Derivatives and standard errors of standardized parameters in the LISREL model

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Assume the following model for a vector of observed variables y has been specified:

$$y = \Lambda\eta + \epsilon \tag{1}$$
$$\eta = B_0\eta + \zeta, \tag{2}$$

where η is a vector of unobserved variables, ζ is a vector of disturbance terms and ϵ is a vector of measurement errors. Model 2 implies the following model $\Sigma_\eta(\theta)$ for the variance-covariance matrix of the unobserved variables:

$$\Sigma_\eta(\theta) = B^{-1}\Phi B^{-T}, \tag{3}$$

where $B \triangleq I - B_0$ is positive definite, and Φ is the variance-covariance matrix of ζ. Model 1 can then be seen to imply the following model $\Sigma_y(\theta)$ for the variance-covariance matrix of the observed variables:

$$\Sigma_y(\theta) = \Lambda B^{-1}\Phi B^{-T}\Lambda' + \Psi, \tag{4}$$

where Ψ is the variance-covariance matrix of ϵ. We assume throughout that both Σ_y and Σ_η are positive definite. In what follows we will write $\Sigma_.$ for $\Sigma_.(\theta)$ in the interest of clarity.

The parameters of the model are collected in a parameter vector

$$\theta \triangleq [\text{vec}\,\Lambda, \text{vech}\,\Phi, \text{vec}\,B_0, \text{vech}\,\Psi].$$

Often, interest focuses not only on the parameters θ, but also on the so-called "standardized" matrices $\tilde{\Lambda}$ and $\tilde{B}_0$. These are defined as:

$$\tilde{\Lambda} \triangleq D_y^{-1}\Lambda D_\eta \tag{5}$$
$$\tilde{B}_0 \triangleq D_\eta^{-1}B_0 D_\eta, \tag{6}$$

where $D_y \triangleq \sqrt{I \circ \Sigma_y}$, and $D_\eta \triangleq \sqrt{I \circ \Sigma_\eta}$. We now derive the differentials of these standardized parameter matrices.

From definition 5,

$$\mathrm{d}\,\mathrm{vec}\,\tilde{\Lambda} = (D_\eta \Lambda' \otimes I_p)\,\mathrm{d}\,\mathrm{vec}\,D_y^{-1} + (D_\eta \otimes D_y^{-1})\,\mathrm{d}\,\mathrm{vec}\,\Lambda + \\ (I_q \otimes D_y^{-1}\Lambda)\,\mathrm{d}\,\mathrm{vec}\,D_\eta, \quad (7)$$

and from definition 6,

$$\mathrm{d}\,\mathrm{vec}\,\tilde{B}_0 = (D_\eta B_0' \otimes I_p)\,\mathrm{d}\,\mathrm{vec}\,D_\eta^{-1} + (D_\eta \otimes D_\eta^{-1})\,\mathrm{d}\,\mathrm{vec}\,B_0 + \\ (I_q \otimes D_\eta^{-1} B_0)\,\mathrm{d}\,\mathrm{vec}\,D_\eta. \quad (8)$$

The differentials of the standardized parameter matrices are, thus, functions of the differentials of the covariance structure models Σ_y and Σ_η. From Neudecker & Satorra (1990), the differential of the implied variance matrix Σ_y of the observed variables is:

$$\begin{aligned} \mathrm{d}\,\mathrm{vec}\,\Sigma_y = (I + K)(\Lambda B^{-1}\Phi B^{-T} \otimes I)\,\mathrm{d}\,\mathrm{vec}\,\Lambda + \\ (I + K)(\Lambda B^{-1}\Phi B^{-T} \otimes \Lambda B^{-1})\,\mathrm{d}\,\mathrm{vec}\,B_0 + \\ (\Lambda B^{-1} \otimes \Lambda B^{-1})P\,\mathrm{d}\,\mathrm{vech}\,\Phi + \\ \mathrm{d}\,\mathrm{vech}\,\Psi, \end{aligned} \quad (9)$$

where the commutation matrix K, the duplication matrix P, and the operators vec and vech are defined in Magnus and Neudecker (1989). Let Δ_λ equal the coefficient matrix of $\mathrm{d}\,\mathrm{vec}\,\Lambda$ in equation 9, Δ_β the coefficient of $\mathrm{d}\,\mathrm{vec}\,B_0$, and Δ_ϕ the coefficient of $\mathrm{d}\,\mathrm{vech}\,\Phi$.

From equation 9 together with equation 3 the differential of the variance matrix of η can be obtained as:

$$\begin{aligned} \mathrm{d}\,\mathrm{vec}\,\Sigma_\eta = (I + K)(B^{-1}\Phi B^{-T} \otimes B^{-1})\,\mathrm{d}\,\mathrm{vec}\,B_0 + \\ (B^{-1} \otimes B^{-1})P\,\mathrm{d}\,\mathrm{vech}\,\Phi. \end{aligned} \quad (10)$$

Let Δ^*_β and Δ^*_ϕ be the coefficients of $\mathrm{d}\,\mathrm{vec}\,B_0$ and $\mathrm{d}\,\mathrm{vech}\,\Phi$ respectively in equation 10.

Also, let

$$C_{y2} \triangleq -(I_p \otimes \frac{1}{2(I_p \circ \Sigma_y)^{\frac{3}{2}}})\mathrm{diag}[\mathrm{vec}\,(I_p)], \quad (11)$$

$$C_{\eta 1} \triangleq (I_q \otimes \frac{1}{2(I_q \circ \Sigma_\eta)^{\frac{1}{2}}})\mathrm{diag}[\mathrm{vec}\,(I_q)], \quad (12)$$

$$C_{\eta 2} \triangleq -(I_q \otimes \frac{1}{2(I_q \circ \Sigma_\eta)^{\frac{3}{2}}})\mathrm{diag}[\mathrm{vec}\,(I_q)]. \quad (13)$$

Then, applying standard operations on 7 and rearranging terms, the differential of the standardized $\tilde{\Lambda}$ matrix is

$$
\begin{aligned}
&\mathrm{d}\,\mathrm{vec}\,\tilde{\Lambda} = \\
&\qquad [(D_\eta \Lambda' \otimes I_p) C_{y2} \Delta_\lambda + (D_\eta \otimes D_y^{-1})]\,\mathrm{d}\,\mathrm{vec}\,\Lambda + \\
&\qquad [(D_\eta \Lambda' \otimes I_p) C_{y2} \Delta_\beta + (I_q \otimes D_y^{-1} \Lambda) C_{\eta 1} \Delta_\beta^*]\,\mathrm{d}\,\mathrm{vec}\,B_0 + \\
&\qquad [(D_\eta \Lambda' \otimes I_p) C_{y2} \Delta_\phi + (I_q \otimes D_y^{-1} \Lambda) C_{\eta 1} \Delta_\phi^*]\,\mathrm{d}\,\mathrm{vec}\,\Phi + \\
&\qquad (D_\eta \Lambda' \otimes I_p) C_{y2}\,\mathrm{d}\,\mathrm{vec}\,\Psi.
\end{aligned}
\tag{14}
$$

And similarly, the differential of the standardized $\tilde{B}_0$ matrix from equation 8 is

$$
\begin{aligned}
&\mathrm{d}\,\mathrm{vec}\,\tilde{B}_0 = \\
&\qquad [\{(D_\eta B_0' \otimes I_q) C_{\eta 2} + (I_q \otimes D_\eta^{-1} B_0) C_{\eta 1}\} \Delta_\beta^* + (D_\eta \otimes D_\eta^{-1})]\,\mathrm{d}\,\mathrm{vec}\,B_0 + \\
&\qquad [(D_\eta B_0' \otimes I_q) C_{\eta 2} + (I_q \otimes D_\eta^{-1} B_0) C_{\eta 1}] \Delta_\phi^*\,\mathrm{d}\,\mathrm{vec}\,\Phi.
\end{aligned}
\tag{15}
$$

From the differentials in equations 14 and 15, we conclude that the derivative matrices $G_{\tilde{\lambda}}$ and $G_{\tilde{\beta}}$ of the standardized parameters vec $\tilde{B}$ and vec $\tilde{\Lambda}$ with respect to the free parameters of the model θ will be the partitioned matrices

$$
\begin{aligned}
G_{\tilde{\lambda}} = [&(D_\eta \Lambda' \otimes I_p) C_{y2} \Delta_\lambda + (D_\eta \otimes D_y^{-1}), \\
&(D_\eta \Lambda' \otimes I_p) C_{y2} \Delta_\phi + (I_q \otimes D_y^{-1} \Lambda) C_{\eta 1} \Delta_\phi^*, \\
&(D_\eta \Lambda' \otimes I_p) C_{y2} \Delta_\beta + (I_q \otimes D_y^{-1} \Lambda) C_{\eta 1} \Delta_\beta^*, \\
&(D_\eta \Lambda' \otimes I_p) C_{y2}]
\end{aligned}
\tag{16}
$$

and

$$
\begin{aligned}
G_{\tilde{\beta}} = [&\mathbf{0}, \{(D_\eta B_0' \otimes I_q) C_{\eta 2} + (I_q \otimes D_\eta^{-1} B_0) C_{\eta 1}\} \Delta_\phi^*, \\
&\{(D_\eta B_0' \otimes I_q) C_{\eta 2} + (I_q \otimes D_\eta^{-1} B_0) C_{\eta 1}\} \Delta_\beta^* + (D_\eta \otimes D_\eta^{-1}), \mathbf{0}].
\end{aligned}
\tag{17}
$$