Derivatives and standard errors of standardized parameters in structural equation models

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Abstract

In structural equation models, often not only the parameters of the model, but also the standardized coefficients are of interest to the researcher. The literature does not, however, provide explicit analytical standard errors for standardized parameters. Due to this lack of availability of analytical derivatives in the literature, standard SEM software that wishes to provide SEM users with standard errors of the standardized parameters requires the use of numerical derivatives or approximate methods.

We derive an explicit expression in terms of the model parameters for derivatives and asymptotic standard errors of standardized parameters in structural equation models, that is straightforward to implement in standard SEM software. The expression is applied to an example application.

1 Introduction

Linear structural equation models (SEM) with latent variables have become a popular tool in different branches of science. Such models encompass as special cases a diverse range of common models of interest such as factor analysis, multivariate regression, errors-in-variables models, growth, and multilevel models (Bollen, 1989). In addition, models for categorical, count, and censored dependent variables as well as complex sampling and other extensions are available (Muthen and Satorra, 1995; Muthén, 2002).

Although the main interest in such models lies in the parameters of the model, researchers' interest often also focuses on the so-called 'standardized' parameters (Bollen, 1989). Typical applications include examination of factor loadings and correlations in factor analysis, and the evaluation of the relative size of regression coefficients, possibly of latent variables.

In spite of the interest in standardized coefficients in structural equation models, standard errors and confidence intervals for these coefficients are usually not provided by the standard software[1]. To our knowledge the literature does not provide any explicit expression for the asymptotic standard errors of standardized coefficients. We remedy this situation by deriving an explicit expression for the asymptotic variance-covariance matrix of the standardized coefficients. The solution requires only the parameter estimates of the model and can be readily implemented in SEM software.

Section 2 provides a motivating example, using a non-recursive SEM model with latent variables where standardized parameters are of interest to the researcher. Section 3 derives the explicit expression for the asymptotic variance-covariance matrix of the standardized estimates. In section 4, this expression is then applied to the example. Finally, the last section discusses the scope and limitations of this proposal and suggests future research.

[1] At the time of writing, the exception is Mplus version 5.2 and above (Muthén and Muthén, 1998).

2 Example SEM with standardized parameters

3 Standard errors of standardized parameters

Assume the following model for a vector of observed variables y has been specified:

$$y = \Lambda\eta + \epsilon \tag{1}$$
$$\eta = B_0\eta + \zeta, \tag{2}$$

where η is a vector of unobserved variables, ζ is a vector of disturbance terms and ϵ is a vector of measurement errors. Model 2 implies the following model $\Sigma_\eta(\theta)$ for the variance-covariance matrix of the unobserved variables:

$$\Sigma_\eta(\theta) = B^{-1}\Phi B^{-T}, \tag{3}$$

where $B \triangleq I - B_0$ is positive definite, and Φ is the variance-covariance matrix of ζ. Model 1 can then be seen to imply thefollowing model $\Sigma_y(\theta)$ for the variance-covariance matrix of the observed variables:

$$\Sigma_y(\theta) = \Lambda B^{-1}\Phi B^{-T}\Lambda' + \Psi, \tag{4}$$

where Ψ is the variance-covariance matrix of ϵ. We assume throughout that both Σ_y and Σ_η are positive definite. In what follows we will write $\Sigma_.$ for $\Sigma_.(\theta)$ in the interest of clarity.

The parameters of the model are collected in a parameter vector

$$\theta \triangleq [\operatorname{vec}\Lambda, \operatorname{vech}\Phi, \operatorname{vec}B_0, \operatorname{vech}\Psi].$$

Often, interest focuses not only on the parameters θ, but also on the so-called "standardized" matrices $\tilde{\Lambda}$ and $\tilde{B}_0$. These are defined as:

$$\tilde{\Lambda} \triangleq D_y^{-1}\Lambda D_\eta \tag{5}$$
$$\tilde{B}_0 \triangleq D_\eta^{-1}B_0 D_\eta, \tag{6}$$

where $D_y \triangleq \sqrt{I \circ \Sigma_y}$, and $D_\eta \triangleq \sqrt{I \circ \Sigma_\eta}$. TODO: Define $\tilde{\theta}$.

By standard application of the Delta method (e.g. Oehlert, 1992), the asymptotic variance of $\tilde{\theta}$ is

$$\left(\frac{d\tilde{\theta}}{d\theta}\right) \operatorname{var}(\theta) \left(\frac{d\tilde{\theta}}{d\theta}\right)', \tag{7}$$

where $\operatorname{var}(\theta)$ is the appropriate asymptotic variance matrix of the free model parameters θ (e.g. Satorra, 1989).

3.1 Derivatives of the standardized parameters

To obtain the full expression for the variance-covariance matrix of standardized parameters, the derivatives of the standardized parameters with respect to the free parameters of the model are needed, as shown in equation **??**. We now derive these differentials.

From definition 5,

$$\mathrm{d}\operatorname{vec}\tilde{\Lambda} = (D_\eta\Lambda' \otimes I_p)\,\mathrm{d}\operatorname{vec}D_y^{-1} + (D_\eta \otimes D_y^{-1})\,\mathrm{d}\operatorname{vec}\Lambda + \\ (I_q \otimes D_y^{-1}\Lambda)\,\mathrm{d}\operatorname{vec}D_\eta, \tag{8}$$

and from definition 6,

$$\mathrm{d}\,\mathrm{vec}\,\tilde{B}_0 = (D_\eta B_0' \otimes I_p)\,\mathrm{d}\,\mathrm{vec}\,D_\eta^{-1} + (D_\eta \otimes D_\eta^{-1})\,\mathrm{d}\,\mathrm{vec}\,B_0 + (I_q \otimes D_\eta^{-1} B_0)\,\mathrm{d}\,\mathrm{vec}\,D_\eta. \quad (9)$$

The differentials of the standardized parameter matrices are, thus, functions of the differentials of the covariance structure models Σ_y and Σ_η. Neudecker and Satorra (1991) derived the differential of the implied variance matrix Σ_y of the observed variables. To ensure completeness of the treatment, we repeat it here:

$$\begin{aligned}\mathrm{d}\,\mathrm{vec}\,\Sigma_y = {} & (I+K)(\Lambda B^{-1}\Phi B^{-T} \otimes I)\,\mathrm{d}\,\mathrm{vec}\,\Lambda + \\ & (I+K)(\Lambda B^{-1}\Phi B^{-T} \otimes \Lambda B^{-1})\,\mathrm{d}\,\mathrm{vec}\,B_0 + \\ & (\Lambda B^{-1} \otimes \Lambda B^{-1})P\,\mathrm{d}\,\mathrm{vech}\,\Phi + \\ & \mathrm{d}\,\mathrm{vech}\,\Psi,\end{aligned} \quad (10)$$

where the commutation matrix K, the duplication matrix P, and the operatorsvec and vech are defined in Magnus and Neudecker (1988) Let Δ_λ equal the coefficient matrix of $\mathrm{d}\,\mathrm{vec}\,\Lambda$ in equation 10, Δ_β the coefficient of $\mathrm{d}\,\mathrm{vec}\,B_0$, and Δ_ϕ the coefficient of $\mathrm{d}\,\mathrm{vech}\,\Phi$.

From equation 10 together with equation 3 the differential of the variance matrix of η can be obtained as:

$$\begin{aligned}\mathrm{d}\,\mathrm{vec}\,\Sigma_\eta = {} & (I+K)(B^{-1}\Phi B^{-T} \otimes B^{-1})\,\mathrm{d}\,\mathrm{vec}\,B_0 + \\ & (B^{-1} \otimes B^{-1})P\,\mathrm{d}\,\mathrm{vech}\,\Phi.\end{aligned} \quad (11)$$

Let Δ_β^* and Δ_ϕ^* be the coefficients of $\mathrm{d}\,\mathrm{vec}\,B_0$ and $\mathrm{d}\,\mathrm{vech}\,\Phi$ respectively in equation 11.

Also, let

$$C_{y2} \triangleq -(I_p \otimes \frac{1}{2(I_p \circ \Sigma_y)^{\frac{3}{2}}})\mathrm{diag}[\mathrm{vec}\,(I_p)], \quad (12)$$

$$C_{\eta 1} \triangleq (I_q \otimes \frac{1}{2(I_q \circ \Sigma_\eta)^{\frac{1}{2}}})\mathrm{diag}[\mathrm{vec}\,(I_q)], \quad (13)$$

$$C_{\eta 2} \triangleq -(I_q \otimes \frac{1}{2(I_q \circ \Sigma_\eta)^{\frac{3}{2}}})\mathrm{diag}[\mathrm{vec}\,(I_q)]. \quad (14)$$

Then, applying standard operations on 8 and rearranging terms, the differential of the standardized $\tilde{\Lambda}$ matrix is

$$\begin{aligned}\mathrm{d}\,\mathrm{vec}\,\tilde{\Lambda} = {} & \\ & [(D_\eta \Lambda' \otimes I_p)C_{y2}\Delta_\lambda + (D_\eta \otimes D_y^{-1})]\,\mathrm{d}\,\mathrm{vec}\,\Lambda + \\ & [(D_\eta \Lambda' \otimes I_p)C_{y2}\Delta_\beta + (I_q \otimes D_y^{-1}\Lambda)C_{\eta 1}\Delta_\beta^*]\,\mathrm{d}\,\mathrm{vec}\,B_0 + \\ & [(D_\eta \Lambda' \otimes I_p)C_{y2}\Delta_\phi + (I_q \otimes D_y^{-1}\Lambda)C_{\eta 1}\Delta_\phi^*]\,\mathrm{d}\,\mathrm{vec}\,\Phi + \\ & (D_\eta \Lambda' \otimes I_p)C_{y2}\,\mathrm{d}\,\mathrm{vec}\,\Psi.\end{aligned} \quad (15)$$

And similarly, the differential of the standardized $\tilde{B}_0$ matrix from equation 9 is

$$\begin{aligned}\mathrm{d}\,\mathrm{vec}\,\tilde{B}_0 = {} & \\ & [\{(D_\eta B_0' \otimes I_q)C_{\eta 2} + (I_q \otimes D_\eta^{-1}B_0)C_{\eta 1}\}\Delta_\beta^* + (D_\eta \otimes D_\eta^{-1})]\,\mathrm{d}\,\mathrm{vec}\,B_0 + \\ & [(D_\eta B_0' \otimes I_q)C_{\eta 2} + (I_q \otimes D_\eta^{-1}B_0)C_{\eta 1}]\Delta_\phi^*\,\mathrm{d}\,\mathrm{vec}\,\Phi.\end{aligned} \quad (16)$$

From the differentials in equations 15 and 16, we conclude that the derivative matrices $G_{\tilde{\lambda}}$ and $G_{\tilde{\beta}}$ of the standardized parameters $\operatorname{vec}\tilde{B}$ and $\operatorname{vec}\tilde{\Lambda}$ with respect to the free parameters of the model θ will be the partitioned matrices

TODO: Lose the G? Just write $d\tilde{\theta}/d\theta$?

$$
\begin{aligned}
G_{\tilde{\lambda}} = [(D_\eta \Lambda' \otimes I_p)C_{y2}\Delta_\lambda + (D_\eta \otimes D_y^{-1}),\\
(D_\eta \Lambda' \otimes I_p)C_{y2}\Delta_\phi + (I_q \otimes D_y^{-1}\Lambda)C_{\eta 1}\Delta^*_\phi,\\
(D_\eta \Lambda' \otimes I_p)C_{y2}\Delta_\beta + (I_q \otimes D_y^{-1}\Lambda)C_{\eta 1}\Delta^*_\beta,\\
(D_\eta \Lambda' \otimes I_p)C_{y2}]
\end{aligned}
\quad (17)
$$

and

$$
\begin{aligned}
G_{\tilde{\beta}} = [\mathbf{0}, \{(D_\eta B_0' \otimes I_q)C_{\eta 2} + (I_q \otimes D_\eta^{-1}B_0)C_{\eta 1}\}\Delta^*_\phi,\\
\{(D_\eta B_0' \otimes I_q)C_{\eta 2} + (I_q \otimes D_\eta^{-1}B_0)C_{\eta 1}\}\Delta^*_\beta + (D_\eta \otimes D_\eta^{-1}), \mathbf{0}].
\end{aligned}
\quad (18)
$$

TODO: Add Φ and Ψ; put it all together in one matrix

4 Application: standard errors of standardized parameters

5 Discussion

5.1 Scope

Limited, categorical variables, effects/correlations between latent variables, factor loadings, error correlations.

5.2 Others

Bollen and Stine (1990) note that an approximate method exists for regression analysis that does not take the estimation of the model-implied standard deviations into account.

References to Delta method but without providing the derivatives.

Likelihood based confidence intervals, bootstrapping.

Our method dispenses with the need for approximate methods or numerical derivatives.

5.3 Future

Examine relative performance of confidence intervals, similar to work done for indirect effects.

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