UNIVERSITY NAME

DOCTORAL THESIS

Hierarchical deterministic wallet

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in the

Research Group Name
Department or School Name

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Declaration of Authorship

I, Daniele FORNARO, declare that this thesis titled, "Hierarchical deterministic wallet" and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
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- I have acknowledged all main sources of help.
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Signed:

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"Thanks to my solid academic training, today I can write hundreds of words on virtually any topic without possessing a shred of information, which is how I got a good job in journalism."

Dave Barry

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Abstract

Faculty Name
Department or School Name

Mathematical Engeneering

Hierarchical deterministic wallet

by Daniele FORNARO

The Thesis Abstract is written here (and usually kept to just this page). The page is kept centered vertically so can expand into the blank space above the title too…

Acknowledgements

The acknowledgments and the people to thank go here, don't forget to include your project advisor…

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List of Abbreviations

LAH	**L**ist **A**bbreviations **H**ere
WSF	**W**hat (it) **S**tands **F**or

Physical Constants

Speed of Light $\quad c_0 = 2.997\,924\,58 \times 10^8\ \mathrm{m\,s^{-1}}$ (exact)

List of Symbols

a	distance	m
P	power	W ($\mathrm{J\,s^{-1}}$)
ω	angular frequency	rad

For/Dedicated to/To my…

Chapter 1

Elliptic Curve Geometry

1.1 Introduction

Bitcoin security is based on public and private key cryptograpy. The main concept is that it is simple to compute the public key, knowing the private, but it is infeasible to calculate the private key, knowing the public.

In order to obtain this result a particular Elliptic Curve is used.

1.2 Elliptic Curve over $\mathbb{F}_p$

A point Q, which coordinates are x and y, belong to an Elliptic Curve if and only if Q satisfies the following equation:

$$y^2 = x^3 + ax + b \quad \text{over } \mathbb{F}_p \tag{1.1}$$

Where $\mathbb{F}_p$ is the finite field defined over the set of integers modulo p and a and b are the coefficients of the curve.

We can rewrite the equation 1.1 in the following way:

$$y^2 = x^3 + ax + b \mod p \tag{1.2}$$

Figure 1.1 shows some examples of Elliptic Curve over $\mathbb{F}_p$ with $a = -7$ and $b = 10$

1.2.1 Operations

A point on the Elliptic Curve has some particular properties:

- Symmetry
- Point addition
- Scalar multiplication

Symmetry

For every point in the x axis exists two points in the y axis. Suppose that a point $P(x, y)$ belongs to the Elliptic Curve, then it must satisfy the equation 1.1. So it is easy to prove that the point $Q(x, p - y)$ belongs to the curve too.

Furthermore we have $P = -Q$, from the moment that $P + Q = 0$ (see addition below).

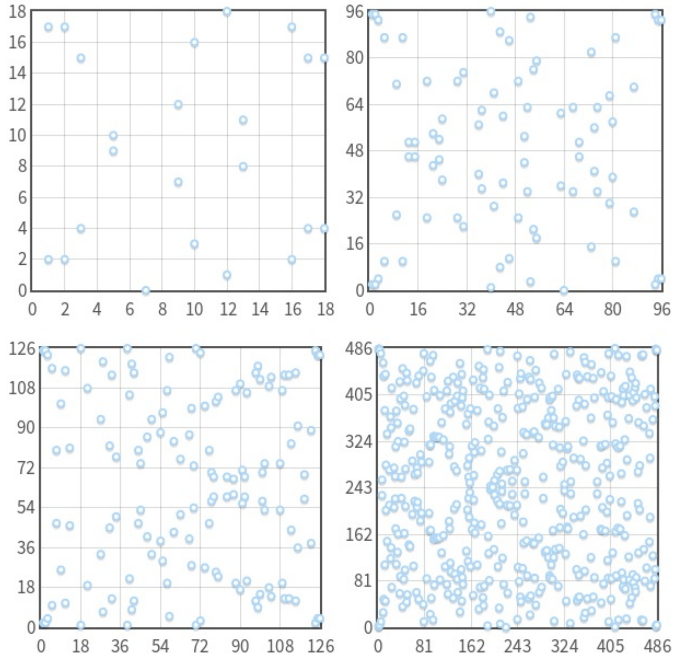

FIGURE 1.1: Points on the Elliptic Curve $y^2 = x^3 - 7x + 10 \mod p$, with $p = 19, 97, 127, 487$

Point addition

We need to change our definition of addition in order to make it works in $\mathbb{F}_p$. In this framework we claim that if three points are aligned over the finite field $\mathbb{F}_p$, then they have zero sum.
So $P + Q = R$ if and only if P, Q and $-R$ are aligned, in the sense shown in figure 1.2

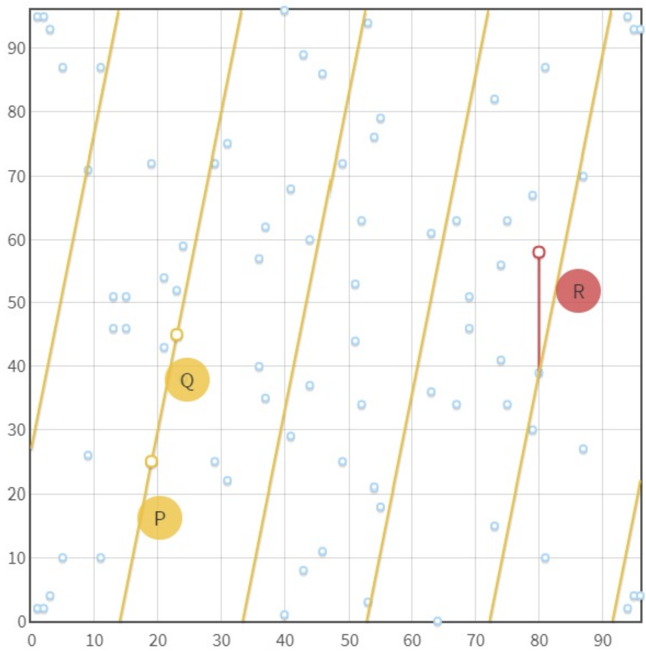

FIGURE 1.2: Elliptic Curve $y^2 = x^3 - 7x + 10 \mod 97$

The equations for calculating point additions are the follow:
Suppose that A and B belong to the Elliptic Curve.

$$A = (x_1, y_1) \quad B = (x_2, y_2)$$

Let's defined $A + B := (x_3, y_3)$
So we have:

$$s = \begin{cases} \dfrac{y_2 - y_1}{x_2 - x_1}, & \text{if } x_1 \neq x_2 \\ \\ \dfrac{3x_1^2 + a}{2y_1}, & \text{if } x_1 = x_2 \end{cases}$$

$$x_3 = s^2 - x_1 - x_2 \mod p$$
$$y_3 = s(x_1 - x_3) - y_1 \mod p$$

Scalar multiplication

Once defined the addition, any multiplication can be defined as:

$$nP = \underbrace{P + P + \cdots + P}_{n \text{ times}}$$

When n is a very large number can be difficult or even infeasible to compute nP in this way, but we can use the *double and add algorithm* in order to perform multiplication in $\mathcal{O}(\log n)$ steps.

1.2.2 Group order

An elliptic curve defined over a finite field is a group and so it has a finite number of points. This number is called order of the group.
If the prime order is a very large number, it is impossible to count all the point in that field, but there is an algorithm that allows to calculate the order of a group in a fast and efficient way: *Schoof's algorithm*.

Cyclic subgroups

Let's consider a generic point P, we have:

$$nP + mP = \underbrace{P + \cdots + P}_{n \text{ times}} + \underbrace{P + \cdots + P}_{m \text{ times}} = \underbrace{P + \cdots + P}_{n+m \text{ times}} = (n+m)P$$

So multiple of P are closed under addition and this is enough to prove that the set of the multiples of P is a cyclic subgroup of the group formed by the elliptic curve.

The point P is called generator of the cyclic subgroup.

Remark *The order of P is linked to the order of the elliptic curve by Lagrange's theorem, which states that the order of a subgroup is a divisor of the order of the parent group.*

Remark *If the order of the group is a prime number, all the point P generate a subgroup with the same order of the group.*

1.2.3 Bitcoin Elliptic Curve

Bitcoin uses a specific Elliptic Curve defined over the finite field of the natural numbers, where $a = 0$ and $b = 7$.

The equation 1.1 becomes:

$$y^2 = x^3 + 7 \mod p \tag{1.3}$$

The *mod p* (modulo prime number) indicates that this curve is over a finite field of prime order $p = 2^{256} - 2^{32} - 2^9 - 2^8 - 2^7 - 2^6 - 2^4 - 1$.

1.3 Bitcoin private-public key cryptography

Appendix A

Frequently Asked Questions

A.1 How do I change the colors of links?

The color of links can be changed to your liking using:
`\hypersetup{urlcolor=red}`, or
`\hypersetup{citecolor=green}`, or
`\hypersetup{allcolor=blue}`.
If you want to completely hide the links, you can use:
`\hypersetup{allcolors=.}`, or even better:
`\hypersetup{hidelinks}`.
If you want to have obvious links in the PDF but not the printed text, use:
`\hypersetup{colorlinks=false}`.

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