

# Investments Class

## Problem set 1

Daniel Grosu, William Martin, Denis Steffen

February 29, 2020

### Exercise 1. (Utility Theory)

From the statement, we know that the random variables  $Y$  and  $\epsilon$  have the following distributions:

$$Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2), \quad \epsilon \sim \mathcal{N}(-\mu_\epsilon, \sigma_\epsilon^2)$$

but they are not independent and have correlation  $\rho$ . The covariance between the two is  $\text{Cov}(Y, \epsilon) = \rho \sqrt{\sigma_Y^2} \sqrt{\sigma_\epsilon^2} = \rho \sigma_Y \sigma_\epsilon$ .

We have to consider the random vector  $\mathbf{Z} = (Y, \epsilon)^\top$  which has multivariate gaussian distribution because every linear combination of the components  $\alpha Y + \beta \epsilon$  is normally distributed. Thus:

$$\mathbf{Z} \sim \mathcal{N}_2 \left( \begin{pmatrix} \mu_Y \\ -\mu_\epsilon \end{pmatrix}, \begin{bmatrix} \sigma_Y^2 & \rho \sigma_Y \sigma_\epsilon \\ \rho \sigma_Y \sigma_\epsilon & \sigma_\epsilon^2 \end{bmatrix} \right)$$

First, we can calculate the relative risk aversion and the absolute risk aversion:

$$\text{ARA}(W) = -\frac{u''(W)}{u'(W)} = \frac{a^2 e^{-aW}}{a e^{-aW}} = a, \quad \text{RRA}(W) = -\frac{W u''(W)}{u'(W)} = -aW$$

since  $u'(W) = a e^{-aW}$  and  $u''(W) = -a^2 e^{-aW}$ .

Then, we want to compute the insurance premium  $\pi$  to insure the risk of  $\epsilon$  such that:

$$\mathbb{E}[u(Y + \epsilon)] = \mathbb{E}[u(Y - \pi)]$$

We can first compute the right hand side expectation using the moments generating function of a multivariate gaussian random variable  $\mathcal{N}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ :

$$\text{MGF}_{\mathbf{X}}(\mathbf{t}) = \mathbb{E}[e^{\mathbf{t}^\top \mathbf{Z}}] = \exp(\mathbf{t}^\top \boldsymbol{\mu} + \frac{1}{2} \mathbf{t}^\top \boldsymbol{\Sigma} \mathbf{t})$$

Thus,

$$\begin{aligned} \mathbb{E}[e^{-a(Y+\epsilon)}] &= \mathbb{E}[e^{-a\mathbf{1}^\top \mathbf{Z}}] = \exp(-a\mathbf{1}^\top \boldsymbol{\mu} + \frac{1}{2} a^2 \mathbf{1}^\top \boldsymbol{\Sigma} \mathbf{1}) \\ &= \exp(-a(\mu_Y - \mu_\epsilon) + \frac{1}{2} a^2 (\sigma_Y^2 + 2\rho \sigma_Y \sigma_\epsilon + \sigma_\epsilon^2)) \end{aligned}$$

On the other side, we can see that:

$$\begin{aligned} \mathbb{E}[e^{-a(Y-\pi)}] &= e^{a\pi} \mathbb{E}[e^{-aY}] \\ &= e^{a\pi} e^{-a\mu_Y + \frac{1}{2} a^2 \sigma_Y^2} \end{aligned}$$

Combining the two, we get:

$$\begin{aligned}
 e^{-a(\mu_Y - \mu_\epsilon) + \frac{1}{2}a^2(\sigma_Y^2 + 2\rho\sigma_Y\sigma_\epsilon + \sigma_\epsilon^2)} &= e^{a\pi} e^{-a\mu_Y + \frac{1}{2}a^2\sigma_Y^2} \\
 \iff e^{a\pi} &= e^{a\mu_\epsilon + \frac{1}{2}a^2(2\rho\sigma_Y\sigma_\epsilon + \sigma_\epsilon^2)} \\
 \iff \pi &= \mu_\epsilon + \frac{1}{2}a(2\rho\sigma_Y\sigma_\epsilon + \sigma_\epsilon^2)
 \end{aligned}$$

Finally, the maximum insurance premium we are willing to pay to insure the risk of  $\epsilon$  is  $\mu_\epsilon + \frac{1}{2}a(2\rho\sigma_Y\sigma_\epsilon + \sigma_\epsilon^2)$ .

Furthermore, we can study how the premium evolves according to the five parameters of the random variable distributions. We differentiate  $\pi$  with respect to these variables:

- $\frac{\partial \pi}{\partial \mu_\epsilon} = 1$
- $\frac{\partial \pi}{\partial a} = \rho\sigma_Y\sigma_\epsilon + \sigma_\epsilon^2$
- $\frac{\partial \pi}{\partial \rho} = a\sigma_Y\sigma_\epsilon$
- $\frac{\partial \pi}{\partial \sigma_\epsilon} = a\rho\sigma_Y + \sigma_\epsilon$
- $\frac{\partial \pi}{\partial \sigma_Y} = a\rho\sigma_\epsilon$

## Exercise 2. (Value at Risk, Expected Shortfall, and Expected Utility)

- (a) In order to compute the moments of the given distribution, one can use the built-in NumPy and SciPy functionality for samples. It is easy to see that by creating a sample in which the prior probabilities are the same as the ones given, the sample and the distribution moments will be the same. This is, however, only applicable for discrete distributions.

```

1  ### Imports
2
3  import pandas as pd;
4  import numpy as np;
5  import matplotlib.pyplot as plt;
6
7  import scipy.stats as stats;
8
9  ### Generating the Sample
10
11 pr = np.array([0.025, 0.05, 0.10, 0.10, 0.10, 0.2, 0.2, 0.2, 0.025]);
12 ret = np.array([-0.4, -0.2, -0.1, 0, 0.05, 0.1, 0.15, 0.20, 0.30]);
13
14
15 sample = np.array([]);
16
17 # Generate the sample with the given priors
18 for i in range(pr.size):
19     # print(pr[i]);
20     for j in range(int(1000*pr[i])):
21         sample = np.append(sample, ret[i])
22
23
24 ### Plotting the sample (visualization)
25

```

```

26 plt.figure();
27 plt.hist(sample);
28 plt.show();
29
30
31 ### Sub-item (a): Compute the moments of the distribution
32
33 print(round(np.mean(sample), 3))
34 print(round(np.std(sample), 3))
35 print(round(stats.skew(sample), 3))
36 print(round(stats.kurtosis(sample), 3))
37

```

As such, the computed values are:

$\mu$	0.072
$\sigma$	0.137
$g$	-1.323
$k$	1.953

Figure 1: The moments of the distribution

From the moments, we observe that the mean of the distribution is positive, but the distribution is negatively-skewed. For example, the lowest (negative) return is bigger in absolute value than the highest possible positive return. The kurtosis of any univariate normal distribution is 3. *Excess kurtosis* is defined as the kurtosis minus 3. In our case the excess kurtosis is negative which classifies this distribution as a *platykurtic* or sometimes called sub-Gaussian distribution. This means that the tales are thinner and sampling from it produces fewer and less extreme outliers than does the normal distribution.

- (b) The Value-at-Risk  $\text{VAR}_p$  is the loss  $l$  such that the probability of a loss higher than  $l$  is bounded by  $p$ . For our discrete distribution this equivalent to the quantile for which the left tail has a total probability of  $p$ . For a discrete distribution, the values at risk are also discrete and as  $p$  increases from 0 to 1, the VAR increases when sum of the left prefix of the probabilities (left tail) becomes smaller than  $p$ . As such,  $\text{VAR}_{1\%} = -.40 \times \$100 = -\$40$  and  $\text{VAR}_{5\%} = -.20 \times \$100 = -\$20$  because  $1\% < 0.025$  and  $0.025 < 5\% < 0.075$ .
- (c) By definition, the Expected Shortfall (ES) or the Conditional Tail Expectation (CTE) is the mean of the values at risk with probability smaller than  $p$ :

$$\text{ES}_p = \frac{1}{p} \int_0^p \text{VAR}_\gamma d\gamma$$

For the distribution of asset returns it holds that:

$$\text{VAR}_\gamma = \begin{cases} -\$40 & \gamma < 0.025 \\ -\$20 & 0.025 < \gamma < 0.075 \end{cases}$$

Hence, the Expected Shortfall at 5% equals:

$$\text{ES}_{5\%} = \frac{1}{5\%} \int_0^{5\%} \text{VAR}_\gamma d\gamma = \frac{1}{5\%} (0.025 \times (-\$40) + 0.025 \times (-\$20)) = -\$30.$$

- (d) The break-even risk-free rate can be computed by equating the utility of the two scenarios: portfolio with all wealth invested in the risky asset or the risk-free rate:

$$\mathbb{E}[u((1+r) \times W_0)] = u(r_f \times W_0)$$

where the Constant Elasticity of Substitution (CES) utility function is

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}, \quad \gamma = 2$$

Solving for  $r_f$ :

$$r_f = \frac{u^{-1}(\mathbb{E}[u((1+r) \times W_0)])}{W_0}$$

The following Python code computes the equivalent risk-free rate to stand at  $r_f = 5.03\%$ . That is, given the customer's risk-averseness, he would be willing to forgo  $\mu - r_f = 2.22\%$  in expected return if the risk is eliminated. The level of risk-free rate above which she would choose to invest all her wealth in the risk-free asset is 5.03%. This cut-off rate decreases with the increase in the risk-aversion coefficient:

$\gamma$	$r_f$
2	5.03%
3	3.59%
4	1.87%
5	-0.14%
6	-2.46%
7	-5.01%
8	-7.68%
9	-10.34%
10	-12.88%
11	-15.21%
12	-17.31%
13	-19.16%
14	-20.79%

The break-even rate decreases because the investor becomes more risk-averse which translates into a lower risk-free rate that he would accept just to eliminate any risk.

```

1  ### Part (d) Utility
2
3  def utility(w, gamma = 2):
4      return np.power(w, 1 - gamma)/(1 - gamma)
5
6  def inverse_utility(u, gamma = 2):
7      return np.exp(np.log(u*(1 - gamma))/(1 - gamma))
8
9
10 # Test the inverse utility function
11 # print(inverse_utility(utility(400)));
12
13 # 100% either in risky or risk-free

```

```

14
15 gamma_range = range(2, 15)
16 rf = pd.Series(data = np.zeros(len(gamma_range)),
17               index = gamma_range,
18               name = "rate")
19
20 W0 = 100
21
22 rf = [inverse_utility(np.mean([utility((1 + x) * W0, ra) for x in sample])
23       , ra)/W0 - 1 for ra in gamma_range]
24
25 """
26 for ra in gamma_range:
27     expected_utility = np.mean([utility((1 + x) * W0, ra) for x in sample
28     ])
29     rf[ra] = inverse_utility(expected_utility, ra) / W0 - 1
30 """
31
32 pf = pd.DataFrame(data = rf, index = gamma_range)
33 pf.columns = ['rate']
34 display(pf.style.format({"rate": lambda x: "{:.2%}".format(x)}))
35
36 print('The breakeven risk-free asset is %.2f%%' % (pf.loc[2] * 100))
37
38 print('The investor is willing to forgo %.2f%% in expected return' % ((mu
39     - pf.loc[2])*100))
40
41

```

- (e) If she can invest a fraction of her wealth in the risk-free asset, the rate at which she will not choose to invest anything in the risk-free asset is between 2% and 2.5% which is lower than the risk-free rate found in the previous sub-item. While the  $r_f = 5.03\%$  risk-free rate generates the same amount of utility as the risky asset alone, the two combined is able to produce more utility because the investor compromises between return and risk by investing in the risk-less asset and mitigating the unattractive risk. At the same time, the investor will choose not to invest in the risk-free rate if its rate is too disadvantageous and will choose to invest all of its wealth in the risk-less asset if its return is at least the expected return of the risky asset.

The rate at which she will not choose to invest anything in the risk-free asset is lower than the risk-free rate found in the previous sub-item because if it were higher or equal ( $r'_f \geq r_f$ ), by choosing not to invest in it at all, the utility generated will be  $E[(1 + r) \times W_0]$  while she could have generated more utility by simply investing all  $W_0$  in the risk-free rate:  $u(r'_f \times W_0) \geq u(r_f \times W_0) = E[(1 + r) \times W_0]$ , hence contradiction.

$r_f$	$\pi^* \times 100\%$
0.000	0.00
0.005	0.00
0.010	0.00
0.015	0.00
0.020	0.00
0.025	5.00
0.030	10.00
0.035	20.00
0.040	30.00
0.045	35.00
0.050	45.00
0.055	55.00
0.060	70.00
0.065	80.00
0.070	95.00
0.075	100.00

Figure 2: The optimal fraction of wealth to invest in the risk-free asset under various risk-free rates