

a) How many possible states of nature are there at time 1?

b) Show that this model is free of arbitrage opportunities by computing the set $\mathcal{M}^e$ of *equivalent* martingale measures.

Solutions

Basic questions

Question 1. Assume that π is self-financing and recall that this means

$$\sum_{k=0}^{n} (\pi_{kt} - \pi_{kt-1}) S_{k,t-1} = 0, \qquad t \geq 1. \tag{2}$$

Using this condition we find that

$$\begin{aligned}\Delta X_t^{\pi} &= \sum_{k=0}^{n} \pi_{kt} S_{kt} - \sum_{k=0}^{n} \pi_{kt-1} S_{kt-1} \\ &= \sum_{k=0}^{n} \pi_{kt} S_{kt} - \sum_{k=0}^{n} \pi_{kt} S_{kt-1} = \sum_{k=0}^{n} \pi_{kt} \Delta S_{kt} \end{aligned} \tag{3}$$

which is the required result. Conversely, if the value process of the strategy π satisfies this difference equation then

$$\begin{aligned}
X_t^\pi &= \sum_{k=0}^{n} \pi_{kt} S_{kt} \\
&= X_{t-1}^\pi + \Delta X_t^\pi \\
&= \sum_{k=0}^{n} \pi_{kt-1} S_{kt-1} + \sum_{k=0}^{n} \pi_{kt} \Delta S_{kt}
\end{aligned}$$

and rearranging the equality between the quantities in blue gives (2).

Like the characterization we derived in class, the interpretation of (3) is that gains and losses on a self-financing trading strategy only result from changes in the prices of the traded securities. But this characterization of self-financing strategies is less useful because it *still **depends on all the components** of the trading strategy*. By contrast, the relation we proved in class only depends on the risky part of the portfolio and thus allows us to eliminate a degree of freedom: We can pick any risky part and define implicitly the riskless part by the requirement that the whole strategy be self-financing given its initial value.

Question 2. To determine whether the given model is arbitrage free let us construct the set $\mathcal{M}^e$. To this end we have to solve the following system

$$\begin{aligned}
q_1 + q_2 + q_3 &= 1 \qquad (\mathbb{Q} \text{ is a probability}) \\
7q_1 + 5q_2 + 3q_3 &= 5 \qquad (\hat{S} \text{ is a } \mathbb{Q}-\text{martingale})
\end{aligned}$$

subject to $q_i \in (0, 1)$. The solution to this system is non empty (so that the market is arbitrage free) and given by

$$\begin{aligned}
\mathcal{M}^e &= \left\{ \mathbb{Q} : \mathbb{Q}(\{\omega_2\}) = q,\, \mathbb{Q}(\{\omega_1\}) = \mathbb{Q}(\{\omega_3\}) = \frac{1-q}{2} \text{ for some } 0 < q < 1 \right\} \\
&= \{\mathbb{Q} : \mathbb{Q} = \mathbb{Q}^q \text{ for some } 0 < q < 1\}
\end{aligned}$$

To show that the extended market is arbitrage free it suffices to construct a probability measure $\mathbb{Q} \sim \mathbb{P}$ such that **both** the discounted stock price and the discounted option price are martingales. By definition such a probability must belong to the set $\mathcal{M}^e$ of EMM for the initial market. Therefore, finding such a probability amounts to finding a constant $q^* \in (0,1)$ such that

$$c_0 = 0.9 = \frac{1}{S_{01}} E^{\mathbb{Q}^{q^*}}\left[\left(S_{11} - \frac{50}{9}\right)^+\right] = \left(\frac{9}{10}\right)\left(\frac{20}{9}\right)\mathbb{Q}^{q^*}(\{\omega_1\}) = 1 - q^*.$$

The unique solution to this equation is given by $q^* = 0.1 \in (0,1)$. This shows that $\mathbb{Q}^{q^*}$ is an EMM for the extended market and it follows that this market is arbitrage free. More generally, the above calculation shows that any price $c_0 \in (0,1)$ for the call leads to an extended market that is arbitrage free.

Question 3. Since the random variable Z is strictly positive (in all states) we have that $\mathbb{Q}$ is equivalent to $\mathbb{P}$. To show that it defines a probability measure it suffices to compute its total mass:

$$\begin{aligned}
\mathbb{Q}(\Omega) &= \sum_{k=1}^{9} \mathbb{Q}(\{\omega_k\}) = \sum_{k=1}^{9} Z(\omega_k)\mathbb{P}(\{\omega_k\}) = E^{\mathbb{P}}[Z] \\
&= \sum_{k=1}^{3} Z(\omega_k)\mathbb{P}(U)\mathbb{P}(\omega_k|U) + \sum_{k=4}^{6} Z(\omega_k)\mathbb{P}(M)\mathbb{P}(\omega_k|M) \\
&+ \sum_{k=7}^{9} Z(\omega_k)\mathbb{P}(D)\mathbb{P}(\omega_k|D) = 1.
\end{aligned}$$

Now compute the required probabilities we start by computing

$$\begin{aligned}
\mathbb{Q}(U) &= \sum_{k=1}^{3} Z(\omega_k)\mathbb{P}(U)\mathbb{P}(\omega_k|U) \\
&= 0.5(1.44 \cdot 0.5 + 1.2 \cdot 0.25 + 0.72 \cdot 0.25) = 0.6 \\
\mathbb{Q}(M) &= \sum_{k=4}^{6} Z(\omega_k)\mathbb{P}(M)\mathbb{P}(\omega_k|M) \\
&= 0.25(1.2 \cdot 0.5 + 1 \cdot 0.25 + 0.6 \cdot 0.25) = 0.25 \\
\mathbb{Q}(D) &= \sum_{k=7}^{9} Z(\omega_k)\mathbb{P}(D)\mathbb{P}(\omega_k|D) \\
&= 0.25(0.72 \cdot 0.5 + 0.6 \cdot 0.25 + 0.36 \cdot 0.25) = 0.15
\end{aligned}$$

and then use Bayes' rule to obtain the conditional probabilities. At the up node of the tree at time 1 we obtain

$$\begin{aligned}
\mathbb{Q}(\omega_1|U) &= \frac{\mathbb{Q}(\{\omega_1\})}{\mathbb{Q}(U)} = \frac{Z(\omega_1)\mathbb{P}(U)\mathbb{P}(\omega_1|U)}{\mathbb{Q}(U)} = \frac{1.44 \cdot 0.5 \cdot 0.5}{0.6} = 0.6 \\
\mathbb{Q}(\omega_2|U) &= \frac{Z(\omega_2)\mathbb{P}(U)\mathbb{P}(\omega_2|U)}{\mathbb{Q}(U)} = \frac{1.20 \cdot 0.5 \cdot 0.25}{0.6} = 0.25 \\
\mathbb{Q}(\omega_3|U) &= \frac{Z(\omega_3)\mathbb{P}(U)\mathbb{P}(\omega_3|U)}{\mathbb{Q}(U)} = \frac{0.72 \cdot 0.5 \cdot 0.25}{0.6} = 0.15
\end{aligned}$$

At the middle node of the tree at time one we obtain:

$$\begin{aligned}
\mathbb{Q}(\omega_4|M) &= \frac{\mathbb{Q}(\{\omega_4\})}{\mathbb{Q}(M)} = \frac{Z(\omega_4)\mathbb{P}(M)\mathbb{P}(\omega_4|M)}{\mathbb{Q}(M)} = \frac{1.2 \cdot 0.25 \cdot 0.5}{0.25} = 0.6 \\
\mathbb{Q}(\omega_5|M) &= \frac{Z(\omega_5)\mathbb{P}(M)\mathbb{P}(\omega_5|M)}{\mathbb{Q}(M)} = \frac{1 \cdot 0.25 \cdot 0.25}{0.25} = 0.25 \\
\mathbb{Q}(\omega_6|M) &= \frac{Z(\omega_6)\mathbb{P}(M)\mathbb{P}(\omega_6|M)}{\mathbb{Q}(M)} = \frac{0.6 \cdot 0.25 \cdot 0.25}{0.25} = 0.15
\end{aligned}$$

Finally, at the down node of the tree at time one we obtain:

$$\begin{aligned}
\mathbb{Q}(\omega_7|D) &= \frac{\mathbb{Q}(\{\omega_7\})}{\mathbb{Q}(D)} = \frac{Z(\omega_7)\mathbb{P}(D)\mathbb{P}(\omega_7|D)}{\mathbb{Q}(D)} = \frac{0.72 \cdot 0.25 \cdot 0.5}{0.15} = 0.6 \\
\mathbb{Q}(\omega_8|D) &= \frac{Z(\omega_8)\mathbb{P}(D)\mathbb{P}(\omega_8|D)}{\mathbb{Q}(D)} = \frac{0.6 \cdot 0.25 \cdot 0.25}{0.15} = 0.25 \\
\mathbb{Q}(\omega_9|D) &= \frac{Z(\omega_9)\mathbb{P}(D)\mathbb{P}(\omega_9|D)}{\mathbb{Q}(D)} = \frac{0.36 \cdot 0.25 \cdot 0.25}{0.15} = 0.15.
\end{aligned}$$

Exercises

Exercise 4.

a) Combining the self-financing condition and the at time $t-1$ and the fact that $\pi_{k,t} = 0$ for $k \geq 2$ shows that we have

$$X_{t-1} = \pi_{0t}S_{0t-1} + \pi_{1,t}S_{t-1} = \pi_{0t}e^{r\Delta(t-1)} + \pi_{1,t}S_{1,t-1}.$$

Using the definition of the trading strategy and solving for the riskless part of the portfolio then gives

$$\pi_{0t}S_{0t-1} = X_{t-1} - \pi_{1,t}S_{t-1} = X_{t-1} - (\alpha - \beta X_{t-1}).$$

and the result now follows by setting $b = -\alpha$ and $a = (1+\beta)$.

b) Using the fact that $\pi_{k,t} = 0$ for $k \geq 2$ shows that the value of the portfolio at time $t \geq 1$ is given by

$$\begin{aligned}
X_t = \pi_{0t}S_{0t} + \pi_{1,t}S_{1,t} &= e^{r\Delta}((1+\beta)X_{t-1} - \alpha) + (\alpha - \beta X_{t-1})\left(S_{1,t}/S_{1,t-1}\right) \\
&= e^{r\Delta}\left[((1+\beta)X_{t-1} - \alpha) + (\alpha - \beta X_{t-1})(1+R_t)\right] \\
&= e^{r\Delta}\left[(1-\beta R_t)X_{t-1} + \alpha R_t\right] \qquad (4)
\end{aligned}$$

and the required result follows by setting $f(x) = e^{r\Delta}\alpha x$ and $g(x) = e^{r\Delta}(1-\beta x)$. Alternatively, using the self-financing condition we have that the discounted value process of the strategy satisfies the difference equation

$$\hat{X}_t = \hat{X}_{t-1} + \pi_{1,t}\left(\hat{S}_{1,t} - \hat{S}_{1,t-1}\right) = \hat{X}_{t-1} + \pi_{1,t}\hat{S}_{1,t-1}R_t$$

and the required result follows by substituting the definition of π_t.

c) Applying (4) recursively shows that

$$X_t = \Phi(x; R_1, \ldots, R_t)$$

with the function

$$\begin{aligned}
\Phi(x; R_1, \ldots, R_t) &= f(R_t) \\
&\quad + x\left(\prod_{i=1}^{t} g(R_i)\right) + \sum_{i=1}^{t-1} f(R_i)\left(\prod_{j=i+1}^{t} g(R_j)\right).
\end{aligned}$$

and it now follows from question a) that the riskless part of the portfolio can be calculated as $\pi_{0t} = -\alpha + (1+\beta)\Phi(x; R_1, \ldots, R_{t-1})$.

Exercise 5. The 1M forward price is

$$F_0(1M) = \frac{S_0 - D_0(1M)}{B_0(1M)} = \frac{100}{0.99} = 101.01$$

where the second equality follows from the fact that $D_0(1M) = 0$. On the other hand, by the put/call parity we have

$$\begin{aligned}
8 - 0.5 &= 100 - 90B_0(3M) - D_0(3M) \\
0.25 - 16.5 &= 100 - 115B_0(3M) - D_0(3M).
\end{aligned}$$

This yields

$$B_0(3M) = 0.95 \text{ and } D_0(3M) = 7$$

and it follows that the 3M forward price is

$$F_0(3M) = \frac{S_0 - D_0(3M)}{B_0(3M)} = \frac{1}{0.95}(100 - 7) = 97.89$$

It follows from the put/call parity that the price of the 1M European call is given by

$$c_0(1M) = p_0(1M) + S_0 - D_0(1M) - B_0(1M)k = 1 + 100 - 100 \times 0.99 = 2$$

Finally, the value of the 3M European put with strike 110 is given by the put/call parity

$$\begin{aligned} p_0(3M) &= c_0(3M) - S_0 + D_0(3M) + B_0(3M)k \\ &= 0.5 - 100 + 7 + 110 \times 0.95 = 12. \end{aligned}$$

The completed table is thus given by

Security	Strike	Maturity	Price
IBM Stock	–	–	100
Zero coupon bond	–	1M	0.99
Zero coupon bond	–	3M	**0.95**
Forward price IBM stock	–	1M	**101.01**
Forward price IBM stock	–	3M	**97.89**
European Call on IBM	100	1M	**2**
European Call on IBM	90	3M	8.0
European Call on IBM	110	3M	0.5
European Call on IBM	115	3M	0.25
European Put on IBM	100	1M	1.0
European Put on IBM	90	3M	0.5
European Put on IBM	110	3M	**12**
European Put on IBM	115	3M	16.5

Exercise 6.

a) This type of trading strategy is referred to as a constant proportions strategy because the fraction

$$\frac{\pi_{kt}S_{k,t-1}}{X_{t-1}^{\pi}} = \eta_k$$

of the portfolio value invested in each of the traded asset is reset to a (asset-specific) constant at each rebalancing date. To establish that the strategy is self financing we need to show that

$$\sum_{k=0}^{n}(\pi_{k,t+1} - \pi_{kt})S_{kt} = 0.$$

To this end we observe that

$$\begin{aligned} \sum_{k=0}^{n}(\pi_{kt} - \pi_{k,t+1})S_{kt} &= \sum_{k=0}^{n}\left(\pi_{kt} - \frac{\eta_k X_t^{\pi}}{S_{kt}}\right)S_{kt} \\ &= X_t^{\pi} - X_t^{\pi}\sum_{k=0}^{n}\eta_k = 0. \end{aligned}$$

where the last equality follows from the fact that $\sum_k \eta_k = 1$.

b) The initial value is $X_0^{\pi} = 1$ by assumption and we have

$$\begin{aligned} X_t^{\pi} &= \sum_{k=0}^{n}\frac{\eta_k X_{t-1}^{\pi}}{S_{k,t-1}}S_{k,t} \\ &= X_{t-1}^{\pi}\sum_{k=0}^{n}\eta_k\left(\frac{S_{k,t} - S_{k,t-1}}{S_{k,t-1}} + 1\right) \\ &= X_{t-1}^{\pi}\left(\sum_{k=0}^{n}\eta_k + \sum_{k=0}^{n}\eta_k R_{k,t}\right) = X_{t-1}^{\pi}\left(1 + \sum_{k=0}^{n}\eta_k R_{k,t}\right) \end{aligned}$$

where the last equality follows from the assumption that $\sum_k \eta_k = 1$.

c) By definition we have that $\hat{X}_0^\pi = 1$. Moreover,

$$\begin{aligned}
\hat{X}_t^\pi &= \sum_{k=0}^{n} \pi_{kt}\hat{S}_{kt} \\
&= \sum_{k=0}^{n} \frac{\eta_k X_{t-1}^\pi}{S_{k,t-1}}\hat{S}_{kt} = \sum_{k=0}^{n} \frac{\eta_k \hat{X}_{t-1}^\pi}{\hat{S}_{k,t-1}}\hat{S}_{kt} \\
&= \hat{X}_{t-1}^\pi \left(\sum_{k=0}^{n} \eta_k + \sum_{k=0}^{n} \eta_k \hat{R}_{k,t} \right) = \hat{X}_{t-1}^\pi \left(1 + \sum_{k=0}^{n} \eta_k \hat{R}_{kt} \right)
\end{aligned}$$

where the last equality follows from the fact that $\sum_k \eta_k = 1$. Iterating this relation then shows that we have

$$\begin{aligned}
\hat{X}_t^\pi &= \hat{X}_{t-1}^\pi \left(1 + \sum_{k=0}^{n} \eta_k \hat{R}_{kt} \right) \\
&= \hat{X}_{t-2}^\pi \left(1 + \sum_{k=0}^{n} \eta_k \hat{R}_{kt-1} \right) \left(1 + \sum_{k=0}^{n} \eta_k \hat{R}_{kt} \right) \\
&= \cdots = \prod_{\tau=1}^{t} \left(1 + \sum_{k=0}^{n} \eta_k \hat{R}_{k,\tau} \right).
\end{aligned}$$

Exercise 7.

a) Suppose that you invest in π_{01} units of the riskless asset and π_{11} units of the stock at time zero. The *cost* of setting up this portfolio is

$$X_0^\pi + \epsilon|\pi_{11}|S_{10} = \pi_{01} + \pi_{11}S_{10} + \epsilon|\pi_{11}|S_{10}$$

where the last term takes into account the transaction cost on the risky asset. At the terminal time the liquidation value of this portfolio is simply

$$X_1^\pi = \pi_{01}S_{01} + \pi_{11}S_{11}$$

since there is no transaction cost at the terminal date (see below for the case where there are). In this setting an arbitrage opportunity is defined as a vector

$\pi = (\pi_{01}, \pi_{11})$ such that

$$X_0^\pi + \epsilon|\pi_{11}|S_{10} = 0 \Longleftrightarrow \pi_{01} = -(\pi_{11} + \epsilon|\pi_{11}|)S_{10}, \tag{6}$$
$$\mathbb{P}[X_1^\pi \geq 0] = 1$$
$$\mathbb{P}[X_1^\pi > 0] > 0.$$

b) Under condition (6) we have that

$$X_1^\pi = \pi_{11}[S_{11}(\omega) - S_{10}(1+R)(1 - \epsilon 1_{\{\pi_{11}<0\}} + \epsilon 1_{\{\pi_{11}\geq 0\}})]$$

and it follows that the model with initial transaction costs admits an arbitrage opportunity if and only if either

$$(A) : \begin{cases} U - S_{10}(1+R)(1+\epsilon) > 0 \\ D - S_{10}(1+R)(1+\epsilon) \geq 0 \end{cases}$$

or

$$(B) : \begin{cases} U - S_{10}(1+R)(1-\epsilon) \leq 0 \\ D - S_{10}(1+R)(1-\epsilon) < 0. \end{cases}$$

Since $U > D$ by assumption we have that

$$(A) \Longleftrightarrow (A') : D - S_{10}(1+R)(1+\epsilon) \geq 0$$
$$(B) \Longleftrightarrow (B') : U - S_{10}(1+R)(1-\epsilon) \leq 0$$

and it follows that the model with initial transaction cost is arbitrage free if and only if the up and down factors are such that

$$D < S_{10}(1+R)(1+\epsilon) = \alpha S_{10}$$
$$U > S_{10}(1+R)(1-\epsilon) = \beta S_{10}.$$

If $S_{10} \leq D/\alpha$ then the return on a long position in the risky asset dominates the return on the riskless asset in all states and an arbitrage can be obtained by borrowing at the riskfree rate and investing the proceeds in the risky asset. Similarly, if $S_{10} > U/\beta$ then the return on a short position in the risky asset is dominated by the return on the riskless in all states and an arbitrage can be obtained by shorting the stock and investing the proceeds in the riskless asset.

If the liquidation of the portfolio at time $t = 1$ incurs a transaction cost of γS_{11} per unit (long or short) of the risky asset then an arbitrage opportunity corresponds to a vector $\pi = (\pi_{01}, \pi_{11})$ such that

$$X_0^\pi + \epsilon|\pi_{11}|S_{10} = 0 \Longleftrightarrow \pi_{01} = -(\pi_{11} + \epsilon|\pi_{11}|)S_{10},$$
$$\mathbb{P}[X_1^\pi - \gamma|\pi_{11}|S_{11} \geq 0] = 1$$
$$\mathbb{P}[X_1^\pi - \gamma|\pi_{11}|S_{11} > 0] > 0.$$

Going through the same steps as in the previous case then shows that the model is arbitrage free if and only the up and down factors are such that

$$D < S_{10}(1+R)\left(\frac{1+\epsilon}{1-\gamma}\right) = \bar{\alpha}S_{10}$$
$$U > S_{10}(1+R)\left(\frac{1-\epsilon}{1+\gamma}\right) = \bar{\beta}S_{10}.$$

Exercise 8.

a) To determine $\mathcal{M}^e$ we have to solve

$$2q_1 + 0.5q_2 = 1$$
$$q_1 + q_2 + q_3 = 1$$

subject to $q_i \in (0,1)$. The solutions are

$$q_1 = \frac{1+q}{3}$$
$$q_2 = \frac{2-4q}{3}$$
$$q_3 = q$$

where $q \in (0, 1/2)$.

b) Assume that the bond is traded at price p. To show that the extended market is arbitrage free it is enough to show that it admits a EMM which is equivalent to

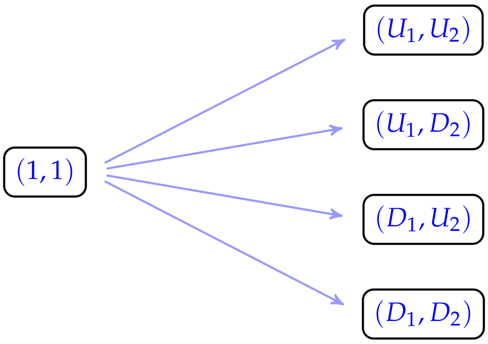


Figure 2: Tree for Exercise 9

the existence of a solution $q \in (0,1)^3$ to the system

$$\begin{aligned} q_1 + q_2 + q_3 &= 1 && (\mathbb{Q} \text{ is a probability measure}) \\ 2q_1 + q_2/2 &= 1 && (\hat{S} \text{ is a } \mathbb{Q}\text{−martingale}) \\ q_1 + q_2 &= p && (\hat{P} \text{ is a } \mathbb{Q}\text{−martingale}) \end{aligned}$$

For a given p the unique solution to this system is

$$q(p) = \left(\frac{2-p}{3}, \frac{4p-2}{3}, 1-p \right)$$

and it follows that the extended market is arbitrage free if and only if the bond is traded at an initial price $0.5 < p < 1$.

Exercise 9.

a) Let $D_k = 1/U_k$. As shown in Figure 2 there are four different states of natures at time 1.

b) We have two risky assets and one riskless asset. As illustrated by Figure 9, there are four possibilities going from date 0 to date 1: (U_1, U_2), (U_1, D_2), (D_1, U_2) and (D_1, D_2). Therefore, the system of equations to be solved for the equivalent martingale measures can be written as

$$\begin{aligned}
q_{uu} + q_{ud} + q_{du} + q_{dd} &= 1 && (\mathbb{Q} \text{ is a probability}), \\
(q_{uu} + q_{ud})U_1 + (q_{du} + q_{dd})\frac{1}{U_1} &= 1 && (\hat{S}_1 \text{ is a } \mathbb{Q}\text{−martingale}), \\
(q_{uu} + q_{du})U_2 + (q_{ud} + q_{dd})\frac{1}{U_2} &= 1 && (\hat{S}_2 \text{ is a } \mathbb{Q}\text{−martingale})
\end{aligned}$$

subject to $0 < q_{kj} < 1$ for $k, j \in \{u, d\}$. Straightforward algebra shows that the solutions to this system are given by

$$\begin{aligned}
q_{uu} &= q, \\
q_{du} &= \frac{1}{1+U_2} - q, \\
q_{ud} &= \frac{1}{1+U_1} - q, \\
q_{dd} &= \frac{U_1}{1+U_1} - \frac{1}{1+U_2} + q
\end{aligned}$$

for any strictly positive constant

$$q < \min\left\{\frac{1}{1+U_1}, \frac{1}{1+U_2}\right\}.$$