Derivatives and standard errors of standardized parameters in the LISREL model

Daniel Oberski
RECSM research paper, Universitat Pompeu Fabra

May 1, 2011

Assume the following model for a vector of observed variables y has been specified:

$$y = \Lambda\eta + \epsilon \tag{1}$$
$$\eta = B_0\eta + \zeta, \tag{2}$$

where η is a vector of unobserved variables, ζ is a vector of disturbance terms and ϵ is a vector of measurement errors. Model 2 implies the following model Σ_η for the variance-covariance matrix of the unobserved variables:

$$\Sigma_\eta = B^{-1}\Phi B^{-T}, \tag{3}$$

where $B \triangleq I - B_0$ is positive definite, and Φ is the variance-covariance matrix of ζ. Model 1 can then be seen to imply the following model Σ_y for the variance-covariance matrix of the observed variables:

$$\Sigma_y = \Lambda B^{-1}\Phi B^{-T}\Lambda' + \Psi, \tag{4}$$

where Ψ is the variance-covariance matrix of ϵ. We assume throughout that both Σ_y and Σ_η are positive definite.

Often, interest focuses not only on the parameter matrices Λ, B_0, Φ, and Ψ, but also on the so-called "standardized" matrices $\tilde{\Lambda}$ and $\tilde{B}_0$. These are defined as:

$$\tilde{\Lambda} \triangleq D_y^{-1}\Lambda D_\eta \tag{5}$$
$$\tilde{B}_0 \triangleq D_\eta^{-1} B_0 D_\eta, \tag{6}$$

where $D_y \triangleq \sqrt{I \circ \Sigma_y}$, and $D_\eta \triangleq \sqrt{I \circ \Sigma_\eta}$. We now derive the differentials of these standardized parameter matrices.

From definition 5,

$$\mathrm{d}\,\mathrm{vec}\,\tilde{\Lambda} = (D_\eta\Lambda' \otimes I_p)\,\mathrm{d}\,\mathrm{vec}\,D_y^{-1} + (D_\eta \otimes D_y^{-1})\,\mathrm{d}\,\mathrm{vec}\,\Lambda + (I_q \otimes D_y^{-1}\Lambda)\,\mathrm{d}\,\mathrm{vec}\,D_\eta, \tag{7}$$

and from definition 6,

$$\mathrm{d}\,\mathrm{vec}\,\tilde{B}_0 = (D_\eta B_0' \otimes I_p)\,\mathrm{d}\,\mathrm{vec}\,D_\eta^{-1} + (D_\eta \otimes D_\eta^{-1})\,\mathrm{d}\,\mathrm{vec}\,B_0 + \\ (I_q \otimes D_\eta^{-1} B_0)\,\mathrm{d}\,\mathrm{vec}\,D_\eta. \quad (8)$$

The differentials of the standardized parameter matrices are, thus, functions of the differentials of the covariance structure models Σ_y and Σ_η. From Neudecker & Satorra (1990), the differential of the implied variance matrix Σ_y of the observed variables is:

$$\begin{aligned} \mathrm{d}\,\mathrm{vec}\,\Sigma_y = (I+K)(\Lambda B^{-1}\Phi B^{-T} \otimes I)\,\mathrm{d}\,\mathrm{vec}\,\Lambda + \\ (I+K)(\Lambda B^{-1}\Phi B^{-T} \otimes \Lambda B^{-1})\,\mathrm{d}\,\mathrm{vec}\,B_0 + \\ (\Lambda B^{-1} \otimes \Lambda B^{-1})E\,\mathrm{d}\,\mathrm{vech}\,\Phi + \\ \mathrm{d}\,\mathrm{vech}\,\Psi, \end{aligned} \quad (9)$$

where the commutation matrix K, the elimination matrix E, and the operators vec and vech are defined in Magnus and Neudecker (1989). Let Δ_λ equal the coefficient matrix of $\mathrm{d}\,\mathrm{vec}\,\Lambda$ in equation 9, Δ_β the coefficient of $\mathrm{d}\,\mathrm{vec}\,B_0$, and Δ_ϕ the coefficient of $\mathrm{d}\,\mathrm{vech}\,\Phi$.

From equation 9 together with equation 3 the differential of the variance matrix of η can be obtained as:

$$\begin{aligned} \mathrm{d}\,\mathrm{vec}\,\Sigma_\eta = (I+K)(B^{-1}\Phi B^{-T} \otimes B^{-1})\,\mathrm{d}\,\mathrm{vec}\,B_0 + \\ (B^{-1} \otimes B^{-1})E\,\mathrm{d}\,\mathrm{vech}\,\Phi. \end{aligned} \quad (10)$$

Let Δ_β^* and Δ_ϕ^* be the coefficients of $\mathrm{d}\,\mathrm{vec}\,B_0$ and $\mathrm{d}\,\mathrm{vech}\,\Phi$ respectively in equation 10.

Also, let

$$C_{y2} \triangleq (I_p \otimes \frac{1}{2(I_p \circ \Sigma_y)^{\frac{3}{2}}})\mathrm{diag}[\mathrm{vec}\,(I_p)], \quad (11)$$

$$C_{\eta 1} \triangleq (I_q \otimes \frac{1}{2(I_q \circ \Sigma_\eta)^{\frac{1}{2}}})\mathrm{diag}[\mathrm{vec}\,(I_q)], \quad (12)$$

$$C_{\eta 2} \triangleq (I_q \otimes \frac{1}{2(I_q \circ \Sigma_\eta)^{\frac{3}{2}}})\mathrm{diag}[\mathrm{vec}\,(I_q)]. \quad (13)$$

Then, applying standard operations on 7 and rearranging terms, the differential of the standardized $\tilde{\Lambda}$ matrix is

$$\begin{aligned} \mathrm{d}\,\mathrm{vec}\,\tilde{\Lambda} = \\ & [(D_\eta \Lambda' \otimes I_p)C_{y2}\Delta_\lambda + (D_\eta \otimes D_y^{-1})]\,\mathrm{d}\,\mathrm{vec}\,\Lambda + \\ & [(D_\eta \Lambda' \otimes I_p)C_{y2}\Delta_\beta + (I_q \otimes D_y^{-1}\Lambda)C_{\eta 1}\Delta_\beta^*]\,\mathrm{d}\,\mathrm{vec}\,B_0 + \\ & [(D_\eta \Lambda' \otimes I_p)C_{y2}\Delta_\phi + (I_q \otimes D_y^{-1}\Lambda)C_{\eta 1}\Delta_\phi^*]\,\mathrm{d}\,\mathrm{vec}\,\Phi + \\ & [(D_\eta \Lambda' \otimes I_p)C_{y2}\,\mathrm{d}\,\mathrm{vec}\,\Psi. \end{aligned} \quad (14)$$

And similarly, the differential of the standardized $\tilde{B}_0$ matrix from equation 8 is

$$\begin{aligned}
\mathrm{d}\,\mathrm{vec}\,\tilde{B}_0 = \\
&[\{(D_\eta B_0' \otimes I_p)C_{\eta 2} + (I_q \otimes D_\eta^{-1} B_0)C_{\eta 1}\}\Delta_\beta^* + (D_\eta \otimes D_\eta^{-1})]\,\mathrm{d}\,\mathrm{vec}\,B_0 + \\
&[(D_\eta B_0' \otimes I_p)C_{\eta 2} + (I_q \otimes D_\eta^{-1} B_0)C_{\eta 1}]\Delta_\phi^*\,\mathrm{d}\,\mathrm{vec}\,\Phi.
\end{aligned} \tag{15}$$