CHAPTER 12

Inductance and Magnetic Circuits

12.1 Inductance

The *inductance L* of a conductor system may be defined as *the ratio of the linking magnetic flux to the current producing the flux*. For static (or, at most, low-frequency) current I and a coil containing N turns, as shown in Fig. 12-1,

$$L = \frac{N\,\Phi}{I}$$

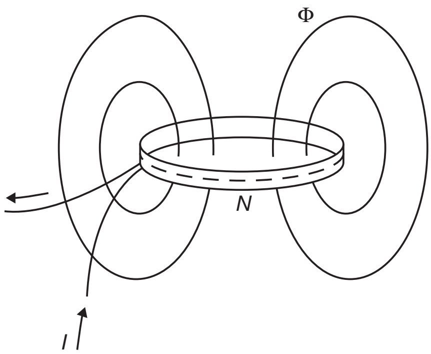


Fig. 12-1

The units on L are *henries*, where 1 H = 1 Wb/A. Inductance is also given by $L = \lambda/I$, where λ, the *flux linkage*, is $N\Phi$ for coils with N turns or simply Φ for other conductor arrangements.

It should be noted that L will always be the product of the permeability μ of the medium (units on μ are H/m) and a geometrical factor having the units of length. Compare the expressions for resistance R (Chapter 7) and capacitance C (Chapter 8).

EXAMPLE 1. Find the inductance per unit length of a coaxial conductor such as that shown in Fig.12-2. Between the conductors,

$$\mathbf{H} = \frac{I}{2\pi r}\,\mathbf{a}_\phi$$

$$\mathbf{B} = \frac{\mu_0 I}{2\pi r}\,\mathbf{a}_\phi$$

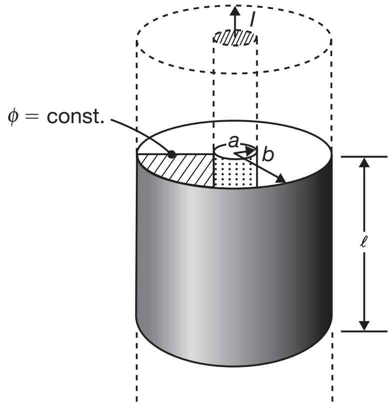


Fig. 12-2

The currents in the two conductors are linked by the flux across the surface ϕ = const. For a length ℓ,

$$\lambda = \int_0^\ell \int_a^b \frac{\mu_0 I}{2\pi r}\,dr\,dz = \frac{\mu_0 I \ell}{2\pi} \ln \frac{b}{a}$$

and

$$\frac{L}{\ell} = \frac{\mu_0}{2\pi} \ln \frac{b}{a} \quad \text{(H/m)}$$

EXAMPLE 2. Find the inductance of an ideal solenoid with 300 turns, $\ell = 0.50$ m, and a circular cross section of radius 0.02 m.

The turns per unit length is $n = 300/0.50 = 600$, so that the axial field is

$$B = \mu_0 H = \mu_0 600 I \quad \text{(Wb/m}^2\text{)}$$

Then

$$\frac{L}{\ell} = \frac{N\Phi}{I} = N\left(\frac{B}{I}\right)A = 300(600\mu_0)\pi(4 \times 10^{-4})$$
$$= 568\ \mu\text{H/m}$$

or $L = 284\ \mu$H.

In Section 6.7 an imagined bringing-in of point charges from infinity was used to derive the energy content of an electric field:

$$W_E = \frac{1}{2}\int_{\text{vol}} \mathbf{D}\cdot\mathbf{E}\,dv$$

There is no equivalent in a magnetic field to the point charge, and consequently no parallel development for its stored energy. However, a more sophisticated approach yields the completely analogous expression

$$W_H = \frac{1}{2}\int_{\text{vol}} \mathbf{B}\cdot\mathbf{H}\,dv$$

Comparing this with the formula $W_H = \frac{1}{2}LI^2$ from circuit analysis yields

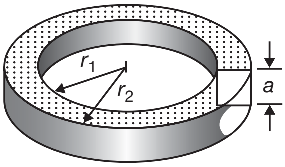
$$L = \int_{\text{vol}} \frac{\mathbf{B}\cdot\mathbf{H}}{I^2}\,dv$$

EXAMPLE 3. Checking Example 1,

$$L = \int_{\text{vol}} \frac{\mathbf{B}\cdot\mathbf{H}}{I^2}\,dv = \frac{\mu_0}{I^2}\int_0^{\ell}\int_0^{2\pi}\int_a^b \left(\frac{I^2}{4\pi^2 r^2}\right) r\,dr\,d\phi\,dz = \frac{\mu_0 \ell}{2\pi}\ln\frac{b}{a}$$

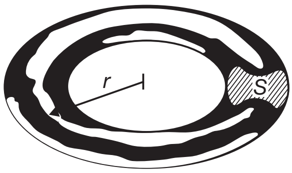
12.2 Standard Conductor Configurations

Figs. 12-3 through 12-7 give exact or approximate inductances of some common noncoaxial arrangements.



$$L = \frac{\mu_0 N^2 a}{2\pi}\ln\frac{r_2}{r_1} \quad \text{(H)}$$

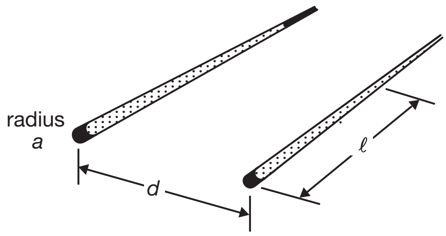
Fig. 12-3 Toroid, square cross section.



$$L \approx \frac{\mu_0 N^2 S}{2\pi r} \quad \text{(H)}$$

(assuming average flux density at average radius r)

Fig. 12-4 Toroid, general cross section S.

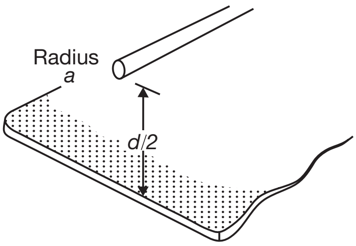


$$\frac{L}{\ell} = \frac{\mu_0}{\pi}\cosh^{-1}\frac{d}{2a} \quad \text{(H/m)}$$

For $d >> a$,

$$\frac{L}{\ell} \approx \frac{\mu_0}{\pi}\ln\frac{d}{a} \quad \text{(H/m)}$$

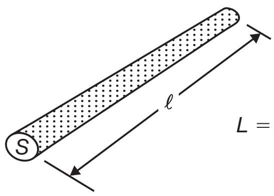
Fig. 12-5 Parallel conductors of radius a.



$$\frac{L}{\ell} = \frac{\mu_0}{2\pi}\cosh^{-1}\frac{d}{2a} \quad \text{(H/m)}$$

$$\approx \frac{\mu_0}{2\pi}\ln\frac{d}{a} \quad \text{(H/m)}$$

Fig. 12-6 Cylindrical conductor parallel to a ground plane.



$$L = \frac{\mu_0 N^2 S}{\ell} \quad \text{(H)}$$

Fig. 12-7 Long solenoid of small cross-sectional area S.

12.3 Faraday's Law and Self-Inductance

Consider an open surface S bounded by a closed contour C. If the magnetic flux ϕ linking S varies with time, then a *voltage v around C* exists; by *Faraday's law*,

$$v = -\frac{d\phi}{dt}$$

As was shown in Chapter 6, the electrostatic potential or voltage, V, is well-defined in space and is associated with a conservative electric field. By contrast, the *induced* voltage v given by Faraday's law is a multivalued function of position and is associated with a nonconservative field (*electromotive force*). More about this in Chapter 13.

Faraday's law holds in particular when the flux through a circuit element is changing *because the current in that same element is changing*:

$$v = -\frac{d\phi}{di}\frac{di}{dt} = -L\frac{di}{dt}$$

In circuit theory, L is called the *self-inductance* of the element and v is called the *voltage of self-inductance* or the back-voltage *in the inductor*.

12.4 Internal Inductance

Magnetic flux occurs within a conductor cross section as well as external to the conductor. This internal flux gives rise to an *internal inductance*, which is often small compared to the external inductance and frequently ignored. In Fig. 12-8(a) a conductor of circular cross section is shown, with a current I assumed to be uniformly distributed over the area. (This assumption is valid only at low frequencies, since *skin effect* at higher frequencies forces the current to be concentrated at the outer surface.) Within the conductor of radius a, Ampère's law gives

$$\mathbf{H} = \frac{Ir}{2\pi a^2}\mathbf{a}_\phi \qquad \text{and} \qquad \mathbf{B} = \frac{\mu_0 Ir}{2\pi a^2}\mathbf{a}_\phi$$

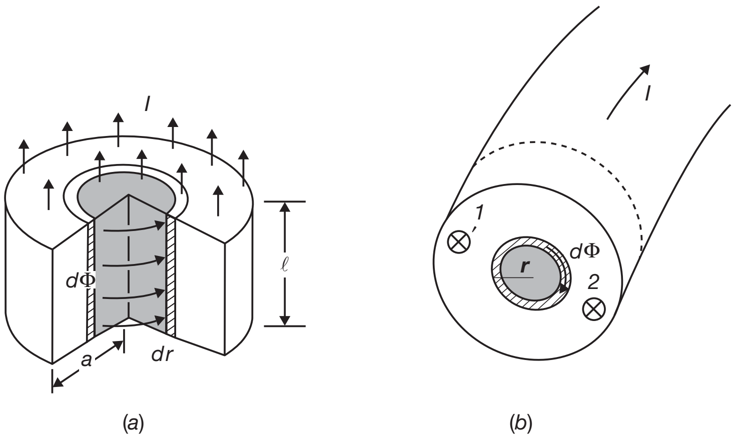


Fig. 12-8

The straight piece of conductor shown in Fig. 12-8(a) must be imagined as a short section of an infinite torus, as suggested in Fig. 12-8(b). The current filaments become circles of infinite radius. The lines of flux $d\Phi$ through the strip $\ell\, dr$ encircle only those filaments whose distance from the conductor axis is smaller than r. Thus, an open surface bounded by one of those filaments is cut once (or an odd number of times) by the lines of $d\Phi$; whereas, for a filament such as *1* or *2*, the surface is cut zero times (or an even number of times). It follows that $d\Phi$ links only with the fraction $\pi r^2/\pi a^2$ of the total current, so that the total flux linkage is given by the weighted "sum"

$$\lambda = \int \left(\frac{\pi r^2}{\pi a^2}\right) d\Phi = \int_0^a \left(\frac{\pi r^2}{\pi a^2}\right) \frac{\mu_0 I r}{2\pi a^2} \ell \, dr = \frac{\mu_0 I \ell}{8\pi}$$

and
$$\frac{L}{\ell} = \frac{\lambda / I}{\ell} = \frac{\mu_0}{8\pi} = \frac{1}{2} \times 10^{-7} \text{ H/m}$$

This result is independent of the conductor radius. The total inductance is the sum of the external and internal inductances. If the external inductance is of the order of $\frac{1}{2} \times 10^{-7}$ H/m, the internal inductance should not be ignored.

12.5 Mutual Inductance

In Fig. 12-9 a part ϕ_{12} of the magnetic flux produced by the current i_1 through coil 1 links the N_2 turns of coil 2. The voltage of *mutual induction* in coil 2 is given by

$$v_2 = N_2 \frac{d\phi_{12}}{dt} \quad \text{(negative sign omitted)}$$

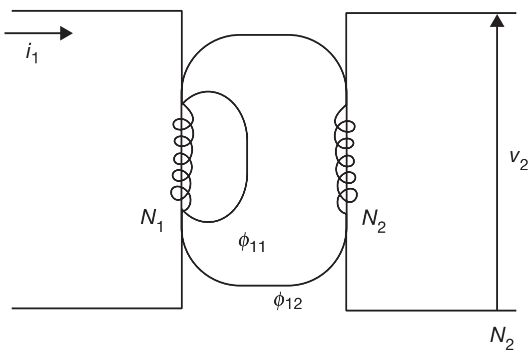


Fig. 12-9

In terms of the *mutual inductance* $M_{12} \equiv N_2 \phi_{12} / I_1$,

$$v_2 = N_2 \frac{d\phi_{12}}{di_1} \frac{di_1}{dt} = M_{12} \frac{di_1}{dt}$$

This mutual inductance will be a product of the permeability μ of the region between the coils and a geometrical length, just like inductance L. If the roles of coils 1 and 2 are reversed,

$$v_1 = M_{21} \frac{di_2}{dt}$$

The following reciprocity relation can be established: $M_{12} = M_{21}$.

EXAMPLE 4. A solenoid with $N_1 = 1000$, $r_1 = 1.0$ cm, and $\ell_1 = 50$ cm is concentric within a second coil of $N_2 = 2000$, $r_2 = 2.0$ cm, and $\ell_2 = 50$ cm. Find the mutual inductance assuming free-space conditions.

For long coils of small cross sections, H may be assumed constant inside the coil and zero for points just outside the coil. With the first coil carrying a current I_1,

$$H = \left(\frac{1000}{0.50}\right) I_1 \quad \text{(A/m)} \qquad \text{(in the axial direction)}$$

$$B = \mu_0 2000 I_1 \quad (\text{Wb/m}^2)$$

$$\Phi = BA = (\mu_0 2000 I_1)(\pi \times 10^{-4}) \quad \text{(Wb)}$$

Since H and B are zero outside the coils, this is the only flux linking the second coil.

$$M_{12} = N_2\left(\frac{\Phi}{I_1}\right) = (2000)(4\pi \times 10^{-7})(2000)(\pi \times 10^{-4}) = 1.58\,\text{mH}$$

12.6 Magnetic Circuits

In Chapter 10, magnetic field intensity **H**, flux **Φ**, and magnetic flux density **B** were examined and various problems were solved where the medium was free space. For example, when Ampère's law is applied to the closed path C through the long, air-core coil shown in Fig. 12-10, the result is

$$\oint \mathbf{H} \cdot d\mathbf{l} = NI$$

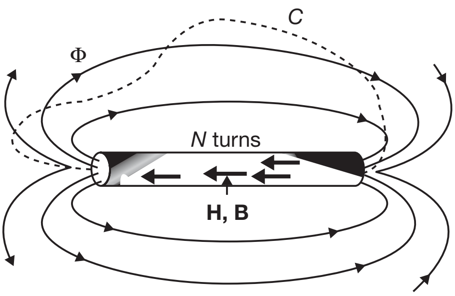


Fig. 12-10

But since the flux lines are widely spread outside of the coil, B is small there. The flux is effectively restricted to the inside of the coil, where

$$H \approx \frac{NI}{\ell}$$

Ferromagnetic materials have relative permeabilities μ_r in the order of thousands. Consequently, the flux density $B = \mu_0\mu_r H$ is, for a given H, much greater than would result in free space. In Fig. 12-11, the coil is not distributed over the iron core. Even so, the NI of the coil causes a flux **Φ** which follows the core. It might be said that the flux prefers the core to the surrounding space by a ratio of several thousand to one. This is so different from the free-space magnetics of Chapter 10 that an entire subject area, known as *iron-core magnetics* or *magnetic circuits*, has developed. This brief introduction to the subject assumes that *all* of the flux is within the core. It is further assumed that the flux is uniformly distributed over the cross section of the core. Core lengths required for calculation of NI drops are mean lengths.

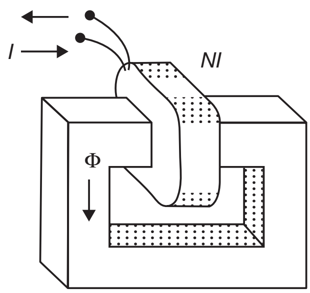


Fig. 12-11

12.7 The *B-H* Curve

A sample of ferromagnetic material could be tested by applying increasing values of H and measuring the corresponding values of flux density B. *Magnetization curves*, or simply *B-H curves*, for some common ferromagnetic materials are given in Figs. 12-12 and 12-13. The relative permeability can be computed from the *B-H* curve by use of $\mu_r = B/\mu_0 H$. Fig. 12-14 shows the extreme nonlinearity of μ_r versus H for silicon steel. This nonlinearity requires that problems be solved graphically.

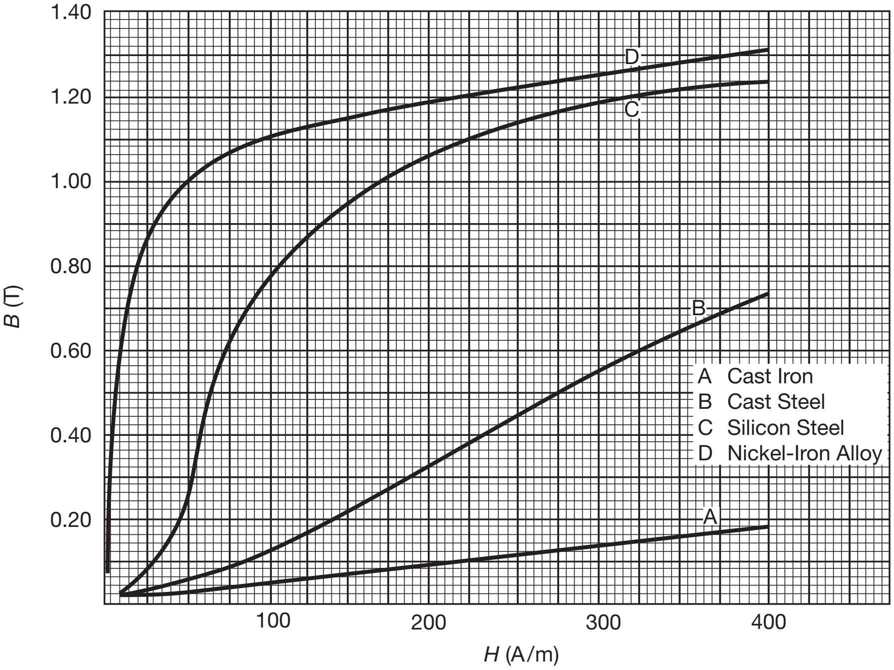


Fig. 12-12 *B-H* curves, $H < 400$ A/m.

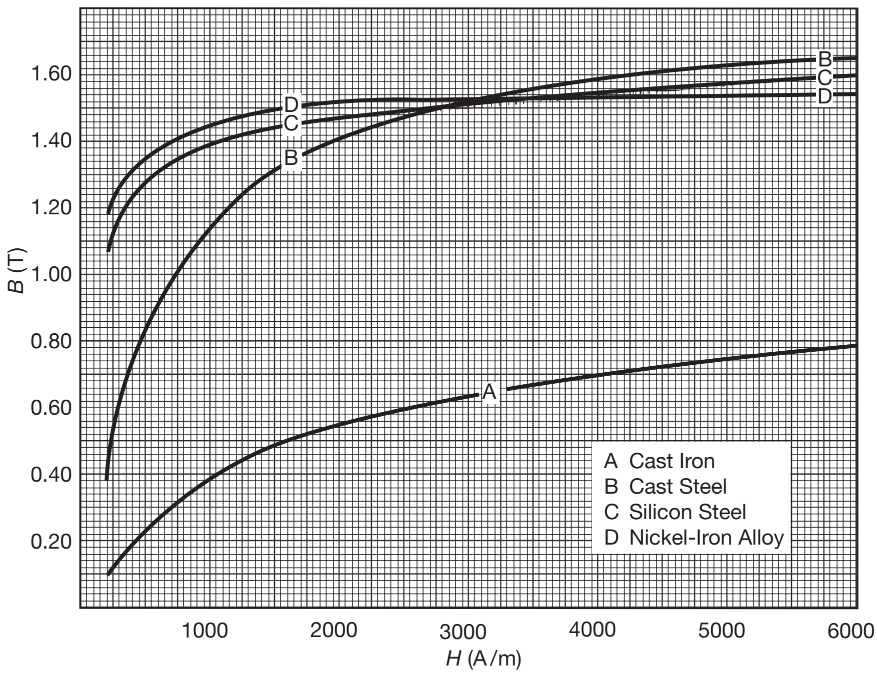


Fig. 12-13 *B-H* curves, $H > 400$ A/m.

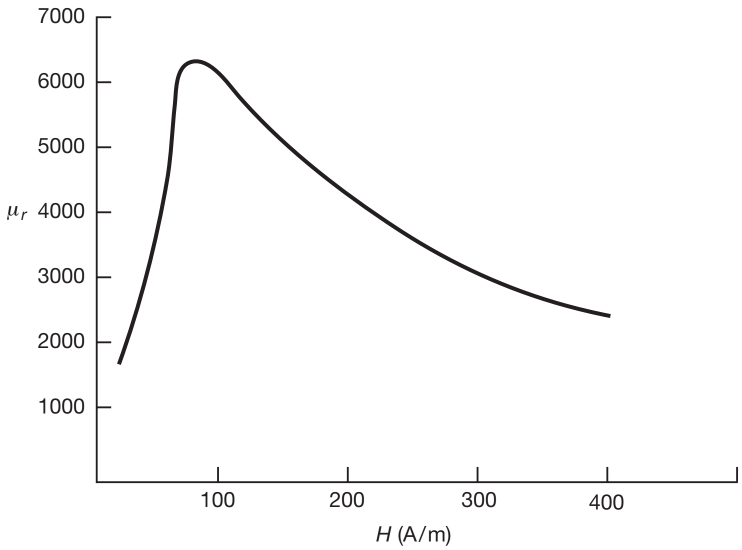
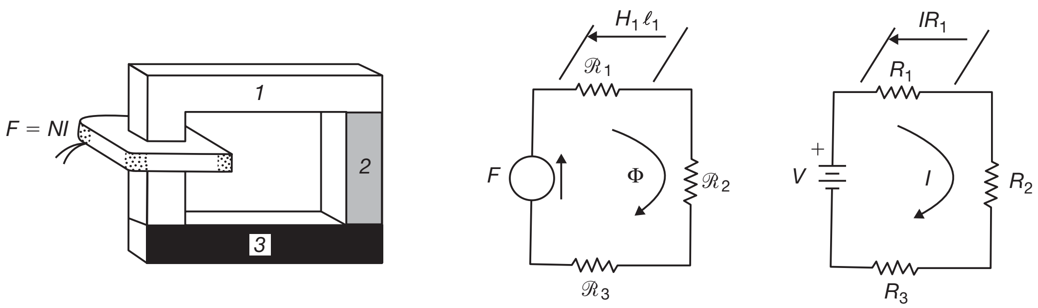


Fig. 12-14

12.8 Ampère's Law for Magnetic Circuits

A coil of N turns and current I around a ferromagnetic core produces a *magnetomotive force* (mmf) given by NI. The symbol F is sometimes used for this mmf; the units are amperes or *ampere turns*. Ampère's law, applied around the path in the center of the core shown in Fig. 12-15(a), gives

$$\begin{aligned} F = NI &= \oint \mathbf{H} \cdot d\mathbf{l} \\ &= \int_1 \mathbf{H} \cdot d\mathbf{l} + \int_2 \mathbf{H} \cdot d\mathbf{l} + \int_3 \mathbf{H} \cdot d\mathbf{l} \\ &= H_1\ell_1 + H_2\ell_2 + H_3\ell_3 \end{aligned}$$



(a) (b) (c)

Fig. 12-15

Comparison with Kirchhoff's law around a single closed loop with three resistors and an emf V,

$$V = V_1 + V_2 + V_3$$

suggests that F can be viewed as an *NI rise* and the $H\ell$ terms considered *NI drops*, in analogy to the voltage rise V and voltage drops V_1, V_2 and V_3. The analogy is developed in Fig. 12-15(b) and (c). Flux Φ in Fig. 12-15(b) is analogous to current I, and *reluctance* $\mathcal{R}$ is analogous to resistance R. An expression for reluctance can be developed as follows.

$$NI \text{ drop} = H\ell = BA\left(\frac{\ell}{\mu A}\right) = \Phi\mathcal{R}$$

Hence,

$$\mathcal{R} = \frac{\ell}{\mu A} \quad (\mathrm{H}^{-1})$$

If the reluctances are known, then the equation

$$F = NI = \Phi(\mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3)$$

can be written for the magnetic circuit of Fig. 12-15(*b*). However, μ_r must be known for each material before its reluctance can be calculated. And only after B or H is known will the value of μ_r be known. This is in contrast to the relation

$$R = \frac{\ell}{\sigma A}$$

(Section 7.7), in which the conductivity σ is independent of the current.

12.9 Cores with Air Gaps

Magnetic circuits with small air gaps are very common. The gaps are generally kept as small as possible, since the *NI* drop of the air gap is often much greater than the drop in the core. The flux fringes outward at the gap, so that the area at the gap exceeds the area of the adjacent core. Provided that the gap length ℓ_a is less than $\frac{1}{10}$ the smaller dimension of the core, an *apparent area*, S_a, of the air gap can be calculated. For a rectangular core of dimensions a and b,

$$S_a = (a + \ell_a)(b + \ell_a)$$

If the total flux in the air gap is known, H_a and $H_a\ell_a$ can be computed directly.

$$H_a = \frac{1}{\mu_0}\left(\frac{\Phi}{S_a}\right) \qquad H_a\ell_a = \frac{\ell_a\Phi}{\mu_0 S_a}$$

For a uniform iron core of length ℓ_i with a single air gap, Ampère's law reads

$$NI = H_i\ell_i + H_a\ell_a = H_i\ell_i + \frac{\ell_a\Phi}{\mu_0 S_a}$$

If the flux Φ is known, it is not difficult to compute the *NI* drop across the air gap, obtain B_i, take H_i from the appropriate *B-H* curve and compute the *NI* drop in the core, $H_i\ell_i$. The sum is the *NI* required to establish the flux Φ. However, with *NI* given, it is a matter of trial and error to obtain B_i and Φ, as will be seen in the problems. Graphical methods of solution are also available.

12.10 Multiple Coils

Two or more coils on a core could be wound such that their mmfs either aid or oppose one another. Consequently, a method of indicating polarity is given in Fig. 12-16. An assumed direction for the resulting flux Φ could be incorrect, just as an assumed current in a dc circuit with two or more voltage sources may be incorrect. A negative result simply means that the flux is in the opposite direction.

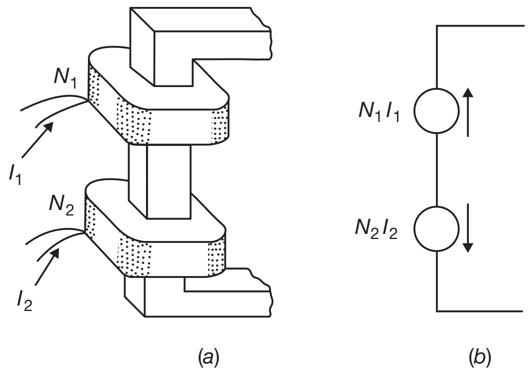


Fig. 12-16

12.11 Parallel Magnetic Circuits

The method of solving a parallel magnetic circuit is suggested by the two-loop equivalent circuit shown in Fig. 12-17(*b*). The leg on the left contains an *NI* rise and an *NI* drop. The *NI* drop between the junctions *a* and *b* can be written for each leg as follows:

$$F - H_1\ell_1 = H_2\ell_2 = H_3\ell_3$$

and the fluxes satisfy

$$\Phi_1 = \Phi_2 + \Phi_3$$

Different materials for the core parts will necessitate working with several *B-H* curves. An air gap in one of the legs would lead to $H_i\ell_i + H_a\ell_a$ for the mmf between the junctions for that leg.

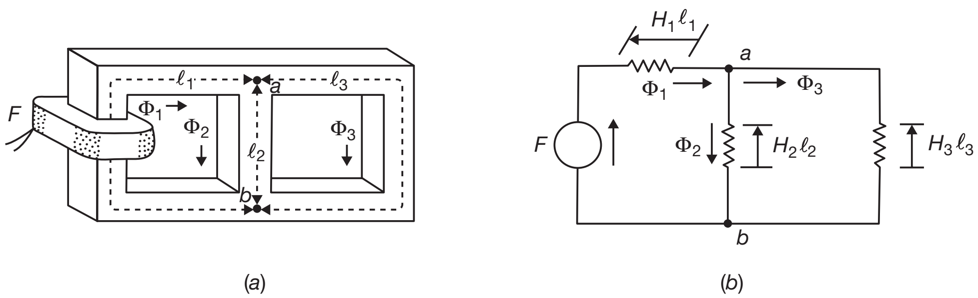


Fig. 12-17

The equivalent magnetic circuit should be drawn for parallel magnetic circuit problems. It is good practice to mark the material types, cross-sectional areas, and mean lengths directly on the diagram. In more complex problems a scheme like Table 12-1 can be helpful. The data are inserted directly into the table, and the remaining quantities are then calculated or taken from the appropriate *B-H* curve.

TABLE 12-1

PART	MATERIAL	AREA	ℓ	Φ	B	H	$H\ell$
1							
2							
3							

SOLVED PROBLEMS

12.1. Find the inductance per unit length of the coaxial cable in Fig. 12-2 if $a = 1$ mm and $b = 3$ mm. Assume $\mu_r = 1$ and omit internal inductance.

$$\frac{L}{\ell} = \frac{\mu}{2\pi} \ln \frac{b}{a} = \frac{4\pi \times 10^{-7}}{2\pi} \ln 3 = 0.22\ \mu\text{H/m}$$

12.2. Find the inductance per unit length of the parallel cylindrical conductors shown in Fig. 12-5, where $d = 25$ ft, $a = 0.803$ in.

$$\frac{L}{\ell} = \frac{\mu_0}{\pi} \cosh^{-1} \frac{d}{2a} = (4 \times 10^{-7}) \cosh^{-1} \frac{25(12)}{2(0.803)} = 2.37\ \mu\text{H/m}$$

The approximate formula gives

$$\frac{L}{\ell} = \frac{\mu_0}{\pi} \ln \frac{d}{a} = 2.37\ \mu\text{H/m}$$

When $d/a \geq 10$, the approximate formula may be used with an error of less than 0.5%.

12.3. A circular conductor with the same radius as in Problem 12.2 is 12.5 ft from an infinite conducting plane. Find the inductance.

$$\frac{L}{\ell} = \frac{\mu_0}{2\pi} \ln \frac{d}{a} = (2 \times 10^{-7}) \ln \frac{25(12)}{0.803} = 1.18\ \mu\text{H/m}$$

This result is $\frac{1}{2}$ that of Problem 12.2. A conducting plane may be inserted midway between the two conductors of Fig. 12-5. The inductance between each conductor and the plane is 1.18 μH/m. Since they are in series, the total inductance is the sum, 2.37 μH/m.

12.4. Assume that the air-core toroid shown in Fig. 12-4 has a circular cross section of radius 4 mm. Find the inductance if there are 2500 turns and the mean radius is $r = 20$ mm.

$$L = \frac{\mu N^2 S}{2\pi r} = \frac{(4\pi \times 10^{-7})(2500)^2 \pi (0.004)^2}{2\pi(0.020)} = 3.14\ \text{mH}$$

12.5. Assume that the air-core toroid in Fig. 12-3 has 700 turns, an inner radius of 1 cm, an outer radius of 2 cm, and height $a = 1.5$ cm. Find L using (*a*) the formula for square cross-section toroids; (*b*) the approximate formula for a general toroid, which assumes a uniform H at a mean radius.

(*a*)
$$L = \frac{\mu_0 N^2 a}{2\pi} \ln \frac{r_2}{r_1} = \frac{(4\pi \times 10^{-7})(700)^2 (0.015)}{2\pi} \ln 2 = 1.02\ \text{mH}$$

(*b*)
$$L = \frac{\mu_0 N^2 S}{2\pi r} = \frac{(4\pi \times 10^{-7})(700)^2 (0.01)(0.015)}{2\pi(0.015)} = 0.98\ \text{mH}$$

With a radius that is larger compared to the cross section, the two formulas yield the same result. See Problem 12.26.

12.6. Use the energy integral to find the internal inductance per unit length of a cylindrical conductor of radius a.

At a distance $r \leq a$ from the conductor axis,

$$\mathbf{H} = \frac{Ir}{2\pi a^2}\mathbf{a}\phi \qquad \mathbf{B} = \frac{\mu_0 Ir}{2\pi a^2}\mathbf{a}\phi$$

whence
$$\mathbf{B}\cdot\mathbf{H} = \frac{\mu_0 I^2}{4\pi^2 a^4} r^2$$

The inductance corresponding to energy storage within a length ℓ of the conductor is then

$$L = \int \frac{(\mathbf{B}\cdot\mathbf{H})\,dv}{I^2} = \frac{\mu_0}{4\pi^2 a^4}\int_0^a r^2 2\pi r\ell\, dr = \frac{\mu_0 \ell}{8\pi}$$

or $L/\ell = \mu_0/8\pi$. This agrees with the result of Section 12.4.

12.7. The cast-iron core shown in Fig. 12-18 has an inner radius of 7 cm and an outer radius of 9 cm. Find the flux Φ if the coil mmf is 500 A.

$$\ell = 2\pi(0.08) = 0.503\ \text{m}$$
$$H = \frac{F}{\ell} = \frac{500}{0.503} = 995\ \text{A/m}$$

From the *B-H* curve for cast iron in Fig. 12-13, $B = 0.40$ T.

$$\Phi = BS = (0.40)(0.02)^2 = 0.16\ \text{m Wb}$$

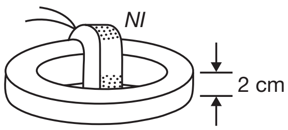


Fig. 12-18

12.8. The magnetic circuit shown in Fig. 12-19 has a C-shaped cast-steel part, *1*, and a cast-iron part, *2*. Find the current required in the 150-turn coil if the flux density in the cast iron is $B_2 = 0.45$ T.

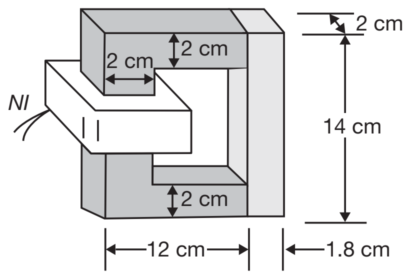


Fig. 12-19

The calculated areas are $S_1 = 4 \times 10^{-4}\ \text{m}^2$ and $S_2 = 3.6 \times 10^{-4}\ \text{m}^2$. The mean lengths are

$$\ell_1 = 0.11 + 0.11 + 0.12 = 0.34\ \text{m}$$
$$\ell_2 = 0.12 + 0.009 + 0.009 = 0.138\ \text{m}$$

From the *B-H* curve for cast iron in Fig. 12-13, $H_2 = 1270$ A/m.

$$\Phi = B_2 S_2 = (0.45)(3.6 \times 10^{-4}) = 1.62 \times 10^{-4}\ \text{Wb}$$
$$B_1 = \frac{\Phi}{S_1} = 0.41\ \text{T}$$

Then, from the cast-steel curve in Fig. 12-12, $H_1 = 233$ A/m.

The equivalent circuit, Fig. 12-20, suggests the equation

$$F = NI = H_1\ell_1 + H_2\ell_2$$
$$150I = 233(0.34) + 1270(0.138)$$
$$I = 1.70\text{ A}$$

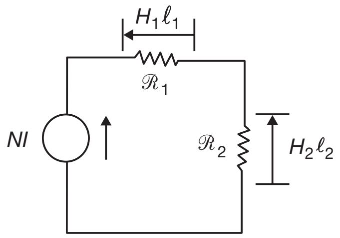


Fig. 12-20

12.9. The magnetic circuit shown in Fig. 12-21 is cast-iron with a mean length $\ell_1 = 0.44$ m and square cross section 0.02×0.02 m. The air-gap length is $\ell_a = 2$ mm and the coil contains 400 turns. Find the current I required to establish an air-gap flux of 0.141 mWb.

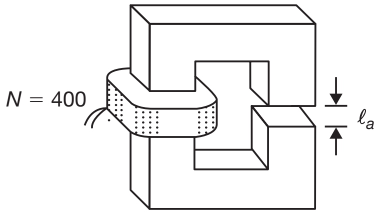


Fig. 12-21

The flux Φ in the air gap is also the flux in the core.

$$B_i = \frac{\Phi}{S_i} = \frac{0.141\times10^{-3}}{4\times10^{-4}} = 0.35\text{ T}$$

From Fig. 12-13, $H_i = 850$ A/m. Then

$$H_i\ell_i = 850(0.44) = 374\text{ A}$$

For the air gap, $S_a = (0.02 + 0.002)^2 = 4.84 \times 10^{-4}\text{ m}^2$, and so

$$H_a\ell_a = \frac{\Phi}{\mu_0 S_a}\ell_a = \frac{0.141\times10^{-3}}{(4\pi\times10^{-7})(4.84\times10^{-4})}(2\times10^{-3}) = 464\text{ A}$$

Therefore, $F = H_i\ell_i + H_a\ell_a = 838$ A and

$$I = \frac{F}{N} = \frac{838}{400} = 2.09\text{ A}$$

12.10. Determine the reluctance of an air gap in a dc machine where the apparent area is $S_a = 4.26 \times 10^{-2}\text{ m}^2$ and the gap length $\ell_a = 5.6$ mm.

$$\mathcal{R} = \frac{\ell_a}{\mu_0 S_a} = \frac{5.6\times10^{-3}}{(4\pi\times10^{-7})(4.26\times10^{-2})} = 1.05\times10^5\text{ H}^{-1}$$

12.11. The cast-iron magnetic core shown in Fig. 12-22 has an area $S_i = 4\ \text{cm}^2$ and a mean length 0.438 m. The 2-mm air gap has an apparent area $S_a = 4.84\ \text{cm}^2$. Determine the air-gap flux Φ.

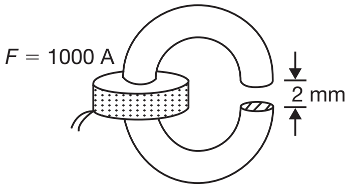


Fig. 12-22

The core is quite long compared to the length of the air gap, and cast iron is not a particularly good magnetic material. As a first estimate, therefore, assume that 600 of the total ampere turns are dropped at the air gap, i.e., $H_a\ell_a = 600$ A.

$$H_a\ell_a = \frac{\Phi}{\mu_0 S_a}\ell_a$$

$$\Phi = \frac{600(4\pi\times10^{-7})(4.84\times10^{-4})}{2\times10^{-3}} = 1.82\times10^{-4}\ \text{Wb}$$

Then $B_i = \Phi/S_i = 0.46$ T, and from Fig. 12-13, $H_i = 1340$ A/m. The core drop is then

$$H_i\ell_i = 1340(0.438) = 587\ \text{A}$$

so that

$$H_i\ell_i + H_a\ell_a = 1187\ \text{A}$$

This sum exceeds the 1000 A mmf of the coil. Consequently, values of B_i lower than 0.46 T should be tried until the sum of $H_i\ell_i$ and $H_a\ell_a$ is 1000 A. The values $B_i = 0.41$ T and $\Phi = 1.64\times10^{-4}$ Wb will result in a sum very close to 1000 A.

12.12. Solve Problem 12.11 using reluctances and the equivalent magnetic circuit, Fig. 12-23.

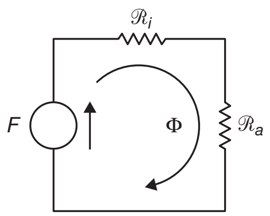


Fig. 12-23

From the values of B_i and H_i obtained in Problem 12.11,

$$\mu_0\mu_r = \frac{B_i}{H_i} = 3.83\times10^{-4}\,\text{H/m}$$

Then, for the core,

$$\mathcal{R}_i = \frac{\ell_i}{\mu_0\mu_r S_i} = \frac{0.438}{(3.83\times10^{-4})(4\times10^{-4})} = 2.86\times10^{6}\ \text{H}^{-1}$$

and for the air gap,

$$\mathcal{R}_a = \frac{\ell_a}{\mu_0 S_a} = \frac{2\times 10^{-3}}{(4\pi\times 10^{-7})(4.84\times 10^{-4})} = 3.29\times 10^6\,\mathrm{H}^{-1}$$

The circuit equation,

$$F = \Phi(\mathcal{R}_i + \mathcal{R}_a)$$

gives

$$\Phi = \frac{1000}{2.86\times 10^6 + 3.29\times 10^6} = 1.63\times 10^{-4}\ \mathrm{Wb}$$

The corresponding flux density in the iron is 0.41 T, in agreement with the results of Problem 12.11. While the air-gap reluctance can be calculated from the dimensions and μ_0, the same is not true for the reluctance of the iron. The reason is that μ_r for the iron depends on the values of B_i and H_i.

12.13. Solve Problem 12.11 graphically with a plot of Φ versus F.

Values of H_i from 700 through 1100 A/m are listed in the first column of Table 12-2; the corresponding values of B_i are found from the cast-iron curve, Fig. 12-13. The values of Φ and $H_i\ell_i$ are computed, and $H_a\ell_a$ is obtained from $\Phi\ell_a/\mu_0 S_a$. Then F is given as the sum of $H_i\ell_i$ and $H_a\ell_a$. Since the air gap is linear, only two points are required. The flux Φ for $F = 1000$ A is seen from Fig. 12-24 to be approximately 1.65×10^{-4} Wb.

This method is simply a plot of the trial and error data used in Problem 12.11. However, it is helpful if several different coils or coil currents are to be examined.

TABLE 12-2

H_i (A/m)	B_i (T)	Φ (Wb)	$H_i\ell_i$ (A)	$H_a\ell_a$ (A)	F (A)
700	0.295	1.18×10^{-4}	307	388	695
800	0.335	1.34×10^{-4}	350	441	791
900	0.365	1.46×10^{-4}	395	480	874
1000	0.400	1.60×10^{-4}	438	526	964
1100	0.420	1.68×10^{-4}	482	552	1034

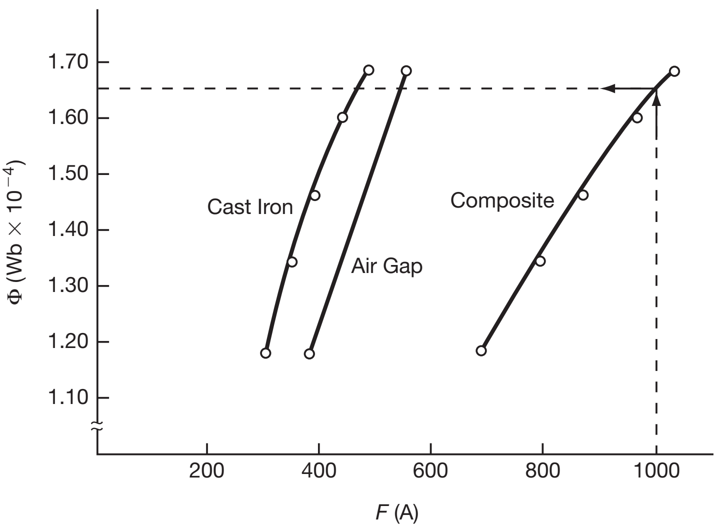


Fig. 12-24

12.14. Determine the fluxes Φ in the core of Problem 12.11 for coil mmfs of 800 and 1200 A. Use a graphical approach and the *negative air-gap line*.

The Φ versus $H_i\ell_i$ data for the cast-iron core, developed in Problem 12.13, are plotted in Fig. 12-25. The air-gap Φ versus F is linear. One end of the negative air-gap line for the coil mmf of 800 A is at $\Phi = 0$, $F = 800$ A. The other end assumes $H_a\ell_a = 800$ A, from which

$$\Phi = \frac{\mu_0 S_a (H_a \ell_a)}{\ell_a} = 2.43 \times 10^{-4}\,\text{Wb}$$

which locates this end at $\Phi = 2.43 \times 10^{-4}$ Wb, $F = 0$.

The intersection of the $F = 800$ A negative air-gap line with the nonlinear Φ versus F curve for the cast-iron core gives $\Phi = 1.34 \times 10^{-4}$ Wb. Other negative air-gap lines have the same negative slope. For a coil mmf of 1000 A, $\Phi = 1.63 \times 10^{-4}$ Wb and for 1200 A, $\Phi = 1.85 \times 10^{-4}$ Wb.

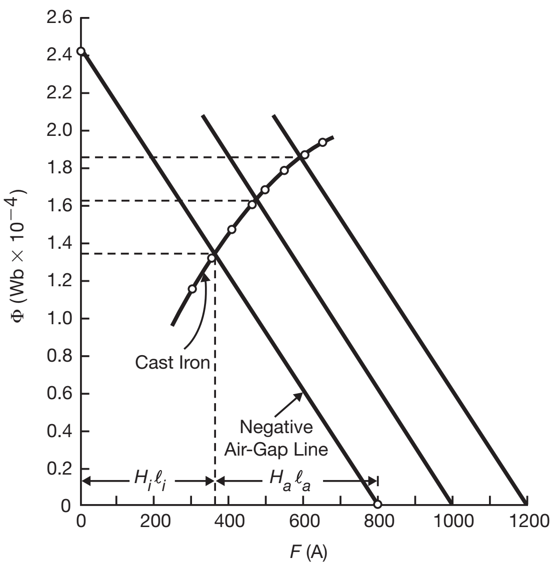


Fig. 12-25

12.15. Solve Problem 12.11 for a coil mmf of 1000 A using the *B-H* curve for cast iron.

This method avoids the construction of an additional curve such as the Φ versus F curves of Problems 12.13 and 12.14. Now, in order to plot the air-gap line on the *B-H* curve of iron, adjustments must be made for the different areas and the different lengths. Table 12-3 suggests the necessary calculations.

$$\frac{F}{\ell_i} = \frac{1000}{0.438} = 2283\ \text{A/m}$$

TABLE 12-3

B_a (T)	H_a (A/m)	$B_a\left(\dfrac{S_a}{S_i}\right)$ (T)	$H_a\left(\dfrac{\ell_a}{\ell_i}\right)$ (A/m)	$\dfrac{F}{\ell_i} - H_a\left(\dfrac{\ell_a}{\ell_i}\right)$ (A/m)
0.10	0.80×10^5	0.12	363	1920
0.30	2.39×10^5	0.36	1091	1192
0.50	3.98×10^5	0.61	1817	466

The data from the third and fifth columns may be plotted directly on the cast-iron B-H curve, as shown in Fig. 12-26. The air gap is linear and only two points are needed. The answer is seen to be $B_i = 0.41$ T. The method can be used with two nonlinear core parts as well (see Problem 12.16).

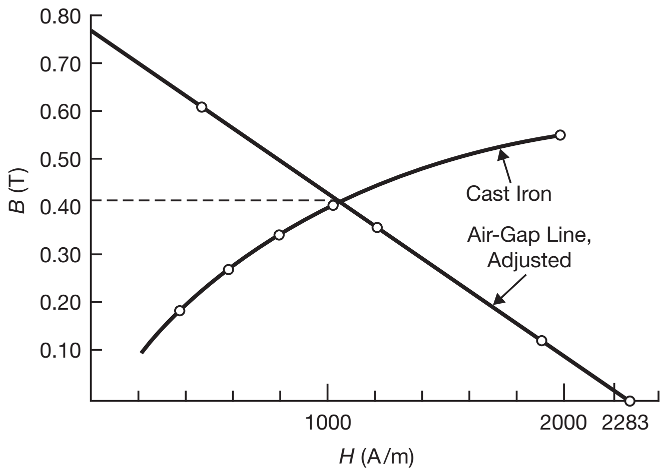


Fig. 12-26

12.16. The magnetic circuit shown in Fig. 12-27 consists of nickel-iron alloy in part *1*, where $\ell_1 = 10$ cm and $S_1 = 2.25$ cm^2, and cast-steel for part *2*, where $\ell_2 = 8$ cm and $S_2 = 3$ cm^2. Find the flux densities B_1 and B_2.

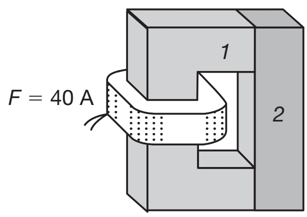


Fig. 12-27

The data for part *2* of cast-steel will be converted and plotted on the B-H curve for part *1* of nickel-iron alloy ($F/\ell_1 = 400$ A/m). Table 12-4 suggests the necessary calculations.

TABLE 12-4

B_2 (T)	H_2 (A/m)	$B_2\left(\frac{S_2}{S_1}\right)$ (T)	$H_2\left(\frac{\ell_2}{\ell_1}\right)$ (A/m)	$\frac{F}{\ell_1} - H_2\left(\frac{\ell_2}{\ell_1}\right)$ (A/m)
0.33	200	0.44	160	240
0.44	250	0.59	200	200
0.55	300	0.73	240	160
0.65	350	0.87	280	120
0.73	400	0.97	320	80
0.78	450	1.04	360	40
0.83	500	1.11	400	0

From the graph, Fig. 12-28, $B_1 = 1.01$ T. Then, since $B_1S_1 = B_2S_2$,

$$B_2 = 1.01\left(\frac{2.25\times10^{-4}}{3\times10^{-4}}\right) = 0.76\ \text{T}$$

These values can be checked by obtaining the corresponding H_1 and H_2 from the appropriate B-H curves and substituting in

$$F = H_1\ell_1 + H_2\ell_2$$

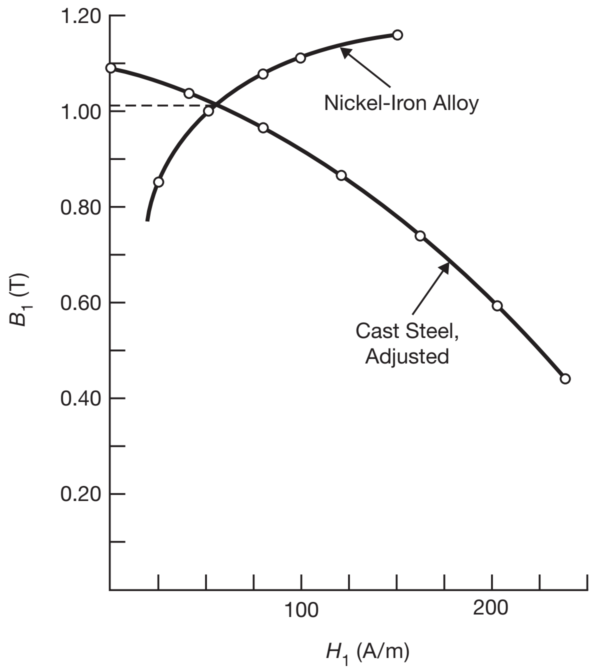


Fig. 12-28

12.17. The cast-steel parallel magnetic circuit in Fig. 12-29(a) has a coil with 500 turns. The mean lengths are $\ell_2 = \ell_3 = 10$ cm, $\ell_1 = 4$ cm. Find the coil current if $\Phi_3 = 0.173$ mWb.

$$\Phi_1 = \Phi_2 + \Phi_3$$

Since the cross-sectional area of the center leg is twice that of the two side legs, the flux density is the same throughout the core, i.e.,

$$B_1 = B_2 = B_3 = \frac{0.173\times10^{-3}}{1.5\times10^{-4}} = 1.15\ \text{T}$$

Corresponding to $B = 1.15$ T, Fig. 12-13 gives $H = 1030$ A/m. The NI drop between points a and b is now used to write the following equation [see Fig. 12-29(b)]:

$$F - H\ell_1 = H\ell_2 = H\ell_3 \qquad \text{or} \qquad F = H(\ell_1 + \ell_2) = 1030(0.14) = 144.2\ \text{A}$$

Then
$$I = \frac{F}{N} = \frac{144.2}{500} = 0.29\ \text{A}$$

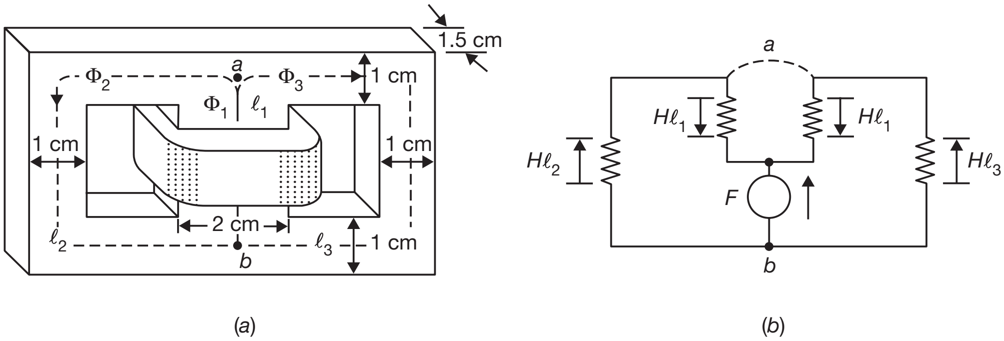


Fig. 12-29

12.18. The same cast-steel core as in Problem 12.17 has identical 500-turn coils on the outer legs, with the winding sense as shown in Fig. 12-30(*a*). If again $\Phi_3 = 0.173$ m Wb, find the coil currents.

The flux densities are the same throughout the core and consequently H is the same. The equivalent circuit in Fig. 12-30(*b*) suggests that the problem can be solved on a *per pole* basis.

$$B = \frac{\Phi_3}{S_3} = 1.15\,\text{T} \quad \text{and} \quad H = 1030\ \text{A/m} \quad (\text{from Fig.12-13})$$

$$F_3 = H(\ell_1 + \ell_3) = 1030(0.14) = 144.2\ \text{A} \qquad I = 0.29\ \text{A}$$

Each coil must have a current of 0.29 A.

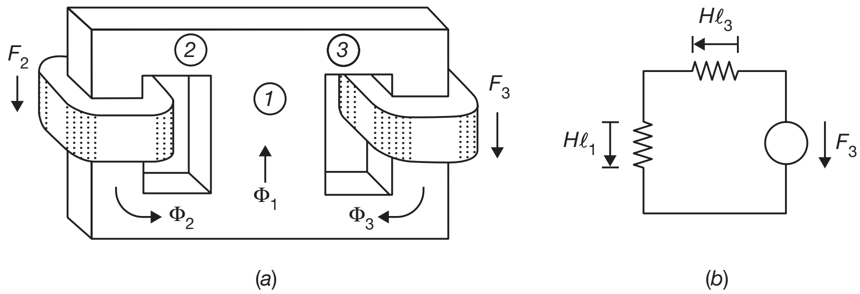


Fig. 12-30

12.19. The parallel magnetic circuit shown in Fig. 12-31(*a*) is silicon steel with the same cross-sectional area throughout, $S = 1.30$ cm^2. The mean lengths are $\ell_1 = \ell_3 = 25$ cm, $\ell_2 = 5$ cm. The coils have 50 turns each. Given that $\Phi_1 = 90\ \mu$ Wb and $\Phi_3 = 120\ \mu$ Wb, find the coil currents.

$$\Phi_2 = \Phi_3 - \Phi_1 = 0.30 \times 10^{-4}\ \text{Wb}$$

$$B_1 = \frac{90 \times 10^{-6}}{1.30 \times 10^{-4}} = 0.69\ \text{T}$$

From Fig. 12-12, $H_1 = 87$ A/m. Then, $H_1\ell_1 = 21.8$ A. Similarly, $B_2 = 0.23$ T, $H_2 = 49$ A/m, $H_2\ell_2 = 2.5$ A and $B_3 = 0.92$ T, $H_3 = 140$ A/m, $H_3\ell_3 = 35.0$ A. The equivalent circuit in Fig. 12-31(*b*) suggests the following equations for the *NI* drop between points *a* and *b*:

$$H_1\ell_1 - F_1 = H_2\ell_2 = F_3 - H_3\ell_3$$

$$21.8 - F_1 = 2.5 = F_3 - 35.0$$

from which $F_1 = 19.3$ A and $F_3 = 37.5$ A. The currents are $I_1 = 0.39$ A and $I_3 = 0.75$ A.

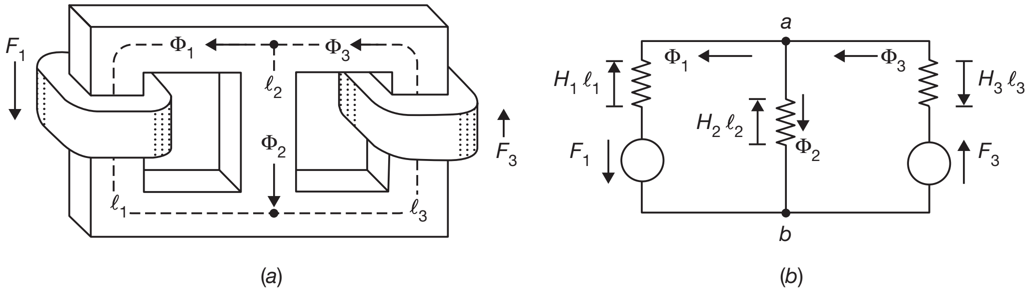


Fig. 12-31

12.20. Obtain the equivalent magnetic circuit for Problem 12.19 using reluctances for three legs, and calculate the flux in the core using $F_1 = 19.3$ A and $F_3 = 37.5$ A.

$$\mathcal{R} = \frac{\ell}{\mu_0\mu_r S}$$

From the values of B and H found in Problem 12.19,

$$\mu_0\mu_{r1} = 7.93 \times 10^{-3}\ \text{H/m} \qquad \mu_0\mu_{r2} = 4.69 \times 10^{-3}\ \text{H/m} \qquad \mu_0\mu_{r3} = 6.57 \times 10^{-3}\ \text{H/m}$$

Now the reluctances are calculated:

$$\mathcal{R}_1 = \frac{\ell_1}{\mu_0\mu_{r1}S_1} = 2.43 \times 10^5\ \text{H}^{-1}$$

$\mathcal{R}_2 = 8.20 \times 10^4\ \text{H}^{-1}$, $\mathcal{R}_3 = 2.93 \times 10^5\ \text{H}^{-1}$. From Fig. 12-32,

$$F_3 = \Phi_3\mathcal{R}_3 + \Phi_2\mathcal{R}_2 \tag{1}$$
$$F_1 = \Phi_1\mathcal{R}_1 - \Phi_2\mathcal{R}_2 \tag{2}$$
$$\Phi_1 + \Phi_2 = \Phi_3 \tag{3}$$

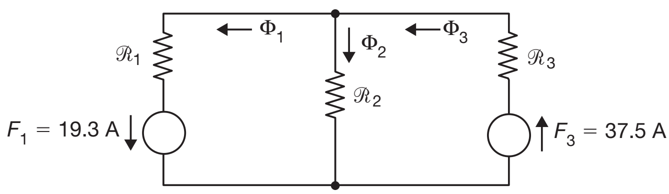


Fig. 12-32

Substituting Φ_2 from (3) into (1) and (2) results in the following set of simultaneous equations in Φ_1 and Φ_3:

$$\begin{aligned} F_1 &= \Phi_1(\mathcal{R}_1 + \mathcal{R}_2) - \Phi_3\mathcal{R}_2 \\ F_3 &= -\Phi_1\mathcal{R}_2 + \Phi_3(\mathcal{R}_2 + \mathcal{R}_3) \end{aligned} \quad \text{or} \quad \begin{aligned} 19.3 &= \Phi_1(3.25 \times 10^5) - \Phi_3(0.82 \times 10^5) \\ 37.5 &= -\Phi_1(0.82 \times 10^5) + \Phi_3(3.75 \times 10^5) \end{aligned}$$

Solving, $\Phi_1 = 89.7\ \mu$Wb, $\Phi_2 = 30.3\ \mu$Wb, $\Phi_3 = 120\ \mu$Wb.

Although the simultaneous equations above and the similarity to a two-mesh circuit problem may be interesting, it should be noted that the flux densities B_1, B_2, and B_3 had to be known before the relative permeabilities and reluctances could be computed. But if B is known, why not find the flux directly from $\Phi = BS$? Reluctance is simply not of much help in solving problems of this type.

SUPPLEMENTARY PROBLEMS

12.21. Find the inductance per unit length of a coaxial conductor with an inner radius $a = 2$ mm and an outer conductor at $b = 9$ mm. Assume $\mu_r = 1$.

12.22. Find the inductance per unit length of two parallel cylindrical conductors, where the conductor radius is 1 mm and the center-to-center separation is 12 mm.

12.23. Two parallel cylindrical conductors separated by 1 m have an inductance per unit length of 2.12 μH/m. What is the conductor radius?

12.24. An air-core solenoid with 2500 evenly spaced turns has a length of 1.5 m and a radius of 2×10^{-2} m. Find the inductance L.

12.25. A square-cross-section, air-core toroid such as that in Fig. 12-3 has inner radius 5 cm, outer radius 7 cm, and height 1.5 cm. If the inductance is 495 μH, how many turns are there in the toroid? Examine the approximate formula and compare the result.

12.26. A square-cross-section toroid such as that in Fig. 12-3 has $r_1 = 80$ cm, $r_2 = 82$ cm, $a = 1.5$ cm, and 700 turns. Find L using both formulas and compare the results. (See Problem 12.5.)

12.27. A coil with 5000 turns, $r_1 = 1.25$ cm, and $\ell_1 = 1.0$ m has a core with $\mu_r = 50$. A second coil of 500 turns, $r_2 = 2.0$ cm, and $\ell_2 = 10.0$ cm is concentric with the first coil, and in the space between the coils $\mu \approx \mu_0$. Find the mutual inductance.

12.28. Determine the relative permeabilities of cast-iron, cast-steel, silicon steel, and nickel-iron alloy at a flux density of 0.4 T. Use Figs. 12-12 and 12-13.

12.29. An air gap of length $\ell_a = 2$ mm has a flux density of 0.4 T. Determine the length of a magnetic core with the same NI drop if the core is of (*a*) cast-iron, (*b*) cast-steel, (*c*) silicon steel.

12.30. A magnetic circuit consists of two parts of the same ferromagnetic material ($\mu_r = 4000$). Part *1* has $\ell_1 = 50$ mm, $S_1 = 104$ mm^2; part *2* has $\ell_2 = 30$ mm, $S_2 = 120$ mm^2. The material is at a part of the curve where the relative permeability is proportional to the flux density. Find the flux Φ if the mmf is 4.0 A.

12.31. A toroid with a circular cross section of radius 20 mm has a mean length 280 mm and a flux $\Phi = 1.50$ mWb. Find the required mmf if the core is silicon steel.

12.32. Both parts of the magnetic circuit in Fig. 12-33 are cast-steel. Part *1* has $\ell_1 = 34$ cm and $S_1 = 6$ cm^2; part *2* has $\ell_2 = 16$ cm and $S_2 = 4$ cm^2. Determine the coil current I_1, if $I_2 = 0.5$ A, $N_1 = 200$ turns, $N_2 = 100$ turns, and $\Phi = 120\ \mu$Wb.

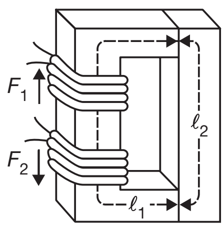


Fig. 12-33

12.33. The silicon steel core shown in Fig. 12-34 has a rectangular cross section 10 mm by 8 mm and a mean length 150 mm. The air-gap length is 0.8 mm and the air-gap flux is 80 μWb. Find the mmf.

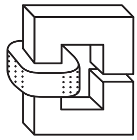


Fig. 12-34

12.34. Solve Problem 12.33 in reverse: the coil mmf is known to be 561.2 A and the air-gap flux is to be determined. Use the trial and error method, starting with the assumption that 90% of the *NI* drop is across the air gap.

12.35. The silicon steel magnetic circuit of Problem 12.33 has an mmf of 600 A. Determine the air-gap flux.

12.36. For the silicon steel magnetic circuit of Problem 12.33, calculate the reluctance of the iron, $\mathcal{R}_i$, and the reluctance of the air gap, $\mathcal{R}_a$. Assume the flux is $\Phi = 80\ \mu\text{Wb}$ and solve for F. See Fig. 12-35.

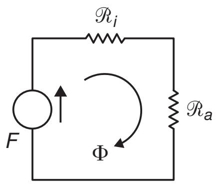


Fig. 12-35

12.37. A silicon steel core such as shown in Fig. 12-34 has a rectangular cross section of area $S_i = 80\ \text{mm}^2$ and an air gap of length $\ell_a = 0.8$ mm with area $S_a = 95\ \text{mm}^2$. The mean length of the core is 150 mm and the mmf is 600 A. Solve graphically for the flux by plotting Φ versus F in the manner of Problem 12.13.

12.38. Solve Problem 12.37 graphically using the negative air-gap line for an mmf of 600 A.

12.39. Solve Problem 12.37 graphically in the manner of Problem 12.15, obtaining the flux density in the core.

12.40. A rectangular ferromagnetic core 40×60 mm has a flux $\Phi = 1.44$ mWb. An air gap in the core is of length $\ell_a = 2.5$ mm. Find the *NI* drop across the air gap.

12.41. A toroid with cross section of radius 2 cm has a silicon steel core of mean length 28 cm and an air gap of length 1 mm. Assume the air-gap area, S_a, is 10% greater than the adjacent core and find the mmf required to establish an air-gap flux of 1.5 mWb.

12.42. The magnetic circuit shown in Fig. 12-36 has an mmf of 500 A. Part *1* is cast-steel with $\ell_1 = 340$ mm and $S_1 = 400\ \text{mm}^2$; part *2* is cast-iron with $\ell_2 = 138$ mm and $S_2 = 360\ \text{mm}^2$. Determine the flux Φ.

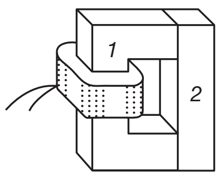


Fig. 12-36

12.43. Solve Problem 12.42 graphically in the manner of Problem 12.16.

12.44. A toroid of square cross section, with $r_1 = 2$ cm, $r_2 = 3$ cm, and height $a = 1$ cm, has a two-part core. Part *1* is silicon steel of mean length 7.9 cm; part *2* is nickel-iron alloy of mean length 7.9 cm. Find the flux that results from an mmf of 17.38 A.

12.45. Solve Problem 12.44 by the graphical method of Problem 12.15. Why is it that the plotting of the second reverse *B-H* curve on the first is not as difficult as might be expected?

12.46. The cast-steel parallel magnetic circuit in Fig. 12-37 has a 500-turn coil in the center leg, where the cross-sectional area is twice that of the remainder of the core. The dimensions are $\ell_a = 1$ mm, $S_2 = S_3 = 150$ mm^2, $S_1 = 300$ mm^2, $\ell_1 = 40$ mm, $\ell_2 = 110$ mm, and $\ell_3 = 109$ mm. Find the coil current required to produce an air-gap flux of 125 μWb. Assume that S_a exceeds S_3 by 17%.

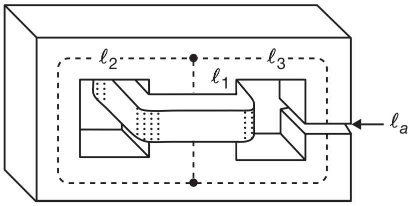


Fig. 12-37

12.47. The cast-iron parallel circuit core in Fig. 12-38 has a 500-turn coil and a uniform cross section of 1.5 cm^2 throughout. The mean lengths are $\ell_1 = \ell_3 = 10$ cm and $\ell_2 = 4$ cm. Determine the coil current necessary to result in a flux density of 0.25 T in leg *3*.

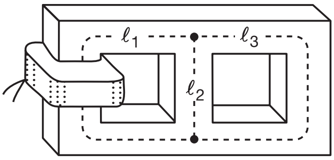


Fig. 12-38

12.48. Two identical 500-turn coils have equal currents and are wound as indicated in Fig. 12-39. The cast-steel core has a flux in leg *3* of 120 μWb. Determine the coil currents and the flux in leg *1*.

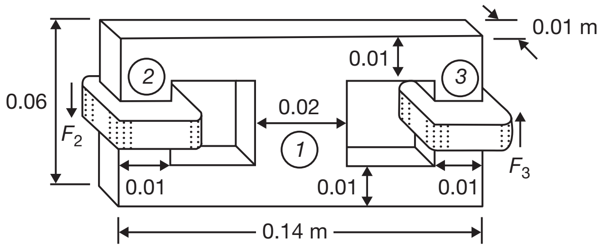


Fig. 12-39

12.49. Two identical coils are wound as indicated in Fig. 12-40. The silicon steel core has a cross section of 6 cm^2 throughout. The mean lengths are $\ell_1 = \ell_3 = 14$ cm and $\ell_2 = 4$ cm. Find the coil mmfs if the flux in leg *1* is 0.7 mWb.

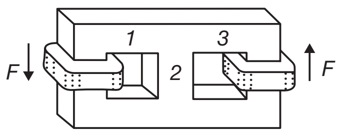


Fig. 12-40

ANSWERS TO SUPPLEMENTARY PROBLEMS

12.21. 0.301 μH/m

12.22. 0.992 μH/m

12.23. 5 mm

12.24. 6.58 mH

12.25. 700, 704

12.26. 36.3 μ H (both formulas)

12.27. 7.71 mH

12.28. 318, 1384, 5305, 42, 440

12.29. (*a*) 0.64 cm; (*b*) 2.77 m; (*c*) 10.6 m

12.30. 26.3 μWb

12.31. 83.2 A

12.32. 0.65 A

12.33. 561.2 A

12.35. 85.2 μWb

12.36. $\mathcal{R}_1 = 0.313\ \mu\text{H}^{-1}, \mathcal{R}_a = 6.70\ \mu\text{H}^{-1}, F = 561$ A

12.37. 85 μWb

12.38. 85 μWb

12.39. 1.06 T

12.40. 1079 A

12.41. 952 A

12.42. 229 μWb

12.43. 229 μWb

12.44. 10^{-4} Wb

12.45. 10^{-4} Wb. The mean lengths and cross-sectional areas are the same.

12.46. 1.34 A

12.47. 1.05 A

12.48. 0.41 A, 0 Wb

12.49. 38.5 A