Investigations into New Lattice Orderings

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Context-theoretic semantics (Clarke, 2007) defines a lattice ordering on the vector space formed from contexts of words. The lattice meet and join are defined by taking component-wise minimums and maximums with respect to the basis defined by contexts: there is a basis vector (dimension) for each context a word can occur in.

However, this ordering is only one that is possible on the vector space. Here we consider two others and evaluate their plausibility for defining entailment between vector representations of meaning.

1 Motivation

In (Clarke, 2012), a context theory is defined as follows:

Definition 1 (Context Theory) *A context theory is a tuple $\langle A, \mathcal{A}, \xi, V, \psi \rangle$, where A is a set (the alphabet), $\mathcal{A}$ is a unital algebra over the real numbers, ξ is a function from A to $\mathcal{A}$, V is an abstract Lebesgue space and ψ is an injective linear map from $\mathcal{A}$ to V.*

The definition is quite complex because the space V in which the ordering is defined is a bigger space than the space $\mathcal{A}$ used to define composition. If we were able to define an ordering on $\mathcal{A}$ instead, then this would significantly simplify the definition, to $\langle A, \mathcal{A}, \xi \rangle$.

The second motivation for investigating new partial orderings is that we can potentially learn them from data, instead of using a fixed ordering defined by a basis. A partial ordering divides the space in a similar way to the planes learnt by support vector machines, so it is feasible that techniques from this area could be generalised to learn cones.

2 Background

A **partially ordered vector space** is a vector space V together with a partial ordering $\leq$ that satisfies:

- If $u \leq v$ then $u + w \leq v + w$

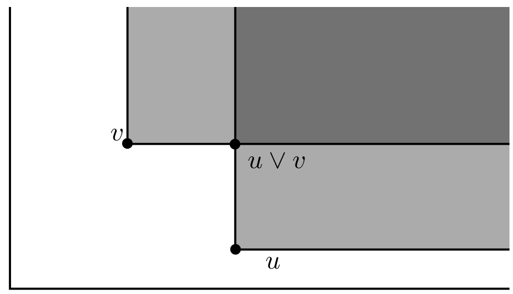

Figure 1: The basis associated with a vector space can defines an ordering. This shows the join of two elements in the plane with the ordering defined by the basis formed from the x and y axes. The shaded areas are $u + C$ and $v + C$ where C is the positive cone defined by this ordering.

- If $u \leq v$ then $\alpha u \leq \alpha v$

for all $u, v, w \in V$ and $\alpha \geq 0$. If the partial ordering is a lattice, then the vector space is called a vector lattice or **Riesz space**.

A **cone** is a subset C of a vector space satisfying

- $C + C \subseteq C$
- $\alpha C \subseteq C$ for all $\alpha \geq 0$
- $C \cap (-C) = \{0\}$

If V is a partially ordered vector space, then the set $V^+ = \{v \in V : v \geq 0\}$ is a cone, called the **positive cone**. Conversely, given any cone C for a vector space V, we can define a partial ordering on V by $u \leq v$ iff $v - u \in C$.

Given a countable set $U = \{u_1, u_2, \ldots\}$ of vectors, the cone generated by U is the set $C_U = \{v : v = \sum_i \alpha_i u_i\}$ for some $\alpha_i \geq 0$.

3 Context Vector Cones

We can define a positive cone on $\mathcal{A}$ to be the cone generated by all context vectors, $\hat{A} = \{\hat{x} : x \in A^*\}$. There are two questions we can ask:

- Is the ordering interesting, or is likely to turn out to be trivial (for example, all strings have meet 0)?
- Does the ordering define a lattice? We need the meet operation in order to define a degree of entailment between strings.

In fact it seems that we can't have our cake and eat it, as if the first question is true, it seems likely the second is false, and vice versa. If the set $\hat{A}$ is linearly independent, then it seems that the first is true, and the second false, while the

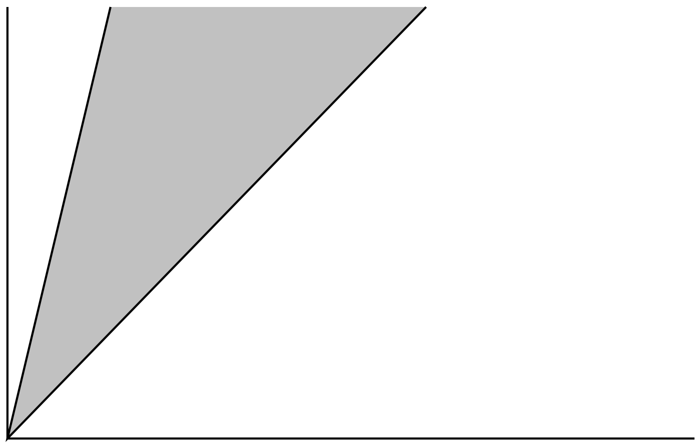

Figure 2: A cone C defining a lattice ordering on the two dimensional vector space in the plane.

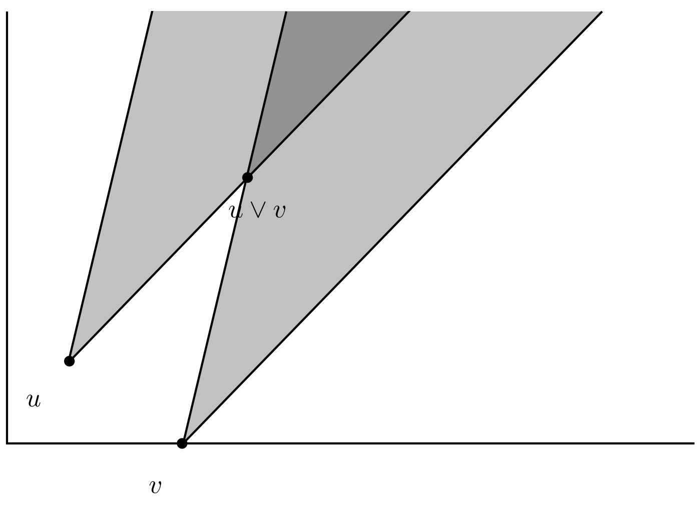

Figure 3: The join of two elements u and v and the spaces $u + C$ and $v + C$ where C is the cone defining the ordering of the space.

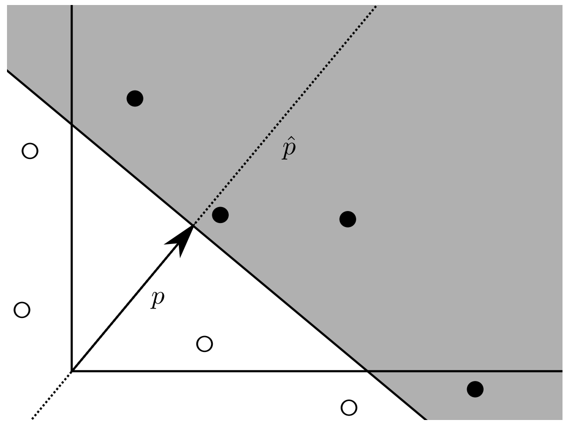

Figure 4: A plane found using a support vector machine can be considered as dividing the vector space, and thus defines a partial ordering.

converse holds in general otherwise. (At the moment, I only have an intuition for this — I've yet to prove it).

4 Plane Orderings

There is an interesting similarity between how a cone divides the vector space and the division of a space introduced by a plane, as used for example in support vector machines. In fact the plane also defines an ordering, if we define one side to be "positive". Then, as before, $x \leq y$ if $y - x$ is on the positive side of the plane.

The ordering is determined entirely by the dimension perpendicular to the plane. Let p be the vector from the origin to the plane, and perpendicular to it, and $\bar{p}$ be the one dimensional subspace generated by p. Then $x \leq y$ iff $\|\hat{p}(y) - \hat{p}(x)\| \geq \|p\|$, where by $\hat{p}(x)$ we mean the projection of x onto the dimension $\hat{p}$, that is, if the projection of $y - x$ onto the subspace generated by p is greater than or equal to p.

In general, this ordering is not a vector space ordering as defined above unless it passes through the origin. The zero vector is either positive or negative depending on which side of the plane it falls. It also does not define a lattice ordering. For example, when the plane does pass through the origin, the plane defines a total ordering between vectors purely depending on their position when projected onto $\hat{p}$. Any vectors on the plane perpendicular to a point on $\hat{p}$ are equivalent under this partial ordering (thus it is not strictly a partial ordering).

5 Relating Strings of Different Lengths with Orderings

We can use vector space orderings to relate strings of different lengths. This is similar to the quotient algebra construction of Clarke et al. (2010), but allows us to specify the asymmetric relation of entailment, instead of equating vectors. In fact the two techniques complement one another: if we need to impose equality between vectors we can use the quotient algebra construction, and if we need to relate two vectors by entailment then we can change the ordering.

For example, if we wish to impose the conjunctive nature of "and" on the vector space, we can do this by requiring that $uav \leq u$ and $uav \leq v$ for all u and v where a is the vector for "and". We can do the same for "or" with $u \leq uov$ and $v \leq uov$ where o is the vector for "or".

Similarly, we can make all adjectives restrict the properties of the nouns they operate on by making $au \leq u$ for all adjective vectors a and all noun space vectors u.

This also gives us a surprising answer for what the sentence space should look like: it doesn't really matter, as long as the ordering between sentences can be specified correctly. For example, we can just use the tensor product between sentences but ensure that $s_1 s_2 \leq s_1$ and $s_1 s_2 \leq s_2$.

In a sense, this is like imposing a logic on top of the vector space structure, where the rules of the logic are encapsulated in the positive cone.

References

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