4

FLOWS ON THE CIRCLE

4.0 Introduction

So far we've concentrated on the equation $\dot{x} = f(x)$, which we visualized as a vector field on the line. Now it's time to consider a new kind of differential equation and its corresponding phase space. This equation,

$$\dot{\theta} = f(\theta),$$

corresponds to a ***vector field on the circle.*** Here θ is a point on the circle and $\dot{\theta}$ is the velocity vector at that point, determined by the rule $\dot{\theta} = f(\theta)$. Like the line, the circle is one-dimensional, but it has an important new property: by flowing in one direction, a particle can eventually return to its starting place (Figure 4.0.1). Thus periodic solutions become possible for the first time in this book! To put it another way, *vector fields on the circle provide the most basic model of systems that can oscillate.*

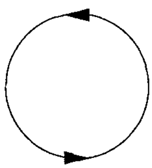


Figure 4.0.1

However, in all other respects, flows on the circle are similar to flows on the line, so this will be a short chapter. We will discuss the dynamics of some simple oscillators, and then show that these equations arise in a wide variety of applications. For example, the flashing of fireflies and the voltage oscillations of superconducting Josephson junctions have been modeled by the same equation, even though their oscillation frequencies differ by about ten orders of magnitude!

4.1 Examples and Definitions

Let's begin with some examples, and then give a more careful definition of vector fields on the circle.

EXAMPLE 4.1.1:

Sketch the vector field on the circle corresponding to $\dot{\theta} = \sin\theta$.

Solution: We assign coordinates to the circle in the usual way, with $\theta = 0$ in the direction of "east," and with θ increasing counterclockwise.

To sketch the vector field, we first find the fixed points, defined by $\dot{\theta} = 0$. These occur at $\theta^* = 0$ and $\theta^* = \pi$. To determine their stability, note that $\sin\theta > 0$ on the upper semicircle. Hence $\dot{\theta} > 0$, so the flow is counterclockwise. Similarly, the flow is clockwise on the lower semicircle, where $\dot{\theta} < 0$. Hence $\theta^* = \pi$ is stable and $\theta^* = 0$ is unstable, as shown in Figure 4.1.1.

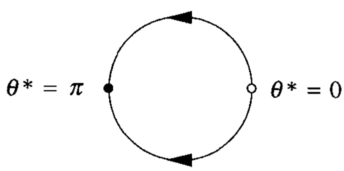


Figure 4.1.1

Actually, we've seen this example before—it's given in Section 2.1. There we regarded $\dot{x} = \sin x$ as a vector field on the *line*. Compare Figure 2.1.1 with Figure 4.1.1 and notice how much clearer it is to think of this system as a vector field on the circle. ■

EXAMPLE 4.1.2:

Explain why $\dot{\theta} = \theta$ cannot be regarded as a vector field on the circle, for θ in the range $-\infty < \theta < \infty$.

Solution: The velocity is not uniquely defined. For example, $\theta = 0$ and $\theta = 2\pi$ are two labels for the same point on the circle, but the first label implies a velocity of 0 at that point, while the second implies a velocity of 2π. ■

If we try to avoid this non-uniqueness by restricting θ to the range $-\pi < \theta \le \pi$, then the velocity vector jumps discontinuously at the point corresponding to $\theta = \pi$. Try as we might, there's no way to consider $\dot{\theta} = \theta$ as a smooth vector field on the entire circle.

Of course, there's no problem regarding $\dot{\theta} = \theta$ as a vector field on the *line*, because then $\theta = 0$ and $\theta = 2\pi$ are different points, and so there's no conflict about how to define the velocity at each of them.

Example 4.1.2 suggests how to define vector fields on the circle. Here's a geometric definition: A ***vector field on the circle*** is a rule that assigns a unique velocity vector to each point on the circle.

In practice, such vector fields arise when we have a first-order system $\dot{\theta} = f(\theta)$, where $f(\theta)$ is a real-valued, 2π*-periodic* function. That is, $f(\theta + 2\pi) = f(\theta)$ for all real θ. Moreover, we assume (as usual) that $f(\theta)$ is smooth enough to guarantee existence and uniqueness of solutions. Although this system could be regarded as a special case of a vector field on the line, it is usually clearer to think of it as a vector field on the circle (as in Example 4.1.1). This means that we don't distinguish between θ's that differ by an integer multiple of 2π. Here's where the periodicity of $f(\theta)$ becomes important—it ensures that the velocity $\dot{\theta}$ is uniquely defined at each point θ on the circle, in the sense that $\dot{\theta}$ is the same, whether we call that point θ or $\theta + 2\pi$, or $\theta + 2\pi k$ for any integer k.

4.2 Uniform Oscillator

A point on a circle is often called an *angle* or a ***phase.*** Then the simplest oscillator of all is one in which the phase θ changes uniformly:

$$\dot{\theta} = \omega$$

where ω is a constant. The solution is

$$\theta(t) = \omega t + \theta_0,$$

which corresponds to uniform motion around the circle at an angular frequency ω. This solution is ***periodic,*** in the sense that $\theta(t)$ changes by 2π, and therefore returns to the same point on the circle, after a time $T = 2\pi/\omega$. We call T the ***period*** of the oscillation.

Notice that we have said nothing about the *amplitude* of the oscillation. There really is no amplitude variable in our system. If we had an amplitude as well as a phase variable, we'd be in a *two-dimensional* phase space; this situation is more complicated and will be discussed later in the book. (Or if you prefer, you can imagine that the oscillation occurs at some *fixed* amplitude, corresponding to the radius of our circular phase space. In any case, amplitude plays no role in the dynamics.)

EXAMPLE 4.2.1:

Two joggers, Speedy and Pokey, are running at a steady pace around a circular track. It takes Speedy T_1 seconds to run once around the track, whereas it takes Pokey $T_2 > T_1$ seconds. Of course, Speedy will periodically overtake Pokey; how long does it take for Speedy to lap Pokey once, assuming that they start together?

Solution: Let $\theta_1(t)$ be Speedy's position on the track. Then $\dot{\theta}_1 = \omega_1$ where $\omega_1 = 2\pi/T_1$. This equation says that Speedy runs at a steady pace and completes a circuit every T_1 seconds. Similarly, suppose that $\dot{\theta}_2 = \omega_2 = 2\pi/T_2$ for Pokey.

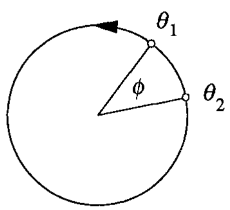


Figure 4.2.1

The condition for Speedy to lap Pokey is that the angle between them has increased by 2π. Thus if we define the ***phase difference*** $\phi = \theta_1 - \theta_2$, we want to find how long it takes for ϕ to increase by 2π (Figure 4.2.1). By subtraction we find $\dot{\phi} = \dot{\theta}_1 - \dot{\theta}_2 = \omega_1 - \omega_2$. Thus ϕ increases by 2π after a time

$$T_{\text{lap}} = \frac{2\pi}{\omega_1 - \omega_2} = \left(\frac{1}{T_1} - \frac{1}{T_2} \right)^{-1}. \blacksquare$$

Example 4.2.1 illustrates an effect called the ***beat phenomenon.*** Two noninteracting oscillators with different frequencies will periodically go in and out of phase with each other. You may have heard this effect on a Sunday morning: sometimes the bells of two different churches will ring simultaneously, then slowly drift apart, and then eventually ring together again. If the oscillators *interact* (for example, if the two joggers try to stay together or the bell ringers can hear each other), then we can get more interesting effects, as we will see in Section 4.5 on the flashing rhythm of fireflies.

4.3 Nonuniform Oscillator

The equation

$$\dot{\theta} = \omega - a\sin\theta \qquad (1)$$

arises in many different branches of science and engineering. Here is a partial list:

Electronics (phase-locked loops)
Biology (oscillating neurons, firefly flashing rhythm, human sleep-wake cycle)
Condensed-matter physics (Josephson junction, charge-density waves)
Mechanics (Overdamped pendulum driven by a constant torque)

Some of these applications will be discussed later in this chapter and in the exercises.

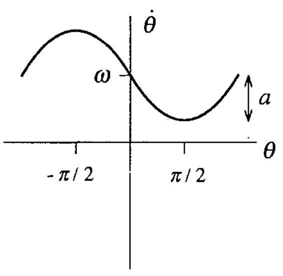


Figure 4.3.1

To analyze (1), we assume that $\omega > 0$ and $a \geq 0$ for convenience; the results for negative ω and a are similar. A typical graph of $f(\theta) = \omega - a\sin\theta$ is shown in Figure 4.3.1. Note that ω is the mean and a is the amplitude.

Vector Fields

If $a = 0$, (1) reduces to the uniform oscillator. The parameter a introduces a *nonuniformity* in the flow around the circle: the flow is fastest at $\theta = -\pi/2$ and slowest at $\theta = \pi/2$ (Figure 4.3.2a). This nonuniformity becomes more pronounced as a increases. When a is slightly less than ω, the oscillation is very jerky: the phase point $\theta(t)$ takes a long time to pass through a ***bottleneck*** near $\theta = \pi/2$, after which it zips around the rest of the circle on a much faster time scale. When $a = \omega$, the system stops oscillating altogether: a half-stable fixed point has been born in a *saddle-node bifurcation* at $\theta = \pi/2$ (Figure 4.3.2b). Finally, when $a > \omega$, the half-stable fixed point splits into a stable and unstable fixed point (Figure 4.3.2c). All trajectories are attracted to the stable fixed point as $t \to \infty$.

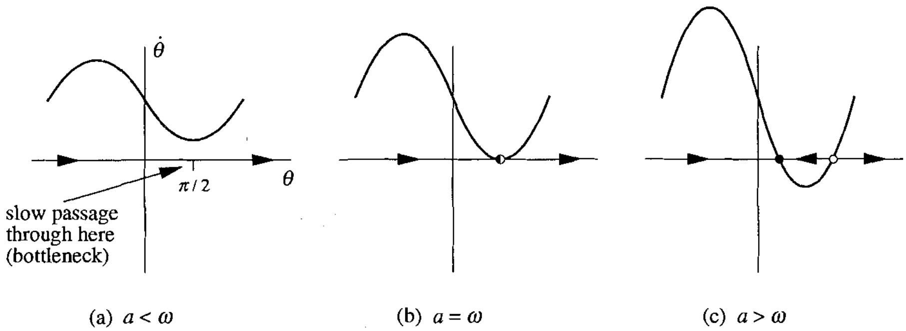


Figure 4.3.2

The same information can be shown by plotting the vector fields on the circle (Figure 4.3.3).

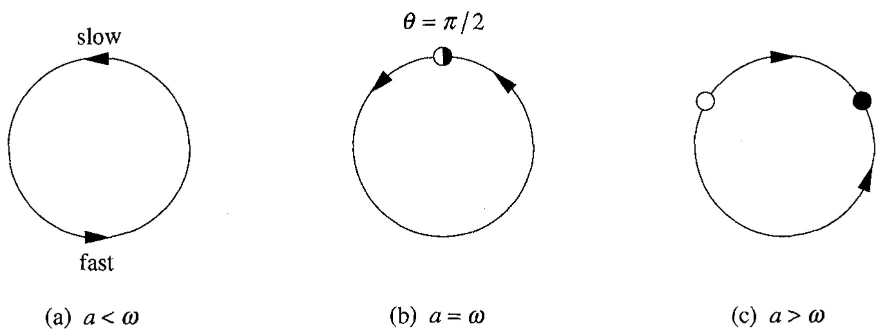


Figure 4.3.3

EXAMPLE 4.3.1:

Use linear stability analysis to classify the fixed points of (1) for $a > \omega$.

Solution: The fixed points θ^* satisfy

$$\sin\theta^* = \omega/a, \qquad \cos\theta^* = \pm\sqrt{1-(\omega/a)^2}.$$

Their linear stability is determined by

$$f'(\theta^*) = -a\cos\theta^* = \mp a\sqrt{1-(\omega/a)^2}\ .$$

Thus the fixed point with $\cos\theta^* > 0$ is the stable one, since $f'(\theta^*) < 0$. This agrees with Figure 4.3.2c. ■

Oscillation Period

For $a < \omega$, the period of the oscillation can be found analytically, as follows: the time required for θ to change by 2π is given by

$$T = \int dt = \int_0^{2\pi} \frac{dt}{d\theta}\, d\theta$$

$$= \int_0^{2\pi} \frac{d\theta}{\omega - a\sin\theta}$$

where we have used (1) to replace $dt/d\theta$. This integral can be evaluated by complex variable methods, or by the substitution $u = \tan\frac{\theta}{2}$. (See Exercise 4.3.2 for details.) The result is

$$T = \frac{2\pi}{\sqrt{\omega^2 - a^2}}\ . \tag{2}$$

Figure 4.3.4 shows the graph of T as a function of a.

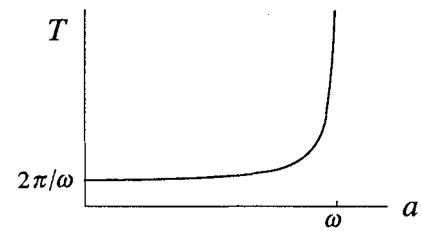


Figure 4.3.4

When $a = 0$, Equation (2) reduces to $T = 2\pi/\omega$, the familiar result for a uniform oscillator. The period increases with a and diverges as a approaches ω from below (we denote this limit by $a \to \omega^-$).

We can estimate the order of the divergence by noting that

$$\begin{aligned}\sqrt{\omega^2 - a^2} &= \sqrt{\omega + a}\sqrt{\omega - a}\\ &\approx \sqrt{2\omega}\sqrt{\omega - a}\end{aligned}$$

as $a \to \omega^-$. Hence

$$T \approx \left(\frac{\pi\sqrt{2}}{\sqrt{\omega}}\right)\frac{1}{\sqrt{\omega - a}}, \tag{3}$$

which shows that T blows up like $(a_c - a)^{-1/2}$, where $a_c = \omega$. Now let's explain the origin of this ***square-root scaling law.***

Ghosts and Bottlenecks

The square-root scaling law found above is a *very general feature of systems that are close to a saddle-node bifurcation.* Just after the fixed points collide, there is a saddle-node remnant or ***ghost*** that leads to slow passage through a bottleneck.

For example, consider $\dot{\theta} = \omega - a\sin\theta$ for decreasing values of a, starting with $a > \omega$. As a decreases, the two fixed points approach each other, collide, and disappear (this sequence was shown earlier in Figure 4.3.3, except now you have to read from right to left.) For a slightly less than ω, the fixed points near $\pi/2$ no longer exist, but they still make themselves felt through a saddle-node ghost (Figure 4.3.5).

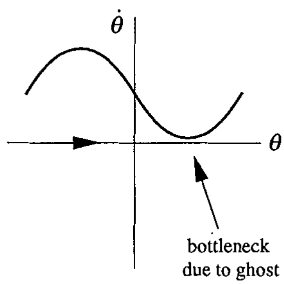


Figure 4.3.5

A graph of $\theta(t)$ would have the shape shown in Figure 4.3.6. Notice how the trajectory spends practically all its time getting through the bottleneck.

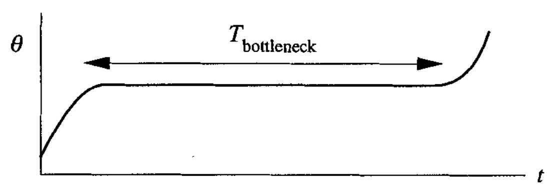


Figure 4.3.6

Now we want to derive a general scaling law for the time required to pass through a bottleneck. The only thing that matters is the behavior of $\dot{\theta}$ in the immediate vicinity of the minimum, since the time spent there dominates all other time scales in the problem. Generically, $\dot{\theta}$ looks *parabolic* near its minimum. Then the problem simplifies tremendously: the dynamics can be reduced to the normal form for a saddle-node bifurcation! By a local rescaling of space, we can rewrite the vector field as

$$\dot{x} = r + x^2$$

where r is proportional to the distance from the bifurcation, and $0 < r << 1$. The graph of $\dot{x}$ is shown in Figure 4.3.7.

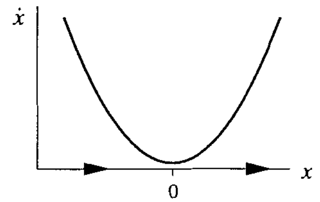


Figure 4.3.7

To estimate the time spent in the bottleneck, we calculate the time taken for x to go from $-\infty$ (all the way on one side of the bottleneck) to $+\infty$ (all the way on the other side). The result is

$$T_{\text{bottleneck}} \approx \int_{-\infty}^{\infty} \frac{dx}{r + x^2} = \frac{\pi}{\sqrt{r}}, \tag{4}$$

which shows the generality of the square-root scaling law. (Exercise 4.3.1 reminds you how to evaluate the integral in (4).)

EXAMPLE 4.3.2:

Estimate the period of $\dot{\theta} = \omega - a\sin\theta$ in the limit $a \to \omega^-$, using the normal form method instead of the exact result.

Solution: The period will be essentially the time required to get through the bottleneck. To estimate this time, we use a Taylor expansion about $\theta = \pi/2$, where the bottleneck occurs. Let $\phi = \theta - \pi/2$, where ϕ is small. Then

$$\begin{aligned}\dot{\phi} &= \omega - a\sin(\phi + \tfrac{\pi}{2}) \\ &= \omega - a\cos\phi \\ &= \omega - a + \tfrac{1}{2}a\phi^2 + \cdots\end{aligned}$$

which is close to the desired normal form. If we let

$$x = (a/2)^{1/2}\phi, \qquad r = \omega - a$$

then $(2/a)^{1/2}\,\dot{x} \approx r + x^2$, to leading order in x. Separating variables yields

$$T \approx (2/a)^{1/2} \int_{-\infty}^{\infty} \frac{dx}{r + x^2} = (2/a)^{1/2} \frac{\pi}{\sqrt{r}}.$$

Now we substitute $r = \omega - a$. Furthermore, since $a \to \omega^-$, we may replace $2/a$ by $2/\omega$. Hence

$$T \approx \left(\frac{\pi\sqrt{2}}{\sqrt{\omega}} \right) \frac{1}{\sqrt{\omega - a}},$$

which agrees with (3). ■

4.4 Overdamped Pendulum

We now consider a simple mechanical example of a nonuniform oscillator: an overdamped pendulum driven by a constant torque. Let θ denote the angle between the pendulum and the downward vertical, and suppose that θ increases counterclockwise (Figure 4.4.1).

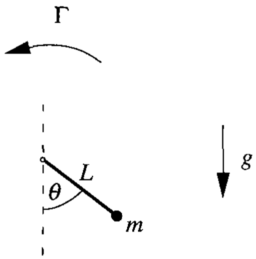


Figure 4.4.1

Then Newton's law yields

$$mL^2\ddot{\theta} + b\dot{\theta} + mgL\sin\theta = \Gamma \qquad (1)$$

where m is the mass and L is the length of the pendulum, b is a viscous damping constant, g is the acceleration due to gravity, and Γ is a constant applied torque. All of these parameters are positive. In particular, $\Gamma > 0$ implies that the applied torque drives the pendulum counterclockwise, as shown in Figure 4.4.1.

Equation (1) is a second-order system, but in the *overdamped limit* of extremely large b, it may be approximated by a first-order system (see Section 3.5 and Exercise 4.4.1). In this limit the inertia term $mL^2\ddot{\theta}$ is negligible and so (1) becomes

$$b\dot{\theta} + mgL\sin\theta = \Gamma. \qquad (2)$$

To think about this problem physically, you should imagine that the pendulum is immersed in molasses. The torque Γ enables the pendulum to plow through its viscous surroundings. Please realize that this is the *opposite* limit from the familiar frictionless case in which energy is conserved, and the pendulum swings back and forth forever. In the present case, energy is lost to damping and pumped in by the applied torque.

To analyze (2), we first nondimensionalize it. Dividing by mgL yields

$$\frac{b}{mgL}\dot{\theta} = \frac{\Gamma}{mgL} - \sin\theta .$$

Hence, if we let

$$\tau = \frac{mgL}{b}t, \qquad \gamma = \frac{\Gamma}{mgL} \tag{3}$$

then

$$\theta' = \gamma - \sin\theta \tag{4}$$

where $\theta' = d\theta/d\tau$.

The dimensionless group γ is the ratio of the applied torque to the maximum gravitational torque. If $\gamma > 1$ then the applied torque can never be balanced by the gravitational torque and *the pendulum will overturn continually.* The rotation rate is nonuniform, since gravity helps the applied torque on one side and opposes it on the other (Figure 4.4.2).

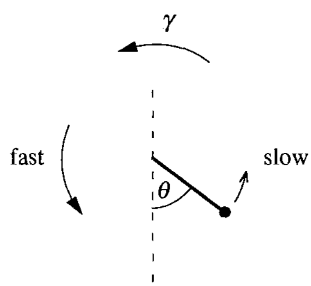


Figure 4.4.2

As $\gamma \to 1^+$, the pendulum takes longer and longer to climb past $\theta = \pi/2$ on the slow side. When $\gamma = 1$ a fixed point appears at $\theta^* = \pi/2$, and then splits into two when $\gamma < 1$ (Figure 4.4.3). On physical grounds, it's clear that the lower of the two equilibrium positions is the stable one.

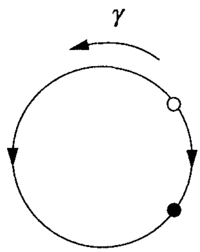


Figure 4.4.3

As γ decreases, the two fixed points move farther apart. Finally, when $\gamma = 0$, the applied torque vanishes and there is an unstable equilibrium at the top (inverted pendulum) and a stable equilibrium at the bottom.

4.5 Fireflies

Fireflies provide one of the most spectacular examples of synchronization in nature. In some parts of southeast Asia, thousands of male fireflies gather in trees at night and flash on and off in unison. Meanwhile the female fireflies cruise overhead, looking for males with a handsome light.

To really appreciate this amazing display, you have to see a movie or videotape of it. A good example is shown in David Attenborough's (1992) television series *The Trials of Life,* in the episode called "Talking to Strangers." See Buck and Buck (1976) for a beautifully written introduction to synchronous fireflies, and Buck (1988) for a more recent review. For mathematical models of synchronous fireflies, see Mirollo and Strogatz (1990) and Ermentrout (1991).

How does the synchrony occur? Certainly the fireflies don't start out synchronized; they arrive in the trees at dusk, and the synchrony builds up gradually as the night goes on. The key is that *the fireflies influence each other*: When one firefly sees the flash of another, it slows down or speeds up so as to flash more nearly in phase on the next cycle.

Hanson (1978) studied this effect experimentally, by periodically flashing a light at a firefly and watching it try to synchronize. For a range of periods close to the firefly's natural period (about 0.9 sec), the firefly was able to match its frequency to the periodic stimulus. In this case, one says that the firefly had been ***entrained*** by the stimulus. However, if the stimulus was too fast or too slow, the firefly could not keep up and entrainment was lost—then a kind of beat phenomenon occurred. But in contrast to the simple beat phenomenon of Section 4.2, the phase difference between stimulus and firefly did not increase uniformly. The phase difference increased slowly during part of the beat cycle, as the firefly struggled in vain to synchronize, and then it increased rapidly through 2π, after which the firefly tried again on the next beat cycle. This process is called *phase walk-through* or ***phase drift.***

Model

Ermentrout and Rinzel (1984) proposed a simple model of the firefly's flashing rhythm and its response to stimuli. Suppose that $\theta(t)$ is the phase of the firefly's flashing rhythm, where $\theta = 0$ corresponds to the instant when a flash is emitted. Assume that in the absence of stimuli, the firefly goes through its cycle at a frequency ω, according to $\dot{\theta} = \omega$.

Now suppose there's a periodic stimulus whose phase Θ satisfies

$$\dot{\Theta} = \Omega, \tag{1}$$

where $\Theta = 0$ corresponds to the flash of the stimulus. We model the firefly's response to this stimulus as follows: If the stimulus is ahead in the cycle, then we assume that the firefly speeds up in an attempt to synchronize. Conversely, the firefly slows down if it's flashing too early. A simple model that incorporates these assumptions is

$$\dot{\theta} = \omega + A\sin(\Theta - \theta) \tag{2}$$

where $A > 0$. For example, if Θ is ahead of θ (i.e., $0 < \Theta - \theta < \pi$) the firefly speeds up ($\dot{\theta} > \omega$). The ***resetting strength*** A measures the firefly's ability to modify its instantaneous frequency.

Analysis

To see whether entrainment can occur, we look at the dynamics of the phase difference $\phi = \Theta - \theta$. Subtracting (2) from (1) yields

$$\dot{\phi} = \dot{\Theta} - \dot{\theta} = \Omega - \omega - A\sin\phi, \tag{3}$$

which is a *nonuniform oscillator* equation for $\phi(t)$. Equation (3) can be nondimensionalized by introducing

$$\tau = At, \quad \mu = \frac{\Omega - \omega}{A}. \tag{4}$$

Then

$$\phi' = \mu - \sin\phi \tag{5}$$

where $\phi' = d\phi/d\tau$. The dimensionless group μ is a measure of the frequency difference, relative to the resetting strength. When μ is small, the frequencies are relatively close together and we expect that entrainment should be possible. This is confirmed by Figure 4.5.1, where we plot the vector fields for (5), for different values of $\mu \ge 0$. (The case $\mu < 0$ is similar.)

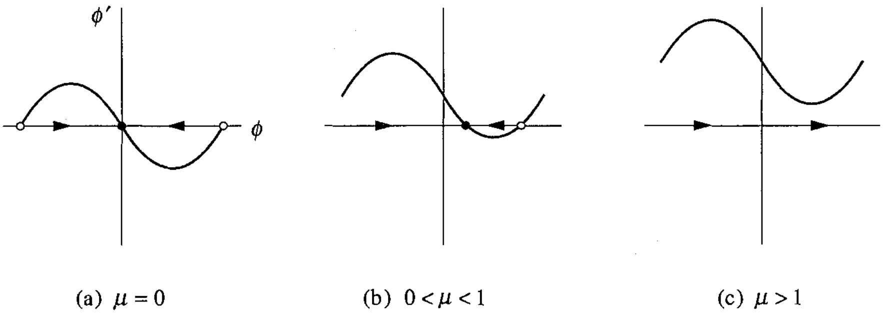


Figure 4.5.1

When $\mu = 0$, all trajectories flow toward a stable fixed point at $\phi^* = 0$ (Figure 4.5.1a). Thus the firefly eventually entrains with *zero phase difference* in the case $\Omega = \omega$. In other words, the firefly and the stimulus flash *simultaneously* if the firefly is driven at its natural frequency.

Figure 4.5.1b shows that for $0 < \mu < 1$, the curve in Figure 4.5.1a lifts up and the stable and unstable fixed points move closer together. All trajectories are still attracted to a stable fixed point, but now $\phi^* > 0$. Since the phase difference approaches a constant, one says that the firefly's rhythm is ***phase-locked*** to the stimulus.

Phase-locking means that the firefly and the stimulus run with the same instantaneous frequency, although they no longer flash in unison. The result $\phi^* > 0$ implies that the stimulus flashes *ahead* of the firefly in each cycle. This makes sense—we assumed $\mu > 0$, which means that $\Omega > \omega$; the stimulus is inherently faster than the firefly, and drives it faster than it wants to go. Thus the firefly falls behind. But it never gets lapped—it always lags in phase by a constant amount ϕ^*.

If we continue to increase μ, the stable and unstable fixed points eventually coalesce in a saddle-node bifurcation at $\mu = 1$. For $\mu > 1$ both fixed points have disappeared and now phase-locking is lost; the phase difference ϕ increases indefinitely, corresponding to *phase drift* (Figure 4.5.1c). (Of course, once ϕ reaches 2π the oscillators are in phase again.) Notice that the phases don't separate at a uniform rate, in qualitative agreement with the experiments of Hanson (1978): ϕ increases most slowly when it passes under the minimum of the sine wave in Figure 4.5.1c, at $\phi = \pi/2$, and most rapidly when it passes under the maximum at $\phi = -\pi/2$.

The model makes a number of specific and testable predictions. Entrainment is predicted to be possible only within a symmetric interval of driving frequencies, specifically $\omega - A \le \Omega \le \omega + A$. This interval is called the ***range of entrainment*** (Figure 4.5.2).

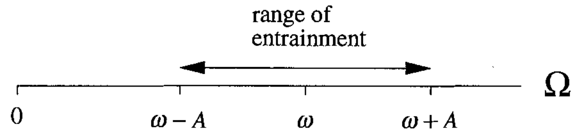


Figure 4.5.2

By measuring the range of entrainment experimentally, one can nail down the value of the parameter A. Then the model makes a rigid prediction for the phase difference during entrainment, namely

$$\sin\phi^* = \frac{\Omega - \omega}{A} \tag{6}$$

where $-\pi/2 \le \phi^* \le \pi/2$ corresponds to the *stable* fixed point of (3).

Moreover, for $\mu > 1$, the period of phase drift may be predicted as follows. The time required for ϕ to change by 2π is given by

$$T_{\text{drift}} = \int dt = \int_0^{2\pi} \frac{dt}{d\phi} d\phi$$

$$= \int_0^{2\pi} \frac{d\phi}{\Omega - \omega - A\sin\phi} \; .$$

To evaluate this integral, we invoke (2) of Section 4.3, which yields

$$T_{\text{drift}} = \frac{2\pi}{\sqrt{(\Omega - \omega)^2 - A^2}} \; . \tag{7}$$

Since A and ω are presumably fixed properties of the firefly, the predictions (6) and (7) could be tested simply by varying the drive frequency Ω. Such experiments have yet to be done.

Actually, the biological reality about synchronous fireflies is more complicated. The model presented here is reasonable for certain species, such as *Pteroptyx cribellata,* which behave as if A and ω were fixed. However, the species that is best at synchronizing, *Pteroptyx malaccae,* is actually able to shift its frequency ω toward the drive frequency Ω (Hanson 1978). In this way it is able to achieve nearly zero phase difference, even when driven at periods that differ from its natural period by ±15 percent! A model of this remarkable effect has been presented by Ermentrout (1991).

4.6 Superconducting Josephson Junctions

Josephson junctions are superconducting devices that are capable of generating voltage oscillations of extraordinarily high frequency, typically 10^{10}–10^{11} cycles per second. They have great technological promise as amplifiers, voltage standards, detectors, mixers, and fast switching devices for digital circuits. Josephson junctions can detect electric potentials as small as one quadrillionth of a volt, and they have been used to detect far-infrared radiation from distant galaxies. For an introduction to Josephson junctions, as well as superconductivity more generally, see Van Duzer and Turner (1981).

Although quantum mechanics is required to explain the *origin* of the Josephson effect, we can nevertheless describe the *dynamics* of Josephson junctions in classical terms. Josephson junctions have been particularly useful for experimental studies of nonlinear dynamics, because the equation governing a single junction is the same as that for a pendulum! In this section we will study the dynamics of a single junction in the overdamped limit. In later sections we will discuss underdamped junctions, as well as arrays of enormous numbers of junctions coupled together.

Physical Background

A Josephson junction consists of two closely spaced superconductors separated by a weak connection (Figure 4.6.1). This connection may be provided by an insulator, a normal metal, a semiconductor, a weakened superconductor, or some other material that weakly couples the two superconductors. The two superconducting regions may be characterized by quantum mechanical wave functions $\psi_1 e^{i\phi_1}$ and $\psi_2 e^{i\phi_2}$ respectively. Normally a much more complicated description would be necessary because there are $\sim 10^{23}$ electrons to deal with, but in the superconducting ground state, these electrons form "Cooper pairs" that can be described by a *single* macroscopic wave function. This implies an astonishing degree of coherence among the electrons. The Cooper pairs act like a miniature version of synchronous fireflies: they all adopt the same phase, because this turns out to minimize the energy of the superconductor.

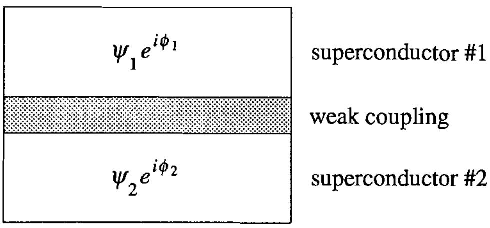


Figure 4.6.1

As a 22-year-old graduate student, Brian Josephson (1962) suggested that it should be possible for a current to pass between the two superconductors, even if there were no voltage difference between them. Although this behavior would be impossible classically, it could occur because of quantum mechanical *tunneling* of Cooper pairs across the junction. An observation of this "Josephson effect" was made by Anderson and Rowell in 1963.

Incidentally, Josephson won the Nobel Prize in 1973, after which he lost interest in mainstream physics and was rarely heard from again. See Josephson (1982) for an interview in which he reminisces about his early work and discusses his more recent interests in transcendental meditation, consciousness, language, and even psychic spoon-bending and paranormal phenomena.

The Josephson Relations

We now give a more quantitative discussion of the Josephson effect. Suppose that a Josephson junction is connected to a dc current source (Figure 4.6.2), so that a constant current $I > 0$ is driven through the junction. Using quantum mechanics, one can show that if this current is less than a certain ***critical current*** I_c, no voltage will be developed across the junction; that is, the junction acts as if it had zero resistance! However, the phases of the two superconductors will be driven apart to a constant phase difference $\phi = \phi_2 - \phi_1$, where ϕ satisfies the *Josephson current-phase relation*

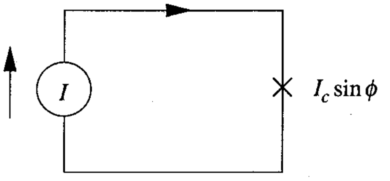


Figure 4.6.2

$$I = I_c \sin\phi. \tag{1}$$

Equation (1) implies that the phase difference increases as the ***bias current*** I increases.

When I exceeds I_c, a constant phase difference can no longer be maintained and a voltage develops across the junction. The phases on the two sides of the junction begin to slip with respect to each other, with the rate of slippage governed by the *Josephson voltage-phase relation*

$$V = \frac{\hbar}{2e}\dot{\phi}. \tag{2}$$

Here $V(t)$ is the instantaneous voltage across the junction, $\hbar$ is Planck's constant divided by 2π, and e is the charge on the electron. For an elementary derivation of the Josephson relations (1) and (2), see Feynman's argument (Feynman et al. (1965), Vol. III), also reproduced in Van Duzer and Turner (1981).

Equivalent Circuit and Pendulum Analog

The relation (1) applies only to the *supercurrent* carried by the electron pairs. In general, the total current passing through the junction will also contain contributions from a *displacement current* and an *ordinary current.* Representing the displacement current by a capacitor, and the ordinary current by a resistor, we arrive at the equivalent circuit shown in Figure 4.6.3, first analyzed by Stewart (1968) and McCumber (1968).

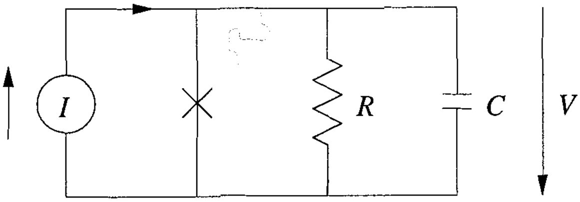


Figure 4.6.3

Now we apply Kirchhoff's voltage and current laws. For this parallel circuit, the voltage drop across each branch must be equal, and hence all the voltages are equal to V, the voltage across the junction. Hence the current through the capacitor equals $C\dot{V}$ and the current through the resistor equals V/R. The sum of these currents and the supercurrent $I_c \sin\phi$ must equal the bias current I; hence

$$C\dot{V} + \frac{V}{R} + I_c \sin\phi = I. \tag{3}$$

Equation (3) may be rewritten solely in terms of the phase difference ϕ, thanks to (2). The result is

$$\frac{\hbar C}{2e}\ddot{\phi} + \frac{\hbar}{2eR}\dot{\phi} + I_c \sin\phi = I, \tag{4}$$

which is precisely analogous to the equation governing a damped pendulum driven by a constant torque! In the notation of Section 4.4, the pendulum equation is

$$mL^2\ddot{\theta} + b\dot{\theta} + mgL\sin\theta = \Gamma.$$

Hence the analogies are as follows:

Pendulum	*Josephson junction*
Angle θ	Phase difference ϕ
Angular velocity $\dot{\theta}$	Voltage $\frac{\hbar}{2e}\dot{\phi}$
Mass m	Capacitance C
Applied torque Γ	Bias current I
Damping constant b	Conductance $1/R$
Maximum gravitational torque mgL	Critical current I_c

This mechanical analog has often proved useful in visualizing the dynamics of Josephson junctions. Sullivan and Zimmerman (1971) actually constructed such a mechanical analog, and measured the average rotation rate of the pendulum as a function of the applied torque; this is the analog of the physically important $I-V$ curve (current–voltage curve) for the Josephson junction.

Typical Parameter Values

Before analyzing (4), we mention some typical parameter values for Josephson junctions. The critical current is typically in the range $I_c \approx 1\ \mu\text{A} - 1$ mA, and a typical voltage is $I_c R \approx 1$ mV. Since $2e/h \approx 4.83 \times 10^{14}$ Hz/V, a typical frequency is on the order of 10^{11} Hz. Finally, a typical length scale for Josephson junctions is around 1 μm, but this depends on the geometry and the type of coupling used.

Dimensionless Formulation

If we divide (4) by I_c and define a dimensionless time

$$\tau = \frac{2eI_cR}{\hbar}t, \tag{5}$$

we obtain the dimensionless equation

$$\beta\phi'' + \phi' + \sin\phi = \frac{I}{I_c} \tag{6}$$

where $\phi' = d\phi/d\tau$. The dimensionless group β is defined by

$$\beta = \frac{2eI_cR^2C}{\hbar}.$$

and is called the ***McCumber parameter.*** It may be thought of as a dimensionless capacitance. Depending on the size, the geometry, and the type of coupling used in the Josephson junction, the value of β can range from $\beta \approx 10^{-6}$ to much larger values ($\beta \approx 10^6$).

We are not yet prepared to analyze (6) in general. For now, let's restrict ourselves to the *overdamped limit* $\beta \ll 1$. Then the term $\beta\phi''$ may be neglected after a rapid initial transient, as discussed in Section 3.5, and so (6) reduces to a nonuniform oscillator:

$$\phi' = \frac{I}{I_c} - \sin\phi. \tag{7}$$

As we know from Section 4.3, the solutions of (7) tend to a stable fixed point when $I < I_c$, and vary periodically when $I > I_c$.

EXAMPLE 4.6.1:

Find the ***current–voltage curve*** analytically in the overdamped limit. In other words, find the average value of the voltage $\langle V \rangle$ as a function of the constant applied current I, assuming that all transients have decayed and the system has reached steady-state operation. Then plot $\langle V \rangle$ vs. I.

Solution: It is sufficient to find $\langle \phi' \rangle$, since $\langle V \rangle = (\hbar/2e)\langle \dot{\phi} \rangle$ from the voltage-phase relation (2), and

$$\langle \dot{\phi} \rangle = \left\langle \frac{d\phi}{dt} \right\rangle = \left\langle \frac{d\tau}{dt} \frac{d\phi}{d\tau} \right\rangle = \frac{2eI_c R}{\hbar} \langle \phi' \rangle ,$$

from the definition of τ in (5); hence

$$\langle V \rangle = I_c R \langle \phi' \rangle . \tag{8}$$

There are two cases to consider. When $I \le I_c$, all solutions of (7) approach a fixed point $\phi^* = \sin^{-1}(I/I_c)$, where $-\pi/2 \le \phi^* \le \pi/2$. Thus $\phi' = 0$ in steady state, and so $\langle V \rangle = 0$ for $I \le I_c$.

When $I > I_c$, all solutions of (7) are periodic with period

$$T = \frac{2\pi}{\sqrt{(I/I_c)^2 - 1}} , \tag{9}$$

where the period is obtained from (2) of Section 4.3, and time is measured in units of τ. We compute $\langle \phi' \rangle$ by taking the average over one cycle:

$$\langle \phi' \rangle = \frac{1}{T} \int_0^T \frac{d\phi}{d\tau} d\tau = \frac{1}{T} \int_0^{2\pi} d\phi = \frac{2\pi}{T} . \tag{10}$$

Combining (8)–(10) yields

$$\langle V \rangle = I_c R \sqrt{(I/I_c)^2 - 1} \qquad \text{for } I > I_c .$$

In summary, we have found

$$\langle V \rangle = \begin{cases} 0 & \text{for } I \le I_c \\ I_c R \sqrt{(I/I_c)^2 - 1} & \text{for } I > I_c . \end{cases} \tag{11}$$

The I–V curve (11) is shown in Figure 4.6.4.

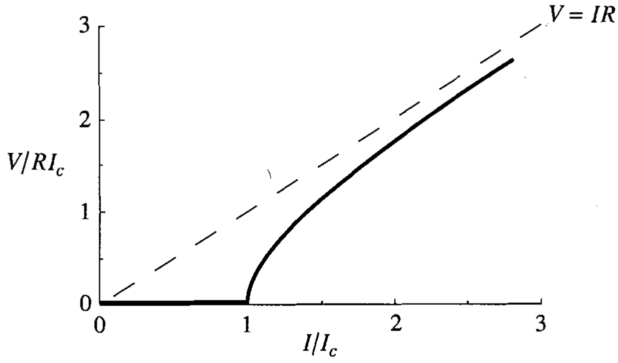


Figure 4.6.4

As I increases, the voltage remains zero until $I > I_c$; then $\langle V \rangle$ rises sharply and eventually asymptotes to the Ohmic behavior $\langle V \rangle \approx IR$ for $I >> I_c$. ■

The analysis given in Example 4.6.1 applies only to the overdamped limit $\beta << 1$. The behavior of the system becomes much more interesting if β is not negligible. In particular, the $I-V$ curve can be ***hysteretic***, as shown in Figure 4.6.5. As the bias current is increased slowly from $I = 0$, the voltage remains at $V = 0$ until $I > I_c$. Then the voltage jumps up to a nonzero value, as shown by the upward arrow in Figure 4.6.5. The voltage increases with further increases of I. However, if we now slowly *decrease* I, the voltage doesn't drop back to zero at I_c—we have to go *below* I_c before the voltage returns to zero.

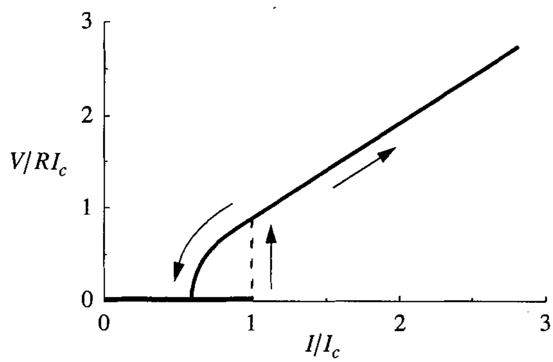


Figure 4.6.5

The hysteresis comes about because the system has ***inertia*** when $\beta \neq 0$. We can make sense of this by thinking in terms of the pendulum analog. The critical current I_c is analogous to the critical torque Γ_c needed to get the pendulum overturning. Once the pendulum has started whirling, its inertia keeps it going so that even if the torque is lowered *below* Γ_c, the rotation continues. The torque has to be lowered even further before the pendulum will fail to make it over the top.

In more mathematical terms, we'll show in Section 8.5 that this hysteresis occurs because a *stable fixed point coexists with a stable periodic solution.* We have never seen anything like *this* before! For vector fields on the line, only fixed points can exist; for vector fields on the circle, both fixed points and periodic solutions can exist, *but not simultaneously.* Here we see just one example of the new kinds of phenomena that can occur in two-dimensional systems. It's time to take the plunge.

EXERCISES FOR CHAPTER 4

4.1 Examples and Definitions

4.1.1 For which real values of a does the equation $\dot{\theta} = \sin(a\theta)$ give a well-defined vector field on the circle?

For each of the following vector fields, find and classify all the fixed points, and sketch the phase portrait on the circle.

4.1.2 $\dot{\theta} = 1 + 2\cos\theta$

4.1.3 $\dot{\theta} = \sin 2\theta$

4.1.4 $\dot{\theta} = \sin^3\theta$

4.1.5 $\dot{\theta} = \sin\theta + \cos\theta$

4.1.6 $\dot{\theta} = 3 + \cos 2\theta$

4.1.7 $\dot{\theta} = \sin k\theta$ where k is a positive integer.

4.1.8 (Potentials for vector fields on the circle)

a) Consider the vector field on the circle given by $\dot{\theta} = \cos\theta$. Show that this system has a single-valued potential $V(\theta)$, i.e., for each point on the circle, there is a well-defined value of V such that $\dot{\theta} = -dV/d\theta$. (As usual, θ and $\theta + 2\pi k$ are to be regarded as the same point on the circle, for each integer k.)
b) Now consider $\dot{\theta} = 1$. Show that there is no single-valued potential $V(\theta)$ for this vector field on the circle.
c) What's the general rule? When does $\dot{\theta} = f(\theta)$ have a single-valued potential?

4.1.9 In Exercises 2.6.2 and 2.7.7, you were asked to give two analytical proofs that periodic solutions are impossible for vector fields on the line. Review these arguments and explain why they *don't* carry over to vector fields on the circle. Specifically, which parts of the argument fail?

4.2 Uniform Oscillator

4.2.1 (Church bells) The bells of two different churches are ringing. One bell rings every 3 seconds, and the other rings every 4 seconds. Assume that the bells have just rung at the same time. How long will it be until the next time they ring together? Answer the question in two ways: using common sense, and using the method of Example 4.2.1.

4.2.2 (Beats arising from linear superpositions) Graph $x(t) = \sin 8t + \sin 9t$ for $-20 < t < 20$. You should find that the amplitude of the oscillations is *modulated*—it grows and decays periodically.

a) What is the period of the amplitude modulations?

b) Solve this problem analytically, using a trigonometric identity that converts sums of sines and cosines to products of sines and cosines.

(In the old days, this beat phenomenon was used to tune musical instruments. You would strike a tuning fork at the same time as you played the desired note on the instrument. The combined sound $A_1 \sin \omega_1 t + A_2 \sin \omega_2 t$ would get louder and softer as the two vibrations went in and out of phase. Each maximum of total amplitude is called a beat. When the time between beats is long, the instrument is nearly in tune.)

4.2.3 (The clock problem) Here's an old chestnut from high school algebra: At 12:00, the hour hand and minute hand of a clock are perfectly aligned. When is the *next* time they will be aligned? (Solve the problem by the methods of this section, and also by some alternative approach of your choosing.)

4.3 Nonuniform Oscillator

4.3.1 As shown in the text, the time required to pass through a saddle-node bottleneck is approximately $T_{\text{bottleneck}} = \int_{-\infty}^{\infty} \frac{dx}{r + x^2}$. To evaluate this integral, let $x = \sqrt{r} \tan \theta$, use the identity $1 + \tan^2 \theta = \sec^2 \theta$, and change the limits of integration appropriately. Thereby show that $T_{\text{bottleneck}} = \pi / \sqrt{r}$.

4.3.2 The oscillation period for the nonuniform oscillator is given by the integral $T = \int_{-\pi}^{\pi} \frac{d\theta}{\omega - a \sin \theta}$, where $\omega > a > 0$. Evaluate this integral as follows.

a) Let $u = \tan \frac{\theta}{2}$. Solve for θ and then express $d\theta$ in terms of u and du.

b) Show that $\sin \theta = 2u/(1 + u^2)$. (Hint: Draw a right triangle with base 1 and height u. Then $\frac{\theta}{2}$ is the angle opposite the side of length u, since $u = \tan \frac{\theta}{2}$ by definition. Finally, invoke the half-angle formula $\sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}$.)

c) Show that $u \to \pm\infty$ as $\theta \to \pm\pi$, and use that fact to rewrite the limits of integration.

d) Express T as an integral with respect to u.

e) Finally, complete the square in the denominator of the integrand of (d), and reduce the integral to the one studied in Exercise 4.3.1, for a suitable choice of x and r.

For each of the following questions, draw the phase portrait as function of the control parameter μ. Classify the bifurcations that occur as μ varies, and find all the bifurcation values of μ.

4.3.3 $\dot{\theta} = \mu \sin\theta - \sin 2\theta$

4.3.4 $\dot{\theta} = \dfrac{\sin\theta}{\mu + \cos\theta}$

4.3.5 $\dot{\theta} = \mu + \cos\theta + \cos 2\theta$

4.3.6 $\dot{\theta} = \mu + \sin\theta + \cos 2\theta$

4.3.7 $\dot{\theta} = \dfrac{\sin\theta}{\mu + \sin\theta}$

4.3.8 $\dot{\theta} = \dfrac{\sin 2\theta}{1 + \mu \sin\theta}$

4.3.9 (Alternative derivation of scaling law) For systems close to a saddle-node bifurcation, the scaling law $T_{\text{bottleneck}} \sim O(r^{-1/2})$ can also be derived as follows.

a) Suppose that x has a characteristic scale $O(r^a)$, where a is unknown for now. Then $x = r^a u$, where $u \sim O(1)$. Similarly, suppose $t = r^b \tau$, with $\tau \sim O(1)$. Show that $\dot{x} = r + x^2$ is thereby transformed to $r^{a-b} \dfrac{du}{d\tau} = r + r^{2a} u^2$.

b) Assume that all terms in the equation have the same order with respect to r, and thereby derive $a = \frac{1}{2}$, $b = -\frac{1}{2}$.

4.3.10 (Nongeneric scaling laws) In deriving the square-root scaling law for the time spent passing through a bottleneck, we assumed that $\dot{x}$ had a quadratic minimum. This is the generic case, but what if the minimum were of higher order? Suppose that the bottleneck is governed by $\dot{x} = r + x^{2n}$, where $n > 1$ is an integer. Using the method of Exercise 4.3.9, show that $T_{\text{bottleneck}} \approx c r^b$, and determine b and c.

(It's acceptable to leave c in the form of a definite integral. If you know complex variables and residue theory, you should be able to evaluate c exactly by integrating around the boundary of the pie-slice $\left\{ z = re^{i\theta} : 0 \le \theta \le \pi/n,\ 0 \le r \le R \right\}$ and letting $R \to \infty$.)

4.4 Overdamped Pendulum

4.4.1 (Validity of overdamped limit) Find the conditions under which it is valid to approximate the equation $mL^2\ddot{\theta} + b\dot{\theta} + mgL\sin\theta = \Gamma$ by its overdamped limit $b\dot{\theta} + mgL\sin\theta = \Gamma$.

4.4.2 (Understanding $\sin\theta(t)$) By imagining the rotational motion of an overdamped pendulum, sketch $\sin\theta(t)$ vs. t for a typical solution of $\theta' = \gamma - \sin\theta$. How does the shape of the waveform depend on γ? Make a series of graphs for different γ, including the limiting cases $\gamma \approx 1$ and $\gamma >> 1$. For the pendulum, what physical quantity is proportional to $\sin\theta(t)$?

4.4.3 (Understanding $\dot{\theta}(t)$) Redo Exercise 4.4.2, but now for $\dot{\theta}(t)$ instead of $\sin\theta(t)$.

4.4.4 (Torsional spring) Suppose that our overdamped pendulum is connected to a torsional spring. As the pendulum rotates, the spring winds up and generates an opposing torque $-k\theta$. Then the equation of motion becomes $b\dot{\theta} + mgL\sin\theta = \Gamma - k\theta$.

a) Does this equation give a well-defined vector field on the circle?
b) Nondimensionalize the equation.
c) What does the pendulum do in the long run?
d) Show that many bifurcations occur as k is varied from 0 to ∞. What kind of bifurcations are they?

4.5 Fireflies

4.5.1 (Triangle wave) In the firefly model, the sinusoidal form of the firefly's response function was chosen somewhat arbitrarily. Consider the alternative model $\dot{\Theta} = \Omega$, $\dot{\theta} = \omega + Af(\Theta - \theta)$, where f is given now by a triangle wave, not a sine wave. Specifically, let

$$f(\phi) = \begin{cases} \phi, & -\frac{\pi}{2} \le \phi \le \frac{\pi}{2} \\ \pi - \phi, & \frac{\pi}{2} \le \phi \le \frac{3\pi}{2} \end{cases}$$

on the interval $-\frac{\pi}{2} \le \phi \le \frac{3\pi}{2}$, and extend f periodically outside this interval.

a) Graph $f(\phi)$.
b) Find the range of entrainment.
c) Assuming that the firefly is phase-locked to the stimulus, find a formula for the phase difference ϕ^*.
d) Find a formula for T_{drift}.

4.5.2 (General response function) Redo as much of the previous exercise as possible, assuming only that $f(\phi)$ is a smooth, 2π-periodic function with a single maximum and minimum on the interval $-\pi \le \phi \le \pi$.

4.5.3 (Excitable systems) Suppose you stimulate a neuron by injecting it with a pulse of current. If the stimulus is small, nothing dramatic happens: the neuron increases its membrane potential slightly, and then relaxes back to its resting potential. However, if the stimulus exceeds a certain threshold, the neuron will "fire" and produce a large voltage spike before returning to rest. Surprisingly, the size of the spike doesn't depend much on the size of the stimulus—anything above threshold will elicit essentially the same response.

Similar phenomena are found in other types of cells and even in some chemical reactions (Winfree 1980, Rinzel and Ermentrout 1989, Murray 1989). These systems are called ***excitable***. The term is hard to define precisely, but roughly speaking, an excitable system is characterized by two properties: (1) it has a unique, globally attracting rest state, and (2) a large enough stimulus can send the system on a long excursion through phase space before it returns to the resting state.

This exercise deals with the simplest caricature of an excitable system. Let $\dot{\theta} = \mu + \sin\theta$, where μ is slightly less than 1.

a) Show that the system satisfies the two properties mentioned above. What object plays the role of the "rest state"? And the "threshold"?

b) Let $V(t) = \cos\theta(t)$. Sketch $V(t)$ for various initial conditions. (Here V is analogous to the neuron's membrane potential, and the initial conditions correspond to different perturbations from the rest state.)

4.6 Superconducting Josephson Junctions

4.6.1 (Current and voltage oscillations) Consider a Josephson junction in the overdamped limit $\beta = 0$.

a) Sketch the supercurrent $I_c \sin\phi(t)$ as a function of t, assuming first that I/I_c is slightly greater than 1, and then assuming that $I/I_c >> 1$. (Hint: In each case, visualize the flow on the circle, as given by Equation (4.6.7).)

b) Sketch the instantaneous voltage $V(t)$ for the two cases considered in (a).

4.6.2 (Computer work) Check your qualitative solution to Exercise 4.6.1 by integrating Equation (4.6.7) numerically, and plotting the graphs of $I_c \sin\phi(t)$ and $V(t)$.

4.6.3 (Washboard potential) Here's another way to visualize the dynamics of an overdamped Josephson junction. As in Section 2.7, imagine a particle sliding down a suitable potential.

a) Find the potential function corresponding to Equation (4.6.7). Show that it is *not* a single-valued function on the circle.

b) Graph the potential as a function of ϕ, for various values of I/I_c. Here ϕ is to be regarded as a real number, not an angle.

c) What is the effect of increasing I?

The potential in (b) is often called the "washboard potential" (Van Duzer and Turner 1981, p. 179) because its shape is reminiscent of a tilted, corrugated washboard.

4.6.4 (Resistively loaded array) ***Arrays*** of coupled Josephson junctions raise many fascinating questions. Their dynamics are not yet understood in detail. The questions are technologically important because arrays can produce much greater power output than a single junction, and also because arrays provide a reasonable model of the (still mysterious) high-temperature superconductors. For an introduction to some of the dynamical questions of current interest, see Tsang et al. (1991) and Strogatz and Mirollo (1993).

Figure 1 shows an array of two identical overdamped Josephson junctions. The junctions are in series with each other, and in parallel with a resistive "load" R.

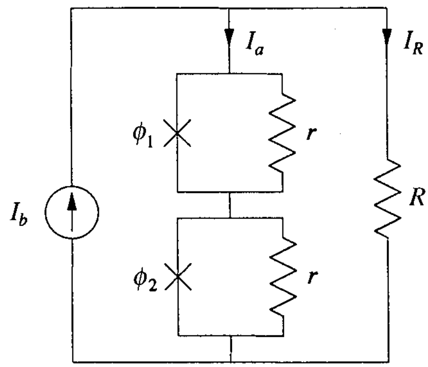


Figure 1

The goal of this exercise is to derive the governing equations for this circuit. In particular, we want to find differential equations for ϕ_1 and ϕ_2.

a) Write an equation relating the dc bias current I_b to the current I_a flowing through the array and the current I_R flowing through the load resistor.

b) Let V_1 and V_2 denote the voltages across the first and second Josephson junctions. Show that $I_a = I_c \sin\phi_1 + V_1/r$ and $I_a = I_c \sin\phi_2 + V_2/r$.

c) Let $k = 1,2$. Express V_k in terms of $\dot{\phi}_k$.

d) Using the results above, along with Kirchhoff's voltage law, show that

$$I_b = I_c \sin\phi_k + \frac{\hbar}{2er}\dot{\phi}_k + \frac{\hbar}{2eR}(\dot{\phi}_1 + \dot{\phi}_2) \text{ for } k = 1,2.$$

e) The equations in part (d) can be written in more standard form as equations for $\dot{\phi}_k$, as follows. Add the equations for $k = 1,2$, and use the result to eliminate the term $(\dot{\phi}_1 + \dot{\phi}_2)$. Show that the resulting equations take the form

$$\dot{\phi}_k = \Omega + a\sin\phi_k + K\sum_{j=1}^{2} \sin\phi_j,$$

and write down explicit expressions for the parameters Ω, a, K.

4.6.5 (N junctions, resistive load) Generalize Exercise 4.6.4 as follows. Instead of the two Josephson junctions in Figure 1, consider an array of N junctions in series. As before, assume the array is in parallel with a resistive load R, and that the junctions are identical, overdamped, and driven by a constant bias current I_b. Show that the governing equations can be written in dimensionless form as

$$\frac{d\phi_k}{d\tau} = \Omega + a\sin\phi_k + \tfrac{1}{N}\sum_{j=1}^{N} \sin\phi_j \ , \text{ for } k = 1, \ldots, N,$$

and write down explicit expressions for the dimensionless groups Ω and a and the dimensionless time τ. (See Example 8.7.4 and Tsang et al. (1991) for further discussion.)

4.6.6 (N junctions, RLC load) Generalize Exercise 4.6.4 to the case where there are N junctions in series, and where the load is a resistor R in series with a capacitor C and an inductor L. Write differential equations for ϕ_k and for Q, where Q is the charge on the load capacitor. (See Strogatz and Mirollo 1993.)

PART TWO

TWO-DIMENSIONAL FLOWS

5

LINEAR SYSTEMS

5.0 Introduction

As we've seen, in one-dimensional phase spaces the flow is extremely confined—all trajectories are forced to move monotonically or remain constant. In higher-dimensional phase spaces, trajectories have much more room to maneuver, and so a wider range of dynamical behavior becomes possible. Rather than attack all this complexity at once, we begin with the simplest class of higher-dimensional systems, namely *linear systems in two dimensions*. These systems are interesting in their own right, and, as we'll see later, they also play an important role in the classification of fixed points of *nonlinear* systems. We begin with some definitions and examples.

5.1 Definitions and Examples

A ***two-dimensional linear system*** is a system of the form

$$\dot{x} = ax + by$$
$$\dot{y} = cx + dy$$

where a, b, c, d are parameters. If we use boldface to denote vectors, this system can be written more compactly in matrix form as

$$\dot{\mathbf{x}} = A\mathbf{x},$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Such a system is ***linear*** in the sense that if $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions, then so is any linear combination $c_1\mathbf{x}_1 + c_2\mathbf{x}_2$. Notice that $\dot{\mathbf{x}} = \mathbf{0}$ when $\mathbf{x} = \mathbf{0}$, so $\mathbf{x}^* = \mathbf{0}$ is always a fixed point for any choice of A.

The solutions of $\dot{\mathbf{x}} = A\mathbf{x}$ can be visualized as trajectories moving on the (x, y) plane, in this context called the ***phase plane***. Our first example presents the phase plane analysis of a familiar system.

EXAMPLE 5.1.1:

As discussed in elementary physics courses, the vibrations of a mass hanging from a linear spring are governed by the linear differential equation

$$m\ddot{x} + kx = 0 \tag{1}$$

where m is the mass, k is the spring constant, and x is the displacement of the mass from equilibrium (Figure 5.1.1). Give a phase plane analysis of this ***simple harmonic oscillator***.

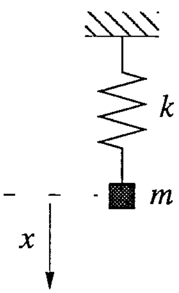


Figure 5.1.1

Solution: As you probably recall, it's easy to solve (1) analytically in terms of sines and cosines. But that's precisely what makes linear equations so special! For the *nonlinear* equations of ultimate interest to us, it's usually impossible to find an analytical solution. We want to develop methods for deducing the behavior of equations like (1) *without actually solving them.*

The motion in the phase plane is determined by a vector field that comes from the differential equation (1). To find this vector field, we note that the ***state*** of the system is characterized by its current position x and velocity v; if we know the values of *both* x and v, then (1) uniquely determines the future states of the system. Therefore we rewrite (1) in terms of x and v, as follows:

$$\dot{x} = v \tag{2a}$$
$$\dot{v} = -\tfrac{k}{m}x. \tag{2b}$$

Equation (2a) is just the definition of velocity, and (2b) is the differential equation (1) rewritten in terms of v. To simplify the notation, let $\omega^2 = k/m$. Then (2) becomes

$$\dot{x} = v \tag{3a}$$
$$\dot{v} = -\omega^2 x. \tag{3b}$$

The system (3) assigns a vector $(\dot{x}, \dot{v}) = (v, -\omega^2 x)$ at each point (x, v), and therefore represents a ***vector field*** on the phase plane.

For example, let's see what the vector field looks like when we're on the x-axis. Then $v = 0$ and so $(\dot{x}, \dot{v}) = (0, -\omega^2 x)$. Hence the vectors point vertically downward for positive x and vertically upward for negative x (Figure 5.1.2). As x gets larger in magnitude, the vectors $(0, -\omega^2 x)$ get longer. Similarly, on the v-axis, the vector field is $(\dot{x}, \dot{v}) = (v, 0)$, which points to the right when $v > 0$ and to the left when $v < 0$. As we move around in phase space, the vectors change direction as shown in Figure 5.1.2.

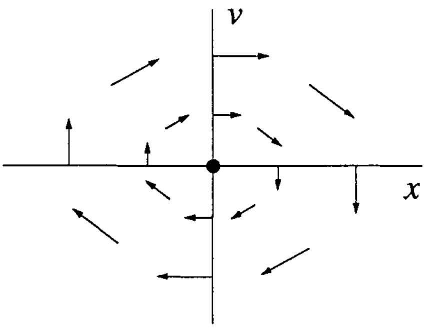


Figure 5.1.2

Just as in Chapter 2, it is helpful to visualize the vector field in terms of the motion of an imaginary fluid. In the present case, we imagine that a fluid is flowing steadily on the phase plane with a local velocity given by $(\dot{x}, \dot{v}) = (v, -\omega^2 x)$. Then, to find the trajectory starting at (x_0, v_0), we place an imaginary particle or ***phase point*** at (x_0, v_0) and watch how it is carried around by the flow.

The flow in Figure 5.1.2 swirls about the origin. The origin is special, like the eye of a hurricane: a phase point placed there would remain motionless, because $(\dot{x}, \dot{v}) = (0,0)$ when $(x, v) = (0,0)$; hence the origin is a ***fixed point***. But a phase point starting anywhere else would circulate around the origin and eventually return to its starting point. Such trajectories form ***closed orbits***, as shown in Figure 5.1.3. Figure 5.1.3 is called the ***phase portrait*** of the system—it shows the overall picture of trajectories in phase space.

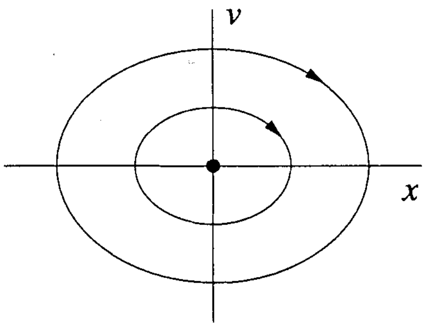
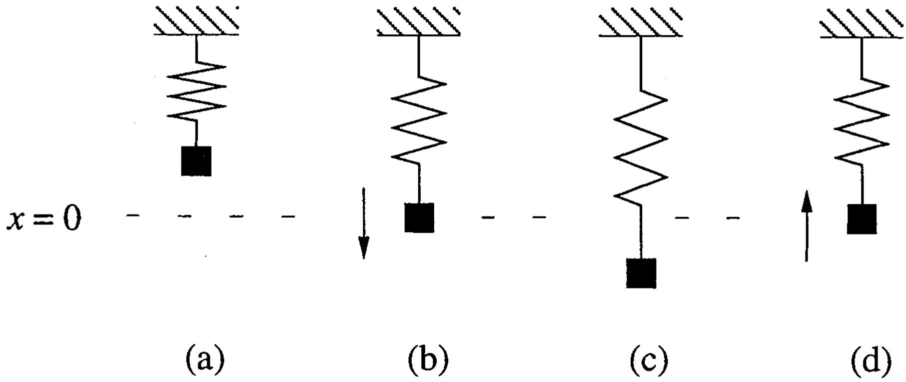


Figure 5.1.3

What do fixed points and closed orbits have to do with the original problem of a mass on a spring? The answers are beautifully simple. The fixed point $(x, v) = (0,0)$ corresponds to static equilibrium of the system: the mass is at rest at its equilibrium position and will remain there forever, since the spring is relaxed. The closed orbits have a more interesting interpretation: they correspond to periodic motions, i.e., oscillations of the mass. To see this, just look at some points on a closed orbit (Figure 5.1.4). When the displacement x is most negative, the velocity v is zero; this corresponds to one extreme of the oscillation, where the spring is most compressed (Figure 5.1.4).



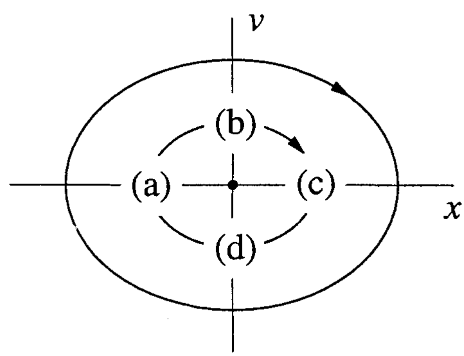


Figure 5.1.4

In the next instant as the phase point flows along the orbit, it is carried to points where x has increased and v is now positive; the mass is being pushed back toward its equilibrium position. But by the time the mass has reached $x = 0$, it has a large positive velocity (Figure 5.1.4b) and so it overshoots $x = 0$. The mass eventually comes to rest at the other end of its swing, where x is most positive and v is zero again (Figure 5.1.4c). Then the mass gets pulled up again and eventually completes the cycle (Figure 5.1.4d).

The shape of the closed orbits also has an interesting physical interpretation. The orbits in Figures 5.1.3 and 5.1.4 are actually *ellipses* given by the equation $\omega^2 x^2 + v^2 = C$, where $C \geq 0$ is a constant. In Exercise 5.1.1, you are asked to derive this geometric result, and to show that it is equivalent to conservation of energy. ■

EXAMPLE 5.1.2:

Solve the linear system $\dot{\mathbf{x}} = A\mathbf{x}$, where $A = \begin{pmatrix} a & 0 \\ 0 & -1 \end{pmatrix}$. Graph the phase portrait as a varies from $-\infty$ to $+\infty$, showing the qualitatively different cases.

Solution: The system is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

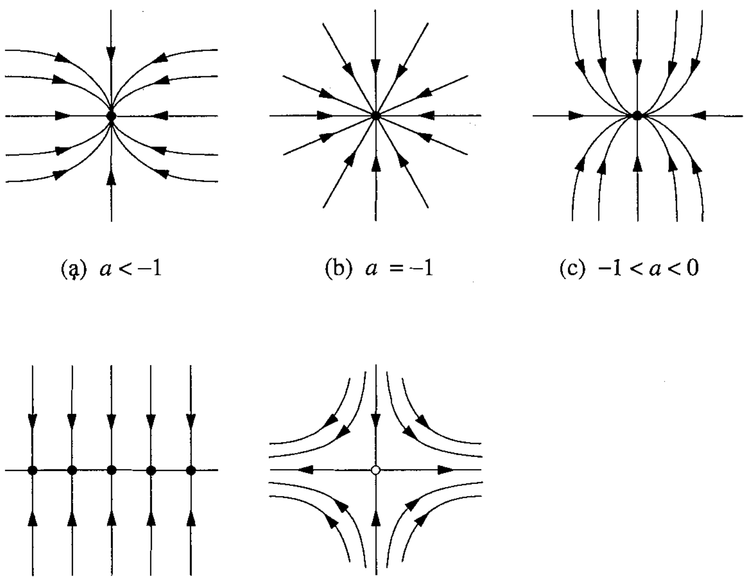
Matrix multiplication yields

$$\begin{aligned} \dot{x} &= ax \\ \dot{y} &= -y \end{aligned}$$

which shows that the two equations are ***uncoupled***; there's no x in the y-equation and vice versa. In this simple case, each equation may be solved separately. The solution is

$$x(t) = x_0 e^{at} \tag{1a}$$
$$y(t) = y_0 e^{-t}. \tag{1b}$$

The phase portraits for different values of a are shown in Figure 5.1.5. In each case, $y(t)$ decays exponentially. When $a < 0$, $x(t)$ also decays exponentially and so all trajectories approach the origin as $t \to \infty$. However, the direction of approach depends on the size of a compared to -1.



(a) $a < -1$ (b) $a = -1$ (c) $-1 < a < 0$

(d) $a = 0$ (e) $a > 0$

Figure 5.1.5

In Figure 5.1.5a, we have $a < -1$, which implies that $x(t)$ decays more rapidly than $y(t)$. The trajectories approach the origin tangent to the *slower* direction (here, the y-direction). The intuitive explanation is that when a is very negative, the trajectory slams horizontally onto the y-axis, because the decay of $x(t)$ is almost instantaneous. Then the trajectory dawdles along the y-axis toward the origin, and so the approach is tangent to the y-axis. On the other hand, if we look *backwards* along a trajectory ($t \to -\infty$), then the trajectories all become parallel to the faster decaying direction (here, the x-direction). These conclusions are easily proved by looking at the slope $dy/dx = \dot{y}/\dot{x}$ along the trajectories; see Exercise 5.1.2. In Figure 5.1.5a, the fixed point $\mathbf{x}^* = \mathbf{0}$ is called a ***stable node***.

Figure 5.1.5b shows the case $a = -1$. Equation (1) shows that $y(t)/x(t) = y_0/x_0 =$ constant, and so all trajectories are straight lines through the origin. This is a very special case—it occurs because the decay rates in the two directions are precisely equal. In this case, $\mathbf{x}^*$ is called a symmetrical node or ***star***.

When $-1 < a < 0$, we again have a node, but now the trajectories approach $\mathbf{x}^*$ along the x-direction, which is the more slowly decaying direction for this range of a (Figure 5.1.5c).

Something dramatic happens when $a = 0$ (Figure 5.1.5d). Now (1a) becomes $x(t) \equiv x_0$ and so there's an entire ***line of fixed points*** along the x-axis. All trajectories approach these fixed points along vertical lines.

Finally when $a > 0$ (Figure 5.1.5e), $\mathbf{x}^*$ becomes unstable, due to the exponential growth in the x-direction. Most trajectories veer away from $\mathbf{x}^*$ and head out to infinity. An exception occurs if the trajectory starts on the y-axis; then it walks a tightrope to the origin. In forward time, the trajectories are asymptotic to the x-axis; in backward time, to the y-axis. Here $\mathbf{x}^* = \mathbf{0}$ is called a ***saddle point***. The y-axis is called the ***stable manifold*** of the saddle point $\mathbf{x}^*$, defined as the set of initial conditions $\mathbf{x}_0$ such that $\mathbf{x}(t) \to \mathbf{x}^*$ as $t \to \infty$. Likewise, the ***unstable manifold*** of $\mathbf{x}^*$ is the set of initial conditions such that $\mathbf{x}(t) \to \mathbf{x}^*$ as $t \to -\infty$. Here the unstable manifold is the x-axis. Note that a typical trajectory asymptotically approaches the unstable manifold as $t \to \infty$, and approaches the stable manifold as $t \to -\infty$. This sounds backwards, but it's right! ■

Stability Language

It's useful to introduce some language that allows us to discuss the stability of different types of fixed points. This language will be especially useful when we analyze fixed points of *nonlinear* systems. For now we'll be informal; precise definitions of the different types of stability will be given in Exercise 5.1.10.

We say that $\mathbf{x}^* = \mathbf{0}$ is an ***attracting*** fixed point in Figures 5.1.5a–c; all trajectories that start near $\mathbf{x}^*$ approach it as $t \to \infty$. That is, $\mathbf{x}(t) \to \mathbf{x}^*$ as $t \to \infty$. In fact $\mathbf{x}^*$ attracts *all* trajectories in the phase plane, so it could be called ***globally attracting***.

There's a completely different notion of stability which relates to the behavior of trajectories for *all* time, not just as $t \to \infty$. We say that a fixed point $\mathbf{x}^*$ is ***Liapunov stable*** if all trajectories that start sufficiently close to $\mathbf{x}^*$ remain close to it for all time. In Figures 5.1.5a–d, the origin is Liapunov stable.

Figure 5.1.5d shows that a fixed point can be Liapunov stable but not attracting. This situation comes up often enough that there is a special name for it. When a fixed point is Liapunov stable but not attracting, it is called ***neutrally stable***. Nearby trajectories are neither attracted to nor repelled from a neutrally stable point. As a second example, the equilibrium point of the simple harmonic oscillator (Figure 5.1.3) is neutrally stable. Neutral stability is commonly encountered in mechanical systems in the absence of friction. Conversely, it's possible for a fixed point to be attracting but not Liapunov stable; thus, neither notion of stability implies the other. An example is given by the following vector field on the circle: $\dot{\theta} = 1 - \cos\theta$ (Figure 5.1.6). Here $\theta^* = 0$ attracts all trajectories as $t \to \infty$, but it is not Liapunov stable; there are trajectories that start infinitesimally close to θ^* but go on a very large excursion before returning to θ^*.

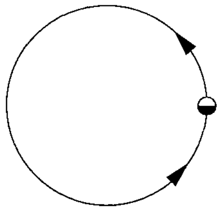


Figure 5.1.6

However, in practice the two types of stability often occur together. If a fixed point is *both* Liapunov stable and attracting, we'll call it ***stable***, or sometimes ***asymptotically stable***.

Finally, $\mathbf{x}^*$ is ***unstable*** in Figure 5.1.5e, because it is neither attracting nor Liapunov stable.

A graphical convention: we'll use open dots to denote unstable fixed points, and solid black dots to denote Liapunov stable fixed points. This convention is consistent with that used in previous chapters.

5.2 Classification of Linear Systems

The examples in the last section had the special feature that two of the entries in the matrix A were zero. Now we want to study the general case of an arbitrary 2×2 matrix, with the aim of classifying all the possible phase portraits that can occur.

Example 5.1.2 provides a clue about how to proceed. Recall that the x and y axes played a crucial geometric role. They determined the direction of the trajectories as $t \to \pm\infty$. They also contained special ***straight-line trajectories***: a trajectory starting on one of the coordinate axes stayed on that axis forever, and exhibited simple exponential growth or decay along it.

For the general case, we would like to find the analog of these straight-line trajectories. That is, we seek trajectories of the form

$$\mathbf{x}(t) = e^{\lambda t}\mathbf{v}, \tag{2}$$

where $\mathbf{v} \neq \mathbf{0}$ is some fixed vector to be determined, and λ is a growth rate, also to be determined. If such solutions exist, they correspond to exponential motion along the line spanned by the vector $\mathbf{v}$.

To find the conditions on $\mathbf{v}$ and λ, we substitute $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ into $\dot{\mathbf{x}} = A\mathbf{x}$, and obtain $\lambda e^{\lambda t}\mathbf{v} = e^{\lambda t}A\mathbf{v}$. Canceling the nonzero scalar factor $e^{\lambda t}$ yields

$$A\mathbf{v} = \lambda\mathbf{v}, \tag{3}$$

which says that the desired straight line solutions exist if $\mathbf{v}$ is an ***eigenvector*** of A with corresponding ***eigenvalue*** λ. In this case we call the solution (2) an ***eigen-solution***.

Let's recall how to find eigenvalues and eigenvectors. (If your memory needs more refreshing, see any text on linear algebra.) In general, the eigenvalues of a matrix A are given by the ***characteristic equation*** $\det(A - \lambda I) = 0$, where I is the identity matrix. For a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

the characteristic equation becomes

$$\det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = 0.$$

Expanding the determinant yields

$$\lambda^2 - \tau\lambda + \Delta = 0 \tag{4}$$

where

$$\tau = \text{trace}(A) = a + d,$$
$$\Delta = \det(A) = ad - bc.$$

Then

$$\lambda_1 = \frac{\tau + \sqrt{\tau^2 - 4\Delta}}{2}, \qquad \lambda_2 = \frac{\tau - \sqrt{\tau^2 - 4\Delta}}{2} \tag{5}$$

are the solutions of the quadratic equation (4). In other words, the eigenvalues depend only on the trace and determinant of the matrix A.

The typical situation is for the eigenvalues to be distinct: $\lambda_1 \neq \lambda_2$. In this case, a theorem of linear algebra states that the corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent, and hence span the entire plane (Figure 5.2.1). In particular, any initial condition $\mathbf{x}_0$ can be written as a linear combination of eigenvectors, say $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$.

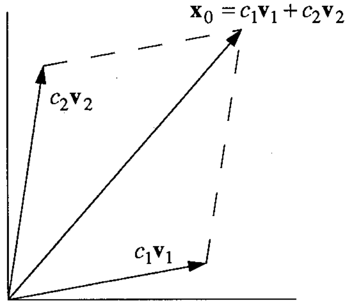


Figure 5.2.1

This observation allows us to write down the general solution for $\mathbf{x}(t)$—it is simply

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 . \tag{6}$$

Why is this the general solution? First of all, it is a linear combination of solutions to $\dot{\mathbf{x}} = A\mathbf{x}$, and hence is itself a solution. Second, it satisfies the initial condition $\mathbf{x}(0) = \mathbf{x}_0$, and so by the existence and uniqueness theorem, it is the *only* solution. (See Section 6.2 for a general statement of the existence and uniqueness theorem.)

EXAMPLE 5.2.1:

Solve the initial value problem $\dot{x} = x + y$, $\dot{y} = 4x - 2y$, subject to the initial condition $(x_0, y_0) = (2, -3)$.

Solution: The corresponding matrix equation is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

First we find the eigenvalues of the matrix A. The matrix has $\tau = -1$ and $\Delta = -6$, so the characteristic equation is $\lambda^2 + \lambda - 6 = 0$. Hence

$$\lambda_1 = 2, \quad \lambda_2 = -3 .$$

Next we find the eigenvectors. Given an eigenvalue λ, the corresponding eigenvector $\mathbf{v} = (v_1, v_2)$ satisfies

$$\begin{pmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For $\lambda_1 = 2$, this yields $\begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, which has a nontrivial solution

$(v_1, v_2) = (1,1)$, or any scalar multiple thereof. (Of course, any multiple of an eigenvector is always an eigenvector; we try to pick the simplest multiple, but any one will do.) Similarly, for $\lambda_2 = -3$, the eigenvector equation becomes $\begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, which has a nontrivial solution $(v_1, v_2) = (1, -4)$. In summary,

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}.$$

Next we write the general solution as a linear combination of eigensolutions. From (6), the general solution is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t}. \tag{7}$$

Finally, we compute c_1 and c_2 to satisfy the initial condition $(x_0, y_0) = (2, -3)$. At $t = 0$, (7) becomes

$$\begin{pmatrix} 2 \\ -3 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -4 \end{pmatrix},$$

which is equivalent to the algebraic system

$$\begin{aligned} 2 &= c_1 + c_2 \,, \\ -3 &= c_1 - 4c_2 \,. \end{aligned}$$

The solution is $c_1 = 1$, $c_2 = 1$. Substituting back into (7) yields

$$\begin{aligned} x(t) &= e^{2t} + e^{-3t}, \\ y(t) &= e^{2t} - 4e^{-3t} \end{aligned}$$

for the solution to the initial value problem. ■

Whew! Fortunately we don't need to go through all this to draw the phase portrait of a linear system. All we need to know are the eigenvectors and eigenvalues.

EXAMPLE 5.2.2:

Draw the phase portrait for the system of Example 5.2.1.

Solution: The system has eigenvalues $\lambda_1 = 2$, $\lambda_2 = -3$. Hence the first eigensolution grows exponentially, and the second eigensolution decays. This means the origin is a *saddle point*. Its stable manifold is the line spanned by the eigenvector $\mathbf{v}_2 = (1, -4)$, corresponding to the decaying eigensolution. Similarly, the unstable manifold is the line spanned by $\mathbf{v}_1 = (1,1)$. As with all saddle points, a typical trajectory approaches the unstable manifold as $t \to \infty$, and the stable manifold as $t \to -\infty$. Figure 5.2.2 shows the phase portrait. ■

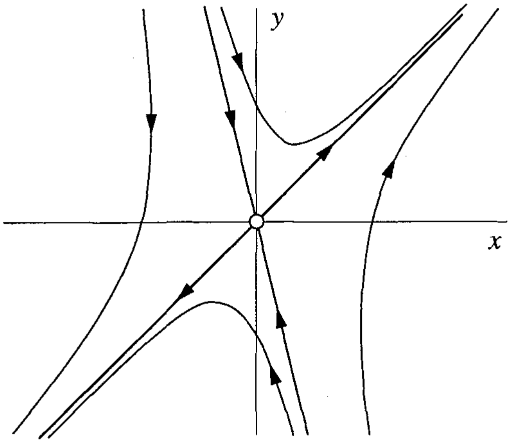


Figure 5.2.2

EXAMPLE 5.2.3:

Sketch a typical phase portrait for the case $\lambda_2 < \lambda_1 < 0$.

Solution: First suppose $\lambda_2 < \lambda_1 < 0$. Then both eigensolutions decay exponentially. The fixed point is a stable node, as in Figures 5.1.5a and 5.1.5c, except now the eigenvectors are not mutually perpendicular, in general. Trajectories typically approach the origin tangent to the ***slow eigendirection***, defined as the direction spanned by the eigenvector with the smaller $|\lambda|$. In backwards time ($t \to -\infty$), the trajectories become parallel to the fast eigendirection. Figure 5.2.3 shows the phase portrait. (If we reverse all the arrows in Figure 5.2.3, we obtain a typical phase portrait for an ***unstable node***.) ■

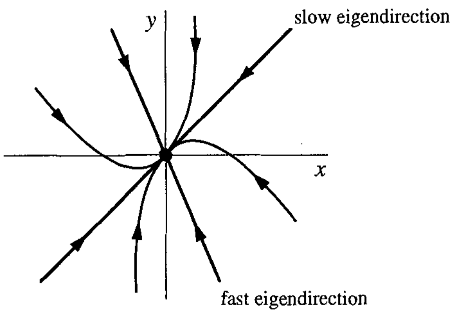
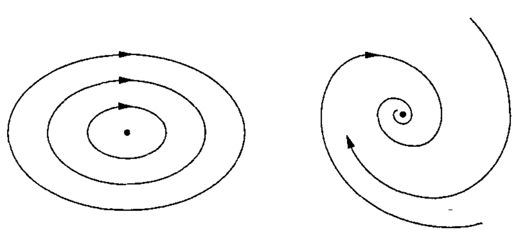


Figure 5.2.3

EXAMPLE 5.2.4:

What happens if the eigenvalues are *complex* numbers?

Solution: If the eigenvalues are complex, the fixed point is either a ***center*** (Figure 5.2.4a) or a ***spiral*** (Figure 5.2.4b). We've already seen an example of a center in the simple harmonic oscillator of Section 5.1; the origin is surrounded by a family of closed orbits. Note that centers are *neutrally stable*, since nearby trajectories are neither attracted to nor repelled from the fixed point. A spiral would occur if the harmonic oscillator were lightly damped. Then the trajectory would just fail to close, because the oscillator loses a bit of energy on each cycle.



(a) center (b) spiral

Figure 5.2.4

To justify these statements, recall that the eigenvalues are $\lambda_{1,2} = \frac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\Delta}\right)$. Thus complex eigenvalues occur when

$$\tau^2 - 4\Delta < 0.$$

To simplify the notation, let's write the eigenvalues as

$$\lambda_{1,2} = \alpha \pm i\omega$$

where

$$\alpha = \tau/2, \qquad \omega = \tfrac{1}{2}\sqrt{4\Delta - \tau^2}\,.$$

By assumption, $\omega \neq 0$. Then the eigenvalues are distinct and so the general solution is still given by

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2\,.$$

But now the c's and $\mathbf{v}$'s are *complex*, since the λ's are. This means that $\mathbf{x}(t)$ involves linear combinations of $e^{(\alpha \pm i\omega)t}$. By Euler's formula, $e^{i\omega t} = \cos\omega t + i\sin\omega t$. Hence $\mathbf{x}(t)$ is a combination of terms involving $e^{\alpha t}\cos\omega t$ and $e^{\alpha t}\sin\omega t$. Such terms represent exponentially *decaying oscillations* if $\alpha = \mathrm{Re}(\lambda) < 0$ and *growing oscillations* if $\alpha > 0$. The corresponding fixed points are ***stable*** and ***unstable spirals***, respectively. Figure 5.2.4b shows the stable case.

If the eigenvalues are pure imaginary ($\alpha = 0$), then all the solutions are periodic with period $T = 2\pi/\omega$. The oscillations have fixed amplitude and the fixed point is a center.

For both centers and spirals, it's easy to determine whether the rotation is clockwise or counterclockwise; just compute a few vectors in the vector field and the sense of rotation should be obvious. ■

EXAMPLE 5.2.5:

In our analysis of the general case, we have been assuming that the eigenvalues are distinct. What happens if the eigenvalues are *equal*?

Solution: Suppose $\lambda_1 = \lambda_2 = \lambda$. There are two possibilities: either there are two independent eigenvectors corresponding to λ, or there's only one.

If there are two independent eigenvectors, then they span the plane and so *every vector is an eigenvector with this same eigenvalue* λ. To see this, write an arbitrary vector $\mathbf{x}_0$ as a linear combination of the two eigenvectors: $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$. Then

$$A\mathbf{x}_0 = A(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1\lambda\mathbf{v}_1 + c_2\lambda\mathbf{v}_2 = \lambda\mathbf{x}_0$$

so $\mathbf{x}_0$ is also an eigenvector with eigenvalue λ. Since multiplication by A simply stretches every vector by a factor λ, the matrix must be a multiple of the identity:

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}.$$

Then if $\lambda \neq 0$, all trajectories are straight lines through the origin ($\mathbf{x}(t) = e^{\lambda t}\mathbf{x}_0$) and the fixed point is a ***star node*** (Figure 5.2.5).

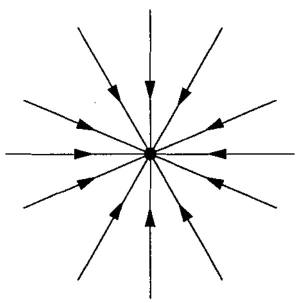


Figure 5.2.5

On the other hand, if $\lambda = 0$, the whole plane is filled with fixed points! (No surprise—the system is $\dot{\mathbf{x}} = \mathbf{0}$.)

The other possibility is that there's only one eigenvector (more accurately, the eigenspace corresponding to λ is one-dimensional.) For example, any matrix of the form $A = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}$, with $b \neq 0$ has only a one-dimensional eigenspace (Exercise 5.2.11).

When there's only one eigendirection, the fixed point is a ***degenerate node***. A

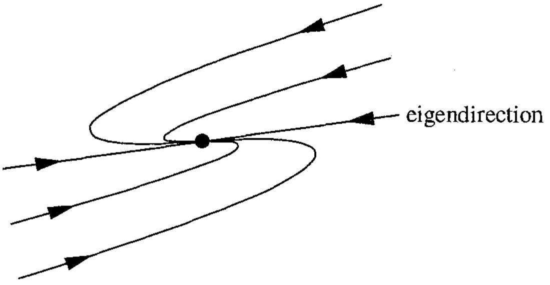


Figure 5.2.6

typical phase portrait is shown in Figure 5.2.6. As $t \to +\infty$ and also as $t \to -\infty$, all trajectories become parallel to the one available eigendirection.

A good way to think about the degenerate node is to imagine that it has been created by deforming an ordinary node. The ordinary node has two independent eigendirections; all trajectories are parallel to the slow eigendirection as $t \to \infty$, and to the fast eigendirection as $t \to -\infty$ (Figure 5.2.7a).

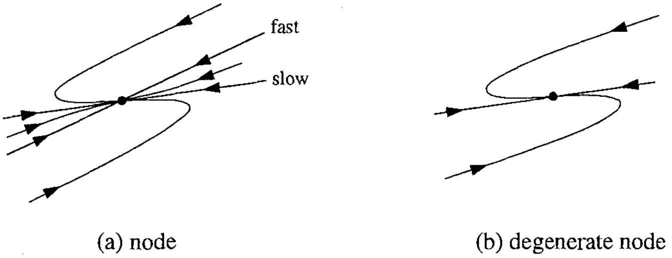


Figure 5.2.7

Now suppose we start changing the parameters of the system in such a way that the two eigendirections are scissored together. Then some of the trajectories will get squashed in the collapsing region between the two eigendirections, while the surviving trajectories get pulled around to form the degenerate node (Figure 5.2.7b).

Another way to get intuition about this case is to realize that the degenerate node is on the *borderline between a spiral and a node.* The trajectories are trying to wind around in a spiral, but they don't quite make it. ■

Classification of Fixed Points

By now you're probably tired of all the examples and ready for a simple classification scheme. Happily, there is one. We can show the type and stability of all the different fixed points on a single diagram (Figure 5.2.8).

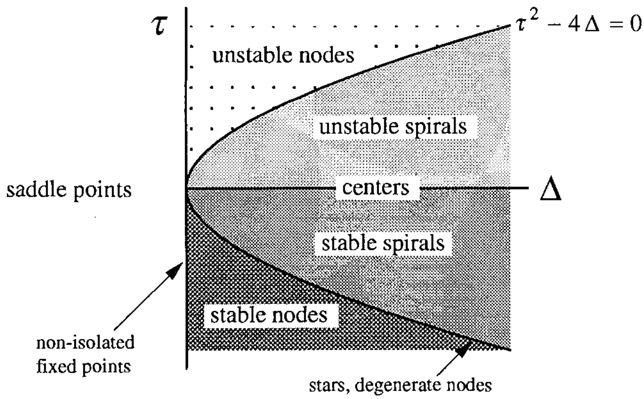


Figure 5.2.8

The axes are the trace τ and the determinant Δ of the matrix A. All of the information in the diagram is implied by the following formulas:

$$\lambda_{1,2} = \tfrac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\Delta}\right), \qquad \Delta = \lambda_1\lambda_2, \qquad \tau = \lambda_1 + \lambda_2.$$

The first equation is just (5). The second and third can be obtained by writing the characteristic equation in the form $(\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - \tau\lambda + \Delta = 0$.

To arrive at Figure 5.2.8, we make the following observations:

If $\Delta < 0$, the eigenvalues are real and have opposite signs; hence the fixed point is a *saddle point*.

If $\Delta > 0$, the eigenvalues are either real with the same sign (*nodes*), or complex conjugate (*spirals* and *centers*). Nodes satisfy $\tau^2 - 4\Delta > 0$ and spirals satisfy $\tau^2 - 4\Delta < 0$. The parabola $\tau^2 - 4\Delta = 0$ is the borderline between nodes and spirals; star nodes and degenerate nodes live on this parabola. The stability of the nodes and spirals is determined by τ. When $\tau < 0$, both eigenvalues have negative real parts, so the fixed point is stable. Unstable spirals and nodes have $\tau > 0$. Neutrally stable centers live on the borderline $\tau = 0$, where the eigenvalues are purely imaginary.

If $\Delta = 0$, at least one of the eigenvalues is zero. Then the origin is not an isolated fixed point. There is either a whole line of fixed points, as in Figure 5.1.5d, or a plane of fixed points, if $A = 0$.

Figure 5.2.8 shows that saddle points, nodes, and spirals are the major types of fixed points; they occur in large open regions of the (Δ, τ) plane. Centers, stars, degenerate nodes, and non-isolated fixed points are ***borderline cases*** that occur along curves in the (Δ, τ) plane. Of these borderline cases, centers are by far the most important. They occur very commonly in frictionless mechanical systems where energy is conserved.

EXAMPLE 5.2.6:

Classify the fixed point $\mathbf{x}^* = \mathbf{0}$ for the system $\dot{\mathbf{x}} = A\mathbf{x}$, where $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

Solution: The matrix has $\Delta = -2$; hence the fixed point is a saddle point. ■

EXAMPLE 5.2.7:

Redo Example 5.2.6 for $A = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$.

Solution: Now $\Delta = 5$ and $\tau = 6$. Since $\Delta > 0$ and $\tau^2 - 4\Delta = 16 > 0$, the fixed point is a node. It is unstable, since $\tau > 0$. ■

5.3 Love Affairs

To arouse your interest in the classification of linear systems, we now discuss a simple model for the dynamics of love affairs (Strogatz 1988). The following story illustrates the idea.

Romeo is in love with Juliet, but in our version of this story, Juliet is a fickle lover. The more Romeo loves her, the more Juliet wants to run away and hide. But when Romeo gets discouraged and backs off, Juliet begins to find him strangely attractive. Romeo, on the other hand, tends to echo her: he warms up when she loves him, and grows cold when she hates him.

Let

$$R(t) = \text{Romeo's love/hate for Juliet at time } t$$
$$J(t) = \text{Juliet's love/hate for Romeo at time } t.$$

Positive values of R, J signify love, negative values signify hate. Then a model for their star-crossed romance is

$$\dot{R} = aJ$$
$$\dot{J} = -bR$$

where the parameters a and b are positive, to be consistent with the story.

The sad outcome of their affair is, of course, a neverending cycle of love and hate; the governing system has a center at $(R, J) = (0, 0)$. At least they manage to achieve simultaneous love one-quarter of the time (Figure 5.3.1).

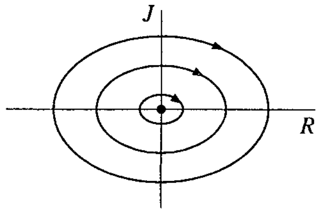


Figure 5.3.1

Now consider the forecast for lovers governed by the general linear system

$$\dot{R} = aR + bJ$$
$$\dot{J} = cR + dJ$$

where the parameters a, b, c, d may have either sign. A choice of signs specifies the romantic styles. As named by one of my students, the choice $a > 0$, $b > 0$ means that Romeo is an "eager beaver"—he gets excited by Juliet's love for him, and is further spurred on by his own affectionate feelings for her. It's entertaining to name the other three romantic styles, and to predict the outcomes for the various pairings. For example, can a "cautious lover" ($a < 0$, $b > 0$) find true love with an eager beaver? These and other pressing questions will be considered in the exercises.

EXAMPLE 5.3.1:

What happens when two identically cautious lovers get together?

Solution: The system is

$$\dot{R} = aR + bJ$$
$$\dot{J} = bR + aJ$$

with $a < 0$, $b > 0$. Here a is a measure of cautiousness (they each try to avoid throwing themselves at the other) and b is a measure of responsiveness (they both get excited by the other's advances). We might suspect that the outcome depends on the relative size of a and b. Let's see what happens.

The corresponding matrix is

$$A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

which has

$$\tau = 2a < 0\,, \qquad \Delta = a^2 - b^2\,, \qquad \tau^2 - 4\Delta = 4b^2 > 0\,.$$

Hence the fixed point $(R, J) = (0,0)$ is a saddle point if $a^2 < b^2$ and a stable node if $a^2 > b^2$. The eigenvalues and corresponding eigenvectors are

$$\lambda_1 = a+b, \qquad \mathbf{v}_1 = (1,1), \qquad \lambda_2 = a-b, \qquad \mathbf{v}_2 = (1,-1).$$

Since $a+b > a-b$, the eigenvector $(1,1)$ spans the unstable manifold when the origin is a saddle point, and it spans the slow eigendirection when the origin is a stable node. Figure 5.3.2 shows the phase portrait for the two cases.

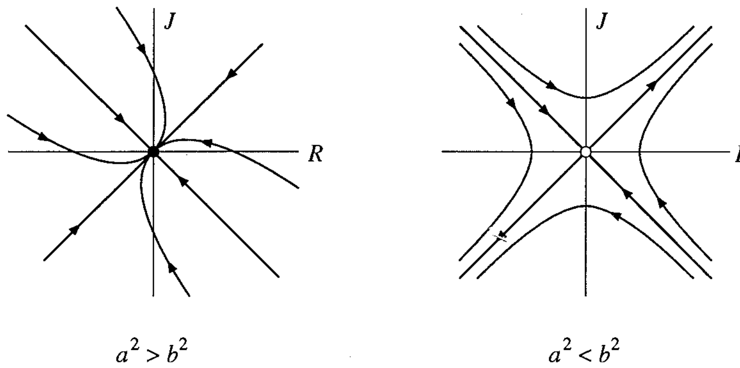


Figure 5.3.2

If $a^2 > b^2$, the relationship always fizzles out to mutual indifference. The lesson seems to be that excessive caution can lead to apathy.

If $a^2 < b^2$, the lovers are more daring, or perhaps more sensitive to each other. Now the relationship is explosive. Depending on their feelings initially, their relationship either becomes a love fest or a war. In either case, all trajectories approach the line $R = J$, so their feelings are eventually mutual. ■

EXERCISES FOR CHAPTER 5

5.1 Definitions and Examples

5.1.1 (Ellipses and energy conservation for the harmonic oscillator) Consider the harmonic oscillator $\dot{x} = v$, $\dot{v} = -\omega^2 x$.

a) Show that the orbits are given by ellipses $\omega^2 x^2 + v^2 = C$, where C is any non-negative constant. (Hint: Divide the $\dot{x}$ equation by the $\dot{v}$ equation, separate the v's from the x's, and integrate the resulting separable equation.)
b) Show that this condition is equivalent to conservation of energy.

5.1.2 Consider the system $\dot{x} = ax$, $\dot{y} = -y$, where $a < -1$. Show that all trajectories become parallel to the y-direction as $t \to \infty$, and parallel to the x-direction as $t \to -\infty$.

(Hint: Examine the slope $dy/dx = \dot{y}/\dot{x}$.)

Write the following systems in matrix form.

5.1.3 $\dot{x} = -y,\ \dot{y} = -x$

5.1.4 $\dot{x} = 3x - 2y,\ \dot{y} = 2y - x$

5.1.5 $\dot{x} = 0,\ \dot{y} = x + y$

5.1.6 $\dot{x} = x,\ \dot{y} = 5x + y$

Sketch the vector field for the following systems. Indicate the length and direction of the vectors with reasonable accuracy. Sketch some typical trajectories.

5.1.7 $\dot{x} = x,\ \dot{y} = x + y$

5.1.8 $\dot{x} = -2y,\ \dot{y} = x$

5.1.9 Consider the system $\dot{x} = -y,\ \dot{y} = -x$.

a) Sketch the vector field.

b) Show that the trajectories of the system are hyperbolas of the form $x^2 - y^2 = C$. (Hint: Show that the governing equations imply $x\dot{x} - y\dot{y} = 0$ and then integrate both sides.)

c) The origin is a saddle point; find equations for its stable and unstable manifolds.

d) The system can be decoupled and solved as follows. Introduce new variables u and v, where $u = x + y$, $v = x - y$. Then rewrite the system in terms of u and v. Solve for $u(t)$ and $v(t)$, starting from an arbitrary initial condition (u_0, v_0).

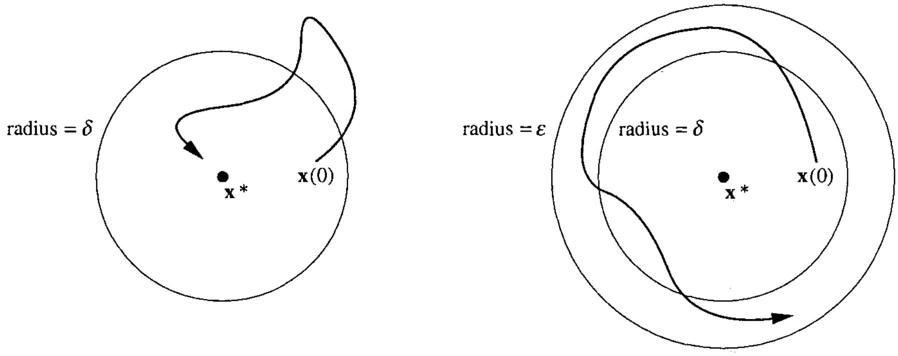
e) What are the equations for the stable and unstable manifolds in terms of u and v?

f) Finally, using the answer to (d), write the general solution for $x(t)$ and $y(t)$, starting from an initial condition (x_0, y_0).

5.1.10 (Attracting and Liapunov stable) Here are the official definitions of the various types of stability. Consider a fixed point $\mathbf{x}^*$ of a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$.

We say that $\mathbf{x}^*$ is ***attracting*** if there is a $\delta > 0$ such that $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{x}^*$ whenever $\|\mathbf{x}(0) - \mathbf{x}^*\| < \delta$. In other words, any trajectory that starts within a distance δ of $\mathbf{x}^*$ is guaranteed to converge to $\mathbf{x}^*$ *eventually*. As shown schematically in Figure 1, trajectories that start nearby are allowed to stray from $\mathbf{x}^*$ in the short run, but they must approach $\mathbf{x}^*$ in the long run.

In contrast, Liapunov stability requires that nearby trajectories remain close for *all* time. We say that $\mathbf{x}^*$ is ***Liapunov stable*** if for each $\varepsilon > 0$, there is a $\delta > 0$ such that $\|\mathbf{x}(t) - \mathbf{x}^*\| < \varepsilon$ whenever $t \geq 0$ and $\|\mathbf{x}(0) - \mathbf{x}^*\| < \delta$. Thus, trajectories that start within δ of $\mathbf{x}^*$ remain within ε of $\mathbf{x}^*$ for all positive time (Figure 1).



Attracting Liapunov stable

Figure 1

Finally, $\mathbf{x}^*$ is ***asymptotically stable*** if it is both attracting and Liapunov stable.

For each of the following systems, decide whether the origin is attracting, Liapunov stable, asymptotically stable, or none of the above.

a) $\dot{x} = y$, $\dot{y} = -4x$.
b) $\dot{x} = 2y$, $\dot{y} = x$
c) $\dot{x} = 0$, $\dot{y} = x$
d) $\dot{x} = 0$, $\dot{y} = -y$
e) $\dot{x} = -x$, $\dot{y} = -5y$
f) $\dot{x} = x$, $\dot{y} = y$

5.1.11 (Stability proofs) Prove that your answers to 5.1.10 are correct, using the definitions of the different types of stability. (You must produce a suitable δ to prove that the origin is attracting, or a suitable $\delta(\varepsilon)$ to prove Liapunov stability.)

5.1.12 (Closed orbits from symmetry arguments) Give a simple proof that orbits are closed for the simple harmonic oscillator $\dot{x} = v$, $\dot{v} = -x$, using *only* the symmetry properties of the vector field. (Hint: Consider a trajectory that starts on the v-axis at $(0, -v_0)$, and suppose that the trajectory intersects the x-axis at $(x, 0)$. Then use symmetry arguments to find the subsequent intersections with the v-axis and x-axis.)

5.1.13 Why do you think a "saddle point" is called by that name? What's the connection to real saddles (the kind used on horses)?

5.2 Classification of Linear Systems

5.2.1 Consider the system $\dot{x} = 4x - y$, $\dot{y} = 2x + y$.

a) Write the system as $\dot{\mathbf{x}} = A\mathbf{x}$. Show that the characteristic polynomial is $\lambda^2 - 5\lambda + 6$, and find the eigenvalues and eigenvectors of A.
b) Find the general solution of the system.
c) Classify the fixed point at the origin.
d) Solve the system subject to the initial condition $(x_0, y_0) = (3, 4)$.

5.2.2 (Complex eigenvalues) This exercise leads you through the solution of a linear system where the eigenvalues are complex. The system is $\dot{x} = x - y$, $\dot{y} = x + y$.

a) Find A and show that it has eigenvalues $\lambda_1 = 1 + i$, $\lambda_2 = 1 - i$, with eigenvectors $\mathbf{v}_1 = (i, 1)$, $\mathbf{v}_2 = (-i, 1)$. (Note that the eigenvalues are complex conjugates, and so are the eigenvectors—this is always the case for real A with complex eigenvalues.)

b) The general solution is $\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$. So in one sense we're done! But this way of writing $\mathbf{x}(t)$ involves complex coefficients and looks unfamiliar. Express $\mathbf{x}(t)$ purely in terms of real-valued functions. (Hint: Use $e^{i\omega t} = \cos\omega t + i\sin\omega t$ to rewrite $\mathbf{x}(t)$ in terms of sines and cosines, and then separate the terms that have a prefactor of i from those that don't.)

Plot the phase portrait and classify the fixed point of the following linear systems. If the eigenvectors are real, indicate them in your sketch.

5.2.3 $\dot{x} = y$, $\dot{y} = -2x - 3y$

5.2.4 $\dot{x} = 5x + 10y$, $\dot{y} = -x - y$

5.2.5 $\dot{x} = 3x - 4y$, $\dot{y} = x - y$

5.2.6 $\dot{x} = -3x + 2y$, $\dot{y} = x - 2y$

5.2.7 $\dot{x} = 5x + 2y$, $\dot{y} = -17x - 5y$

5.2.8 $\dot{x} = -3x + 4y$, $\dot{y} = -2x + 3y$

5.2.9 $\dot{x} = 4x - 3y$, $\dot{y} = 8x - 6y$

5.2.10 $\dot{x} = y$, $\dot{y} = -x - 2y$.

5.2.11 Show that any matrix of the form $A = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}$, with $b \neq 0$, has only a one-dimensional eigenspace corresponding to the eigenvalue λ. Then solve the system $\dot{\mathbf{x}} = A\mathbf{x}$ and sketch the phase portrait.

5.2.12 (*LRC* circuit) Consider the circuit equation $L\ddot{I} + R\dot{I} + I/C = 0$, where $L, C > 0$ and $R \geq 0$.

a) Rewrite the equation as a two-dimensional linear system.

b) Show that the origin is asymptotically stable if $R > 0$ and neutrally stable if $R = 0$.

c) Classify the fixed point at the origin, depending on whether $R^2C - 4L$ is positive, negative, or zero, and sketch the phase portrait in all three cases.

5.2.13 (Damped harmonic oscillator) The motion of a damped harmonic oscillator is described by $m\ddot{x} + b\dot{x} + kx = 0$, where $b > 0$ is the damping constant.

a) Rewrite the equation as a two-dimensional linear system.

b) Classify the fixed point at the origin and sketch the phase portrait. Be sure to show all the different cases that can occur, depending on the relative sizes of the parameters.

c) How do your results relate to the standard notions of overdamped, critically damped, and underdamped vibrations?

5.2.14 (A project about random systems) Suppose we pick a linear system at random; what's the probability that the origin will be, say, an unstable spiral? To be more specific, consider the system $\dot{\mathbf{x}} = A\mathbf{x}$, where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Suppose we pick the entries a, b, c, d independently and at random from a uniform distribution on the interval $[-1,1]$. Find the probabilities of all the different kinds of fixed points.

To check your answers (or if you hit an analytical roadblock), try the *Monte Carlo method.* Generate millions of random matrices on the computer and have the machine count the relative frequency of saddles, unstable spirals, etc.

Are the answers the same if you use a normal distribution instead of a uniform distribution?

5.3 Love Affairs

5.3.1 (Name-calling) Suggest names for the four romantic styles, determined by the signs of a and b in $\dot{R} = aR + bJ$.

5.3.2 Consider the affair described by $\dot{R} = J$, $\dot{J} = -R + J$.

a) Characterize the romantic styles of Romeo and Juliet.

b) Classify the fixed point at the origin. What does this imply for the affair?

c) Sketch $R(t)$ and $J(t)$ as functions of t, assuming $R(0) = 1$, $J(0) = 0$.

In each of the following problems, predict the course of the love affair, depending on the signs and relative sizes of a and b.

5.3.3 (Out of touch with their own feelings) Suppose Romeo and Juliet react to each other, but not to themselves: $\dot{R} = aJ$, $\dot{J} = bR$. What happens?

5.3.4 (Fire and water) Do opposites attract? Analyze $\dot{R} = aR + bJ$, $\dot{J} = -bR - aJ$.

5.3.5 (Peas in a pod) If Romeo and Juliet are romantic clones ($\dot{R} = aR + bJ$, $\dot{J} = bR + aJ$), should they expect boredom or bliss?

5.3.6 (Romeo the robot) Nothing could ever change the way Romeo feels about Juliet: $\dot{R} = 0$, $\dot{J} = aR + bJ$. Does Juliet end up loving him or hating him?