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UNIVERSITY OF TEHRAN

Engineering Mathematics

Report for Computer Assignment 2

Partial Differential Equations

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Part 1 : Solving the Heat Equation

The equation that we are trying to solve is this :(M = 2, N = 2, O = 3)

$$u_t = O^2 u_{xx}, \qquad 0 \le x \le M\pi, \qquad t \ge 0$$

$$\begin{cases} u_x(0,t) = e^{-Mt} \\ u_x(2\pi,t) = e^{-Nt} \end{cases}, \qquad u(x,0) = \sin^2(x)$$

Equation

Before we can code the equation, we need to rewrite it in a form that the pdepe solver expects. The standard form that pdepe expects is :

$$c\left(x,t,u,\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial t} = x^{-m}\frac{\partial}{\partial x}\left(x^m f\left(x,t,u,\frac{\partial u}{\partial x}\right)\right) + s\left(x,t,u,\frac{\partial u}{\partial x}\right).$$

```
function [c,f,s] = pdefun1(x,t,u,dudx)
c = 1;
f = 9*dudx;
s = 0;
end
```

Boundary Conditions

For problems posed on the interval $a \le x \le b$, the boundary conditions apply for all t and either $x=a$ or $x=b$. The standard form for the boundary conditions expected by the solver is :

$$p(x,t,u) + q(x,t)\,f\left(x,t,u,\frac{\partial u}{\partial x}\right) = 0.$$

```
function [pl, ql, pr, qr] = bcfun(xl, ul, xr, ur, t)
pl = -9*exp(-2*t);
ql = 1;
pr = -9*exp(-2*t);
qr = 1;
end
```

`

Initial Conditions

$$u(x, t_0) = u_0(x)$$

```
function u = icfun(x)
u = (sin(x))^2;
end
```

Select Solution Mesh

```
x = linspace(0,1,51);
t = linspace(0,1,51);
```

Solve Equation

```
m = 0;
sol = pdepe(m,@pdefun,@icfun,@bcfun,x,t);
```

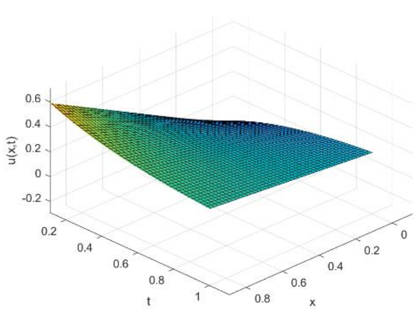
Extract the first solution component from sol.

```
u = sol(:,:,1);
```

Plot Solution

```
surf(x,t,u)
xlabel('x')
ylabel('t')
zlabel('u(x,t)')

xlim([-0.111 0.889])
ylim([0.112 1.112])
zlim([-0.290 0.710])
view([165.396 3.125])
```



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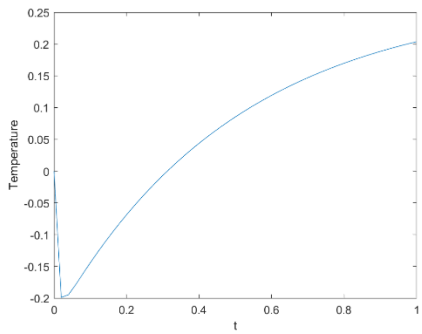
Temperature at the center of the rod

```
theta = u(:,1);
plot(t,theta)
ylabel('Temperature')
xlabel('t')
```



Solving Heat Equation Theoretically

$u_t = 9 u_{nn} \qquad 0 \le n \le 2\pi \qquad t \ge 0$

$$\text{BC:}\begin{cases} u_n(0,t) = e^{-2t} \\ u_n(2\pi,t) = e^{-2t} \end{cases} \qquad \text{IC:}\begin{cases} u(n,0) = \sin^2 n \end{cases}$$

$$u(n,t) = w(n,t) + v(n,t)$$

$$w(n,t) = n\,a(t) + \frac{n^2}{2L}\left[b(t) - a(t)\right]$$

$$= n e^{-2t} + \frac{n^2}{2(2\pi)}\left[\cancel{e^{-2t} - e^{-2t}}_{0}\right] = n e^{-2t}$$

$$u(n,t) = n e^{-2t} + v(n,t)$$

$$\begin{cases} u_t(n,t) = -2n e^{-2t} + v_t(n,t) \\ u_{nn}(n,t) = v_{nn}(n,t) \end{cases}$$

$$u_t = 9 u_{nn} \Rightarrow v_t(n,t) - 2n e^{-2t} = 9 v_{nn}(n,t) \Longrightarrow$$

$$v_t - 9v_{nn}(n,t) = 2n e^{-2t}$$

Initial Condition of the new equation:

$$u(n,0) = \sin^2 n \Rightarrow v(n,0) = \sin^2 n - n$$

$$\boxed{\begin{array}{l} v_t - 9v_{nn}(n,t) = 2n e^{-2t} \\ \text{BC:}\begin{cases} u_n(0,t) = 0 \\ u_n(2\pi,t) = 0 \end{cases} \qquad \text{IC:}\begin{cases} v(n,0) = \sin^2 n - n \end{cases} \end{array}}$$

$$V(x,t) = G_0(t) + \sum_{n=1}^{+\infty} G_n(t)\cos\left(\frac{n\pi x}{L}\right)$$

$$= G_0(t) + \sum_{n=1}^{+\infty} G_n(t)\cos\left(\frac{n\pi x}{2\pi}\right)$$

$$\begin{cases} V_t = \dot{G}_0(t) + \sum_{n=1}^{+\infty} \dot{G}_n(t)\cos\left(\frac{nx}{2}\right) \\ V_{xx} = \sum_{n=1}^{+\infty} -\left(\frac{n}{2}\right)^2 G_n(t)\cos\left(\frac{nx}{2}\right) \end{cases}$$

$$\dot{G}_0(t) + \sum_{n=1}^{+\infty}\left[\dot{G}_n(t) + 9\left(\frac{n}{2}\right)^2 G_n(t)\right]\cos\left(\frac{nx}{2}\right) = 2xe^{-2t}$$

$$\dot{G}_0(t) = \frac{1}{\pi}\int_0^{2\pi} 2xe^{-2t}\cdot dx = \frac{1}{\pi}x^2e^{-2t}\Big|_0^{2\pi} = 4\pi e^{-2t} \Rightarrow \int$$

$$\boxed{G_0(t) = -2\pi e^{-2t} + C}$$

$$\dot{G}_n + \left(\frac{3n}{2}\right)^2 G_n(t) = \frac{2}{2\pi}\int_0^{2\pi} 2xe^{-2t}\cos\left(\frac{nx}{2}\right)\cdot dx$$

$$= \frac{2e^{-2t}}{\pi}\cdot\frac{4\left[\pi n\,\cancel{\sin(\pi n)}^{\,0} + \cos(\pi n) - 1\right]}{n^2}$$

$$= \boxed{\frac{8e^{-2t}}{\pi}\cdot\frac{\left((-1)^n - 1\right)}{n^2}} \quad (\text{ODE}(t))$$

$$\mu(t) = e^{\int\left(\frac{3n}{2}\right)^2\cdot dt} = e^{\left(\frac{3n}{2}\right)^2 t}$$

$$G_n(t) = e^{-\left(\frac{3n}{2}\right)^2 t}\left(\int \frac{8}{\pi}\cdot\frac{(-1)^n - 1}{n^2}e^{-2t}\cdot dt + D_n\right)$$

`

$$G_n(t) = \frac{4}{\pi n^2}\left(-1+(-1)^{n+1}\right) e^{\left[-\left(\frac{3n}{2}\right)^2-2\right]t} + D_n e^{-\left(\frac{3n}{2}\right)^2 t}$$

$$V(x,t) = \left(2\pi e^{-2t} + C\right) + \sum_{n=1}^{+\infty}\left[\underbrace{\frac{4}{\pi n^2}\left((-1)^{n+1}-1\right)}_{A_n} e^{\left(-\left(\frac{3n}{2}\right)^2-2\right)t} + D_n e^{-\left(\frac{3n}{2}\right)^2 t}\right]\cos\left(\frac{nx}{2}\right)$$

$$V(x,0) = \sin^2 x - x = \left(C+2\pi\right) + \sum_{n=1}^{+\infty}\left(A_n + D_n\right)\cos\left(\frac{nx}{2}\right)$$

$$C+2\pi = \frac{1}{\pi}\int_0^{2\pi}\left(\sin^2 x - x\right)\cdot dx = 1-2\pi \Rightarrow \boxed{C = 1-4\pi}$$

$$A_n + D_n = \frac{2}{2\pi}\int_0^{2\pi}\left(\sin^2 x - x\right)\cos\left(\frac{nx}{2}\right)\cdot dx \Rightarrow$$

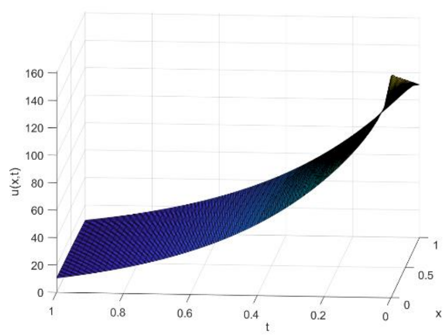
$$D_n = \frac{1}{\pi}\cdot(-4)\frac{\left[\left(n^2-16\right)(-1)^n - n^2+16\right]}{n^2\left(n^2-16\right)} - \frac{4}{\pi n^2}\left((-1)^{n+1}-1\right)$$

$$A_n = \frac{4}{\pi n^2}\left((-1)^{n+1}-1\right)$$

$$u(x,t) = x e^{-2t} + \left(2\pi e^{-2t} + (1-4\pi)\right) + \sum_{n=1}^{+\infty}\left[A_n e^{\left(-\left(\frac{3n}{2}\right)^2-2\right)t} + D_n e^{-\left(\frac{3n}{2}\right)^2 t}\right]\cos\left(\frac{nx}{2}\right)$$

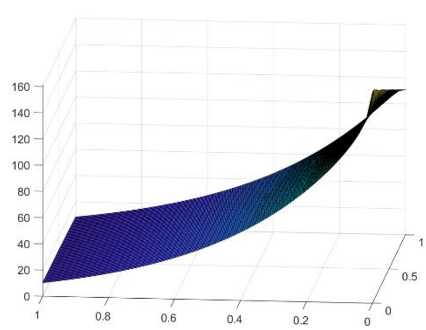
`

```
xRange = 0:0.02:1;
tRange = 0:0.02:1;
[x,t] = meshgrid(xRange,tRange);
ut = zeros(size(x));
ut = ut + x*exp(-2*t) + (2*pi*exp(-2*t) + (1-4*pi));
for n=1:50
    if n == 4
        continue;
    end
    An = (4*( (-1)^(n+1)-1 ) / (pi*n^2));
    Dn = -(4*((pi*n^3+(4-16*pi)*n)*sin(pi*n)+(n^2-16)*cos(pi*n)-n^2+16))/(pi*n^2*(n^2-16))-An;
    ut = ut + (An*exp((-(3*n/2)^2-2)*t)+Dn*exp((-(3*n/2)^2)*t))*cos(n*x/2);
end
surf(x,t,ut)
xlabel('x')
ylabel('t')
zlabel('u(x,t)')
view([-85 9])
```



Difference between u and ut

```
surf(x,t,ut-u)
```



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Helper Functions

Listed here are the local helper functions that the PDE solver pdepe calls to calculate the solution :

```
function [c,f,s] = pdefun(x,t,u,dudx)
c = 1;
f = 9*dudx;
s = 0;
end
%---------------------------------------------
function [pl, ql, pr, qr] = bcfun(xl, ul, xr, ur, t)
pl = -9*exp(-2*t);
ql = 1;
pr = -9*exp(-2*t);
qr = 1;
end
%---------------------------------------------
function u = icfun(x)
u = (sin(x))^2;
end
%---------------------------------------------
```

`

Part 2 : Finite Difference Method Laplace Equation

```
Lx=1; Ly=1; %rectangle dimensions
Nx = 50; Ny=50; %number of intervals in x,y directions
nx=Nx+1; ny=Ny+1; %number of gridpoints in x,y directions
dx=Lx/Nx; dy=Ly/Ny; %grid length in x,y directions
x=(0:Nx)*dx; y=(0:Ny)*dy; %x,y values on the grid
```

Constructin A matrix

Next, the A matrix is constructed. The k indices of the boundary points are defined in the vector boundary_index, the five diagonals are placed in the A matrix using the Matlab spdiags function, and the rows associated with the boundary points are replaced by the corresponding rows of the identity matrix using speye:

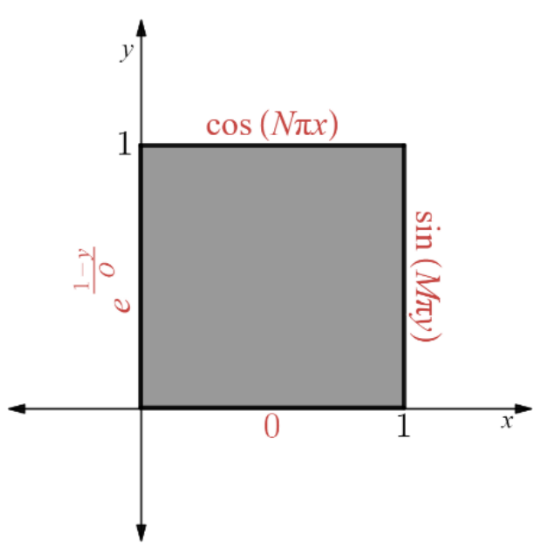
```
boundary_index=[1:nx, 1:nx:1+(ny-1)*nx, 1+(ny-1)*nx:nx*ny, nx:nx:nx*ny];
diagonals = [4*ones(nx*ny,1), -ones(nx*ny,4)];
A=spdiags(diagonals,[0 -1 1 -nx nx], nx*ny, nx*ny);
I=speye(nx*ny);
A(boundary_index,:)=I(boundary_index,:);
```

Constructin U matrix

Interior rows are zero :

```
u=zeros(nx,ny);
```

Boundary Conditions (M = 2, N = 2, O = 3)



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```
u(:,1)=0; %bottom
u(1,:)=1-y/3; %left
u(:,ny)=cos(2*pi*x); %top
u(nx,:)= sin(2*pi*y); %right
```

Make column vector using natural ordering [same as b=b(:)]

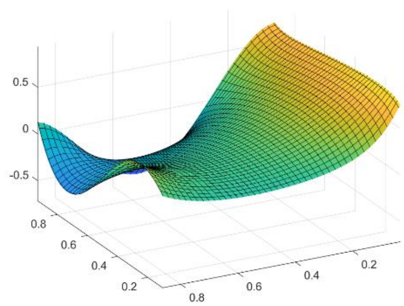
```
z=reshape(u,nx*ny,1);
```

Solving the system of equations

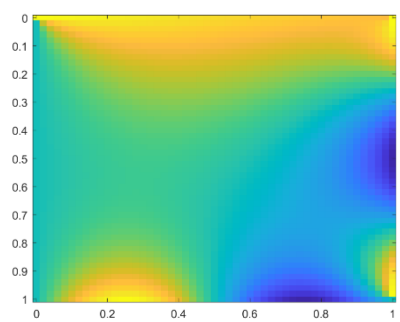
```
Phi=A\z; %solution by Gaussian elimination
Phi=reshape(Phi,nx,ny); %back to (i,j) indexing
```

Plotting the result

```
surf(x,y,Phi)
xlim([0.102 0.929])
ylim([0.035 0.862])
zlim([-0.72 0.93])
view([-117.78 32.40])
```



```
imagesc(x,y,Phi);
```



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```
Phi
```

Phi = 51×51

Rows 42:51 | Columns 36:51

-0.3056	-0.3141	-0.3161	-0.3113	-0.2991	-0.2792	-0.2510	⋯
-0.3419	-0.3506	-0.3519	-0.3454	-0.3305	-0.3067	-0.2735	
-0.3851	-0.3944	-0.3955	-0.3878	-0.3707	-0.3437	-0.3061	
-0.4358	-0.4464	-0.4478	-0.4395	-0.4209	-0.3914	-0.3500	
-0.4950	-0.5074	-0.5098	-0.5016	-0.4821	-0.4507	-0.4066	
-0.5636	-0.5785	-0.5825	-0.5749	-0.5553	-0.5228	-0.4769	
-0.6424	-0.6605	-0.6667	-0.6604	-0.6411	-0.6084	-0.5616	
-0.7325	-0.7544	-0.7633	-0.7588	-0.7405	-0.7081	-0.6611	
-0.8350	-0.8613	-0.8734	-0.8711	-0.8540	-0.8222	-0.7757	
-0.9511	-0.9823	-0.9980	-0.9980	-0.9823	-0.9511	-0.9048	
⋮							