

Derivatives and standard errors of standardized parameters in the LISREL model

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Assume the following model for a vector of observed variables y has been specified:

$$y = \Lambda\eta + \epsilon \quad (1)$$

$$\eta = B_0\eta + \zeta, \quad (2)$$

where η is a vector of unobserved variables, ζ is a vector of disturbance terms and ϵ is a vector of measurement errors. Model 2 implies the following model Σ_η for the variance-covariance matrix of the unobserved variables:

$$\Sigma_\eta = B^{-1}\Phi B^{-T}, \quad (3)$$

where $B \triangleq I - B_0$ is positive definite, and Φ is the variance-covariance matrix of ζ . Model 1 can then be seen to imply the following model Σ_y for the variance-covariance matrix of the observed variables:

$$\Sigma_y = \Lambda B^{-1}\Phi B^{-T}\Lambda' + \Psi, \quad (4)$$

where Ψ is the variance-covariance matrix of ϵ . We assume throughout that both Σ_y and Σ_η are positive definite.

Often, interest focuses not only on the parameter matrices Λ , B_0 , Φ , and Ψ , but also on the so-called “standardized” matrices $\Lambda^{(s)}$ and $B_0^{(s)}$. These are defined as:

$$\Lambda^{(s)} \triangleq D_y^{-1}\Lambda D_\eta \quad (5)$$

$$B_0^{(s)} \triangleq D_\eta^{-1}B_0 D_\eta, \quad (6)$$

where $D_y \triangleq \sqrt{I \circ \Sigma_y}$, and $D_\eta \triangleq \sqrt{I \circ \Sigma_\eta}$. We now derive the differentials of these standardized parameter matrices.

From definition 5,

$$\begin{aligned} d \operatorname{vec} \Lambda^{(s)} = & (D_\eta \Lambda' \otimes I_p) d \operatorname{vec} D_y^{-1} + (D_\eta \otimes D_y^{-1}) d \operatorname{vec} \Lambda + \\ & (I_q \otimes D_y^{-1} \Lambda) d \operatorname{vec} D_\eta, \end{aligned} \quad (7)$$

and from definition 6,

$$\begin{aligned} d \text{vec } B_0^{(s)} = & (D_\eta B'_0 \otimes I_p) d \text{vec } D_\eta^{-1} + (D_\eta \otimes D_\eta^{-1}) d \text{vec } B_0 + \\ & (I_q \otimes D_\eta^{-1} B_0) d \text{vec } D_\eta. \end{aligned} \quad (8)$$

The differentials of the standardized parameter matrices are, thus, functions of the differentials of the covariance structure models Σ_y and Σ_η . From Neudecker & Satorra (1990), the differential of the implied variance matrix Σ_y of the observed variables is:

$$\begin{aligned} d \text{vec } \Sigma_y = & (I + K)(\Lambda B^{-1} \Phi B^{-T} \otimes I) d \text{vec } \Lambda + \\ & (I + K)(\Lambda B^{-1} \Phi B^{-T} \otimes \Lambda B^{-1}) d \text{vec } B_0 + \\ & (\Lambda B^{-1} \otimes \Lambda B^{-1}) E d \text{vech } \Phi + \\ & d \text{vech } \Psi, \end{aligned} \quad (9)$$

where the commutation matrix K , the elimination matrix E , and the operators vec and vech are defined in Magnus and Neudecker (1989). Let Δ_λ equal the coefficient matrix of $d \text{vec } \Lambda$ in equation 9, Δ_β the coefficient of $d \text{vec } B_0$, and Δ_ϕ the coefficient of $d \text{vech } \Phi$.

From equation 9 together with equation 3 the differential of the variance matrix of η can be obtained as:

$$\begin{aligned} d \text{vec } \Sigma_\eta = & (I + K)(B^{-1} \Phi B^{-T} \otimes B^{-1}) d \text{vec } B_0 + \\ & (B^{-1} \otimes B^{-1}) E d \text{vech } \Phi. \end{aligned} \quad (10)$$

Let Δ_β^* and Δ_ϕ^* be the coefficients of $d \text{vec } B_0$ and $d \text{vech } \Phi$ respectively in equation 10.

Also, let

$$C_{y2} \triangleq (I_p \otimes \frac{1}{2(I_p \circ \Sigma_y)^{\frac{3}{2}}}) \text{diag}[\text{vec}(I_p)], \quad (11)$$

$$C_{\eta 1} \triangleq (I_q \otimes \frac{1}{2(I_q \circ \Sigma_\eta)^{\frac{1}{2}}}) \text{diag}[\text{vec}(I_q)], \quad (12)$$

$$C_{\eta 2} \triangleq (I_q \otimes \frac{1}{2(I_q \circ \Sigma_\eta)^{\frac{3}{2}}}) \text{diag}[\text{vec}(I_q)]. \quad (13)$$

Then, applying standard operations on 7 and rearranging terms, the differential of the standardized $\Lambda^{(s)}$ matrix is

$$\begin{aligned} d \text{vec } \Lambda^{(s)} = & [(D_\eta \Lambda' \otimes I_p) C_{y2} \Delta_\lambda + (D_\eta \otimes D_\eta^{-1})] d \text{vec } \Lambda + \\ & [(D_\eta \Lambda' \otimes I_p) C_{y2} \Delta_\beta + (I_q \otimes D_\eta^{-1} \Lambda) C_{\eta 1} \Delta_\beta^*] d \text{vec } B_0 + \\ & [(D_\eta \Lambda' \otimes I_p) C_{y2} \Delta_\phi + (I_q \otimes D_\eta^{-1} \Lambda) C_{\eta 1} \Delta_\phi^*] d \text{vec } \Phi + \\ & (D_\eta \Lambda' \otimes I_p) C_{y2} d \text{vec } \Psi. \end{aligned} \quad (14)$$

And similarly, the differential of the standardized $B_0^{(s)}$ matrix from equation 8 is

$$\begin{aligned} \text{d vec } B_0^{(s)} = & [\{(D_\eta B'_0 \otimes I_p)C_{\eta 2} + (I_q \otimes D_\eta^{-1} B_0)C_{\eta 1}\}\Delta_\beta^* + (D_\eta \otimes D_\eta^{-1})] \text{d vec } B_0 + \\ & [(D_\eta B'_0 \otimes I_p)C_{\eta 2} + (I_q \otimes D_\eta^{-1} B_0)C_{\eta 1}]\Delta_\phi^* \text{d vec } \Phi. \end{aligned} \tag{15}$$