Assignment 3

1. **Efficient portfolios (20 points).**

   Consider an investor who wants to invest in N risky assets with return $R_i \ \forall i = 1, \dots, N$ with expected return $E[R_i] = \mu_i$ and variance $V[R_i] = \sigma_i^2$, and in a risk-free asset with return R_f. The investor seeks a N-risky asset portfolio weight vector w (and a weight $1 - w^\top \mathbf{1}$ in the risk-free asset), such that her portfolio return $R_p = R_f + w^\top (R - R_f \mathbf{1})$ maximizes her mean-variance objective function $U(w) = E[R_p] - \frac{\gamma}{2} V[R_p]$.

   - Show that an optimal portfolio weight vector w is such that for the corresponding mean-variance efficient portfolio return R_p we have

     $$\mu_i - R_f = \gamma\, cov[R_i, R_p] \ \ \forall i = 1, \dots N$$

   - Show that for any such mean-variance efficient portfolio, we have

     $$\mu_i - R_f = \beta_{i,P}(\mu_P - R_f)$$

     where $\beta_{i,P} = \frac{cov(R_i, R_P)}{\sigma_P^2}$ is the linear regression coefficient of return R_i on the mean-variance efficient portfolio return R_P.

   - In turn, show that this implies that, if R_p is the return to a mean-variance efficient portfolio, then for any return i we have

     $$R_i = R_f + \beta_i (R_P - R_f) + \epsilon_i$$

     where $cov(R_P, \epsilon_i) = 0$ and $E(\epsilon_i) = 0$.

     Hint: use the definition of a linear regression

   - Show that all mean-variance efficient portfolios have the same Sharpe ratio where we define its Sharpe ratio as $SR_p = \frac{\mu_p - R_f}{\sigma_p}$.

2. **Portfolio math (10 points).**

   - Show that any risky-asset only minimum variance frontier portfolio w can be rewritten as a convex combination of any two arbitrary minimum variance frontier portfolios w_a, w_b in the sense that $w = \alpha w_a + (1 - \alpha) w_b$.

- Let R_{min} denote the return on the global minimum-variance portfolio of risky assets. Let R be the return on any risky asset or portfolio of risky assets, efficient or not. Show that $Cov(R, R_{min}) = Var(R_{min})$. *Hint*: Consider a portfolio consisting of a fraction w in this risky asset. and a fraction $(1 - w)$ in the global minimum-variance portfolio. Compute the variance of the return on this portfolio and realize that the variance has to be minimized for $w = 0$.

3. **Risk Parity (30 points).** The optimal mean-variance portfolio is a complex function of estimated means, volatilities, and correlations of asset returns. There are many parameters to estimate. Optimized mean-variance portfolios can give poor results when these inputs are imperfectly estimated. In their paper "leverage aversion and risk-parity" (posted on moodle) Asness, Frazzini, and Pedersen (AFP) suggest that **risk-parity** allocation, which has become widely popular and ignores information in sample-means, dominates the standard mean-variance portfolio because it exploits leverage aversion of investors. Here we will try to replicate some of their findings.

   - Following AFP download from CRSP the monthly value weighted CRSP Stock index using

     ```
     db.raw_sql("select date,vwretd "
     "from crsp.msi " "where date>='1960-01-01'"
     "and date<='2019-12-31'", date_cols=['date'])
     ```

     and the value weighted Bond index since 1960, using

     ```
     bonds=db.raw_sql("select caldt, b2ret "
     "from crsp.mcti "
     "where caldt>='1960-01-01'"
     "and caldt<='2019-12-31'", date_cols=['caldt'])
     ```

     as well as the 1-month Treasury Bill return. Use

     ```
     db.raw_sql("select mcaldt, tmytm from crsp.tfz_mth_rf where kytreasnox
     = 2000001 and mcaldt>='1960-01-01' and <='2019-12-31'")
     ```

     to select the one-month Treasury Bill rate for the years from 1960 to 2019.[1]

[1]See `http://www.crsp.com/files/treasury_guide_0.pdf` for more infos on the CRSP Treasury risk-free rate database. Note that "mcaldt" denotes the last quotation date of the relevant bond in a given month, i.e. the the yield corresponding to January 31 is the yield that was promised in the beginning of January. Note that the CRSP monthly risk-free rates are continuously compounded and annualized. Thus, you need

Compute the mean return, standard deviation and correlation matrix for these returns over the entire sample period. Using the 1-month T-Bill as the risk-free rate to compute excess returns. To compute excess returns assume that the risk-free rate is constant at the sample average of the T-Bill rate. Find the Tangency portfolio and give its mean, standard deviation and Sharpe ratio. Also compute mean, standard deviation and Sharpe ratio of a portfolio that invests 60% in stocks and 40% in bonds (the 60/40 portfolio).

- Compute the mean, standard deviation and Sharpe ratio of a risk-parity (RP) portfolio that holds stocks and bonds in proportion to the inverse of their (full-sample) volatility such that the portfolio's (full sample) volatility is equal to the volatility of the 60/40 portfolio. Following AFP, also compute the mean, standard deviation and Sharpe ratio of the RP-unlevered portfolio whose weights in stock and bond are rescaled by a constant so that they sum up to 1 (so that the portfolio does not hold the risk-free T-Bill). Plot the RP and RP-unlevered portfolios on the efficient frontier along with the Tangency portfolio and the 60/40 portfolio. What explains the difference between the RP and RP-unlevered portfolio performance?
- Following the notes of figure 1 in AFP (or of table 2 in AFP), note that their RP-strategy is actually computed by rebalancing the portfolio at every month setting the weights in each asset class equal to the inverse of its volatility, estimated by using three-year monthly excess returns up to month $t-1$. Compute the returns to this "rolling-window" RP-strategy. Choose portfolio weights again such that the portfolio's full sample volatility is equal to the volatility of the 60/40 portfolio. Also compute the RP-unlevered strategy returns following AFP. How do the performances of these "rolling-window" RP strategies compare with that of the "full-window" strategies you estimated previously? Are they identical? Why?
- Consider an investor who has mean-variance utility $U = \mu_p - \frac{a}{2}\sigma_p^2$ and a risk aversion coefficient a of 6. Using the full-sample estimates of the means and covariance matrix stocks and bonds, what is her optimal portfolio? What is its expected return, standard deviation, and Sharpe ratio of her optimal portfolio?
- If the mean-variance investor could in addition invest in either the "full-window"

to transform them into simple and monthly rates, in order to use them in your portfolio construction (the stock returns are already simple monthly returns).

RP strategy or the ”rolling-window” RP strategy, would she want to do so? Explain what is AFP’s argument why risk-parity is actually a useful strategy.