Derivatives and standard errors of standardized parameters in structural equation models

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Abstract

In structural equation models (SEM), often not only the standard errors of parameters of the model, but also of the standardized coefficients are of interest to the researcher. Examples are the comparison of reliability coefficients across different groups, and meta-analysis of standardized quantities. The literature does not, however, provide explicit analytical standard errors for standardized parameters.

We derive an explicit expression in terms of the model parameters for derivatives and the asymptotic variance-covariance matrix of standardized parameters in structural equation models, that is straightforward to implement in standard SEM software. The solution is applied to an analysis of the reliability of self-rated health in groups with different levels of education.

1 Introduction

Linear structural equation models (SEM) with latent variables have become a popular tool in the behavioral sciences. Such models encompass as special cases a diverse range of common models of interest such as factor analysis, multivariate regression, errors-in-variables models, growth, and multilevel and multigroup models (Bollen, 1989). Extensions are available for categorical, count, and censored dependent variables as well as complex sampling (Muthen and Satorra, 1995; Muthén, 2002).

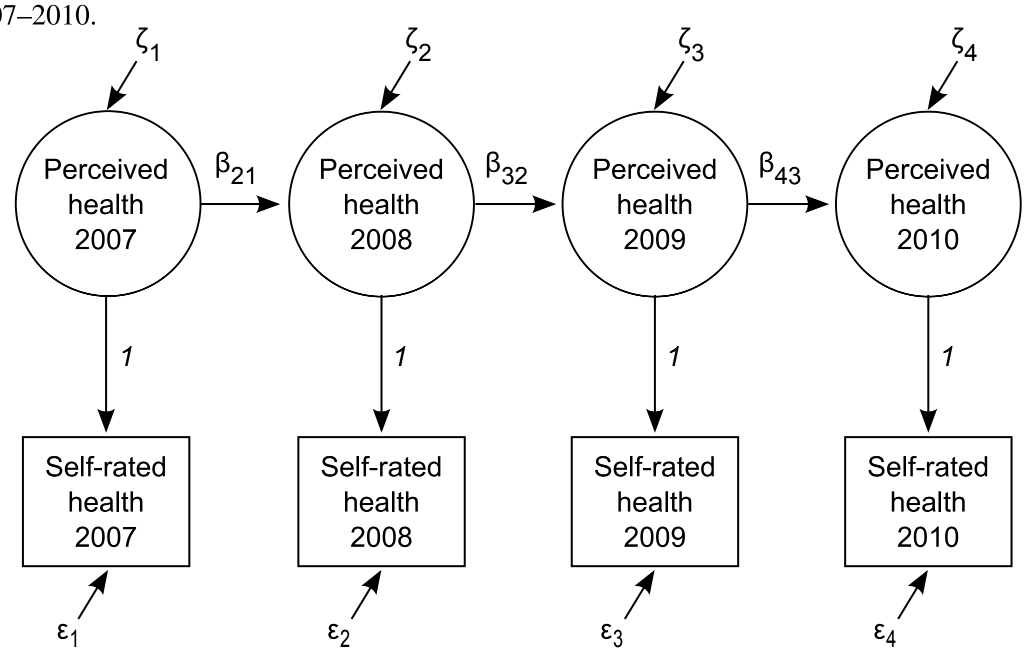
Although the main interest in such models lies in the parameters of the model, researchers' interest often also focuses on the so-called 'standardized' parameters (Bollen, 1989). Typical applications include examination of factor loadings and correlations in factor analysis, as well as the evaluation of the relative size of regression coefficients, possibly of latent variables.

In spite of the interest in standardized coefficients in structural equation models, to our knowledge the literature does not provide any explicit expression for the asymptotic standard errors of standardized coefficients. As a consequence, standard errors and confidence intervals for these coefficients are usually not provided by the standard software[1]. We remedy this situation by deriving an explicit expression, in terms of the unstandardized parameter estimates, for the asymptotic variance-covariance matrix of the standardized coefficients. The solution requires only the parameter estimates of the model and can be readily implemented in SEM software.

Section 2 provides a motivating example, using a SEM model with latent variables where standardized parameters and their standard errors and confidence intervals are of interest to the researcher. Section 3 derives the explicit expression for the asymptotic variance-covariance matrix of the standardized estimates. Section 4 applies this expression to the example. The last section discusses the scope and limitations of this proposal and suggests future research.

[1] At the time of writing, the exception is Mplus version 5.2 and above (Muthén and Muthén, 1998).

Figure 1: Quasi-simplex model for fours repeated measures of self-rated health in the LISS panel 2007–2010.



2 Example SEM with interest in standardized parameters

The study of differences across societal groups in self-rated health is of interest to researchers in public health. For example, von dem Knesebeck et al. (2006) compare people with different incomes, age groups, and level of education. It is well-known, however, that in order to be able to compare correlations across groups it is necessary for the reliability of the measures to be the same (e.g. Saris and Gallhofer, 2007). Therefore not only across-group comparison of the levels of self-rated health are of interest, but also the evaluation of differences between the groups in reliability (Lundberg and Manderbacka, 1996).

Different types of designs exist to estimate the reliability of survey measures. One such design is the repeated measures design, wherein the same question is asked at least three times with a certain interval. One can then apply the so-called 'quasi-simplex' model in different groups to the estimate reliability, and compare reliabilities across groups (Heise and Bohrnstedt, 1970; Wiley and Wiley, 1970). For an overview of other designs we refer to Alwin (2007).

The quasi-simplex model can be formulated and estimated as a multiple-group structural equation model for groups with different levels of highest completed education. This model is shown for four repetitions in figure 1. Following Wiley and Wiley (1970), the unstandardized error variances are restricted to be equal across the four repetitions. The reliability coefficients of interest are then the standardized loadings. This yields the structural equation model shown as a path diagram in figure 1.

We estimate this model using data from the LISS panel study in the Netherlands. The LISS panel is a random probability sample of 8000 Dutch citizens. The respondents answer questionnaires over the web. For more information about the LISS we refer to Scherpenzeel (2011). The panel contains a study that included the commonly used self-rated health question[2]. The question was asked as follows:

[2]The original question was asked in Dutch. See http://www.lissdata.nl/dataarchive/question_constructs/view/600

		Year							
		2007		2009		2009		2010	
	n	Mean	sd	Mean	sd	Mean	sd	Mean	sd
Education level									
Primary	279	2.83	(0.69)	2.91	(0.73)	2.84	(0.74)	2.82	(0.72)
Lower secondary	940	2.98	(0.73)	3.08	(0.73)	3.01	(0.71)	2.97	(0.73)
Middle secondary	782	3.21	(0.80)	3.24	(0.76)	3.27	(0.81)	3.20	(0.75)
Upper secondary	369	3.17	(0.75)	3.18	(0.74)	3.16	(0.72)	3.10	(0.72)
Lower tertiary	799	3.28	(0.73)	3.28	(0.76)	3.22	(0.72)	3.22	(0.74)
Upper tertiary	256	3.33	(0.86)	3.35	(0.86)	3.32	(0.82)	3.29	(0.83)

Table 1: Means and standard deviations (in brackets) of the self-rated health question across the four repetitions, in groups with different levels of education.

	Group: education level					
Par.	Primary	Lower secondary	Middle secondary	Upper secondary	Lower tertiary	Upper tertiary
ϕ_{11}	0.50 (0.049)	0.59 (0.068)	0.35 (0.026)	0.38 (0.030)	0.31 (0.044)	0.40 (0.029)
ϕ_{22}	0.13 (0.027)	0.17 (0.040)	0.10 (0.018)	0.03 (0.019)	0.05 (0.031)	0.12 (0.021)
ϕ_{33}	0.10 (0.026)	0.08 (0.027)	0.05 (0.012)	0.04 (0.013)	0.02 (0.024)	0.08 (0.015)
ϕ_{44}	0.06 (0.027)	0.06 (0.035)	-0.02 (0.017)	0.02 (0.019)	0.02 (0.030)	0.08 (0.020)
β_{21}	0.78 (0.051)	0.85 (0.062)	0.84 (0.045)	0.96 (0.050)	0.99 (0.098)	0.89 (0.043)
β_{32}	0.99 (0.055)	0.87 (0.052)	0.89 (0.039)	0.89 (0.041)	1.01 (0.074)	0.83 (0.034)
β_{43}	0.84 (0.044)	0.95 (0.057)	1.07 (0.041)	0.96 (0.044)	0.92 (0.065)	0.94 (0.039)
ψ_{ii}	0.13 (0.015)	0.14 (0.019)	0.17 (0.010)	0.18 (0.010)	0.17 (0.017)	0.13 (0.010)

Table 2: Unstandardized parameter estimates and standard errors for the multiple group quasi-simplex model shown in figure 1. Chi-square: 6.923 on 12 degrees of freedom ($p = 0.863$), SRMR: 0.0041, RMSEA: 0.000

How would you describe your health, generally speaking?

1. Poor
2. Moderate
3. Good
4. Very good
5. Excellent

This question was asked of 3425 LISS respondents in the years 2007, 2008, 2009, and 2010. Splitting the sample into groups with different levels of education, the means and standard deviations shown in table 1 are obtained. Using the data from all four years, we estimate the quasi-simplex model shown in figure 1 as a SEM to yield four standardized loadings for each educational group, which can be interpreted as the reliability coefficients for each group[3]. The unstandardized parameter estimates and model fit measures are shown in table 2, while standardized loadings (reliability coefficients) are shown in table 3.

Table 3 shows that there appear to be quite some differences across the groups in reliability. For the lowest educational level in 2007, the reliability coefficient of self-rated health is 0.799, while for the highest level it is 0.896. There also appear to be some differences between the years,

[3]For the estimation we used `R` 2.13.0 64-bit (R Development Core Team, 2011) and `lavaan` 0.4.9 (Rosseel, 2011)

	n	Year			
		2007	2008	2009	2010
Education level					
Primary	279	0.799	0.818	0.828	0.816
Lower secondary	940	0.821	0.821	0.812	0.822
Middle secondary	782	0.891	0.879	0.896	0.877
Upper secondary	369	0.822	0.820	0.808	0.805
Lower tertiary	799	0.869	0.878	0.863	0.874
Upper tertiary	256	0.896	0.898	0.886	0.887

Table 3: Reliability (standardized Λ) of self-rated health in the Netherlands 2007-2010 for groups with different levels of education.

although the differences across years are much smaller than those across different educational groups. Finally, the values of all reliability coefficients in table 3 are quite a bit higher than those reported by Lundberg and Manderbacka (1996) for self-rated health. This may be due to differences in the question and/or mode of interviewing, but it may also be due to sampling.

One would require standard errors or confidence intervals to properly study the differences between people with different levels of attained education in reliability. The same is true if one would like to analyze these results together with those of Lundberg and Manderbacka (1996) as a meta-analysis. It is clear, therefore, that to answer the questions that are of interest to the researcher, standard errors and confidence intervals for the reliability coefficients would be useful. The next section derives these standard errors for general structural equation models.

3 Standard errors of standardized parameters

Let y be a p-vector of observed variables, from which a sample is obtained. The following SEM for y is specified:

$$y = \Lambda\eta + \epsilon \tag{1}$$

$$\eta = B_0\eta + \zeta, \tag{2}$$

where η is a vector of unobserved variables, ζ is a vector of disturbance terms and ϵ is a vector of measurement errors. Model 2 implies the following model $\Sigma_\eta(\theta)$ for the variance-covariance matrix of the unobserved variables as a function of a parameter vector θ:

$$\Sigma_\eta(\theta) = B^{-1}\Phi B^{-T}, \tag{3}$$

where $B \triangleq I - B_0$ is positive definite, and Φ is the variance-covariance matrix of ζ. Model 1 can then be seen to imply the following model $\Sigma_y(\theta)$ for the variance-covariance matrix of the observed variables:

$$\Sigma_y(\theta) = \Lambda B^{-1}\Phi B^{-T}\Lambda' + \Psi, \tag{4}$$

where Ψ is the variance-covariance matrix of ϵ. We assume throughout that both Σ_y and Σ_η are positive definite. In what follows we will write $\Sigma_.$ for $\Sigma_.(\theta)$ in the interest of clarity.

The parameters of the model are collected in a parameter vector

$$\theta \triangleq [\text{vec}\,\Lambda, \text{vech}\,\Phi, \text{vec}\,B_0, \text{vech}\,\Psi].$$

Interest focuses not only on the parameter vector θ, but also on the so-called "standardized" parameter vector, denoted $\tilde{\theta} \triangleq [\tilde{\Lambda}, \tilde{B}_0]$, where:

$$\tilde{\Lambda} \triangleq D_y^{-1} \Lambda D_\eta \tag{5}$$

$$\tilde{B}_0 \triangleq D_\eta^{-1} B_0 D_\eta, \tag{6}$$

and $D_y \triangleq \sqrt{I \circ \Sigma_y}$, and $D_\eta \triangleq \sqrt{I \circ \Sigma_\eta}$.

By standard application of the Delta method (e.g. Oehlert, 1992), the asymptotic variance of $\tilde{\theta}$ can be shown to equal

$$\left(\frac{d\tilde{\theta}}{d\theta} \right) \mathrm{var}(\theta) \left(\frac{d\tilde{\theta}}{d\theta} \right)'. \tag{7}$$

Here $\mathrm{var}(\theta)$ is the appropriate asymptotic variance matrix of the free model parameters θ (e.g. Satorra, 1989). The choice of the appropriate $\mathrm{var}(\theta)$ matrix allows for incorporation of non-normal and complex sample data with possibly missing observations.

3.1 Derivatives of the standardized parameters

The derivatives $d\tilde{\theta}/d\theta$ in equation 7 are not available in the literature and are derived here.

From definition 5,

$$\mathrm{d}\,\mathrm{vec}\,\tilde{\Lambda} = (D_\eta \Lambda' \otimes I_p)\,\mathrm{d}\,\mathrm{vec}\,D_y^{-1} + (D_\eta \otimes D_y^{-1})\,\mathrm{d}\,\mathrm{vec}\,\Lambda + \\ (I_q \otimes D_y^{-1}\Lambda)\,\mathrm{d}\,\mathrm{vec}\,D_\eta, \tag{8}$$

and from definition 6,

$$\mathrm{d}\,\mathrm{vec}\,\tilde{B}_0 = (D_\eta B_0' \otimes I_p)\,\mathrm{d}\,\mathrm{vec}\,D_\eta^{-1} + (D_\eta \otimes D_\eta^{-1})\,\mathrm{d}\,\mathrm{vec}\,B_0 + \\ (I_q \otimes D_\eta^{-1} B_0)\,\mathrm{d}\,\mathrm{vec}\,D_\eta. \tag{9}$$

The differentials of the standardized parameter matrices are, thus, functions of the differentials of the covariance structure models Σ_y and Σ_η. Neudecker and Satorra (1991) derived the differential of the implied variance matrix Σ_y of the observed variables. To ensure completeness of the treatment, we repeat it here:

$$\begin{aligned} \mathrm{d}\,\mathrm{vec}\,\Sigma_y = (I + K)(\Lambda B^{-1} \Phi B^{-T} \otimes I)\,\mathrm{d}\,\mathrm{vec}\,\Lambda + \\ (I + K)(\Lambda B^{-1} \Phi B^{-T} \otimes \Lambda B^{-1})\,\mathrm{d}\,\mathrm{vec}\,B_0 + \\ (\Lambda B^{-1} \otimes \Lambda B^{-1}) P\,\mathrm{d}\,\mathrm{vech}\,\Phi + \\ \mathrm{d}\,\mathrm{vech}\,\Psi, \end{aligned} \tag{10}$$

where the commutation matrix K, the duplication matrix P, and the operators vec and vech are defined in Magnus and Neudecker (1988) Let Δ_λ equal the coefficient matrix of $\mathrm{d}\,\mathrm{vec}\,\Lambda$ in equation 10, Δ_β the coefficient of $\mathrm{d}\,\mathrm{vec}\,B_0$, and Δ_ϕ the coefficient of $\mathrm{d}\,\mathrm{vech}\,\Phi$.

The differential of the variance matrix of η can be obtained as:

$$\begin{aligned} \mathrm{d}\,\mathrm{vec}\,\Sigma_\eta = (I + K)(B^{-1} \Phi B^{-T} \otimes B^{-1})\,\mathrm{d}\,\mathrm{vec}\,B_0 + \\ (B^{-1} \otimes B^{-1}) P\,\mathrm{d}\,\mathrm{vech}\,\Phi. \end{aligned} \tag{11}$$

Let Δ_β^* and Δ_ϕ^* be the coefficients of $\mathrm{d}\,\mathrm{vec}\,B_0$ and $\mathrm{d}\,\mathrm{vech}\,\Phi$ respectively in equation 11.

Also, let

$$C_{y2} \triangleq -(I_p \otimes \frac{1}{2(I_p \circ \Sigma_y)^{\frac{3}{2}}})\mathrm{diag}[\mathrm{vec}\,(I_p)], \tag{12}$$

$$C_{\eta 1} \triangleq (I_q \otimes \frac{1}{2(I_q \circ \Sigma_\eta)^{\frac{1}{2}}})\mathrm{diag}[\mathrm{vec}\,(I_q)], \tag{13}$$

$$C_{\eta 2} \triangleq -(I_q \otimes \frac{1}{2(I_q \circ \Sigma_\eta)^{\frac{3}{2}}})\mathrm{diag}[\mathrm{vec}\,(I_q)]. \tag{14}$$

Then, applying standard operations on 8 and rearranging terms, the differential of the standardized $\tilde{\Lambda}$ matrix is

$$\begin{aligned}\mathrm{d}\,\mathrm{vec}\,\tilde{\Lambda} =&[(D_\eta\Lambda' \otimes I_p)C_{y2}\Delta_\lambda + (D_\eta \otimes D_y^{-1})]\,\mathrm{d}\,\mathrm{vec}\,\Lambda+\\ &[(D_\eta\Lambda' \otimes I_p)C_{y2}\Delta_\beta + (I_q \otimes D_y^{-1}\Lambda)C_{\eta 1}\Delta^*_\beta]\,\mathrm{d}\,\mathrm{vec}\,B_0+\\ &[(D_\eta\Lambda' \otimes I_p)C_{y2}\Delta_\phi + (I_q \otimes D_y^{-1}\Lambda)C_{\eta 1}\Delta^*_\phi]\,\mathrm{d}\,\mathrm{vec}\,\Phi+\\ &(D_\eta\Lambda' \otimes I_p)C_{y2}\,\mathrm{d}\,\mathrm{vec}\,\Psi.\end{aligned} \tag{15}$$

And similarly, the differential of the standardized $\tilde{B}_0$ matrix from equation 9 is

$$\begin{aligned}\mathrm{d}\,\mathrm{vec}\,\tilde{B}_0 =&[\{(D_\eta B_0' \otimes I_q)C_{\eta 2} + (I_q \otimes D_\eta^{-1}B_0)C_{\eta 1}\}\Delta^*_\beta + (D_\eta \otimes D_\eta^{-1})]\,\mathrm{d}\,\mathrm{vec}\,B_0+\\ &[(D_\eta B_0' \otimes I_q)C_{\eta 2} + (I_q \otimes D_\eta^{-1}B_0)C_{\eta 1}]\Delta^*_\phi\,\mathrm{d}\,\mathrm{vec}\,\Phi.\end{aligned} \tag{16}$$

From the differentials in equations 15 and 16, we conclude that the derivative matrices $G_{\tilde{\lambda}}$ and $G_{\tilde{\beta}}$ of the standardized parameters $\mathrm{vec}\,\tilde{B}$ and $\mathrm{vec}\,\tilde{\Lambda}$ with respect to the free parameters of the model θ will be the partitioned matrices

$$\begin{aligned}\frac{d\tilde{\lambda}}{d\theta} = [&(D_\eta\Lambda' \otimes I_p)C_{y2}\Delta_\lambda + (D_\eta \otimes D_y^{-1}),\\ &(D_\eta\Lambda' \otimes I_p)C_{y2}\Delta_\phi + (I_q \otimes D_y^{-1}\Lambda)C_{\eta 1}\Delta^*_\phi,\\ &(D_\eta\Lambda' \otimes I_p)C_{y2}\Delta_\beta + (I_q \otimes D_y^{-1}\Lambda)C_{\eta 1}\Delta^*_\beta,\\ &(D_\eta\Lambda' \otimes I_p)C_{y2}]\end{aligned} \tag{17}$$

and

$$\begin{aligned}\frac{d\tilde{\beta}_0}{d\theta} = [&\mathbf{0}, \{(D_\eta B_0' \otimes I_q)C_{\eta 2} + (I_q \otimes D_\eta^{-1}B_0)C_{\eta 1}\}\Delta^*_\phi,\\ &\{(D_\eta B_0' \otimes I_q)C_{\eta 2} + (I_q \otimes D_\eta^{-1}B_0)C_{\eta 1}\}\Delta^*_\beta + (D_\eta \otimes D_\eta^{-1}), \mathbf{0}].\end{aligned} \tag{18}$$

TODO: Add Φ and Ψ; put it all together in one matrix

4 Application: standard errors of standardized parameters

Estimating the

5 Discussion

5.1 Scope

Limited, categorical variables, effects/correlations between latent variables, factor loadings, error correlations.

	n	*Year* 2007	2008	2009	2010
Education level					
Primary	279	0.799 (0.029)	0.818 (0.024)	0.828 (0.023)	0.816 (0.027)
Lower secondary	940	0.821 (0.014)	0.821 (0.013)	0.812 (0.014)	0.822 (0.014)
Middle secondary	782	0.891 (0.016)	0.879 (0.017)	0.896 (0.015)	0.877 (0.018)
Upper secondary	369	0.822 (0.015)	0.820 (0.014)	0.808 (0.015)	0.805 (0.017)
Lower tertiary	799	0.869 (0.013)	0.878 (0.012)	0.863 (0.013)	0.874 (0.013)
Upper tertiary	256	0.896 (0.017)	0.898 (0.017)	0.886 (0.018)	0.887 (0.019)

Table 4: Reliability of self-rated health in the Netherlands 2007-2010 for different educational groups. Asymptotic standard errors based on the method described in this paper are also given. For direct tests of differences between the groups in reliability, see table 5.

5.2 Others

Bollen and Stine (1990) note that an approximate method exists for regression analysis that does not take the estimation of the model-implied standard deviations into account.

References to Delta method but without providing the derivatives.

Likelihood based confidence intervals, bootstrapping.

Our method dispenses with the need for approximate methods or numerical derivatives.

5.3 Future

Examine relative performance of confidence intervals, similar to work done for indirect effects.

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Comparison	t_{dif}	p_{dif}	$p_{\text{dif,adj}}$	
Primary <> vmbo	-0.68	0.25	0.96	
Primary <> havo/vwo	-2.76	0.00	0.03	*
Primary <> mbo	-0.71	0.24	0.96	
Primary <> hbo	-2.18	0.01	0.10	
Primary <> wo	-2.86	0.00	0.02	*
vmbo <> havo/vwo	-3.25	0.00	0.01	*
vmbo <> mbo	-0.06	0.48	0.96	
vmbo <> hbo	-2.46	0.01	0.06	
vmbo <> wo	-3.35	0.00	0.01	*
havo/vwo <> mbo	3.12	0.00	0.01	*
havo/vwo <> hbo	1.07	0.14	0.71	
havo/vwo <> wo	-0.22	0.41	0.96	
mbo <> hbo	-2.33	0.01	0.08	
mbo <> wo	-3.22	0.00	0.01	*
hbo <> wo	-1.26	0.10	0.62	

Table 5: Multiple comparison between educational groups for the reliability in 2007. The groups that are compared on reliability are shown in the first column. The second column shows the t-value for the test of no difference between the groups in reliability. The last two columns show the corresponding p-value and p-value adjusted for multiple comparisons (Holm, 1979), respectively.

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	Unstandardized free parameters θ										
	ψ_{11}	ψ_{22}	ψ_{33}	ψ_{44}	ϕ_{11}	ϕ_{22}	ϕ_{33}	ϕ_{44}	β_{21}	β_{32}	β_{43}
$\lambda_{s,11}$	-0.733	.	.	.	0.277	.	.	.	.	.	.
$\lambda_{s,22}$	.	-0.739	.	.	0.217	0.282	.	.	0.209	.	.
$\lambda_{s,33}$	.	.	-0.755	.	0.188	0.244	0.300	.	0.180	0.225	.
$\lambda_{s,33}$	.	.	.	-0.754	0.173	0.225	0.276	0.298	0.166	0.207	0.230
$\beta_{s,21}$	.	.	.	.	0.232	-1.059	.	.	0.223	.	.
$\beta_{s,32}$	.	.	.	.	0.131	0.171	-1.147	.	0.126	0.157	.
$\beta_{s,43}$	.	.	.	.	0.058	0.076	0.093	-1.193	0.056	0.070	0.077

Table 6: Derivatives of the standardized parameters Λ_s and B_s with respect to the free parameters of the model in the quasi simplex example.

	ψ	ϕ_{11}	ϕ_{22}	ϕ_{33}	ϕ_{44}	β_{21}	β_{32}	β_{43}
ψ	0.2×10^{-4}							
ϕ_{11}	-0.2×10^{-4}	2.2×10^{-4}						
ϕ_{22}	-0.3×10^{-4}	0.4×10^{-4}	0.9×10^{-4}					
ϕ_{33}	-0.2×10^{-4}	0.2×10^{-4}	0.1×10^{-4}	0.5×10^{-4}				
ϕ_{44}	-0.3×10^{-4}	0.3×10^{-4}	0.3×10^{-4}	0.1×10^{-4}	0.8×10^{-4}			
β_{21}	0.4×10^{-4}	-1.5×10^{-4}	-0.9×10^{-4}	-0.1×10^{-4}	-0.5×10^{-4}	4.3×10^{-4}		
β_{32}	0.2×10^{-4}	-0.2×10^{-4}	-0.4×10^{-4}	-0.4×10^{-4}	-0.1×10^{-4}	-0.9×10^{-4}	3.3×10^{-4}	
β_{43}	0.3×10^{-4}	-0.3×10^{-4}	-0.3×10^{-4}	-0.3×10^{-4}	-0.7×10^{-4}	0.4×10^{-4}	-1.0×10^{-4}	3.4×10^{-4}

Table 7: Variance-covariance matrix of the parameter estimates.

A Calculations for the example analysis

B Extension to multigroup SEM

In multigroup SEM, let each matrix in the above equations equal the block diagonal of the corresponding matrices in each group.

Then define the stacking matrix $Z_{p \times pg} \triangleq [I_p, I_p, ..., I_p]$. The derivatives are then

$$(I_q \otimes Z)G$$

Parameter	Estimate	s.e.	z
$\lambda_{s,11}$	0.852	0.006	134
$\lambda_{s,22}$	0.851	0.006	139
$\lambda_{s,33}$	0.846	0.006	136
$\lambda_{s,44}$	0.847	0.007	128
$\beta_{s,21}$	0.882	0.012	75
$\beta_{s,32}$	0.920	0.009	103
$\beta_{s,43}$	0.960	0.011	85

Table 8: Standardized parameter estimates and asymptotic standard errors for the example ($n = 3425$).