# Derivatives and standard errors of standardized parameters in the LISREL model

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Assume the following model for a vector of observed variables $y$ has been specified:

$$y = \Lambda\eta + \epsilon \tag{1}$$
$$\eta = B_0\eta + \zeta, \tag{2}$$

where $\eta$ is a vector of unobserved variables, $\zeta$ is a vector of disturbance terms and $\epsilon$ is a vector of measurement errors. Model 2 implies the following model $\Sigma_\eta$ for the variance-covariance matrix of the unobserved variables:

$$\Sigma_\eta = B^{-1}\Phi B^{-T}, \tag{3}$$

where $B \triangleq I - B_0$ is positive definite, and $\Phi$ is the variance-covariance matrix of $\zeta$. Model 1 can then be seen to imply the following model $\Sigma_y$ for the variance-covariance matrix of the observed variables:

$$\Sigma_y = \Lambda B^{-1}\Phi B^{-T}\Lambda' + \Psi, \tag{4}$$

where $\Psi$ is the variance-covariance matrix of $\epsilon$. We assume throughout that both $\Sigma_y$ and $\Sigma_\eta$ are positive definite.

Often, interest focuses not only on the parameter matrices $\Lambda$, $B_0$, $\Phi$, and $\Psi$, but also on the so-called "standardized" matrices $\Lambda^{(s)}$ and $B_0^{(s)}$. These are defined as:

$$\Lambda^{(s)} \triangleq D_y^{-1}\Lambda D_\eta \tag{5}$$
$$B_0^{(s)} \triangleq D_\eta^{-1}B_0 D_\eta, \tag{6}$$

where $D_y \triangleq \sqrt{I \circ \Sigma_y}$, and $D_\eta \triangleq \sqrt{I \circ \Sigma_\eta}$. We now derive the differentials of these standardized parameter matrices.

From definition 5,

$$\mathrm{d}\,\mathrm{vec}\,\Lambda^{(s)} = (D_\eta\Lambda' \otimes I_p)\,\mathrm{d}\,\mathrm{vec}\,D_y^{-1} + (D_\eta \otimes D_y^{-1})\,\mathrm{d}\,\mathrm{vec}\,\Lambda + (I_q \otimes D_y^{-1}\Lambda)\,\mathrm{d}\,\mathrm{vec}\,D_\eta. \tag{7}$$

The differentials of the standardized parameter matrices are, thus, functions of the differentials of the covariance structure models $\Sigma_y$ and $\Sigma_\eta$. From Neudecker & Satorra (1990), the differential of the implied variance matrix $\Sigma_y$ of the observed variables is:

$$\begin{aligned} \mathrm{d}\,\mathrm{vec}\,\Sigma_y = (I+K)(\Lambda B^{-1}\Phi B^{-T} \otimes I)\,\mathrm{d}\,\mathrm{vec}\,\Lambda - \\ (I+K)(\Lambda B^{-1}\Phi B^{-T} \otimes \Lambda B^{-1})\,\mathrm{d}\,\mathrm{vec}\,B + \\ (\Lambda B^{-1} \otimes \Lambda B^{-1})E\,\mathrm{d}\,\mathrm{vech}\,\Phi + \\ \mathrm{d}\,\mathrm{vech}\,\Psi, \end{aligned} \tag{8}$$

where the commutation matrix $K$, the elimination matrix $E$, and the operators vec and vech are defined in Magnus and Neudecker (1989). Let $\Delta_\lambda$ equal the coefficient matrix of $\mathrm{d}\,\mathrm{vec}\,\Lambda$ in equation 8, $\Delta_\beta$ the coefficient of $\mathrm{d}\,\mathrm{vec}\,B$, and $\Delta_\phi$ the coefficient of $\mathrm{d}\,\mathrm{vech}\,\Phi$.

From equation 8 together with equation 3 the differential of the variance matrix of $\eta$ can be obtained as:

$$\begin{aligned} \mathrm{d}\,\mathrm{vec}\,\Sigma_\eta = -(I+K)(B^{-1}\Phi B^{-T} \otimes B^{-1})\,\mathrm{d}\,\mathrm{vec}\,B + \\ (B^{-1} \otimes B^{-1})E\,\mathrm{d}\,\mathrm{vech}\,\Phi. \end{aligned} \tag{9}$$

Let $\Delta^*_\beta$ and $\Delta^*_\phi$ be the coefficients of $\mathrm{d}\,\mathrm{vec}\,B$ and $\mathrm{d}\,\mathrm{vech}\,\Phi$ respectively in equation 9.

Let

$$C_{y2} \triangleq (I_p \otimes \frac{1}{2(I_p \circ \Sigma_y)^{\frac{3}{2}}})\mathrm{diag}[\mathrm{vec}\,(I_p)], \tag{10}$$

$$C_{\eta 1} \triangleq (I_q \otimes \frac{1}{2(I_q \circ \Sigma_\eta)^{\frac{1}{2}}})\mathrm{diag}[\mathrm{vec}\,(I_q)], \tag{11}$$

$$C_{\eta 2} \triangleq (I_q \otimes \frac{1}{2(I_q \circ \Sigma_\eta)^{\frac{3}{2}}})\mathrm{diag}[\mathrm{vec}\,(I_q)]. \tag{12}$$