Modeling credit spreads with the Cheyette model and its application to credit default swaptions

Article *in* Journal of Credit Risk · March 2009

DOI: 10.21314/JCR.2009.084

Martin Krekel
UniCredit Bank AG, Munich, Germany

The Journal of Credit Risk (47–71) Volume 5/Number 1, Spring 2009

Modeling credit spreads with the Cheyette model and its application to credit default swaptions

Kalina Natcheva-Acar
Fraunhofer Institut Techno- und Wirtschaftsmathematik (ITWM),
Fraunhofer-Platz 1, D-67663, Kaiserslautern, Germany;
email: kalina.natcheva@itwm.fraunhofer.de

Sarp Kaya Acar
Fraunhofer Institut Techno- und Wirtschaftsmathematik (ITWM),
Fraunhofer-Platz 1, D-67663, Kaiserslautern, Germany;
email: sarp_kaya.acar@itwm.fraunhofer.de

Martin Krekel
UniCredit Markets and Investment Banking, Bayerische Hypo- und
Vereinsbank AG, Arabellastrasse 12, D-81925, Munich, Germany;
email: martin.krekel@unicreditgroup.de

In this paper we apply Cheyette's Markov representation of the Heath–Jarrow–Morton framework to the modeling of stochastic credit spreads. As an application of this framework, the volatility of the credit spread process is modeled by considering the constant elasticity of variance approach of Ritchken and Sankarasubramanian and the Andersen–Andreasen displaced approach. To examine the practicability of this approach, we calibrate the model to market prices of credit default swaptions. Thereby we use Monte Carlo simulation and the alternating direction implicit finite-difference method.

1 INTRODUCTION

In recent years the credit derivatives market has grown significantly. Liquid markets of credit default swaps have been developed, allowing the construction of an implied term structure of default probabilities. In this paper we present a framework for the modeling of the stochastic nature of the full term structure of the credit spreads. Our framework is inspired by Heath, Jarrow and Morton (HJM) (see Heath *et al* (1992)), Cheyette (1994) and Schönbucher (2000).

In their seminal paper, Heath, Jarrow and Morton choose the entire term structure of the forward rates as their state variable and use the observed current forward rate curve as an initial condition to derive an arbitrage-free framework for the stochastic evolution of the state variable, where the dynamics of the forward rates are fully specified through their instantaneous volatility structures. Their general framework permits an arbitrary term structure volatility and covariance of forward rates across maturities. However, the main criticism of their framework is the non-Markovian character of the forward and short-rate processes. By choosing the entire term structure of forward rates as a state variable, their model can be viewed as a joint Markov process in an infinite number of forward rates leading to an infinite-dimensional state space. Under this general framework, it is not possible both to keep the no-arbitrage environment and to describe the evolution of the term structure as a Markovian process by means of a finite number of state variables. Therefore, since the dynamics of the short-rate process depend on its whole history, plenty of practical difficulties arise in its numerical implementation, such as non-recombining trees, costly Monte Carlo simulations, failure to use finite-difference methods, etc.

In order to overcome the drawbacks of the HJM framework, Cheyette (1994) and Ritchken and Sankarasubramanian (1995) offer a limiting form of the volatility of the forward rate process while keeping the desired generality of the forward rate representation. Cheyette proves that it is possible to approximate a large class of HJM models with an arbitrage-free Markov model in a finite number of state variables up to an arbitrary accuracy by limiting the class of volatility functions of the forward rate process. Independently, Ritchken and Sankarasubramanian (RS) also identify the necessary and sufficient condition for capturing the path dependence in the short-rate process by a single additional condition. Further, Li *et al* (1995) offer and investigate the lattice construction needed for the approximation of the short-rate process. In this paper, when using the lattice method for pricing credit risk sensitive claims we shall apply a lattice construction of a Li *et al* (1995) type.

The aim of this paper is to apply the Cheyette model to the modeling of credit risk and to exploit the advantages of this application. Therefore we adapt the Cheyette model and analyze its usability for two types of Cheyette processes for the example of credit default swaptions. More precisely, we choose the instantaneous credit spread as the state variable and model its stochastic nature by Cheyette-type diffusion processes. Moreover, the stochastic volatility of the credit spread process is modeled by considering the RS approach and the Andersen and Andreasen (2002) displaced (AA-DP) approach. Since it is not possible to derive closed-form solutions (or even approximations) for the credit default swaption prices under the Cheyette framework, we invoke Monte Carlo simulations to price credit default swaptions within our model assumptions. Moreover, in the case of RS-type volatility we are also able to implement a finite-difference method: namely, the modified alternating direction implicit (ADI) method[1] of Craig and Sneyd (1988).

[1] In Appendix C we present the modified ADI method to solve the corresponding two-dimensional partial differential equation.

Other approaches to the pricing of credit default swaptions can be found, for example, in Brigo and Alfonsi (2005), Schönbucher (2004), Hull and White (2003) and Krekel and Wenzel (2006).

The rest of the paper is structured as follows. In the following section we will introduce the notation that we will use throughout the paper and also detail the basic facts that our framework relies on. In Section 3, the dynamics of the credit spread process under the pricing measure Q and specifications of the dynamics of the stochastic volatility are introduced. In Section 4, we modify the forward survival measure technique for our pricing purposes and introduce the notion of the survival measure, where the numeraire is no longer the defaultable forward rate as in Schönbucher (2000) but is in fact the defaultable money market account. Section 5 covers the valuation of credit default swaptions in our framework. Section 6 contains the calibration results for credit default swaptions. Finally, in Section 7, we draw our conclusions.

2 NOTATION AND MODEL SETUP

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geqslant 0}, Q)$ be our filtered probability space, where Q stands for a risk-neutral probability measure. The default time on the filtered probability space is denoted by τ and it is a stopping time. In the filtration $\mathbb{F} := (\mathcal{F}_t)_{t \geqslant 0}$, there are two information flows: one coming from the diffusion and the other from the default event.

First, we introduce a right continuous process H by setting $H(t) = 1_{\{\tau \leqslant t\}}$ and we denote by $\mathbb{H}$ the associated filtration: $\mathcal{H}_t = \sigma(H_u : u \leqslant t)$. Then, let $\mathbb{F}^W$ be the filtration generated by the Q-Brownian motion W, ie, $\mathcal{F}_t^W = \sigma(W_u : u \leqslant t)$. In what follows, we set the filtration of our probability space to be $\mathbb{F} = \mathbb{H} \vee \mathbb{F}^W$. We assume that all filtrations satisfy the usual conditions of right continuity and completeness (for more details see Jacod and Shiryaev (1988)) and that the time horizon is large but finite.

We work directly with the risk-neutral probability measure Q, rather than starting with the subjective measure P, which does not take risk premia into account and therefore cannot be used for pricing. The main tool for changing between some subjective measure P and a risk-neutral measure Q in the credit risk models is given in Appendix B. In addition to the usual drift change in the Brownian motion, this change of measure implies a significantly higher default intensity h_Q under Q, which reflects the high risk premium on default risk. In our notation, we drop the Q subscripts and superscripts while working under the measure Q. The intensity of the default time τ is the non-negative adapted process h such that:

$$M(t) = H(t) - \int_0^{t \wedge \tau} h(u) \, \mathrm{d}u$$

is a Q-martingale.

The following definition introduces the basic rates and prices that are used in the framework.

DEFINITION 2.1

1) At any time t, there are default-free and defaultable zero-coupon bonds of all maturities $T > t$. Their prices at time t are denoted by:[2]

$$B(t, T), \qquad I(t)\bar{B}(t, T)$$

where $I(t) := 1 - H(t)$.[3]

2) The default risk factor at time t for maturity T is:

$$D(t, T) = \frac{\bar{B}(t, T)}{B(t, T)}$$

3) The instantaneous default-free forward rate at time t for date T is defined as:

$$f(t, T) = -\frac{\partial}{\partial T} \ln(B(t, T))$$

4) The instantaneous defaultable forward rate at time t for date T is defined as:

$$\bar{f}(t, T) = -\frac{\partial}{\partial T} \ln(\bar{B}(t, T))$$

5) The instantaneous default-free short rate and defaultable short rate at time t are defined by:

$$r(t) := f(t, t), \qquad \bar{r}(t) := \bar{f}(t, t)$$

6) The instantaneous forward credit spread at time t for date T is defined as:

$$\mathcal{S}(t, T) = \bar{f}(t, T) - f(t, T) = -\frac{\partial}{\partial T} \ln(D(t, T))$$

7) The instantaneous credit spread at time t is defined as:

$$s(t) = \mathcal{S}(t, t) = \bar{r}(t) - r(t) \tag{1}$$

8) The default-free money market account and the defaultable money market account are defined, respectively, as follows:

$$M(t) = \exp\left(\int_0^t r(s)\,\mathrm{d}s\right), \qquad I(t)\bar{M}(t) = I(t)\exp\left(\int_0^t \bar{r}(s)\,\mathrm{d}s\right)$$

---

[2] In our notation, the overbar implies that the quantity is subject to default risk.

[3] Therefore, these defaultable zero-coupon bonds have zero recovery in default. We assume zero recovery throughout the whole paper. $\bar{B}(t, T)$ need not jump to zero at default because $I(t)$ already does so.

3 MODELING THE CREDIT SPREAD

3.1 Introduction

In their celebrated articles, Cheyette (1994) and Ritchken and Sankarasubramanian (1995) independently introduced the Markov representation of the forward rate by limiting the forward credit spread volatility term structure to a specific form. More precisely, they used the HJM dynamics of the forward rate:

$$\mathrm{d}f(t,T) = \mu_f(t,T)\,\mathrm{d}t + \sigma_f(t,T)\,\mathrm{d}W_1(t) \tag{2}$$

and specified the volatility of the forward rate by:

$$\sigma_f(t,T) = \frac{\alpha_f(T)}{\alpha_f(t)}\sigma_f(t,t) \tag{3}$$

where $\alpha_f(t) = \mathrm{e}^{-\kappa_f t}$, with a constant κ_f and where $\sigma_f(t,t)$ is some adapted stochastic process. The default-free short-rate Q-dynamics is then given by:

$$\left.\begin{aligned} r(t) &= X(t) + f(0,t) \\ \mathrm{d}X(t) &= (-\kappa_f X(t) + \Phi(t))\,\mathrm{d}t + \sigma_f(t,t)\,\mathrm{d}W_1(t), \quad X(0) = 0 \end{aligned}\right\} \tag{4}$$

where $\Phi(t)$ is the accumulated variance of the forward rate up to time t:

$$\Phi(t) = \int_0^t \sigma_f^2(u,t)\,\mathrm{d}u$$

which can also be written as a solution to the following general first-order linear ordinary differential equation:

$$\mathrm{d}\Phi(t) = \sigma_f(t,t)^2\,\mathrm{d}t - 2\kappa_f\Phi(t)\,\mathrm{d}t, \quad \Phi(0) = 0$$

3.2 The Markovianity of the credit spread process

Inspired by the above result and because the Markovianity of a credit spread process is a desired feature for pricing credit spread sensitive products, we model the instantaneous credit spread by using the Cheyette (1994) Markov representation of the HJM model.

Let the evolution of the forward credit spread for every maturity T be given by a diffusion process of the form:

$$\mathrm{d}\mathcal{S}(t,T) = \mu_{\mathcal{S}}(t,T)\,\mathrm{d}t + \sigma_{\mathcal{S}}(t,T)\,\mathrm{d}W_2(t) \tag{5}$$

where $\mu_{\mathcal{S}}(t,T)$ and $\sigma_{\mathcal{S}}(t,T)$ are the drift and volatility parameters and where $W_2(t)$ is a Brownian motion under Q. The no-arbitrage drift condition for the forward credit spread is proved by Schönbucher (1998) to have the following form:

$$\begin{aligned} \mu_{\mathcal{S}}(t,T) = \sigma_{\mathcal{S}}(t,T)&\int_t^T \sigma_f(t,u)\,\mathrm{d}u \\ &+ \sigma_f(t,T)\int_t^T \sigma_{\mathcal{S}}(t,u)\,\mathrm{d}u + \sigma_{\mathcal{S}}(t,T)\int_t^T \sigma_{\mathcal{S}}(t,u)\,\mathrm{d}u \end{aligned} \tag{6}$$

where $\sigma_f(u, v)$ is the volatility of the HJM forward rate $f(u, v)$. By assuming the independence of $\mathcal{S}(t, T_1)$ and $f(t, T_2)$ for all $t \leqslant T_1, t \leqslant T_2$, the no-arbitrage condition (6) takes the form:

$$\mu_{\mathcal{S}}(t, T) = \sigma_{\mathcal{S}}(t, T) \int_t^T \sigma_{\mathcal{S}}(t, u)\, \mathrm{d}u \tag{7}$$

In the following proposition, we shall state the Markovianity of the credit spread process.

PROPOSITION 3.1 (Markov representation of the credit spread process)

1) *Assume that the no-arbitrage drift condition is given by Equation (7), ie, that the short rate and the credit spread are uncorrelated. Let the volatility of the forward credit spread have the following form:*

$$\sigma_{\mathcal{S}}(t, T) = \frac{\alpha_{\mathcal{S}}(T)}{\alpha_{\mathcal{S}}(t)} \sigma_{\mathcal{S}}(t, t) \tag{8}$$

where $\alpha_{\mathcal{S}}(t) = \mathrm{e}^{-\kappa_s t}$*, with a constant* κ_s*, and where* $\sigma_{\mathcal{S}}(t, t)$ *is some adapted stochastic process. The instantaneous credit spread process can then be represented as a two-dimensional Markov process* $(X_s(t), \Phi_s(t))$*, given as follows:*

$$\left.\begin{aligned} s(t) &= X_s(t) + \mathcal{S}(0, t) \\ \mathrm{d}X_s(t) &= (-\kappa_s X_s(t) + \Phi_s(t))\, \mathrm{d}t + \sigma_{\mathcal{S}}(t, t)\, \mathrm{d}W_2(t), \quad X_s(0) = 0 \end{aligned}\right\} \tag{9}$$

where $\Phi_s(t)$ *is the accumulated variance for the forward credit spread up to time* t*, ie:*

$$\Phi_s(t) = \int_0^t \sigma_{\mathcal{S}}^2(u, t)\, \mathrm{d}u \tag{10}$$

It can be written in differential form as:

$$\mathrm{d}\Phi_s(t) = (\sigma_{\mathcal{S}}^2(t, t) - 2\kappa_s \Phi_s(t))\, \mathrm{d}t, \quad \Phi_s(0) = 0$$

2) *Assume that the no-arbitrage drift condition is given by Equation (6), ie, that the short rate and the credit spread are correlated and that the volatility of the short rate and the volatility of the forward credit spread have the forms given in Equations (3) and (8), respectively. The instantaneous credit spread process can then be represented as a four-dimensional Markov process* $(X_s(t), \Phi_s(t), \Phi_{sf}(t), \Upsilon_{sf}(t))$*, given as follows:*

$$\left.\begin{aligned} s(t) &= X_s(t) + \mathcal{S}(0, t) \\ \mathrm{d}X_s(t) &= \biggl(- \kappa_s X_s(t) + \Phi_s(t) \\ &\quad + \left(2 - \frac{\kappa_f - \kappa_s}{\kappa_s}\right) \Phi_{sf}(t) - \frac{\kappa_f - \kappa_s}{\mathrm{e}^{-\kappa_s t}} \Upsilon_{sf}(t) \biggr)\, \mathrm{d}t + \sigma_{\mathcal{S}}(t, t)\, \mathrm{d}W_2(t) \end{aligned}\right\} \tag{11}$$

where $\Phi_s(t)$ is given by Equation (10) and where $\Phi_{sf}(t)$ and $\Upsilon_{sf}(t)$ are defined as follows:

$$\Phi_{sf}(t) := \int_0^t \sigma_{\mathcal{S}}(u,t)\sigma_f(u,t)\,\mathrm{d}u$$
$$\Upsilon_{sf}(t) := -\frac{1}{\kappa_S}\int_0^t \sigma_{\mathcal{S}}(u,t)\sigma_f(u,t)\mathrm{e}^{-\kappa_s u}\,\mathrm{d}u$$

PROOF See Appendix A. □

Let us note that $\sigma_{\mathcal{S}}(t,t)$ and κ_s appear to be the volatility and the mean-reversion rate of the credit spread process, respectively. Further, under the assumption of independence, the stochastic differential equation (11) reduces to the stochastic differential equation (9). Throughout the rest of the paper, we assume that the short-rate and the credit spread processes are independent.

The following corollaries are direct consequences of Proposition 3.1 and postulate the Markov representation of the forward credit spread and the formula for the default risk factor (see Cheyette (1994)).

COROLLARY 3.2 *The forward credit spread is given as follows:*

$$\mathcal{S}(t,T) = \mathcal{S}(0,T) + \mathrm{e}^{-\kappa_s(T-t)}X_s(t) + \mathrm{e}^{-\kappa_s(T-t)}\beta_{\mathcal{S}}(t,T)\Phi_s(t)$$

where:

$$\beta_{\mathcal{S}}(t,T) = \int_t^T \mathrm{e}^{-\kappa_s(u-t)}\,\mathrm{d}u$$

COROLLARY 3.3 *The default risk factor at time t with maturity T is given as follows:*

$$D(t,T) = \frac{D(0,T)}{D(0,t)}\exp(-0.5\beta_{\mathcal{S}}^2(t,T)\Phi_s(t) - \beta_{\mathcal{S}}(t,T)X_s(t))$$

3.3 The volatility dynamics of the credit spread process

As we have already stated under the independence assumption of forward rates and forward credit spreads, the resulting model is Markovian in only two state variables: $X_s(t)$ and $\Phi_s(t)$. The first state variable can be interpreted as a credit spread curve factor, with instantaneous credit spread given by:

$$s(t) = X_s(t) + \mathcal{S}(0,t)$$

The second state variable $\Phi_s(t)$ can be seen as a convexity correction term, ensuring that the model is arbitrage-free. The model reduces to a single state variable one only when the credit spread volatility, $\sigma_{\mathcal{S}}(t,t)$, is deterministic. In this paper we consider the stochastic nature of the volatility, first by modeling it as a function of the credit spread process, namely the RS approach, and second by modeling it as a function of a Heston-type stochastic volatility process, namely the AA-DP approach.

The RS approach:

$$\sigma_{\mathcal{S}}(t,t) = \sigma(t)s(t)^m \tag{12}$$

where $\sigma(t)$ is a deterministic time-dependent function and $m > 0$. In fact, expression (12) is a constant elasticity of variance approach. The elasticity parameter m must be chosen such that the process $s(t)$ is always positive. For $m = 0.5$, Feller's criterion, $2\kappa\theta > \sigma^2$, ensures that the process is positive, if started from a positive value. For the $m = 1$ case, we derive a sufficient condition for the credit spread to be positive in Appendix D.[4]

The AA-DP approach: the approach is essentially a shifted Heston stochastic volatility model:

$$\sigma_{\mathcal{S}}(t,t) = \sqrt{\sigma(t)}[m\xi(t) + (1-m)\xi(0)] \tag{13}$$

where $\xi(t)$ is the (credit) swap rate,[5] which will be defined in Section 5, and $\sigma(t)$ is a stochastic process, whose dynamics are given by the following CIR process:

$$\mathrm{d}\sigma(t) = \kappa[\Theta - \sigma(t)]\,\mathrm{d}t + \varepsilon\sqrt{\sigma(t)}\,\mathrm{d}W_3(t) \tag{14}$$

where m, κ, Θ and ε are constants[6] and $W_3(t)$ is a standard Q-Brownian motion. It is assumed that $W_1(t)$, $W_2(t)$ and $W_3(t)$ are mutually independent.

4 CHANGE OF MEASURE

In this section, we will adapt the forward survival measure technique of Schönbucher (2000) to (instantaneous) probability measures and introduce the notion of the survival measure.

4.1 The survival measure

Let us consider a contingent claim paying a random payout at time T if the obligor is still alive at time T. The payout can then be written as $XI(T)$ and the time-t price of the contingent claim under the equivalent martingale measure is given as follows:

$$\mathbb{E}_Q\left(\exp\left(-\int_t^T r(u)\,\mathrm{d}u\right)XI(T)\;\middle|\;\mathcal{F}_t\right) \tag{15}$$

[4] Thanks to Holger Kraft for useful discussion on this.
[5] Note that $\sigma_{\mathcal{S}}(t,t)$ is a function of some swap rate $\xi(t)$ that is specific to the instrument that is priced (see Andersen and Andreasen (2002)). The specification of $\xi(t)$ in the pricing issue of credit default swaptions is the underlying forward swap rate.
[6] The effects of the parameters on the volatility smile are explained in Andreasen (2003). The slope of the smile is controlled by the m parameter. The volatility of the volatility process, ε, controls the curvature of the smile. Decreasing the mean-reversion rate decreases the rate at which the curvature of the smile decays with expiry.

We introduce a new measure $\bar{Q}$, namely the survival measure, which we use to price defaultable contingent claims. The numeraire of this new measure is the defaultable money market account, which was introduced in Definition 2.1.

The survival measure is defined by its Radon–Nikodym density:

$$L(t) := \frac{\mathrm{d}\bar{Q}}{\mathrm{d}Q}\bigg|_{\mathcal{F}_t} = \frac{\bar{M}(t)}{\bar{M}(0)}\frac{M(0)}{M(t)}I(t) = \exp\biggl(\int_0^t s(u)\,\mathrm{d}u\biggr)I(t) \tag{16}$$

The Radon–Nikodym density $L(t) = \bar{M}(t)I(t)/M(t)$ is a Q-martingale, since it can be seen as a (tradable) security $\bar{M}(t)I(t)$ discounted with the default-free money market account. Note that $\bar{Q}$ is not an equivalent martingale measure to Q since it attaches zero probability to events after default. It is, however, absolutely continuous to Q, and hence Girsanov's theorem is applicable.

The drift change coming from the measure change is equal to zero, since the numeraire of Q and $\bar{Q}$ are the money market account and the defaultable money market account, respectively, and have locally deterministic returns, ie, zero diffusion coefficients.

The dynamics of the credit spread process under the survival measure $\bar{Q}$ is then given as follows:

$$\left.\begin{aligned} s(t) &= X_s(t) + \mathcal{S}(0,t) \\ \mathrm{d}X_s(t) &= (-\kappa_s X_s(t) + \Phi_s(t))\,\mathrm{d}t + \sigma_{\mathcal{S}}(t,t)\,\mathrm{d}\bar{W}(t), \quad X_s(0) = 0 \end{aligned}\right\} \tag{17}$$

Changing the measure by using Equation (16), the time-t price of the above contingent claim can be rewritten as follows:

$$\mathbb{E}_Q\biggl(\exp\biggl(-\int_t^T r(u)\,\mathrm{d}u\biggr)XI(T)\biggm|\mathcal{F}_t\biggr) = \mathbb{E}_{\bar{Q}}\biggl(\exp\biggl(-\int_t^T \bar{r}(u)\,\mathrm{d}u\biggr)X\biggm|\mathcal{F}_t\biggr) \tag{18}$$

where $\bar{r}(t) = r(t) + s(t)$. Notice that the formula of the default factor $D(t,T)$, given in Corollary 3.3, remains unchanged under the survival measure $\bar{Q}$.

5 THE VALUATION OF CREDIT DEFAULT SWAPTIONS

In this section we will introduce the valuation of a credit default swaption (CDSwaption) into our setup. A CDSwaption is an option written on a (forward) credit default swap (CDS).

DEFINITION 5.1 A forward CDS, starting at time T_k, consists of two payment legs:

1) the fixed (or premium) leg, paying $\xi\delta_i$ at T_{i+1} for all $i = k,\dots,n-1$ if no default happened before T_{i+1}; and

2) the floating (or protection) leg, paying $(1-R)$ at T_{i+1} for all $i = k,\dots,n-1$ if a default happens between T_i and T_{i+1}.

As before, ξ is the premium payment rate (swap rate) and R is the recovery rate.

The value at time 0 of the fixed leg of a forward CDS is therefore:

$$V_{\text{fixed}}(0) = \xi \sum_{i=k}^{n-1} \delta_i \mathbb{E}_{\bar{Q}}\bigg(\exp\bigg(-\int_0^{T_{i+1}} \bar{r}(s)\,\mathrm{d}s \bigg)\bigg) = \xi \sum_{i=k}^{n-1} \delta_i \bar{B}(0, T_{i+1})$$

Therefore, the value of the floating leg of the forward CDS is:

$$\begin{aligned} V_{\text{floating}}(0) = (1-R) \sum_{i=k}^{n-1} \mathbb{E}_{\bar{Q}}\bigg(\exp\bigg(-\int_0^{T_i} \bar{r}(s)\,\mathrm{d}s \bigg) \\ \times \bigg(B(T_i, T_{i+1}) - \exp\bigg(-\int_{T_i}^{T_{i+1}} \bar{r}(s)\,\mathrm{d}s \bigg)\bigg)\bigg) \end{aligned}$$

The forward credit swap rate $\xi_{k,n}(0)$, which makes the value of the fixed leg equal the value of the floating leg, is given as:

$$\xi_{k,n}(0) = (1-R) \frac{\sum_{i=k}^{n-1} [\mathbb{E}_{\bar{Q}}(\exp(-\int_0^{T_i} \bar{r}(u)\,\mathrm{d}u) B(T_i, T_{i+1})) - \bar{B}(0, T_{i+1})]}{\sum_{i=k}^{n-1} \delta_i \bar{B}(0, T_{i+1})}$$

In general, for $t \leqslant T_k$:

$$\xi_{k,n}(t) = (1-R) \frac{\sum_{i=k}^{n-1} [\mathbb{E}_{\bar{Q}}(\exp(-\int_t^{T_i} \bar{r}(u)\,\mathrm{d}u) B(T_i, T_{i+1}) \mid \mathcal{F}_t) - \bar{B}(t, T_{i+1})]}{\sum_{i=k}^{n-1} \delta_i \bar{B}(t, T_{i+1})} \tag{19}$$

REMARK 5.2 Notice that in the case of the deterministic default-free short-rate process $r(t)$, we would have the fair forward credit swap rate of the simplified form:

$$\xi_{k,n}(t) = (1-R) \frac{\sum_{i=k}^{n-1} [D(t, T_i) B(t, T_{i+1}) - \bar{B}(t, T_{i+1})]}{\sum_{i=k}^{n-1} \delta_i \bar{B}(t, T_{i+1})} \tag{20}$$

DEFINITION 5.3 A payer (receiver) CDSwaption with maturity T_k and strike ξ is a call (put) option on a k-forward CDS. It gives the owner of this option at time T_k the right to enter a long (short) position in a credit default swap at the predefined default swap rate if there has been no default event until time T_k.

The payout function of the payer CDSwaption at time T_k is:

$$\text{Payoff} = (V_{\text{floating}}(T_k) - V_{\text{fixed}}(T_k))^+ = \sum_{i=k}^{n-1} \bar{B}(T_k, T_{i+1})(\xi_{k,n}(T_k) - \xi)^+$$

Hence, the price of the payer CDSwaption at time 0 is:

$$V_{\text{payer}}(0) = \mathbb{E}_{\bar{Q}}\bigg(\exp\bigg(-\int_0^{T_k} \bar{r}(s)\,\mathrm{d}s \bigg) \sum_{i=k}^{n-1} \bar{B}(T_k, T_{i+1})(\xi_{k,n}(T_k) - \xi)^+ \bigg) \tag{21}$$

TABLE 1 iTRAXX S10 IG five-year CDSwaption market quotes (Ref: 135bps).

Maturity	December 20, 2008			March 20, 2009		
Strike	**Bid**	**Ask**	**MidVol (%)**	**Bid**	**Ask**	**MidVol (%)**
80	272	280	114.0	305	317	110.9
90	237	245	114.0	275	287	110.9
100	204	212	112.8	249	261	110.4
110	175	183	112.2	224	236	110.1
120	149	157	112.0	203	215	110.0
130	127	136	112.2	184	196	110.1
140	109	117	112.6	167	179	110.3
150	94	102	113.2	153	165	110.5
160	81	89	114.1	139	151	110.9
170	69	77	114.1	129	139	110.9

TABLE 2 iTRAXX S10 Xover five-year CDSwaption market quotes (Ref: 735bps).

Maturity	December 20, 2008			March 20, 2009		
Strike	**Bid**	**Ask**	**MidVol (%)**	**Bid**	**Ask**	**MidVol (%)**
600	634	644	67.5	801	815	65.5
625	567	577	67.5	741	755	65.5
650	507	517	67.5	685	699	65.5
675	451	461	67.6	633	647	65.5
700	402	412	67.7	585	599	65.6
725	357	367	67.9	541	555	65.7
750	317	327	68.1	500	514	65.8
775	282	292	63.3	463	477	65.9
800	250	260	68.6	429	443	66.0
825	222	232	68.6	398	412	66.0

6 NUMERICAL RESULTS AND CALIBRATION

In this section we calibrate the parameters of the credit spread process $s(t)$ to credit default swaption market prices written on the iTRAXX IG and Xover indexes.[7] The market quotes are stated in Tables 1 and 2. For the purposes of calibration we assume a deterministic default-free short rate. We assume that the initial term structures of

[7] Note that credit default swaptions on indexes are not embedded with the knock-out feature, which means that the buyer of a payer option has default protection prior to option expiry. We use the market standard method to account for this feature, ie, adjusting the forward spread by the quotient of the value of the front-end protection and the forward DV01.

defaultable forward rates are flat.[8] We present the calibration results by considering the Q-dynamics of the credit spread, given by Equation (9), where the term structure of volatility is modeled by the RS and AA-DP approaches. In the RS case, we choose $m = 1$, ie, $\sigma_S(t,t) = \sigma s(t)$, where σ is constant. Thus, the parameters that have to be calibrated are the volatility, σ, and the mean reversion, κ_S. To calibrate the parameters we have used the finite-difference method. For the derivation of the corresponding partial differential equation and for the corresponding finite-difference method, see Appendix C.

In the AA-DP case, we choose $\xi(t)$ to be the underlying forward swap rate[9] of the CDSwaption contract. The parameters that have to be calibrated are the volatility, σ_s, the mean reversion of the credit spread process, κ_s, the volatility, ϵ, the mean reversion of the stochastic volatility process, κ, and the displacement coefficient, m. Due to the complexity of this model,[10] we implemented only the Monte Carlo simulation. For the numerical calibration, we have further set the number of time steps to 100 and the number of simulations to 20,000.

As the calibration target function we used the root mean square error (RMSE):

$$\mathrm{RMSE}(\vec{p}) = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(\hat{s}_i - s_i(\vec{p}))^2}$$

where the $\hat{s}_i$ are the mid market spreads, the $s_i(\vec{p})$ are the model spreads and $\vec{p}$ is the parameter vector. So our objective is to find:

$$\vec{p}_o = \arg\min(\mathrm{RMSE}(\vec{p}))$$

The market quotes in Tables 1 and 2 are the market prices of payer swaptions from October 17, 2008 for an iTRAXX S10 IG five-year CDS and an iTRAXX S10 Xover five-year CDS. The maturity date of iTRAXX S10 five-year CDSs is December 20, 2013. All market prices are quoted in basis points. The "MidVol" column shows the Black volatilities that match mid market quotes (the mean of the bid and ask quotes). Note that the longer the maturity and the wider the underlying credit spread, the wider the bid–ask spreads are. The option maturities are December 20, 2008 and March 20, 2009, respectively. For each maturity we have 10 data points, giving a total of 20 data points in all. We calibrate the market data for both maturities with a common set of model parameters. For root solving we used the solver embedded in Excel, which uses a conjugate gradient method.

---

[8] This is the common market assumption, when credit default swaptions are quoted.
[9] Note that this swap rate covers 20 tenor dates, each of which has a length of a quarter of a year.
[10] What we mean by the "complexity" of the model is that for each time step of the numerical procedure we have to calculate the corresponding forward swap rate in order to simulate the stochastic volatility of the credit spread, which does not allow a fine-grid Monte Carlo simulation.

6.1 Calibration results for the Ritchken–Sankarasubramanian model

In Tables 3 and 4 we present the calibration results of the RS model for the payer swaptions market data from Tables 1 and 2.

The calibration results for RS are very satisfactory. For iTRAXX IG we got an RMSE of 3.45 and for iTRAXX Xover we got 7.67. The iTRAXX IG model quotes are located within the bid–ask spreads except for two exceptions; for Xover we have five exceptions. The calculation time for each option price is roughly one second. The optimal model parameters for iTRAXX IG are $\sigma = 0.94$ and $\kappa_s = -0.1$; for iTRAXX Xover we calibrated as optimal parameters $\sigma = 1.39$ and $\kappa_s = 0.39$. For comparison, the quoted Black–Scholes volatilities for iTRAXX IG are approximately 110% and for iTRAXX Xover they are approximately 67%. That the volatility for iTRAXX IG is higher than it is for Xover can be explained by the fact that the spread proportion accounting for liquidity risk or hedge risk is greater in iTRAXX IG spreads. Therefore, iTraxx IG spreads are more volatile. Overall, the good performance can be explained by the fact that the Black volatility skew is almost flat and the Black formula is based on the assumption that the forward CDS spread is driven by a constant elasticity of variance (CEV) process with $m = 1$ (or geometric Brownian motion). In the RS model we made the assumption that the instantaneous credit spread is also driven by a CEV process with $m = 1$.

We calculated the surface of the target function (since the RS model has just two parameters). We observed that the volatility σ and the mean reversion κ_s have a reversed effect on the pricing function. For instance, if we subtract 4% from the optimal σ for iTRAXX IG and add 1.5% to the optimal κ, we can get similar good calibration results. So the the error surface of iTRAXX IG with respect to σ and κ_s has a "valley" located diagonal to the axes with the deepest point at $\sigma = 0.94$ and $\kappa_s = -0.1$.

6.2 Calibration results for the Andersen–Andreasen displaced model

In Tables 5 and 6 we state the calibration results for the AA-DP model. The calculation time for one price was around 10 seconds. Despite trying much harder for good calibration results here than we did for the RS model, the results are worse. The reasons for this are threefold: first, using Monte Carlo makes the calibration slow and unstable; second, we have to deal with six parameters, so the calibration is generally more difficult (counteracting parameters, local minima); third, the spread dynamics does not seem to reflect the market prices.

7 CONCLUSION

In this paper we laid down the theoretical fundamentals for the modeling of credit spreads in the HJM framework, where the volatility is specified to be of Cheyette type. The

TABLE 3 iTRAXX S10 IG five-year payer swaption RS calibration results.

Maturity	December 20, 2008			March 20, 2009		
Strike	**Bid**	**Ask**	**Model**	**Bid**	**Ask**	**Model**
80	272	280	273.65	305	317	302.14
90	237	245	240.38	275	287	276.61
100	204	212	209.89	249	261	252.64
110	175	183	182.31	224	236	231.38
120	149	157	157.61	203	215	211.58
130	127	136	135.69	184	196	193.18
140	109	117	116.39	167	179	176.98
150	94	102	99.54	153	165	161.85
160	81	89	84.92	139	151	147.93
170	69	77	72.31	129	139	135.66
Optimal parameters	$\sigma = 0.94$, $\kappa_s = -0.10$					
RMSE	3.45					

TABLE 4 iTRAXX S10 Xover five-year payer swaption RS calibration results.

Maturity	December 20, 2008			March 20, 2009		
Strike	**Bid**	**Ask**	**Model**	**Bid**	**Ask**	**Model**
600	634	644	662.10	801	815	816.04
625	567	577	589.53	741	755	750.37
650	507	517	523.47	685	699	692.37
675	451	461	463.94	633	647	637.66
700	402	412	410.70	585	599	590.11
725	357	367	363.38	541	555	545.58
750	317	327	321.46	500	514	505.88
775	282	292	284.40	463	477	469.59
800	250	260	251.69	429	443	436.46
825	222	232	222.88	398	412	406.92
Optimal parameters	$\sigma = 1.39$, $\kappa_s = 0.39$					
RMSE	7.67					

motivation for this undertaking was to transfer the benefits of the Cheyette model, which are mainly its Markovian nature and the existence of closed-form solutions for bond prices, to the modeling of credit risk. We also examined the usability of this approach.

TABLE 5 iTRAXX S10 IG five-year payer swaption AA-DP calibration results.

Maturity	December 20, 2008			March 20, 2009		
Strike	**Bid**	**Ask**	**Model**	**Bid**	**Ask**	**Model**
80	272	280	272.36	305	317	303.24
90	237	245	240.10	275	287	276.67
100	204	212	205.56	249	261	247.77
110	175	183	181.20	224	236	231.14
120	149	157	154.92	203	215	210.68
130	127	136	133.28	184	196	187.52
140	109	117	115.54	167	179	172.54
150	94	102	99.52	153	165	158.75
160	81	89	82.43	139	151	135.61
170	69	77	67.87	129	139	125.66
$X(t)$	$\kappa_s = 0.0$, $m = 0.80$					
$\sigma(t)$	$\sigma(0) = 1.15$, $\kappa = -0.05$, $\Theta = 1.15$, $\varepsilon = 0.6$					
RMSE	4.20					

TABLE 6 iTRAXX S10 Xover five-year payer swaption AA-DP calibration results.

Maturity	December 20, 2008			March 20, 2009		
Strike	**Bid**	**Ask**	**Model**	**Bid**	**Ask**	**Model**
600	634	644	670.13	801	815	826.16
625	567	577	593.32	741	755	767.50
650	507	517	510.25	685	699	705.80
675	451	461	456.80	633	647	640.44
700	402	412	395.16	585	599	589.24
725	357	367	339.88	541	555	541.87
750	317	327	296.04	500	514	485.80
775	282	292	249.73	463	477	438.17
800	250	260	206.29	429	443	393.85
825	222	232	171.33	398	412	359.25
$X(t)$	$\kappa_s = 0.23$, $m = 0.80$					
$\sigma(t)$	$\sigma(0) = 1.04$, $\kappa = 0.15$, $\Theta = 0.79$, $\varepsilon = 0.69$					
RMSE	28.08					

For this purpose we have chosen market prices of instruments, which clearly depend on the credit spread dynamics, namely credit default swaptions, and calibrated them successfully with our model.

For the calibration we considered first RS-type volatility. We calibrated the payer swaption market prices to the model prices, obtained from the finite-difference ADI method, by changing the volatility parameter σ and the mean reversion parameter κ_s. The calibration time and the results are very satisfactory: we could minimize the root mean squared error to 3.45 basis points for iTraxx IG and 7.67 basis points for iTraxx Xover. Note that the obtained volatility parameter is very high but the actual volatility is $\sigma s(t)$, which is reasonable at each time step.

Second, the volatility structure of AA-DP has been considered and further numerical results, based on Monte Carlo simulation, have been obtained. The implementation of a finite-difference method or a tree method was not possible in this case. Therefore, we had to rely on the Monte Carlo simulation for the calibration results. The calibration time and the results were less satisfactory: an RMSE of 4.20 basis points for iTraxx IG and an RMSE of 28.08 basis points for iTraxx Xover.

We applied the Cheyette model to the credit world. Initial tests with credit default swaptions and RS-type volatility were very promising. In our opinion, it therefore seems beneficial to undertake further research in this direction.

APPENDIX A: PROOF OF PROPOSITION 3.1

For the first part of the proof, we refer to Ritchken and Sankarasubramanian (1995).

THE PROOF OF THE SECOND PART By using the no-arbitrage drift condition (6), the forward credit spread can be represented as follows:

$$\begin{aligned} S(t,T) = S(0,T) &+ \int_0^t \sigma_{\mathcal{S}}(u,T) \int_u^T \sigma_f(u,v)\,\mathrm{d}v\,\mathrm{d}u \\ &+ \int_0^t \sigma_f(u,T) \int_u^T \sigma_{\mathcal{S}}(u,v)\,\mathrm{d}v\,\mathrm{d}u \\ &+ \int_0^t \sigma_{\mathcal{S}}(u,T) \int_u^T \sigma_{\mathcal{S}}(u,v)\,\mathrm{d}v\,\mathrm{d}u \\ &+ \int_0^t \sigma_{\mathcal{S}}(u,T)\,\mathrm{d}W(u) \end{aligned} \tag{A.1}$$

Recall that the volatilities of the forward rates have the following separable form:

$$\sigma_i(t,T) = \frac{\alpha_i(T)}{\alpha_i(t)}\sigma_i(t,t)$$

where $i = \{f, \mathcal{S}\}$. Let us define $\mathrm{d}A_i(t) := \alpha_i(t)\,\mathrm{d}t$. Equation (A.1) then becomes:

$$\begin{aligned} S(t,T) = S(0,T) &+ \int_0^t \frac{\sigma_{\mathcal{S}}(u,u)\sigma_f(u,u)}{\alpha_{\mathcal{S}}(u)\alpha_f(u)}\alpha_{\mathcal{S}}(T)(A_f(T) - A_f(u))\,\mathrm{d}u \\ &+ \int_0^t \frac{\sigma_{\mathcal{S}}(u,u)\sigma_f(u,u)}{\alpha_{\mathcal{S}}(u)\alpha_f(u)}\alpha_f(T)(A_{\mathcal{S}}(T) - A_{\mathcal{S}}(u))\,\mathrm{d}u \\ &+ \int_0^t \frac{\sigma_{\mathcal{S}}^2(u,u)}{\alpha_{\mathcal{S}}^2(u)}\alpha_{\mathcal{S}}(T)(A_{\mathcal{S}}(T) - A_{\mathcal{S}}(u))\,\mathrm{d}u \\ &+ \int_0^t \frac{\sigma_{\mathcal{S}}(u,u)}{\alpha_{\mathcal{S}}(u)}\alpha_{\mathcal{S}}(T)\,\mathrm{d}W(u) \end{aligned}$$

Therefore, the credit spread process is obtained as follows:

$$s(t) := S(t,t) = S(0,t) + I(t) + \int_0^t \frac{\sigma_{\mathcal{s}}(u,u)}{\alpha_{\mathcal{s}}(u)} \alpha_{\mathcal{s}}(t)\,\mathrm{d}W(u) \tag{A.2}$$

where:

$$\begin{aligned} I(t) = &\int_0^t \frac{\sigma_{\mathcal{s}}(u,u)\sigma_f(u,u)}{\alpha_{\mathcal{s}}(u)\alpha_f(u)} \alpha_{\mathcal{s}}(t)(A_f(t) - A_f(u))\,\mathrm{d}u \\ &+ \int_0^t \frac{\sigma_{\mathcal{s}}(u,u)\sigma_f(u,u)}{\alpha_{\mathcal{s}}(u)\alpha_f(u)} \alpha_f(t)(A_{\mathcal{s}}(t) - A_{\mathcal{s}}(u))\,\mathrm{d}u \\ &+ \int_0^t \frac{\sigma_{\mathcal{s}}^2(u,u)}{\alpha_{\mathcal{s}}^2(u)} \alpha_{\mathcal{s}}(t)(A_{\mathcal{s}}(t) - A_{\mathcal{s}}(u))\,\mathrm{d}u \end{aligned}$$

By rearranging the terms, we obtain:

$$\int_0^t \frac{\sigma_{\mathcal{s}}(u,u)}{\alpha_{\mathcal{s}}(u)}\,\mathrm{d}W(u) = \frac{s(t) - S(0,t)}{\alpha_{\mathcal{s}}(t)} - \frac{1}{\alpha_{\mathcal{s}}(t)} I(t) \tag{A.3}$$

By differentiating $I(t)$ we obtain:

$$\begin{aligned} I'(t) = 2\Phi_{sf}(t) + \Phi_s(t) &+ \alpha'_{\mathcal{s}}(t) \int_0^t \frac{\sigma_{\mathcal{s}}(u,u)\sigma_f(u,u)}{\alpha_{\mathcal{s}}(u)\alpha_f(u)} (A_f(t) - A_f(u))\,\mathrm{d}u \\ &+ \alpha'_f(t) \int_0^t \frac{\sigma_{\mathcal{s}}(u,u)\sigma_f(u,u)}{\alpha_{\mathcal{s}}(u)\alpha_f(u)} (A_{\mathcal{s}}(t) - A_{\mathcal{s}}(u))\,\mathrm{d}u \\ &+ \alpha'_{\mathcal{s}}(t) \int_0^t \frac{\sigma_{\mathcal{s}}^2(u,u)}{\alpha_{\mathcal{s}}^2(u)} (A_{\mathcal{s}}(t) - A_{\mathcal{s}}(u))\,\mathrm{d}u \end{aligned} \tag{A.4}$$

From Equation (A.2) one can derive:

$$\mathrm{d}s(t) = (S'(0,t) + I'(t))\,\mathrm{d}t + \left(\alpha'_{\mathcal{s}}(t) \int_0^t \frac{\sigma_{\mathcal{s}}(u,u)}{\alpha_{\mathcal{s}}(u)}\,\mathrm{d}W(u)\right)\mathrm{d}t + \sigma_{\mathcal{s}}(t,t)\,\mathrm{d}W(t)$$

As a consequence, by substituting Equations (A.3) and (A.4) into the above equation and arranging the terms we obtain the required result. □

APPENDIX B: CHANGE OF MEASURE

THEOREM B 1 *Assume that the default process has an intensity $h(t)$. Let $\alpha(t)$ be a predictable process and let $\phi(t)$ be a strictly positive predictable process with:*

$$\int_0^t \alpha(s)^2\,\mathrm{d}s < \infty, \qquad \int_0^t |\phi(s) - 1|\,\mathrm{d}s < \infty$$

for finite t. Define the process L by $L(0) = 1$ and:

$$\frac{\mathrm{d}L(t)}{L(t-)} = \alpha(t)\,\mathrm{d}W(t) + (\phi(t) - 1)\,\mathrm{d}N(t)$$

Assume that $\mathbb{E}(L(t)) < \infty$ for finite t.

There is then a probability measure Q equivalent to P with:

$$\mathrm{d}Q(t) = L(t)\,\mathrm{d}P(t)$$

such that:

$$\mathrm{d}W^{Q}(t) = \mathrm{d}W(t) - \alpha(t)\,\mathrm{d}t, \qquad h_{Q}(t) = \phi(t)h(t)$$

defines W^{Q} as a Q-Brownian motion and h_Q as the intensity of the default indicator process under Q. Furthermore, every probability measure that is equivalent to P can be represented in the way given above.

PROOF See Jacod and Shiryaev (1988, Sections III.3 and III.5). □

APPENDIX C: ON THE NUMERICAL SOLUTION OF THE CHEYETTE MODEL WITH RS-TYPE VOLATILITY

In this section we consider the diffusion modeling of credit spread processes with RS-type volatility and we introduce contingent-claim pricing by solving the corresponding partial differential equation with the modified ADI scheme of Craig and Sneyd (1988).

Let us first derive the pricing partial differential equation. Under the survival measure, $\bar{Q}$, the fair price $V(\cdot)$ of some contingent claim with payout $F(X_s(T), \Phi_s(T))$ is given by:

$$V(t, X_s(t), \Phi_s(t)) = \mathbb{E}_{\bar{Q}}\biggl[\exp\biggl(-\int_t^T \bar{r}(s)\,\mathrm{d}s\biggr)F(X_s(T), \Phi_s(T)) \biggm| \mathcal{F}_t\biggr]$$

Due to Feynman–Kac this corresponds to the solution of:

$$\left.\begin{aligned} \frac{\partial V}{\partial t} + \mathcal{D}_x V + \mathcal{D}_\Phi V &= 0 \\ V(T, X_s(T), \Phi_s(T)) &= F(X_s(T), \Phi_s(T)) \end{aligned}\right\} \tag{C.1}$$

with:

$$\begin{aligned} \mathcal{D}_x &= -\tfrac{1}{2}\bar{r} + (\Phi_s(t) - \kappa_s X_s(t))\frac{\partial}{\partial x} + \tfrac{1}{2}\sigma_{\mathcal{S}}^2(t,t)\frac{\partial^2}{\partial x^2} \\ \mathcal{D}_\Phi &= -\tfrac{1}{2}\bar{r} + (\sigma_{\mathcal{S}}^2(t,t) - 2\kappa_s\Phi_s(t))\frac{\partial}{\partial \Phi_s} \end{aligned}$$

For simplicity we omitted the arguments of $V = V(t, X_s(t), \Phi_s(t))$ in Equation (C.1). We assume that the credit spread process has RS-type volatility, ie, $\sigma_{\mathcal{S}}^2(t,t) = \sigma(t)(X_s(t) + S(0,t))^m$, where $\sigma(t)$ is a deterministic function and $m > 0$. To solve this two-dimensional[11] problem, we used the modified *alternating direction implicit* method of Craig and Sneyd (1988). In what follows we denote by u_{ij}^k the finite-difference solution at the note point $(k\,\triangle t, i\,\triangle x + x_0, j\,\triangle y + y_0)$, where $i = 0, \ldots, N_x$, $j = 0, \ldots, N_y$

[11] Note that if we consider the AA-DP type volatility, we obtain a three-dimensional problem.

and $k = 0, \dots, N_t$. For simplicity of notation we switched from Φ_s to y. We denote by D_x and D_y the finite-difference versions of the operators $\mathcal{D}_x$ and $\mathcal{D}_\Phi$. We will perform our calculation for y with five-point discretization:

$$D_x u_{ij}^k = -\tfrac{1}{2} r u_{ij}^k + (y - \kappa_s x)\frac{u_{i+1j}^k - u_{i-1j}^k}{2\triangle x} + \tfrac{1}{2}\sigma_{\mathcal{S}}^2 \frac{u_{i+1j}^k - 2u_{ij}^k + u_{i-1j}^k}{\triangle x^2} \quad (C.2)$$

$$D_y u_{ij}^k = -\tfrac{1}{2} r u_{ij}^k + (\sigma_{\mathcal{S}}^2 - 2\kappa_s y)\frac{1}{\triangle y}[-\tfrac{1}{12}u_{ij+2}^k + \tfrac{2}{3}u_{ij+1}^k - \tfrac{2}{3}u_{ij-1}^k + \tfrac{1}{12}u_{ij-2}^k] \quad (C.3)$$

The basic scheme we consider can be written in the following form:

$$Au^k = (A + B)u^{k+1} \quad (C.4)$$

where:

$$A = (1 - \theta\triangle t D_x)(1 - \theta\triangle t D_y)$$
$$B = \triangle t(D_x + D_y)$$

and $\theta \in [0, 1]$. Note that this is equivalent to:

$$\frac{-(u^{k+1} - u^k)}{\triangle t} - (D_x + D_y)u^{k+1} = \theta(-D_x - D_y + \theta\triangle t D_x D_y)(u^{k+1} - u^k)$$

Note that for $\theta = 0$ we have a fully explicit scheme and for $\theta = 1$ a fully implicit scheme. Craig and Sneyd (1988) suggest using $\theta = 0.5$ and solving Equation (C.4) iteratively in the following way:

$$(1 - \theta\triangle t D_x)v^k = [(1 - \theta\triangle t D_x) + \triangle t D_x + \triangle t D_y]u^{k+1} \quad (C.5)$$

$$(1 - \theta\triangle t D_y)u^k = v^k - \theta\triangle t D_y u^{k+1} \quad (C.6)$$

A.1 First step

In the following we try to make the calculations of Equation (C.5) as precise as possible. By defining:

$$\begin{aligned} m_{ij}^k &:= [(1 - \theta\triangle t D_x) + \triangle t D_x + \triangle t D_y]u_{ij}^{k+1} \\ &= u_{ij}^{k+1} + \triangle t(1-\theta)\bigg[-\tfrac{1}{2}\bar{r}u_{ij}^{k+1} + (y - \kappa_s x)\frac{u_{i+1j}^{k+1} - u_{i-1j}^{k+1}}{2\triangle x} \\ &\qquad + \tfrac{1}{2}\sigma_{\mathcal{S}}^2\frac{u_{i+1j}^{k+1} - 2u_{ij}^{k+1} + u_{i-1j}^{k+1}}{\triangle x^2}\bigg] \\ &\quad + \triangle t\bigg[-\tfrac{1}{2}\bar{r}u_{ij}^{k+1} \\ &\qquad + (\sigma_{\mathcal{S}}^2 - 2\kappa_s y)\frac{1}{\triangle y}(-\tfrac{1}{12}u_{ij+2}^{k+1} + \tfrac{2}{3}u_{ij+1}^{k+1} - \tfrac{2}{3}u_{ij-1}^{k+1} + \tfrac{1}{12}u_{ij-2}^{k+1})\bigg] \end{aligned}$$

Equation (C.5) can be rewritten as:

$$\alpha_{ij} v^k_{i-1j} + \beta_{ij} v^k_{ij} + \gamma_{ij} v^k_{i+1j} = m^k_{ij}$$

where:

$$\begin{aligned}
\alpha_{ij} &= \theta \triangle t \left[\frac{(y - \kappa_s x)}{2\triangle x} - \frac{\sigma^2_{\mathcal{S}}}{2\triangle x^2} \right] \\
\beta_{ij} &= \left[1 + \theta \triangle t \left(\frac{\sigma^2_{\mathcal{S}}}{\triangle x^2} + \tfrac{1}{2}\bar{r} \right) \right] \\
\gamma_{ij} &= \theta \triangle t \left[-\frac{(y - \kappa_s x)}{2\triangle x} - \frac{\sigma^2_{\mathcal{S}}}{2\triangle x^2} \right]
\end{aligned}$$

This equation holds for all $i = 1, \ldots, N_x - 1$, $j = 0, \ldots, N_y$ and $k = 0, \ldots, N_t$. Therefore, we obtain a tridiagonal equation system. Hence we have for all $j = 1, \ldots, N_y - 1$ the following:

$$\boldsymbol{M}^k_{\cdot j} \boldsymbol{v}^k_{\cdot j} = \boldsymbol{m}^k_{\cdot j} \tag{C.7}$$

where $\boldsymbol{M}^k_{\cdot j}$ is equal to:

$$\begin{bmatrix}
(2\alpha_{1j} + \beta_{1j}) & (\gamma_{1j} - \alpha_{1j}) & 0 & & \cdots & 0 \\
\alpha_{2j} & \beta_{2j} & \gamma_{2j} & 0 & \cdots & 0 \\
0 & \alpha_{3j} & \beta_{3j} & \gamma_{3j} & 0 & \vdots \\
\vdots & 0 & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & \alpha_{N_x^- - 1j} & \beta_{N_x^- - 1j} & \gamma_{N_x^- - 1j} \\
0 & \cdots & & 0 & (\alpha_{N_x^- j} - \gamma_{N_x^- j}) & (\beta_{N_x^- j} + 2\gamma_{N_x^- j})
\end{bmatrix}$$

$$\boldsymbol{v}^k_{\cdot j} = \begin{bmatrix} v^k_{1j} \\ v^k_{2j} \\ \vdots \\ v^k_{N_x^- - 1j} \\ v^k_{N_x^- j} \end{bmatrix}, \qquad \boldsymbol{m}^k_{\cdot j} = \begin{bmatrix} m^k_{1j} \\ m^k_{2j} \\ \vdots \\ m^k_{N_x^- - 1j} \\ m^k_{N_x^- j} \end{bmatrix}$$

Note that we used N_x^- as an abbreviation for $N_x - 1$. The computational cost of solving this equation with the LU-decomposition method is $\mathcal{O}(N_x^-)$, hence all together $\mathcal{O}(N_x^- \times N_y)$.

We calculate the missing values v^k_{0j} and $v^k_{N_x j}$ with the so-called soft boundary condition. Let us define:

$$v_{xx}(t, x_0, y) := f(t, x_0, y) \quad \text{and} \quad v_{xx}(t, x_N, y) := f(t, x_N, y)$$

Then the boundary conditions are given as follows:

$$\left.\begin{aligned} \frac{v^k_{N_x j} - 2v^k_{N_x^- j} + v^k_{N_x^- -1j}}{\triangle x^2} &= f^k_{N_x^- j} \iff v^k_{N_x j} = 2v^k_{N_x^- j} - v^k_{N_x^- -1j} + \triangle x^2 f^k_{N_x^- j} \\ \frac{v^k_{2j} - 2v^k_{1j} + v^k_{0j}}{\triangle x^2} &= f^k_{1j} \iff v^k_{0j} = 2v^k_{1j} - v^k_{2j} + \triangle x^2 f^k_{1j} \end{aligned}\right\} \tag{C.8}$$

We apply the so-called soft boundary condition, which means that $f \equiv 0$, hence:

$$v_{xx}(t, x_0, y) := 0 \quad \text{and} \quad v_{xx}(t, x_N, y) := 0$$

A.2 Second step

In this section we explain the implicit scheme in the y direction. By defining:

$$\begin{aligned} n^k_{ij} &:= v^k_{ij} - \theta \triangle t D_y u^{k+1}_{ij} \\ &= v^k_{ij} - \theta \triangle t \bigg[-\tfrac{1}{2}\bar{r} u^{k+1}_{ij} \\ &\qquad + (\sigma^2_{\mathcal{S}} - 2\kappa_s y)\frac{1}{\triangle y}(-\tfrac{1}{12}u^{k+1}_{ij+2} + \tfrac{2}{3}u^{k+1}_{ij+1} - \tfrac{2}{3}u^{k+1}_{ij-1} + \tfrac{1}{12}u^{k+1}_{ij-2})\bigg] \end{aligned}$$

Equation (C.6) can be rewritten as:

$$\begin{aligned} c_{ij}u^k_{ij} + e_{ij}u^k_{ij+2} + d_{ij}u^k_{ij+1} + b_{ij}u^k_{ij-1} + a_{ij}u^k_{ij-2} &= n^k_{ij} \\ a_{ij} &= \theta \triangle t \left[-\frac{(\sigma^2_{\mathcal{S}} - 2\kappa_s y)}{12\triangle y} \right] \\ b_{ij} &= \theta \triangle t \left[\frac{2(\sigma^2_{\mathcal{S}} - 2\kappa_s y)}{3\triangle y} \right] \\ c_{ij} &= 1 + \tfrac{1}{2}\bar{r}\theta \triangle t \\ d_{ij} &= \theta \triangle t \left[-\frac{2(\sigma^2_{\mathcal{S}} - 2\kappa_s y)}{3\triangle y} \right] \\ e_{ij} &= \theta \triangle t \left[\frac{(\sigma^2_{\mathcal{S}} - 2\kappa_s y)}{12\triangle y} \right] \end{aligned}$$

Again we use the soft condition (with $f \equiv 0$). Since we have a five-point discretization, we have two points left at the boundary. We will estimate the inner one with a three-point discretization and the outer one with a five-point discretization.

The upper boundary conditions are:

$$u^k_{iN_y-1} = 2u^k_{iN_y-2} - u^k_{iN_y-3} \tag{C.9}$$

$$u^k_{iN_y} = 2u^k_{iN_y-2} - u^k_{iN_y-4} \tag{C.10}$$

Putting this in the above equation yields:

$$\begin{aligned}\tilde{n}^k_{iN_y-2} &= \underbrace{(a_{iN_y-2} - e_{iN_y-2})}_{a^*_{iN_y-2}} u^k_{iN_y-4} + \underbrace{(b_{iN_y-2} - d_{iN_y-2})}_{b^*_{iN_y-2}} u^k_{iN_y-3} \\ &\quad + \underbrace{(c_{iN_y-2} + 2d_{iN_y-2} + 2e_{iN_y-2})}_{c^*_{iN_y-2}} u^k_{iN_y-2} \qquad \text{(C.11)}\\ \tilde{n}^k_{iN_y-3} &= \underbrace{a_{iN_y-3}}_{a^*_{iN_y-3}} u^k_{iN_y-5} + \underbrace{b_{iN_y-3}}_{b^*_{iN_y-3}} u^k_{iN_y-4} + \underbrace{(c_{iN_y-3} - e_{iN_y-3})}_{c^*_{iN_y-3}} u^k_{iN_y-3} \\ &\quad + \underbrace{(d_{iN_y-3} + 2e_{iN_y-3})}_{d^*_{iN_y-3}} u^k_{iN_y-2} \qquad \text{(C.12)}\end{aligned}$$

The lower boundary conditions are:

$$u^k_{i1} = 2u^k_{i2} - u^k_{i3} \qquad \text{(C.13)}$$

$$u^k_{i0} = 2u^k_{i2} - u^k_{i4} \qquad \text{(C.14)}$$

These result in the following equations:

$$\begin{aligned}\tilde{n}^k_{i2} &= \underbrace{(c_{i2} + 2a_{i2} + 2b_{i2})}_{c^*_{i2}} u^k_{i2} + \underbrace{(d_{i2} - b_{i2})}_{d^*_{i2}} u^k_{i3} + \underbrace{(e_{i2} - a_{i2})}_{\mathrm{e}^*_{i2}} u^k_{i4} \\ \tilde{n}^k_{i3} &= \underbrace{(b_{i3} + 2a_{i3})}_{b^*_{i3}} u^k_{i2} + \underbrace{(c_{i3} - a_{i3})}_{c^*_{i3}} u^k_{i3} + \underbrace{d_{i3}}_{d^*_{i3}} u^k_{i4} + \underbrace{e_{i3}}_{\mathrm{e}^*_{i3}} u^k_{i5} \qquad \text{(C.15)}\end{aligned}$$

All together we get following linear system of equations:

$$\boldsymbol{N}^k_{i\cdot} \boldsymbol{u}^k_{i\cdot} = \boldsymbol{n}^k_{i\cdot} \qquad \text{(C.16)}$$

with:

$$\boldsymbol{N}^k_{i\cdot} = \begin{bmatrix} c^*_{i2} & d^*_{i2} & \mathrm{e}^*_{i2} & 0 & & \cdots & 0 \\ b^*_{i3} & c^*_{i3} & d^*_{i3} & \mathrm{e}^*_{i3} & 0 & \cdots & 0 \\ a_{i4} & b_{i4} & c_{i4} & d_{i4} & e_{i4} & 0 & \vdots \\ 0 & a_{i5} & b_{i5} & c_{i5} & d_{i5} & e_{i5} & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & 0 & a_{iN_y-4} & b_{iN_y-4} & c_{iN_y-4} & d_{iN_y-4} & e_{iN_y-4} \\ 0 & \cdots & 0 & a^*_{iN_y-3} & b^*_{iN_y-3} & c^*_{iN_y-3} & d^*_{iN_y-3} \\ 0 & \cdots & & 0 & a^*_{iN_y-2} & b^*_{iN_y-2} & c^*_{iN_y-2} \end{bmatrix}$$

$$\boldsymbol{u}_{i\cdot}^k = \begin{bmatrix} u_{i2}^k \\ u_{i3}^k \\ \vdots \\ u_{iN_y-3}^k \\ u_{iN_y-2}^k \end{bmatrix}, \qquad \boldsymbol{n}_{i\cdot}^k = \begin{bmatrix} n_{i2}^k \\ n_{i2}^k \\ \vdots \\ n_{iN_y-3}^k \\ n_{iN_y-2}^k \end{bmatrix}$$

APPENDIX D: A SUFFICIENT CONDITION FOR A POSITIVE CREDIT SPREAD PROCESS IN THE RS APPROACH

In this section we will prove a sufficient condition under which it is ensured that the credit spread process will stay positive. We do this for the credit spreads whose dynamics follow Equation (9) and whose volatility dynamics relays on the RS approach with $m = 1$, ie, for $\sigma_{\mathcal{S}}(t,t) = \sigma s(t)$. Recall the dynamics of the credit spread $s(t)$:

$$\begin{aligned} s(t) &= X_s(t) + \mathcal{S}(0,t) \\ \mathrm{d}X_s(t) &= (-\kappa_s(t)X_s(t) + \Phi_s(t))\,\mathrm{d}t + \sigma_{\mathcal{S}}(t,t)\,\mathrm{d}W_2(t), \quad X_s(0) = 0 \\ \mathrm{d}\Phi_s(t) &= \sigma_{\mathcal{S}}^2(t,t)\,\mathrm{d}t - 2\kappa_s(t)\Phi_s(t)\,\mathrm{d}t, \quad \varphi(0) = 0 \end{aligned}$$

which can be rewritten as:

$$\begin{aligned} \mathrm{d}s(t) &= \mu(t, s(t), \Phi_s(t))\,\mathrm{d}t + \sigma_{\mathcal{S}}(t,t)\,\mathrm{d}W_2(t), \quad s(0) = s_0 \\ \mu(t, s(t), \Phi_s(t)) &= \kappa_s(t)(\mathcal{S}(0,t) - s(t)) + \Phi_s(t) + \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{S}(0,t) \\ \mathrm{d}\Phi_s(t) &= \sigma_{\mathcal{S}}^2(t,t)\,\mathrm{d}t - 2\kappa_s(t)\Phi_s(t)\,\mathrm{d}t, \quad \varphi(0) = 0 \end{aligned}$$

We define a process $\tilde{s}(t)$ such that:

$$\mathrm{d}\tilde{s}(t) = \tilde{\mu}(t, \tilde{s}(t))\,\mathrm{d}t + \sigma_{\mathcal{S}}(t,t)\,\mathrm{d}W_2(t), \quad \tilde{s}(0) = s_0 \tag{D.1}$$

$$\tilde{\mu}(t, \tilde{s}(t)) = \kappa_s(t)(\mathcal{S}(0,t) - \tilde{s}(t)) + \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{S}(0,t) \leqslant \mu(t, s(t), \Phi_s(t)) \tag{D.2}$$

Then, since the processes $s(t)$ and $\tilde{s}(t)$ satisfy the conditions (i)–(v) of Proposition 2.18 in Karatzas and Shreve (2000), the following relationship holds:

$$P[s(t) \geqslant \tilde{s}(t),\ \forall 0 \leqslant t < \infty] = 1$$

Condition (iv), ie, (D.2), holds since, by definition, we have that:

$$\Phi_s(t) = \int_0^t \sigma_{\mathcal{S}}^2(u,t)\,\mathrm{d}u \geqslant 0$$

Condition (v) is fulfilled in the Li *et al* (1995) case due to $m = 1$, see Example 2.14 in Karatzas and Shreve (2000).

TABLE 7 Default-free and defaultable discount factors at the valuation date: October 17, 2008.

Date	DF (EUR) $B(0,T)$	S10 IG $D(0,T)$	S10 Xover $D(0,T)$
October 17, 2008	1.000	1.000	1.000
December 20, 2008	0.9911	0.9975	0.9870
March 20, 2009	0.9789	0.9941	0.9683
June 20, 2009	0.9665	0.9906	0.9499
September 20, 2009	0.9569	0.9872	0.9318
December 20, 2009	0.9496	0.9837	0.9141
March 20, 2010	0.9422	0.9803	0.8968
June 20, 2010	0.9344	0.9769	0.8797
September 20, 2010	0.9267	0.9735	0.8630
December 20, 2010	0.9157	0.9701	0.8466
March 20, 2011	0.9078	0.9668	0.8305
June 20, 2011	0.8980	0.9634	0.8146
September 20, 2011	0.8884	0.9601	0.7993
December 20, 2011	0.8788	0.9567	0.7841
March 20, 2012	0.8694	0.9534	0.7691
June 20, 2012	0.8600	0.9501	0.7545
September 20, 2012	0.8507	0.9468	0.7402
December 20, 2012	0.8415	0.9435	0.7262
March 20, 2012	0.8326	0.9403	0.7123
June 20, 2013	0.8236	0.9369	0.6988
September 20, 2013	0.8146	0.9337	0.6854
December 20, 2013	0.8056	0.9304	0.6725

Due to the variations of constants formula (see, for example, Korn and Korn (2001, p. 62)), $\tilde{s}(t)$ has the following unique solution:

$$\tilde{s}(t) = \mathrm{e}^{-(\kappa_s+0.5\sigma^2)t+\sigma W_2(t)}\left[\tilde{s}_0 + \int_0^t \mathrm{e}^{-(\kappa_s+0.5\sigma^2)t+\sigma W_2(t)} a(u)\,\mathrm{d}u\right]$$

where:

$$a(t) := \kappa_s \mathcal{S}(0,t) + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathcal{S}(0,t)}{\kappa_s}$$

We notice that a sufficient condition for $\tilde{s}(t) \geqslant 0$ is $a(t) \geqslant 0$, which is equivalent to:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{S}(0,t) \geqslant -\kappa_s^2 \mathcal{S}(0,t)$$

or to:

$$\mathcal{S}(0,t) \geqslant S_0 \mathrm{e}^{-\kappa_s^2 t}$$

Finally, notice that the sufficient condition for the credit spread model to stay positive is a very weak one.

APPENDIX E: DATA

The discount and default risk factors used in this study are collected in Table 7.

REFERENCES

Andersen, L., and Andreasen, J. (2002). Volatile volatilities. *Risk Magazine* **15**(12), 163–168.

Andreasen, J. (2003). Back to the future. *Risk Magazine* **18**(9), 104–109.

Brigo, D., and Alfonsi, A. (2005). Credit default swap calibration and derivatives pricing with the SSRD stochastic intensity model. *Finance and Stochastics* **9**(1), 29–42.

Cheyette, O. (1994). Markov representation of the Heath–Jarrow–Morton model. Presented at the UCLA Workshop on the Future of Fixed Income Financial Theory April 8 and 9, 1994.

Craig, I. J. D., and Sneyd, A. D. (1988). An alternating-direction implicit scheme for parabolic equations with mixed derivatives. *Computational Mathematics Applied* **16**, 341–350.

Heath, D., Jarrow, R., and Morton, A. (1992). Bond princing and the term structure of interest rates: a new methodology for contingent claim valuation. *Econometrica* **60**(1), 77–105.

Hull, J., and White, A. (2003). The valuation of credit default swap options. *Journal of Derivatives* **8**(3), 40–50.

Jacod, J., and Shiryaev, A. N. (1988). *Limit Theorems for Stochastic Processes*. Springer.

Karatzas, I., and Shreve, S. E. (2000). *Brownian Motion and Stochastic Calculus*, 2nd edn. Springer.

Korn, R., and Korn, E. (2001). *Option Pricing and Portfolio Optimization*. American Mathematical Society, Providence, RI.

Krekel, M., and Wenzel, J. (2006). A unified approach to CDSwaption and CMCDS valuation. Technical report, Fraunhofer ITWM, Kaiserslautern.

Li, A., Ritchken, P., and Sankarasubramanian, L. (1995). Lattice models for pricing American interest rate claims. *The Journal of Finance* **50**, 719–737.

Ritchken, P., and Sankarasubramanian, L. (1995). Volatility structures of forward rates and dynamics of the term structure. *Mathematical Finance* **5**, 55–72.

Schönbucher, P. (1998). Term structure modeling of defaultable bonds. *Review of Derivatives Research* **2**, 161–192.

Schönbucher, P. (2000). A LIBOR market model with default risk. Bonn Econ Discussion Paper Series No. 15/2001, University of Bonn.

Schönbucher, P. (2004). A measure of survival. *Risk Magazine* **17**(8), 79–85.