UMN, Spring 2017, Math 5707: Lecture 8 (Vandermonde determinant using tournaments)

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Contents

1. Tournaments and the Vandermonde determinant

The goal of this lecture is to demonstrate a curious application of digraphs: a combinatorial proof of the Vandermonde determinant identity. This proof – which was found by Gessel in 1979 – is neither the simplest nor the shortest proof (several others can be found in [Grinbe15, §6.7] and in most textbooks on linear algebra), but it illustrates several techniques in enumerative combinatorics and in the application of combinatorics to other fields.

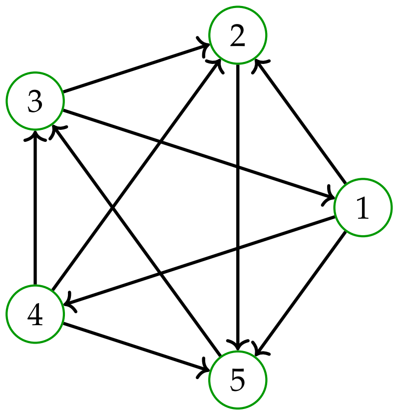
1.1. 3-cycles in tournaments

We shall use the notations introduced in [lec7, §1.1 and §1.4]. In particular, we use the word "*digraph*" as shorthand for "simple digraph". When i and j are two vertices of a digraph, we sometimes use the notation ij for the pair (i, j). We recall that a *tournament* is a loopless simple digraph D that satisfies the *tournament axiom*: For any two distinct vertices u and v of D, **exactly** one of the two pairs $(u, v) = uv$ and $(v, u) = vu$ is an arc of D.

We shall focus on counting certain triples of vertices, which we will refer to as "3-cycles":

Definition 1.1.1. Let $D = (V, A)$ be a simple digraph. A 3-*cycle* of D shall mean a triple (u, v, w) of distinct vertices $u, v, w \in V$ satisfying $uv, vw, wu \in A$.

Example 1.1.2. Consider the following digraph:



This digraph has nine 3-cycles:

$$\begin{array}{lllll} (1,4,3), & (1,5,3), & (2,5,3), & (3,1,4), & (3,1,5), \\ (3,2,5), & (4,3,1), & (5,3,1), & (5,3,2). & \end{array}$$

(We do not count the two 3-cycles $(1, 4, 3)$ and $(4, 3, 1)$ as identical, even though they are just cyclic rotations of one another.) On the other hand, the triple $(1, 2, 3)$ is not a 3-cycle (since 23 is not an arc). The triple $(1, 3, 4)$ is not a 3-cycle either (since none of 13, 34 and 41 is an arc).

We note that our notion of 3-cycles is essentially equivalent to the notion of cycles[1] of length 3. Indeed, if (u, v, w) is a 3-cycle of a simple digraph D, then (u, v, w, u) is a cycle of length 3. Conversely, if (u, v, w, u) is a cycle of length 3, then (u, v, w) is a 3-cycle.

Definition 1.1.3. Let $D = (V, A)$ be a digraph, and let (u, v) be an arc of D. To *reverse* this arc (u, v) means to replace this arc (u, v) by (v, u) in the arc set of D. The result of this operation is a new digraph $(V, \ (A \setminus \{uv\}) \cup \{vu\})$.

We note that if D is a tournament, then the new digraph $(V, \ (A \setminus \{uv\}) \cup \{vu\})$ obtained by reversing the arc (u, v) will again be a tournament.

[1]See [lec7, Definition 1.5.1 **(b)**] for the definition of our notion of cycles.

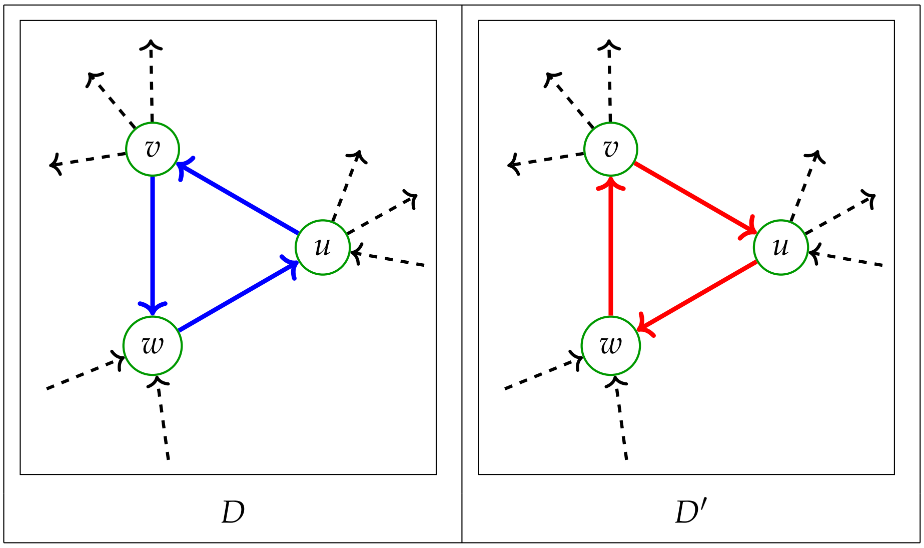
Convention 1.1.4. In the following, the symbol "#" stands for the word "number" (as in "the number of"). For example,

$$(\# \text{ of subsets of } \{1,2,3\}) = 2^3 = 8.$$

Proposition 1.1.5. Let D be a tournament. Let (u, v, w) be a 3-cycle of D. Let D' be the tournament obtained from D by reversing the arcs uv, vw and wu (this means replacing them by vu, wv and uw). Then,

$$(\# \text{ of 3-cycles of } D') = (\# \text{ of 3-cycles of } D).$$

Here is an illustration for Proposition 1.1.5 showing D on the left and D' on the right (the arcs uv, vw and wu of D are painted blue; the arcs vu, wv and uw of D' are painted red; all other arcs are the same in D and in D'):



We shall give two proofs of Proposition 1.1.5. The first relies on the notion of indegrees, which we shall now recall along with that of outdegrees:

Definition 1.1.6. Let $D = (V, A)$ be a digraph. Let $v \in V$ be any vertex. Then:

(a) The *outdegree* of v denotes the number of arcs of D whose source is v. This outdegree is denoted $\deg^+ v$.

(b) The *indegree* of v denotes the number of arcs of D whose target is v. This indegree is denoted $\deg^- v$.

For instance, if D is the digraph in Example 1.1.2, then $\deg^+ 2 = 1$ and $\deg^- 2 = 3$.

We also recall the following fact (Exercise 5 on homework set #2; see [hw2s] for a solution):

Proposition 1.1.7. Let $D = (V, A)$ be a tournament. Set $n = |V|$ and $m = \sum_{v \in V} \binom{\deg^- v}{2}$. Then:

(a) We have $m = \sum_{v \in V} \binom{\deg^+ v}{2}$.

(b) The number of 3-cycles in D is $3\left(\binom{n}{3} - m\right)$.

First proof of Proposition 1.1.5. Proposition 1.1.7 **(b)** yields that the # of 3-cycles in D is $3\left(\binom{n}{3} - m\right)$, where $m = \sum_{v \in V} \binom{\deg^- v}{2}$. Thus, the # of 3-cycles of D depends only on V and the indegrees $\deg^- v$ of the vertices $v \in V$. But these indegrees do not change when we reverse the arcs uv, vw and wu (since each of the vertices u, v and w loses one incoming arc[2] and gains another). Hence, the # of 3-cycles, too, does not change when we reverse these arcs. This proves Proposition 1.1.5. □

Second proof of Proposition 1.1.5. Write the digraph D in the form $D = (V, A)$.

The 3-cycles of D can be classified into the following three types:

- **Type 1:** those 3-cycles that contain **at most one** of the vertices u, v and w.
- **Type 2:** those 3-cycles that contain **precisely two** of the vertices u, v and w.
- **Type 3:** those 3-cycles that contain **all three** of the vertices u, v and w.

The 3-cycles of Type 2 can be classified further: Any 3-cycle of Type 2 contains precisely two of the vertices u, v and w and one further vertex. If this further vertex is x, then we call this 3-cycle a "3-cycle of Type 2_x". Thus, a 3-cycle of Type 2_x (for a vertex $x \in V \setminus \{u, v, w\}$) means a 3-cycle that contains the vertex x and precisely two of the vertices u, v and w.

The 3-cycles of D' can be classified in the exact same way.

Now, when we reverse the arcs uv, vw and wu of the digraph D, all 3-cycles of Type 1 are preserved, and no new 3-cycles of Type 1 are created. Thus,

$$\begin{aligned} &\left(\text{\# of 3-cycles of } D' \text{ of Type 1}\right) \\ &= \left(\text{\# of 3-cycles of } D \text{ of Type 1}\right). \end{aligned} \tag{1}$$

[2] An "incoming arc" of a vertex $r \in V$ means an arc whose target is r. The number of such arcs is the indegree $\deg^- r$ of r.

Furthermore, for each $x \in V \setminus \{u, v, w\}$, we have

$$\begin{aligned} & \left(\text{\# of 3-cycles of } D' \text{ of Type } 2_x\right) \\ & = \left(\text{\# of 3-cycles of } D \text{ of Type } 2_x\right). \end{aligned} \tag{2}$$

[*Proof of (2):* Let $x \in V \setminus \{u, v, w\}$. We must prove that the two numbers

$$\left(\text{\# of 3-cycles of } D' \text{ of Type } 2_x\right) \qquad \text{and} \qquad \left(\text{\# of 3-cycles of } D \text{ of Type } 2_x\right)$$

are equal. These numbers depend only on the presence or absence of the pairs (x, u), (x, v), (x, w), (u, x), (v, x) and (w, x) in the set A (since a 3-cycle of Type 2_x must consist entirely of vertices from the set $\{u, v, w, x\}$, and thus its existence or non-existence depends only on the arcs that join the four vertices in this set); thus, we only have finitely many cases to check. Moreover, since D is a tournament, it suffices to know which of the three pairs (x, u), (x, v) and (x, w) belong to A, because the tournament axiom shows that (e.g.) the pair (u, x) belongs to A if and only if the pair (x, u) does not. Thus, we have eight cases left to consider:

Case 1: We have $(x, u) \in A$ and $(x, v) \in A$ and $(x, w) \in A$ (and thus $(u, x) \notin A$ and $(v, x) \notin A$ and $(w, x) \notin A$).

Case 2: We have $(x, u) \in A$ and $(x, v) \in A$ and $(x, w) \notin A$ (and thus $(u, x) \notin A$ and $(v, x) \notin A$ and $(w, x) \in A$).

Case 3: We have $(x, u) \in A$ and $(x, v) \notin A$ and $(x, w) \in A$ (and thus $(u, x) \notin A$ and $(v, x) \in A$ and $(w, x) \notin A$).

Case 4: We have $(x, u) \in A$ and $(x, v) \notin A$ and $(x, w) \notin A$ (and thus $(u, x) \notin A$ and $(v, x) \in A$ and $(w, x) \in A$).

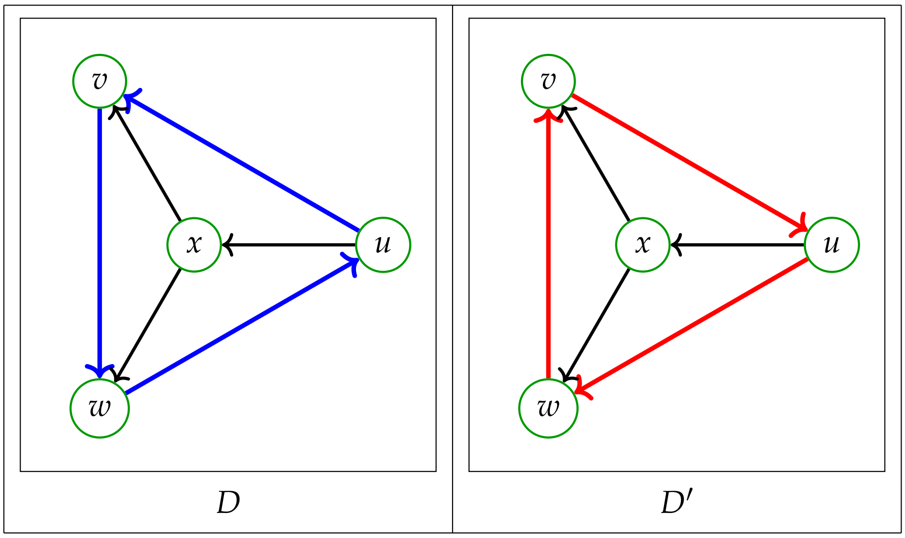
Case 5: We have $(x, u) \notin A$ and $(x, v) \in A$ and $(x, w) \in A$ (and thus $(u, x) \in A$ and $(v, x) \notin A$ and $(w, x) \notin A$).

Case 6: We have $(x, u) \notin A$ and $(x, v) \in A$ and $(x, w) \notin A$ (and thus $(u, x) \in A$ and $(v, x) \notin A$ and $(w, x) \in A$).

Case 7: We have $(x, u) \notin A$ and $(x, v) \notin A$ and $(x, w) \in A$ (and thus $(u, x) \in A$ and $(v, x) \in A$ and $(w, x) \notin A$).

Case 8: We have $(x, u) \notin A$ and $(x, v) \notin A$ and $(x, w) \notin A$ (and thus $(u, x) \in A$ and $(v, x) \in A$ and $(w, x) \in A$).

All eight cases are straightforward. For example, in Case 5, the relevant parts of the digraphs D and D' (that is, the vertices u, v, w, x and the arcs that join them) look as follows:



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In this case, the 3-cycles of D' of Type 2_x are (x, v, u), (v, u, x) and (u, x, v) (this is, of course, essentially just one 3-cycle and its cyclic rotations), whereas the 3-cycles of D of Type 2_x are (x, w, u), (w, u, x) and (u, x, w). The number of the former equals the number of the latter (namely, both numbers are 3). This proves (2) in Case 5. A similar argument works in each of the other seven cases. Thus, (2) is proved.]

Finally, the only 3-cycles of D' of Type 3 are (u, w, v), (w, v, u) and (v, u, w), whereas those of D are (u, v, w), (v, w, u) and (w, u, v). Thus, we have

$$\begin{aligned} &(\text{\# of 3-cycles of } D' \text{ of Type } 3) \\ &= (\text{\# of 3-cycles of } D \text{ of Type } 3) \end{aligned} \tag{3}$$

(since both of these numbers equal 3).

Now, recall that each 3-cycle of D has exactly one of the three Types 1, 2 and 3; moreover, if it has Type 2, then it has Type 2_x for a unique $x \in V \setminus \{u, v, w\}$. Thus,

$$\begin{aligned} &(\text{\# of 3-cycles of } D) \\ &= (\text{\# of 3-cycles of } D \text{ of Type } 1) + \sum_{x \in V \setminus \{u,v,w\}} (\text{\# of 3-cycles of } D \text{ of Type } 2_x) \\ &\qquad + (\text{\# of 3-cycles of } D \text{ of Type } 3) . \end{aligned} \tag{4}$$

The same argument can be made for D' instead of D, and thus we obtain

$$\begin{aligned} &(\text{\# of 3-cycles of } D') \\ &= (\text{\# of 3-cycles of } D' \text{ of Type } 1) + \sum_{x \in V \setminus \{u,v,w\}} (\text{\# of 3-cycles of } D' \text{ of Type } 2_x) \\ &\qquad + (\text{\# of 3-cycles of } D' \text{ of Type } 3) . \end{aligned} \tag{5}$$

The equalities (1), (2) and (3) show that the right hand side of (5) equals the right hand side of (4). Hence, the left hand side of (5) equals the left hand side of (4). In other words, we have

$$\left(\text{\# of 3-cycles of } D'\right) = \left(\text{\# of 3-cycles of } D\right).$$

This proves Proposition 1.1.5 again. $\square$

1.2. Reminders on permutations

Now, let us recall the definitions and a few basic properties of permutations and symmetric groups:

Definition 1.2.1. A *permutation* of a set X means a bijection from X to X.

Definition 1.2.2. For each $n \in \mathbb{N}$, we let S_n denote the set of all permutations of $\{1, 2, \ldots, n\}$. Note that $|S_n| = n!$.

There are several ways to represent (i.e., write down) a permutation $\sigma \in S_n$ for a given $n \in \mathbb{N}$:

- *One-line notation:* We can represent σ by the n-tuple $[\sigma(1), \sigma(2), \ldots, \sigma(n)]$ of all its values. (Note that we would normally use parentheses rather than square brackets here, but it is a habit of combinatorialists to use square brackets in this specific situation.)
- *Two-line notation:* We can represent σ by the $2 \times n$-table $\begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix}$.
- *Cycle digraph:* We can represent σ by the digraph

$$(\{1, 2, \ldots, n\}, \ \{(i, \sigma(i)) \ \mid \ i \in \{1, 2, \ldots, n\}\}).$$

  This is the digraph whose vertices are $1, 2, \ldots, n$, and whose arcs are the pairs $(i, \sigma(i))$ for all $i \in \{1, 2, \ldots, n\}$. This digraph is called the *cycle digraph* of σ.

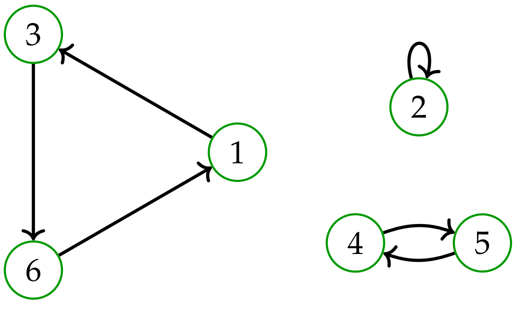
Example 1.2.3. Let σ be the permutation of $\{1, 2, 3, 4, 5, 6\}$ that sends

$$1 \mapsto 3, \qquad 2 \mapsto 2, \qquad 3 \mapsto 6, \qquad 4 \mapsto 5, \qquad 5 \mapsto 4, \qquad 6 \mapsto 1.$$

Then:

- The one-line notation for σ is $[3, 2, 6, 5, 4, 1]$. (Some combinatorialists would drop the brackets and commas, and shorten this to 326541; but we have no need for this much shorthand.)
- The two-line notation for σ is $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 6 & 5 & 4 & 1 \end{pmatrix}$.

- The cycle digraph of σ is



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Remark 1.2.4. Let $n \in \mathbb{N}$, and let $\sigma \in S_n$. Then, the cycle digraph of σ has the property that each vertex v satisfies

$$\deg^- v = 1 \qquad \text{and} \qquad \deg^+ v = 1.$$

From this fact, it follows easily that this digraph is a disjoint union of cycles (including cycles of length 1). This is easily seen to be equivalent to the classical result that the permutation σ can be uniquely written as a composition of disjoint cycles. We will not need this fact, but we found it worth mentioning; details can be found in [Grinbe21, proof of Theorem 5.5.2].

For the theory of determinants, the most important feature of a permutation is its sign. Let us recall its definition:

Definition 1.2.5. Let $n \in \mathbb{N}$, and let $\sigma \in S_n$.

(a) An *inversion* of σ means a pair (i, j) of integers i and j such that

$$1 \leq i < j \leq n \qquad \text{and} \qquad \sigma(i) > \sigma(j).$$

(b) The *length* of σ is defined to be the number of inversions of σ. It is denoted by $\ell(\sigma)$.

(c) The *sign* of σ is defined to be the number $(-1)^{\ell(\sigma)}$. It is denoted by $(-1)^\sigma$ or $\operatorname{sign}\sigma$ or $\operatorname{sgn}\sigma$ or $\varepsilon(\sigma)$. (We will use the notation $\operatorname{sign}\sigma$.)

For example, the permutation $\sigma \in S_6$ from Example 1.2.3 has inversions $(1,2)$, $(1,6)$, $(2,6)$, $(3,4)$, $(3,5)$, $(3,6)$, $(4,5)$, $(4,6)$ and $(5,6)$, and thus has length $\ell(\sigma) = 9$ and sign $\operatorname{sign}\sigma = (-1)^{\ell(\sigma)} = (-1)^9 = -1$.

Here are some well-known properties of signs and lengths:

Proposition 1.2.6. Let $n \in \mathbb{N}$.

(a) For any $\sigma \in S_n$, we have $\operatorname{sign} \sigma \in \{1, -1\}$.

(b) The identity permutation $\operatorname{id} \in S_n$ has sign $\operatorname{sign}(\operatorname{id}) = 1$.

(c) If $\tau \in S_n$ is any transposition, then $\operatorname{sign} \tau = -1$.

(d) For any two permutations $\sigma, \tau \in S_n$, we have $\operatorname{sign}(\sigma \circ \tau) = \operatorname{sign} \sigma \cdot \operatorname{sign} \tau$.

(e) For any $\sigma \in S_n$, we have $\operatorname{sign}\left(\sigma^{-1}\right) = \operatorname{sign} \sigma$.

(f) If the cycle digraph of σ has r cycles (counted up to rotation – so that, e.g., the cycle digraph in Example 1.2.3 has 3 cycles), then $\operatorname{sign} \sigma = (-1)^{n-r}$.

(g) We have $\operatorname{sign} \sigma = \prod\limits_{1 \leq i < j \leq n} \dfrac{\sigma(i) - \sigma(j)}{i - j}$.

(h) If you write down the one-line notation for σ, and sort it into increasing order by repeatedly swapping adjacent entries (this way of sorting a tuple is called "bubblesort", or more precisely, is a more general version of bubblesort), then $\ell(\sigma)$ is the smallest possible number of swaps you will need. (Actually, it is the exact number of swaps you will need if you don't waste time by swapping pairs of entries that already are in increasing order.)

Proofs of the claims of Proposition 1.2.6 can be found in [Grinbe15, §5.1–§5.3], [Conrad], [Day16, Chapter 6.B], [Strick20, Appendix B] and various other sources (including most serious texts on linear algebra or introductory abstract algebra). We will not need them here, however.

1.3. Determinants

Using the notion of signs, we now recall the definition of a determinant:

Definition 1.3.1. Let A be an $n \times n$-matrix (say, with real entries – more generally, it can have entries from an arbitrary commutative ring).

For all $i, j \in \{1, 2, \ldots, n\}$, we let $a_{i,j}$ be the (i, j)-th entry of A (that is, the entry of A in the i-th row and the j-th column).

Then, the *determinant* of A is the number $\det A$ defined by

$$\det A := \sum_{\sigma \in S_n} \operatorname{sign} \sigma \cdot \prod_{i=1}^{n} a_{i,\sigma(i)}. \tag{6}$$

The formula (6) is known as the *Leibniz formula*. Among many equivalent definitions of the determinant, it is the most explicit one.

1.4. The Vandermonde determinant

We can now state the theorem we shall apply our graph-theoretical machinery to:

Theorem 1.4.1 (Vandermonde determinant)**.** Let $n \in \mathbb{N}$.

Let $x_1, x_2, \ldots, x_n$ be n numbers (or, more generally, n elements of a commutative ring).

Let V be the $n \times n$-matrix whose (i, j)-th entry is x_j^{i-1} for all $i, j \in \{1, 2, \ldots, n\}$. That is, let

$$V := \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \\ x_1^2 & x_2^2 & x_3^2 & \cdots & x_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \cdots & x_n^{n-1} \end{pmatrix}.$$

Then,

$$\det V = \prod_{1 \leq i < j \leq n} \left(x_j - x_i \right). \tag{7}$$

This theorem is one of several equivalent versions of the "Vandermonde determinant". Some algebraic proofs can be found in [Grinbe15, §6.7] and in [Grinbe21, Theorem 6.4.31]. We shall here give a combinatorial proof using tournaments. (This proof is a mild variation of the proof found by Ira Gessel, published in [Gessel79].)

1.5. Tournaments on $\{1, 2, \ldots, n\}$

Definition 1.5.1. The *vertex set* of a digraph $D = (V, A)$ is defined to be the set V. This is the set of vertices of D.

Convention 1.5.2. We fix $n \in \mathbb{N}$, and we also fix n numbers $x_1, x_2, \ldots, x_n$.

We let $\mathcal{T}$ be the set of all tournaments with vertex set $\{1, 2, \ldots, n\}$.

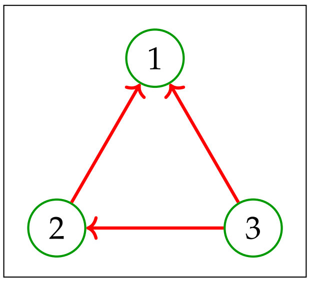
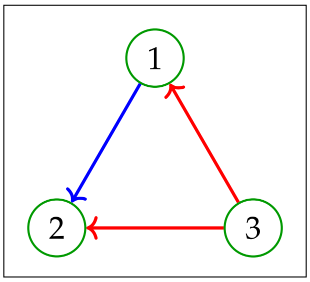
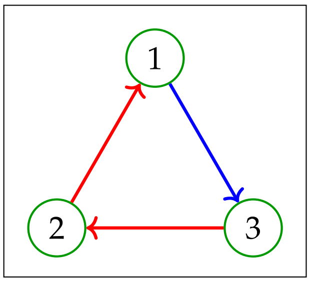
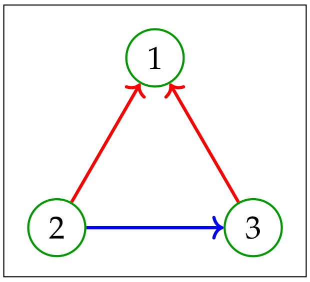
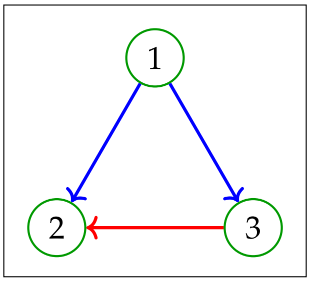
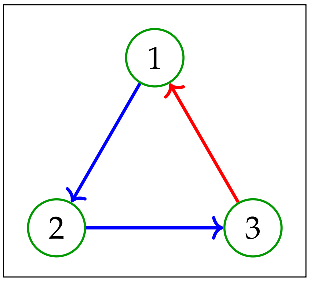
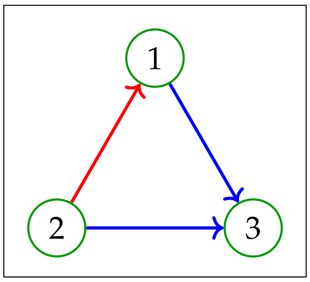
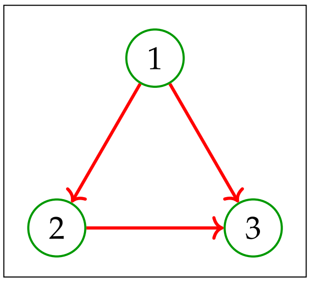
Thus, a tournament $D \in \mathcal{T}$ has vertices $1, 2, \ldots, n$. We introduce another convenient notion:

Definition 1.5.3. Let i and j be two elements of $\{1, 2, \ldots, n\}$. If $i < j$, then the pair (i, j) is said to be *increasing*. If $i > j$, then the pair (i, j) is said to be *decreasing*.

Thus, for each increasing pair (i, j), there is a corresponding decreasing pair (j, i). There are exactly $n(n-1)/2$ many increasing pairs:

$$\begin{array}{cccc} (1,2), & (1,3), & \ldots, & (1,n), \\ & (2,3), & \ldots, & (2,n), \\ & & \ddots & \vdots \\ & & & (n-1,n). \end{array}$$

Example 1.5.4. The following table shows all eight tournaments $D \in \mathcal{T}$ in the case when $n = 3$ (where we are drawing all increasing arcs in blue and all decreasing arcs in red):

 $\{\}$	 $\{(1,2)\}$	 $\{(1,3)\}$
 $\{(2,3)\}$	 $\{(1,2),(1,3)\}$	 $\{(1,2),(2,3)\}$
 $\{(1,3),(2,3)\}$	 $\{(1,2),(1,3),(2,3)\}$	

Underneath each tournament, we have written down the set of all increasing arcs of this tournament.

Now, let $D \in \mathcal{T}$ be a tournament. Then, D is loopless (by definition), so that each of its arcs is either increasing or decreasing. Moreover, the tournament axiom shows that for any increasing pair (i,j), exactly one of the two pairs (i,j) and (j,i) is an arc of D. In other words, an increasing pair (i,j) is an arc of D if and only if the corresponding decreasing pair (j,i) is not. Thus, D is uniquely determined if we know which increasing pairs are arcs of D.

Forget that we fixed D. We thus have shown that a tournament $D \in \mathcal{T}$ is uniquely determined if we know which increasing pairs are arcs of D. In other words, a tournament $D \in \mathcal{T}$ is uniquely determined by the set of all increasing arcs of D. Moreover, for any set S of increasing pairs, there is a unique tournament $D \in \mathcal{T}$ such that S is the set of all increasing arcs of D. Thus, the map

$$\begin{aligned} \mathcal{T} &\to \{\text{all sets of increasing pairs}\}, \\ D &\mapsto \{\text{all increasing arcs of } D\} \end{aligned}$$

is a bijection from the set $\mathcal{T}$ to the set of all sets of increasing pairs. The bijection principle[3] therefore yields

$$|\mathcal{T}| = |\{\text{all sets of increasing pairs}\}| = 2^{n(n-1)/2}$$

(since there are exactly $n(n-1)/2$ many increasing pairs).

Convention 1.5.5. We shall use the *Iverson bracket notation*: If $\mathcal{A}$ is any logical statement, then $[\mathcal{A}]$ will mean the number $\begin{cases} 1, & \text{if } \mathcal{A} \text{ is true;} \\ 0, & \text{if } \mathcal{A} \text{ is false.} \end{cases}$

For instance, $[2+2=4] = 1$ and $[2+2=5] = 0$.

The number $[\mathcal{A}]$ is called the *truth value* of the statement $\mathcal{A}$.

Definition 1.5.6. Let $D \in \mathcal{T}$ be a tournament. Then:

(a) The *sign* of D is defined to be the integer

$$\operatorname{sign} D := \prod_{\substack{(i,j) \text{ is an} \\ \text{arc of } D}} (-1)^{[i>j]} \in \{1, -1\}. \tag{8}$$

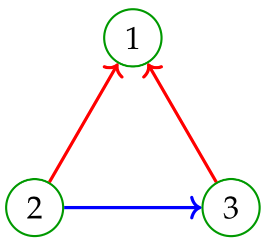
(b) The *x-weight* of D is defined to be the number

$$w(D) := \prod_{\substack{(i,j) \text{ is an} \\ \text{arc of } D}} \left((-1)^{[i>j]} x_j \right). \tag{9}$$

Example 1.5.7. For $n = 3$, the tournament

$$(\{1,2,3\}, \{(2,1), (2,3), (3,1)\})$$

can be drawn as follows:



.

[3]The *bijection principle* says that if $f : X \to Y$ is a bijection from a set X to a set Y, then $|X| = |Y|$.

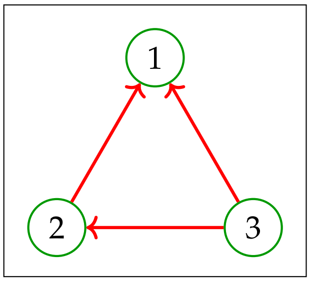
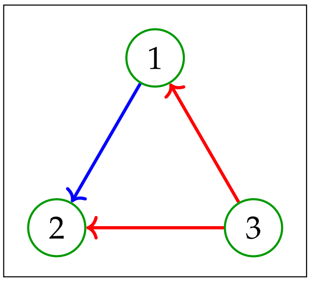
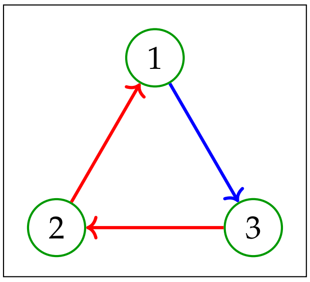
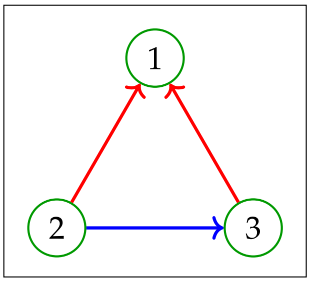
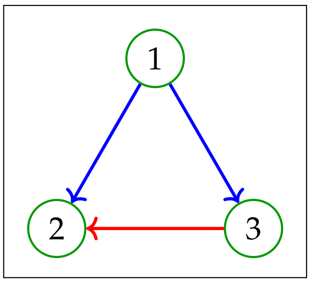
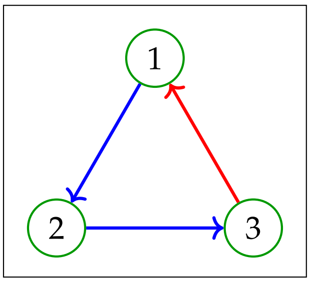
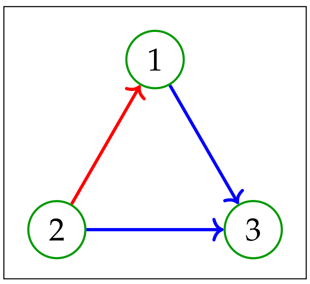
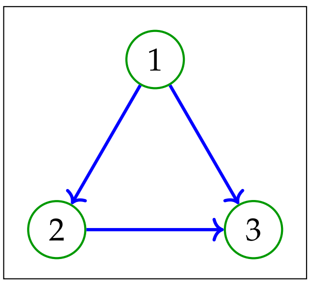
Its sign is

$$\underbrace{(-1)^{[2>1]}}_{=(-1)^1=-1} \cdot \underbrace{(-1)^{[2>3]}}_{=(-1)^0=1} \cdot \underbrace{(-1)^{[3>1]}}_{=(-1)^1=-1} = (-1) \cdot 1 \cdot (-1) = 1.$$

Its x-weight is

$$\underbrace{(-1)^{[2>1]}}_{=(-1)^1=-1} x_1 \cdot \underbrace{(-1)^{[2>3]}}_{=(-1)^0=1} x_3 \cdot \underbrace{(-1)^{[3>1]}}_{=(-1)^1=-1} x_1 = (-1) x_1 \cdot 1 x_3 \cdot (-1) x_1 = x_1^2 x_3.$$

Example 1.5.8. For $n = 3$, here are all the tournaments $D \in \mathcal{T}$ listed along with their x-weights $w(D)$:

 $-x_1^2 x_2$	 $x_1 x_2^2$	 $x_1 x_2 x_3$
 $x_1^2 x_3$	 $-x_2^2 x_3$	 $-x_1 x_2 x_3$
 $-x_1 x_3^2$	 $x_2 x_3^2$	

The equality (8) can be rewritten as follows:

Proposition 1.5.9. Let $D \in \mathcal{T}$ be a tournament. Then,

$$\operatorname{sign} D = (-1)^{(\text{\# of decreasing arcs of } D)}.$$

Proof. Let us simplify the product $\prod\limits_{\substack{(i,j) \text{ is an} \\ \text{arc of } D}} (-1)^{[i>j]}$. The factor $(-1)^{[i>j]}$ of this product equals 1 if the arc (i,j) is increasing (because in this case, we have $i < j$, and thus $[i > j] = 0$, and therefore $(-1)^{[i>j]} = (-1)^0 = 1$), and equals -1 if the arc (i,j) is decreasing (because in this case, we have $i > j$, and thus $[i > j] = 1$, and therefore $(-1)^{[i>j]} = (-1)^1 = -1$). Since any arc (i,j) of D is either increasing or decreasing (but cannot be both at the same time), we thus conclude that

$$\prod_{\substack{(i,j) \text{ is an} \\ \text{arc of } D}} (-1)^{[i>j]} = \underbrace{\left(\prod_{\substack{(i,j) \text{ is an} \\ \text{increasing} \\ \text{arc of } D}} 1 \right)}_{=1} \cdot \underbrace{\left(\prod_{\substack{(i,j) \text{ is a} \\ \text{decreasing} \\ \text{arc of } D}} (-1) \right)}_{=(-1)^{(\text{\# of decreasing arcs of } D)}}$$
$$= (-1)^{(\text{\# of decreasing arcs of } D)}.$$

Hence, (8) becomes

$$\operatorname{sign} D = \prod_{\substack{(i,j) \text{ is an} \\ \text{arc of } D}} (-1)^{[i>j]} = (-1)^{(\text{\# of decreasing arcs of } D)}.$$

This proves Proposition 1.5.9. $\square$

The equality (9) can be rewritten as follows:

Proposition 1.5.10. Let $D \in \mathcal{T}$ be a tournament. Then,

$$w(D) = (\operatorname{sign} D) \cdot \prod_{j=1}^{n} x_j^{\deg^- j},$$

where $\deg^- j$ denotes the indegree of j as a vertex of D.

Proof. From (9), we obtain

$$w(D) = \prod_{\substack{(i,j) \text{ is an} \\ \text{arc of } D}} \left((-1)^{[i>j]} x_j \right) = \left(\prod_{\substack{(i,j) \text{ is an} \\ \text{arc of } D}} (-1)^{[i>j]} \right) \left(\prod_{\substack{(i,j) \text{ is an} \\ \text{arc of } D}} x_j \right).$$

Now, let us consider the product $\prod_{\substack{(i,j) \text{ is an} \\ \text{arc of } D}} x_j$. How often does the factor x_j (for a given $j \in \{1, 2, \ldots, n\}$) appear in this product? It appears once for each arc of D whose target is j. Thus, in total, it appears $\deg^- j$ many times (since $\deg^- j$ is the # of arcs of D whose target is j). Hence, the product $\prod_{\substack{(i,j) \text{ is an} \\ \text{arc of } D}} x_j$ contains each x_j exactly $\deg^- j$ many times (and contains no further factors). Consequently,

$$\prod_{\substack{(i,j) \text{ is an} \\ \text{arc of } D}} x_j = \prod_{j=1}^{n} x_j^{\deg^- j}. \tag{10}$$

Now,

$$w(D) = \underbrace{\left(\prod_{\substack{(i,j) \text{ is an} \\ \text{arc of } D}} (-1)^{[i>j]} \right)}_{\substack{= \operatorname{sign} D \\ \text{(by (8))}}} \underbrace{\left(\prod_{\substack{(i,j) \text{ is an} \\ \text{arc of } D}} x_j \right)}_{\substack{= \prod_{j=1}^{n} x_j^{\deg^- j} \\ \text{(by (10))}}} = (\operatorname{sign} D) \cdot \prod_{j=1}^{n} x_j^{\deg^- j}.$$

This proves Proposition 1.5.10. $\square$

On the other hand, the x-weights of the tournaments $D \in \mathcal{T}$ are precisely the terms that appear in the expansion of the product $\prod_{1 \leq i < j \leq n} (x_j - x_i)$:

Proposition 1.5.11. We have

$$\prod_{1 \leq i < j \leq n} (x_j - x_i) = \sum_{D \in \mathcal{T}} w(D).$$

We shall derive this proposition from the following more general formula:

Lemma 1.5.12. For each pair $(i, j) \in \{1, 2, \ldots, n\}^2$, let $y_{(i,j)}$ be a number. Then,

$$\prod_{1 \leq i < j \leq n} \left(y_{(i,j)} + y_{(j,i)} \right) = \sum_{D \in \mathcal{T}} \prod_{\substack{a \text{ is an} \\ \text{arc of } D}} y_a.$$

Proof of Lemma 1.5.12. The idea is to expand the product $\prod_{1 \leq i < j \leq n} \left(y_{(i,j)} + y_{(j,i)} \right)$ (which has $n(n-1)/2$ many factors) into a huge sum (a sum of $2^{n(n-1)/2}$ many addends), and to match up the resulting addends with the tournaments $D \in \mathcal{T}$.

Before we explain this in the general case, let us first explore the case $n = 3$ as an example. In this case, we have

$$\begin{aligned}
&\prod_{1\leq i<j\leq n}\left(y_{(i,j)}+y_{(j,i)}\right)\\
&=\prod_{1\leq i<j\leq 3}\left(y_{(i,j)}+y_{(j,i)}\right)\\
&=\left(y_{(1,2)}+y_{(2,1)}\right)\left(y_{(1,3)}+y_{(3,1)}\right)\left(y_{(2,3)}+y_{(3,2)}\right)\\
&=y_{(1,2)}y_{(1,3)}y_{(2,3)}+y_{(1,2)}y_{(1,3)}y_{(3,2)}+y_{(1,2)}y_{(3,1)}y_{(2,3)}+y_{(1,2)}y_{(3,1)}y_{(3,2)}\\
&\qquad+y_{(2,1)}y_{(1,3)}y_{(2,3)}+y_{(2,1)}y_{(1,3)}y_{(3,2)}+y_{(2,1)}y_{(3,1)}y_{(2,3)}+y_{(2,1)}y_{(3,1)}y_{(3,2)}.
\end{aligned}$$

The right hand side of this is a sum of 8 addends, and each of these addends has the form $\prod_{\substack{a \text{ is an} \\ \text{arc of } D}} y_a$ for some tournament $D \in \mathcal{T}$. For example, the addend $y_{(2,1)}y_{(1,3)}y_{(3,2)}$ comes from the tournament $D \in \mathcal{T}$ whose arcs are $(2,1)$, $(1,3)$ and $(3,2)$.

Here is the argument in the general case:

We first recall that if F is any finite set, and if $\left(a_f\right)_{f\in F}$ and $\left(b_f\right)_{f\in F}$ are two families of numbers, then

$$\prod_{f\in F}\left(a_f+b_f\right)=\sum_{S \text{ is a subset of } F}\ \prod_{f\in F}\begin{cases} a_f, & \text{if } f\in S;\\ b_f, & \text{if } f\notin S.\end{cases}$$

Applying this fact to $F = \{\text{increasing pairs}\}$ and $a_{(i,j)} = y_{(i,j)}$ and $b_{(i,j)} = y_{(j,i)}$, we obtain

$$\begin{aligned}
&\prod_{(i,j)\in\{\text{increasing pairs}\}}\left(y_{(i,j)}+y_{(j,i)}\right)\\
&=\sum_{\substack{S \text{ is a set of}\\ \text{increasing pairs}}}\ \prod_{(i,j)\in\{\text{increasing pairs}\}}\begin{cases} y_{(i,j)}, & \text{if } (i,j)\in S;\\ y_{(j,i)}, & \text{if } (i,j)\notin S.\end{cases}
\end{aligned}$$

Recalling the definition of an increasing pair, we can rewrite this as follows:

$$\begin{aligned}
&\prod_{1\leq i<j\leq n}\left(y_{(i,j)}+y_{(j,i)}\right)\\
&=\sum_{\substack{S \text{ is a set of}\\ \text{increasing pairs}}}\ \prod_{1\leq i<j\leq n}\begin{cases} y_{(i,j)}, & \text{if } (i,j)\in S;\\ y_{(j,i)}, & \text{if } (i,j)\notin S.\end{cases}
\end{aligned}\tag{11}$$

In the discussion following Example 1.5.4, we have found a bijection from the set $\mathcal{T}$ to the set of all sets of increasing pairs. Specifically, this bijection sends each tournament $D \in \mathcal{T}$ to the set of all increasing arcs of D. Let us denote this bijection by Φ. Let us now substitute $\Phi(D)$ for S in the sum on the right hand side of (11). We thus obtain

$$\sum_{\substack{S \text{ is a set of} \\ \text{increasing pairs}}} \prod_{1 \leq i < j \leq n} \begin{cases} y_{(i,j)}, & \text{if } (i,j) \in S; \\ y_{(j,i)}, & \text{if } (i,j) \notin S \end{cases}$$
$$= \sum_{D \in \mathcal{T}} \prod_{1 \leq i < j \leq n} \begin{cases} y_{(i,j)}, & \text{if } (i,j) \in \Phi(D); \\ y_{(j,i)}, & \text{if } (i,j) \notin \Phi(D). \end{cases}$$

Hence, we can rewrite (11) as

$$\prod_{1 \leq i < j \leq n} \left(y_{(i,j)} + y_{(j,i)} \right)$$
$$= \sum_{D \in \mathcal{T}} \prod_{1 \leq i < j \leq n} \begin{cases} y_{(i,j)}, & \text{if } (i,j) \in \Phi(D); \\ y_{(j,i)}, & \text{if } (i,j) \notin \Phi(D). \end{cases} \tag{12}$$

Now, let $D \in \mathcal{T}$ be a tournament. Recall that $\Phi(D)$ is the set of all increasing arcs of D (by the definition of Φ). Thus,

$$\prod_{\substack{1 \leq i < j \leq n; \\ (i,j) \in \Phi(D)}} y_{(i,j)} = \prod_{\substack{1 \leq i < j \leq n; \\ (i,j) \text{ is an increasing arc of } D}} y_{(i,j)} = \prod_{(i,j) \text{ is an increasing arc of } D} y_{(i,j)}$$

(since each increasing arc (i,j) of D automatically satisfies $1 \leq i < j \leq n$).

Recall again that $\Phi(D)$ is the set of all increasing arcs of D. Thus,

$$\prod_{\substack{1 \leq i < j \leq n; \\ (i,j) \notin \Phi(D)}} y_{(j,i)} = \prod_{\substack{1 \leq i < j \leq n; \\ (i,j) \text{ is not an increasing arc of } D}} y_{(j,i)} = \prod_{\substack{1 \leq i < j \leq n; \\ (i,j) \text{ is not an arc of } D}} y_{(j,i)}$$

(here, we have dropped the "increasing" condition under the product sign, since any arc (i,j) satisfying $1 \leq i < j \leq n$ is automatically increasing). However, if i and j are two integers satisfying $1 \leq i < j \leq n$, then the condition "(i,j) is not an arc of D" is equivalent to the condition "(j,i) is an arc of D" (by the tournament axiom, since the vertices i and j of D are distinct[4]). Hence, we can replace the condition "(i,j) is not an arc of D" under the summation sign $\prod\limits_{\substack{1 \leq i < j \leq n; \\ (i,j) \text{ is not an arc of } D}}$ by the equivalent condition "(j,i) is an arc of D". We thus obtain

$$\prod_{\substack{1 \leq i < j \leq n; \\ (i,j) \text{ is not an arc of } D}} y_{(j,i)} = \prod_{\substack{1 \leq i < j \leq n; \\ (j,i) \text{ is an arc of } D}} y_{(j,i)}$$
$$= \prod_{\substack{1 \leq j < i \leq n; \\ (i,j) \text{ is an arc of } D}} y_{(i,j)} \qquad \left(\begin{array}{c} \text{here, we have renamed the} \\ \text{index } (i,j) \text{ as } (j,i) \end{array} \right)$$
$$= \prod_{\substack{(i,j) \text{ is an arc of } D; \\ j < i}} y_{(i,j)} = \prod_{(i,j) \text{ is a decreasing arc of } D} y_{(i,j)}$$

[4] because $i < j$

(because an arc (i,j) of D satisfying $j < i$ is the same as a decreasing arc of D). Now,

$$\begin{aligned}
&\prod_{1\leq i<j\leq n} \begin{cases} y_{(i,j)}, & \text{if } (i,j) \in \Phi(D); \\ y_{(j,i)}, & \text{if } (i,j) \notin \Phi(D) \end{cases} \\
&= \underbrace{\left(\prod_{\substack{1\leq i<j\leq n; \\ (i,j)\in\Phi(D)}} y_{(i,j)} \right)}_{= \prod_{(i,j) \text{ is an increasing arc of } D} y_{(i,j)}} \cdot \underbrace{\left(\prod_{\substack{1\leq i<j\leq n; \\ (i,j)\notin\Phi(D)}} y_{(j,i)} \right)}_{\substack{= \prod_{\substack{1\leq i<j\leq n; \\ (i,j) \text{ is not an arc of } D}} y_{(j,i)} \\ = \prod_{(i,j) \text{ is a decreasing arc of } D} y_{(i,j)}}} \\
&= \left(\prod_{(i,j) \text{ is an increasing arc of } D} y_{(i,j)} \right) \cdot \left(\prod_{(i,j) \text{ is a decreasing arc of } D} y_{(i,j)} \right) \\
&= \prod_{(i,j) \text{ is an arc of } D} y_{(i,j)}
\end{aligned} \tag{13}$$

(since any arc (i,j) of D is either increasing or decreasing, but cannot be both at the same time).

Forget that we fixed D. We thus have proved (13) for each $D \in \mathcal{T}$. Thus, (12) becomes

$$\begin{aligned}
\prod_{1\leq i<j\leq n} \left(y_{(i,j)} + y_{(j,i)} \right) &= \sum_{D\in\mathcal{T}} \underbrace{\prod_{1\leq i<j\leq n} \begin{cases} y_{(i,j)}, & \text{if } (i,j) \in \Phi(D); \\ y_{(j,i)}, & \text{if } (i,j) \notin \Phi(D) \end{cases}}_{\substack{= \prod_{(i,j) \text{ is an arc of } D} y_{(i,j)} \\ \text{(by (13))}}} \\
&= \sum_{D\in\mathcal{T}} \underbrace{\prod_{(i,j) \text{ is an arc of } D} y_{(i,j)}}_{= \prod_{a \text{ is an arc of } D} y_a} = \sum_{D\in\mathcal{T}} \prod_{\substack{a \text{ is an} \\ \text{arc of } D}} y_a .
\end{aligned}$$

This proves Lemma 1.5.12. □

Proof of Proposition 1.5.11. For each pair $(i,j) \in \{1,2,\ldots,n\}^2$, we define a number

$$y_{(i,j)} := (-1)^{[i>j]} x_j .$$

Then, Lemma 1.5.12 yields

$$\prod_{1\leq i<j\leq n} \left(y_{(i,j)} + y_{(j,i)} \right) = \sum_{D\in\mathcal{T}} \prod_{\substack{a \text{ is an} \\ \text{arc of } D}} y_a . \tag{14}$$

However, if (i,j) is a pair of integers satisfying $1 \leq i < j \leq n$, then

$$\begin{aligned} y_{(i,j)} + y_{(j,i)} &= (-1)^{[i>j]} x_j + (-1)^{[j>i]} x_i && \left(\text{by the definition of } y_{(i,j)} \text{ and of } y_{(j,i)}\right) \\ &= (-1)^0 x_j + (-1)^1 x_i && \left(\begin{array}{c} \text{since } [i>j] = 0 \text{ (because we don't} \\ \text{have } i>j \text{ (since } i<j\text{)) and } [j>i]=1 \\ \text{(since } j>i \text{ (because } i<j\text{))} \end{array}\right) \\ &= 1x_j + (-1) x_i = x_j - x_i. \end{aligned}$$

Thus, we can rewrite (14) as

$$\begin{aligned} \prod_{1 \leq i < j \leq n} (x_j - x_i) &= \sum_{D \in \mathcal{T}} \underbrace{\prod_{a \text{ is an arc of } D} y_a}_{= \prod_{(i,j) \text{ is an arc of } D} y_{(i,j)}} = \sum_{D \in \mathcal{T}} \prod_{(i,j) \text{ is an arc of } D} \underbrace{y_{(i,j)}}_{\substack{=(-1)^{[i>j]} x_j \\ \text{(by the definition} \\ \text{of } y_{(i,j)})}} \\ &\qquad \left(\begin{array}{c} \text{here, we have renamed the index } a \\ \text{as } (i,j) \text{ in the product} \end{array}\right) \\ &= \sum_{D \in \mathcal{T}} \underbrace{\prod_{(i,j) \text{ is an arc of } D} \left((-1)^{[i>j]} x_j\right)}_{\substack{=w(D) \\ \text{(by (9))}}} = \sum_{D \in \mathcal{T}} w(D). \end{aligned}$$

This proves Proposition 1.5.11. □

1.6. Tournaments with no 3-cycles

Recall that our goal is to prove the equality (7). Proposition 1.5.11 interprets the right hand side of this equality in terms of tournaments. We shall next find a similar interpretation for its left hand side.

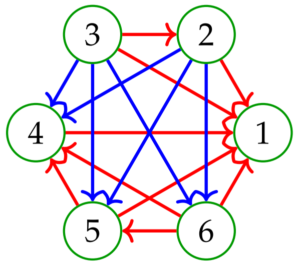
To this purpose, we shall study the tournaments $D \in \mathcal{T}$ that have no 3-cycles. As we will soon see, they have a rather specific form:

Definition 1.6.1. Let $\sigma \in S_n$ be a permutation. Then, we define a digraph T_σ by

$$T_\sigma := \left(\{1, 2, \ldots, n\}, \ \{(\sigma(i), \sigma(j)) \mid i \text{ and } j \text{ are integers with } 1 \leq i < j \leq n\}\right).$$

Thus, the vertices of the digraph T_σ are $1, 2, \ldots, n$, and its arcs are the pairs $(\sigma(i), \sigma(j))$, where i and j range over integers satisfying $1 \leq i < j \leq n$.

Example 1.6.2. If $n = 6$, and if $\sigma \in S_6$ is the permutation from Example 1.2.3, then T_σ is the following digraph (again, we draw the increasing arcs blue and the decreasing arcs red):



The usefulness of these digraphs T_σ for us is due to the following theorem:

Theorem 1.6.3.

(a) For each permutation $\sigma \in S_n$, the digraph T_σ is a tournament in $\mathcal{T}$ and has no 3-cycles.

(b) The tournaments $D \in \mathcal{T}$ that have no 3-cycles are precisely the digraphs of the form T_σ with $\sigma \in S_n$.

(c) If σ and τ are two distinct permutations in S_n, then the digraphs T_σ and T_τ are distinct. (In other words, any permutation $\sigma \in S_n$ can be uniquely reconstructed from T_σ.)

(d) Let $\sigma \in S_n$. Then,

$$\operatorname{sign}(T_\sigma) = \operatorname{sign}\sigma \tag{15}$$

and

$$w(T_\sigma) = \operatorname{sign}\sigma \cdot \prod_{i=1}^{n} x_{\sigma(i)}^{i-1}. \tag{16}$$

Also, each vertex v of T_σ has indegree

$$\deg^- v = \sigma^{-1}(v) - 1. \tag{17}$$

Proof of Theorem 1.6.3. **(a)** Let $\sigma \in S_n$ be a permutation. The arcs of the digraph T_σ are the pairs $(\sigma(i), \sigma(j))$, where i and j range over integers satisfying $1 \leq i < j \leq n$. Such an arc $(\sigma(i), \sigma(j))$ cannot be a loop (since $i < j$ entails $i \neq j$ and therefore $\sigma(i) \neq \sigma(j)$ [5]). Hence, the digraph T_σ is loopless.

For any pair $(u, v) \in \{1, 2, \ldots, n\}^2$, we have the following equivalence:

$$((u, v) \text{ is an arc of } T_\sigma) \iff \left(\sigma^{-1}(u) < \sigma^{-1}(v)\right). \tag{18}$$

[*Proof of (18):* Let $(u, v) \in \{1, 2, \ldots, n\}^2$ be a pair. We must prove the equivalence (18). We shall prove its "$\Longrightarrow$" and "$\Longleftarrow$" directions separately:

[5] because σ is injective

$\Longrightarrow$: Assume that (u,v) is an arc of T_σ. We must prove that $\sigma^{-1}(u) < \sigma^{-1}(v)$.

The arcs of the digraph T_σ are the pairs $(\sigma(i), \sigma(j))$, where i and j range over integers satisfying $1 \leq i < j \leq n$. Hence, (u,v) is such a pair (since (u,v) is an arc of T_σ). In other words, $(u,v) = (\sigma(i), \sigma(j))$ for some integers i and j satisfying $1 \leq i < j \leq n$. Consider these i and j. From $(u,v) = (\sigma(i), \sigma(j))$, we obtain $u = \sigma(i)$ and $v = \sigma(j)$. Thus, $i = \sigma^{-1}(u)$ and $j = \sigma^{-1}(v)$. Hence, the inequality $i < j$ (which we know to be true) can be rewritten as $\sigma^{-1}(u) < \sigma^{-1}(v)$. Thus, $\sigma^{-1}(u) < \sigma^{-1}(v)$ is true. This proves the "$\Longrightarrow$" direction of the equivalence (18).

$\Longleftarrow$: Assume that $\sigma^{-1}(u) < \sigma^{-1}(v)$. We must prove that (u,v) is an arc of T_σ.

The arcs of the digraph T_σ are the pairs $(\sigma(i), \sigma(j))$, where i and j range over integers satisfying $1 \leq i < j \leq n$. Thus, in particular, one of these arcs is $\left(\sigma\left(\sigma^{-1}(u)\right), \sigma\left(\sigma^{-1}(v)\right)\right)$ (obtained by setting $i = \sigma^{-1}(u)$ and $j = \sigma^{-1}(v)$), since $\sigma^{-1}(u)$ and $\sigma^{-1}(v)$ are two integers satisfying $1 \leq \sigma^{-1}(u) < \sigma^{-1}(v) \leq n$. In other words, one of these arcs is (u,v) (since $\sigma\left(\sigma^{-1}(u)\right) = u$ and $\sigma\left(\sigma^{-1}(v)\right) = v$). Thus, (u,v) is an arc of T_σ. This proves the "$\Longleftarrow$" direction of the equivalence (18).

Thus, the proof of the equivalence (18) is complete.]

Now, it is easy to see that this digraph T_σ is a tournament.

[*Proof:* Since we know that T_σ is loopless, we only need to verify the tournament axiom. In other words, we need to show that for any two distinct vertices u and v of T_σ, **exactly** one of the two pairs (u,v) and (v,u) is an arc of T_σ.

Let u and v be two distinct vertices of T_σ. Thus, u and v are two distinct elements of $\{1,2,\ldots,n\}$. We must show that **exactly** one of the two pairs (u,v) and (v,u) is an arc of T_σ.

We WLOG assume that $\sigma^{-1}(u) \leq \sigma^{-1}(v)$ (since otherwise, we can swap u with v). Combining this with $\sigma^{-1}(u) \neq \sigma^{-1}(v)$ (this follows from $u \neq v$), we obtain $\sigma^{-1}(u) < \sigma^{-1}(v)$. Thus, (u,v) is an arc of T_σ (by (18)). Moreover, we do **not** have $\sigma^{-1}(v) < \sigma^{-1}(u)$ (since this would contradict $\sigma^{-1}(u) < \sigma^{-1}(v)$). However, from (18) (applied to (v,u) instead of (u,v)), we obtain the equivalence

$$((v,u) \text{ is an arc of } T_\sigma) \Longleftrightarrow \left(\sigma^{-1}(v) < \sigma^{-1}(u)\right).$$

Thus, (v,u) is not an arc of T_σ (since we do **not** have $\sigma^{-1}(v) < \sigma^{-1}(u)$).

We now know that (u,v) is an arc of T_σ, but (v,u) is not. Therefore, **exactly** one of the two pairs (u,v) and (v,u) is an arc of T_σ. This completes the proof of the tournament axiom for T_σ. Hence, T_σ is a tournament.]

Since T_σ is a tournament with vertex set $\{1,2,\ldots,n\}$, we have $T_\sigma \in \mathcal{T}$. It remains to show that T_σ has no 3-cycles.

Indeed, assume the contrary. Thus, T_σ has a 3-cycle (u,v,w). Consider this (u,v,w). Since (u,v,w) is a 3-cycle, all three pairs uv, vw and wu are arcs of T_σ. In particular, uv is an arc of T_σ. In other words, (u,v) is an arc of T_σ. By (18), we thus conclude that $\sigma^{-1}(u) < \sigma^{-1}(v)$. Similarly, $\sigma^{-1}(v) < \sigma^{-1}(w)$ and $\sigma^{-1}(w) < \sigma^{-1}(u)$. Therefore, $\sigma^{-1}(u) < \sigma^{-1}(v) < \sigma^{-1}(w) < \sigma^{-1}(u)$, which is absurd. This contradiction shows that our assumption was wrong. Hence, we have shown that T_σ has no 3-cycles. This completes the proof of Theorem 1.6.3 **(a)**.

(b) Theorem 1.6.3 **(a)** tells us that each digraph of the form T_σ with $\sigma \in S_n$ is a tournament $D \in \mathcal{T}$ that has no 3-cycles. It thus remains to prove the converse: i.e., that each tournament $D \in \mathcal{T}$ that has no 3-cycles is a digraph of the form T_σ with $\sigma \in S_n$.

So let $D \in \mathcal{T}$ be a tournament that has no 3-cycles. We must find a permutation $\sigma \in S_n$ such that $D = T_\sigma$.

We first show the following:

> *Claim 1:* Let u and v be two distinct vertices of D such that $\deg^- v \leq \deg^- u$. Then, vu is an arc of D.

[*Proof of Claim 1:* Assume the contrary. Thus, vu is not an arc of D.

The tournament axiom shows that exactly one of the two pairs uv and vu is an arc of D. Hence, uv is an arc of D (since vu is not an arc of D).

Let X be the set of all vertices z of D for which zu is an arc of D. Thus, the vertices in X are in 1-to-1 correspondence with the arcs of D that have target u. Hence, $|X|$ equals the number of such arcs. But the latter number is $\deg^- u$ (by the definition of indegrees). Hence, we have shown that $|X| = \deg^- u$.

Let Y be the set of all vertices z of D for which zv is an arc of D. Then, $|Y| = \deg^- v$ (indeed, we can show this in the same way as we showed $|X| = \deg^- u$).

The digraph D is loopless (since it is a tournament); thus, uu is not an arc of D. In other words, we have $u \notin X$ (by the definition of X). However, we have $u \in Y$ (since uv is an arc of D). Hence, $X \neq Y$ (because if we had $X = Y$, then $u \notin X = Y$ would contradict $u \in Y$).

Now, $|Y| = \deg^- v \leq \deg^- u$, so that $\deg^- u \geq |Y|$. Thus, $|X| = \deg^- u \geq |Y|$. However, if X was a subset of Y, then X would be a **proper** subset of Y (since $X \neq Y$), which would entail $|X| < |Y|$; but this would contradict $|X| \geq |Y|$. Thus, X is not a subset of Y. Hence, there exists a vertex $w \in X$ such that $w \notin Y$. Consider this w.

Since $w \in X$, the pair wu is an arc of D (by the definition of X). As a consequence, $w \neq u$ (since uu is not an arc of D) and $w \neq v$ (since vu is not an arc of D). Combining this with $u \neq v$, we see that the three vertices u, v and w are distinct.

Since $w \notin Y$, the pair wv is not an arc of D. However, $w \neq v$; thus, the tournament axiom shows that exactly one of the two pairs wv and vw is an arc of D. Since wv is not an arc of D, we thus conclude that vw is an arc of D.

We now know that u, v and w are three distinct vertices of D and that uv, vw and wu are arcs of D. In other words, (u, v, w) is a 3-cycle. But this contradicts the fact that D has no 3-cycles. This contradiction shows that our assumption was false; hence, Claim 1 is proven.]

Pick a permutation $\tau \in S_n$ of $\{1, 2, \ldots, n\}$ that sorts the vertices of D in the order of increasing outdegree – i.e., that satisfies

$$\deg^- (\tau(1)) \leq \deg^- (\tau(2)) \leq \cdots \leq \deg^- (\tau(n)). \tag{19}$$

(Such a permutation τ exists, since we can sort any n numbers in increasing order.) We shall show that $D = T_\tau$.

Indeed, the digraphs D and T_τ have the same vertex set (namely, $\{1, 2, \ldots, n\}$); thus, we only need to show that they have the same arcs. To do so, we will first show the following two claims:

Claim 2: Any arc of T_τ is an arc of D.

[*Proof of Claim 2:* Let (v, u) be an arc of T_τ. We must prove that (v, u) is an arc of D.

However, the arcs of the digraph T_τ are the pairs $(\tau(i), \tau(j))$, where i and j range over integers satisfying $1 \leq i < j \leq n$ (by the definition of T_τ). Hence, (v, u) is such a pair (since (v, u) is an arc of T_τ). In other words, $(v, u) = (\tau(i), \tau(j))$ for some two integers i and j satisfying $1 \leq i < j \leq n$. Consider these i and j. From $(v, u) = (\tau(i), \tau(j))$, we obtain $v = \tau(i)$ and $u = \tau(j)$. From $i < j$, we obtain $\deg^-(\tau(i)) \leq \deg^-(\tau(j))$ (by (19)). In other words, $\deg^- v \leq \deg^- u$ (since $v = \tau(i)$ and $u = \tau(j)$). Moreover, the vertices i and j are distinct (since $i < j$); thus, the vertices $\tau(i)$ and $\tau(j)$ are distinct as well (since τ is injective). In other words, v and u are distinct (since $v = \tau(i)$ and $u = \tau(j)$). In other words, u and v are distinct. Hence, Claim 1 yields that vu is an arc of D. In other words, (v, u) is an arc of D. This completes the proof of Claim 2.]

Claim 3: Any arc of D is an arc of T_τ.

[*Proof of Claim 3:* Let (u, v) be an arc of D. We must prove that (u, v) is an arc of T_τ.

Assume the contrary. Thus, (u, v) is not an arc of T_τ.

The digraph D is loopless (since it is a tournament). Thus, its arc (u, v) cannot be a loop. In other words, u and v are distinct. Hence, exactly one of the two pairs (u, v) and (v, u) is an arc of D (by the tournament axiom, since D is a tournament). Therefore, the pair (v, u) is not an arc of D (since (u, v) is an arc of D).

However, T_τ is a tournament (by Theorem 1.6.3 **(a)**, applied to $\sigma = \tau$), and u and v are two distinct vertices. Hence, exactly one of the two pairs (u, v) and (v, u) is an arc of T_τ (by the tournament axiom). Therefore, (v, u) is an arc of T_τ (because (u, v) is not an arc of T_τ). By Claim 2, this entails that (v, u) is an arc of D. But this contradicts the fact that (v, u) is not an arc of D. This contradiction shows that our assumption was false. Hence, (u, v) is an arc of T_τ. This proves Claim 3.]

Claim 2 and Claim 3 (combined) yield that the arcs of D are precisely the arcs of T_τ. In other words, the digraphs D and T_τ have the same arcs. Since they also have the same vertex set, we thus conclude that they are equal. In other words, $D = T_\tau$. Hence, D is a digraph of the form T_σ with $\sigma \in S_n$ (namely, $\sigma = \tau$).

Forget that we fixed D. We thus have shown that each tournament $D \in \mathcal{T}$ that has no 3-cycles is a digraph of the form T_σ with $\sigma \in S_n$. This completes the proof of Theorem 1.6.3 **(b)**.

(d) Let $\sigma \in S_n$. Let v be a vertex of T_σ. Thus, $v \in \{1, 2, \ldots, n\}$.

Set $k := \sigma^{-1}(v)$. By the definition of an indegree, we have

$$\begin{aligned}
&\deg^- v \\
&= (\text{\# of arcs of } T_\sigma \text{ whose target is } v) \\
&= (\text{\# of arcs of } T_\sigma \text{ that have the form } (u, v) \text{ for some } u \in \{1, 2, \ldots, n\}) \\
&\qquad \left(\begin{array}{c} \text{since an arc of } T_\sigma \text{ whose target is } v \text{ is the same as an} \\ \text{arc of } T_\sigma \text{ that has the form } (u, v) \text{ for some } u \in \{1, 2, \ldots, n\} \end{array} \right) \\
&= (\text{\# of } u \in \{1, 2, \ldots, n\} \text{ such that } (u, v) \text{ is an arc of } T_\sigma) \\
&= \left(\text{\# of } u \in \{1, 2, \ldots, n\} \text{ such that } \sigma^{-1}(u) < \sigma^{-1}(v) \right) \\
&\qquad \left(\begin{array}{c} \text{here, we have replaced the condition "} (u, v) \text{ is an arc of } T_\sigma \text{"} \\ \text{by the equivalent condition "} \sigma^{-1}(u) < \sigma^{-1}(v) \text{"} \\ \text{(the equivalence follows from (18))} \end{array} \right) \\
&= \left(\text{\# of } u \in \{1, 2, \ldots, n\} \text{ such that } \sigma^{-1}(u) < k \right) \qquad \left(\text{since } \sigma^{-1}(v) = k \right) \\
&= (\text{\# of } j \in \{1, 2, \ldots, n\} \text{ such that } j < k) \\
&\qquad \left(\begin{array}{c} \text{here, we have substituted } j \text{ for } \sigma^{-1}(u) \text{, since the} \\ \text{map } \sigma^{-1} : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\} \text{ is a bijection} \end{array} \right) \\
&= k - 1
\end{aligned}$$

(since the numbers $j \in \{1, 2, \ldots, n\}$ such that $j < k$ are precisely the numbers $1, 2, \ldots, k-1$, and thus there are $k-1$ of them). In view of $k = \sigma^{-1}(v)$, this rewrites as $\deg^- v = \sigma^{-1}(v) - 1$. This proves (17).

Forget that we fixed v. We thus have proved (17) for each vertex v of T_σ.

Next, we shall prove (15). Indeed, the definition of $\operatorname{sign}(T_\sigma)$ yields

$$\operatorname{sign}(T_\sigma) = \prod_{\substack{(i,j) \text{ is an} \\ \text{arc of } T_\sigma}} (-1)^{[i>j]} = \prod_{\substack{(u,v) \text{ is an} \\ \text{arc of } T_\sigma}} (-1)^{[u>v]}$$

(here, we have renamed the index (i, j) as (u, v) in the product). However, the arcs of T_σ are the pairs $(\sigma(i), \sigma(j))$, where i and j range over integers satisfying $1 \leq i < j \leq n$ (by the definition of T_σ). Thus, we can rewrite the product $\prod_{\substack{(u,v) \text{ is an} \\ \text{arc of } T_\sigma}} (-1)^{[u>v]}$ as follows:

$$\prod_{\substack{(u,v) \text{ is an} \\ \text{arc of } T_\sigma}} (-1)^{[u>v]} = \prod_{1 \leq i < j \leq n} (-1)^{[\sigma(i) > \sigma(j)]}$$

(here, we have tacitly used the fact that the pairs $(\sigma(i), \sigma(j))$ for all pairs of integers (i, j) satisfying $1 \leq i < j \leq n$ are distinct[6], and therefore each arc of T_σ can be

[6] This is clear because σ is injective.

written in the form $(\sigma(i), \sigma(j))$ for a **unique** pair (i,j)). Hence,

$$\begin{aligned}
\operatorname{sign}(T_\sigma) &= \prod_{\substack{(u,v) \text{ is an} \\ \text{arc of } T_\sigma}} (-1)^{[u>v]} = \prod_{1\leq i<j\leq n} (-1)^{[\sigma(i)>\sigma(j)]} \\
&= \left(\prod_{\substack{1\leq i<j\leq n; \\ \sigma(i)>\sigma(j)}} (-1)^{[\sigma(i)>\sigma(j)]} \right) \cdot \left(\prod_{\substack{1\leq i<j\leq n; \\ \textbf{not } \sigma(i)>\sigma(j)}} (-1)^{[\sigma(i)>\sigma(j)]} \right)
\end{aligned}$$

(since each pair (i,j) in our product either satisfies $\sigma(i) > \sigma(j)$ or doesn't). In view of

$$\begin{aligned}
&\prod_{\substack{1\leq i<j\leq n; \\ \sigma(i)>\sigma(j)}} \underbrace{(-1)^{[\sigma(i)>\sigma(j)]}}_{\substack{=(-1)^1 \\ (\text{since } \sigma(i)>\sigma(j) \\ \text{and thus } [\sigma(i)>\sigma(j)]=1)}} \\
&= \prod_{\substack{1\leq i<j\leq n; \\ \sigma(i)>\sigma(j)}} \underbrace{(-1)^1}_{=-1} = \prod_{\substack{1\leq i<j\leq n; \\ \sigma(i)>\sigma(j)}} (-1) \\
&= (-1)^{(\#\text{ of pairs } (i,j) \text{ of integers satisfying } 1\leq i<j\leq n \text{ and } \sigma(i)>\sigma(j))} \\
&= (-1)^{(\#\text{ of inversions of } \sigma)} \qquad \left(\begin{array}{c} \text{since the pairs } (i,j) \text{ of integers} \\ \text{satisfying } 1 \leq i < j \leq n \text{ and } \sigma(i) > \sigma(j) \\ \text{are known as the inversions of } \sigma \end{array} \right) \\
&= (-1)^{\ell(\sigma)} \qquad (\text{since the \# of inversions of } \sigma \text{ is called } \ell(\sigma)) \\
&= \operatorname{sign} \sigma \qquad \left(\text{because } \operatorname{sign} \sigma \text{ is defined to be } (-1)^{\ell(\sigma)}\right)
\end{aligned}$$

and

$$\prod_{\substack{1\leq i<j\leq n; \\ \textbf{not } \sigma(i)>\sigma(j)}} \underbrace{(-1)^{[\sigma(i)>\sigma(j)]}}_{\substack{=(-1)^0 \\ (\text{since we don't} \\ \text{have } \sigma(i)>\sigma(j), \text{ and thus} \\ \text{we have } [\sigma(i)>\sigma(j)]=0)}} = \prod_{\substack{1\leq i<j\leq n; \\ \textbf{not } \sigma(i)>\sigma(j)}} \underbrace{(-1)^0}_{=1} = 1,$$

we can simplify this to

$$\operatorname{sign}(T_\sigma) = \underbrace{\left(\prod_{\substack{1\leq i<j\leq n; \\ \sigma(i)>\sigma(j)}} (-1)^{[\sigma(i)>\sigma(j)]} \right)}_{=\operatorname{sign}\sigma} \cdot \underbrace{\left(\prod_{\substack{1\leq i<j\leq n; \\ \textbf{not } \sigma(i)>\sigma(j)}} (-1)^{[\sigma(i)>\sigma(j)]} \right)}_{=1} = \operatorname{sign} \sigma.$$

This proves (15).

It remains to prove (16). We first consider the product $\prod_{j=1}^{n} x_j^{\deg^- j}$, where $\deg^- j$ denotes the indegree of j as a vertex of T_σ. We can substitute $\sigma(i)$ for j in this product (since σ is a bijection from $\{1,2,\ldots,n\}$ to $\{1,2,\ldots,n\}$). Thus, we obtain

$$\prod_{j=1}^{n} x_j^{\deg^- j} = \prod_{i=1}^{n} x_{\sigma(i)}^{\deg^-(\sigma(i))}. \tag{20}$$

However, each $i \in \{1,2,\ldots,n\}$ satisfies

$$\deg^-(\sigma(i)) = \underbrace{\sigma^{-1}(\sigma(i))}_{=i} - 1 \qquad \text{(by (17), applied to } v = \sigma(i))$$
$$= i - 1.$$

In view of this, we can rewrite (20) as

$$\prod_{j=1}^{n} x_j^{\deg^- j} = \prod_{i=1}^{n} x_{\sigma(i)}^{i-1}. \tag{21}$$

Now, Proposition 1.5.10 (applied to $D = T_\sigma$) yields

$$w(T_\sigma) = \underbrace{(\operatorname{sign}(T_\sigma))}_{\substack{=\operatorname{sign}\sigma \\ \text{(by (15))}}} \cdot \underbrace{\prod_{j=1}^{n} x_j^{\deg^- j}}_{\substack{=\prod_{i=1}^{n} x_{\sigma(i)}^{i-1} \\ \text{(by (21))}}} = \operatorname{sign}\sigma \cdot \prod_{i=1}^{n} x_{\sigma(i)}^{i-1},$$

and thus (16) is proven. This completes the proof of Theorem 1.6.3 **(d)**.

(c) Let σ and τ be two distinct permutations in S_n. We must prove that the digraphs T_σ and T_τ are distinct.

Assume the contrary. Thus, $T_\sigma = T_\tau$. Hence, for each $v \in \{1,2,\ldots,n\}$, the indegree of v as a vertex of T_σ equals the indegree of v as a vertex of T_τ. Thus, we can allow ourselves to denote both of these indegrees by $\deg^- v$. From (17), we know that $\deg^- v = \sigma^{-1}(v) - 1$. Similarly, $\deg^- v = \tau^{-1}(v) - 1$. Comparing these two equalities, we find $\sigma^{-1}(v) - 1 = \tau^{-1}(v) - 1$. Hence, $\sigma^{-1}(v) = \tau^{-1}(v)$.

Forget that we fixed v. We thus have shown that $\sigma^{-1}(v) = \tau^{-1}(v)$ for each $v \in \{1,2,\ldots,n\}$. In other words, $\sigma^{-1} = \tau^{-1}$. Hence, $\sigma = \tau$. This contradicts the fact that σ and τ are distinct. This contradiction shows that our assumption was false. Thus, Theorem 1.6.3 **(c)** is proved. □

Remark 1.6.4. One alternative way to prove Theorem 1.6.3 **(b)** uses the fact (known as Rédei's Little Theorem) that any tournament has a Hamiltonian path (see, e.g., [lec7, Theorem 1.4.9] for a proof). Indeed, once this fact is known, we can pick a Hamiltonian path $(\tau(1), \tau(2), \ldots, \tau(n))$ of D, and argue (by strong induction on $j - i$) that each pair $(\tau(i), \tau(j))$ with $i < j$ must be an arc of D. But the above proof is more elementary.

Remark 1.6.5. It can be shown that for a tournament $D \in \mathcal{T}$, the following four statements are equivalent:

1. The tournament D has no 3-cycles.

2. The tournament D has no cycles of length 3.

3. The tournament D has no cycles (of any length).

4. The tournament D has the form T_σ for some $\sigma \in S_n$.

Indeed, the equivalence $1 \Longleftrightarrow 4$ is furnished by Theorem 1.6.3 **(b)**, whereas the implications $4 \Longrightarrow 3 \Longrightarrow 2 \Longrightarrow 1$ are easy to check.

A tournament D satisfying the four equivalent statements 1, 2, 3, 4 is said to be *nontransitive*.

We can now represent the left hand side of (7) as a sum that looks enticingly like the right hand side in Proposition 1.5.11:

Proposition 1.6.6. Define the $n \times n$-matrix V as in Theorem 1.4.1. Then,

$$\det V = \sum_{\substack{D \in \mathcal{T}; \\ D \text{ has no 3-cycles}}} w(D).$$

Proof. The (i,j)-th entry of the matrix V is x_j^{i-1} for all $i,j \in \{1,2,\ldots,n\}$. Thus, the definition of a determinant yields

$$\det V = \sum_{\sigma \in S_n} \underbrace{\operatorname{sign} \sigma \cdot \prod_{i=1}^{n} x_{\sigma(i)}^{i-1}}_{\substack{=w(T_\sigma) \\ \text{(by (16))}}} = \sum_{\sigma \in S_n} w(T_\sigma). \tag{22}$$

However, Theorem 1.6.3 **(b)** yields that the tournaments $D \in \mathcal{T}$ that have no 3-cycles are precisely the digraphs of the form T_σ with $\sigma \in S_n$. Furthermore, Theorem 1.6.3 **(c)** yields that each such tournament can be written as T_σ for a **unique** permutation $\sigma \in S_n$ (since distinct permutations σ lead to distinct tournaments T_σ). Thus,

$$\sum_{\substack{D \in \mathcal{T}; \\ D \text{ has no 3-cycles}}} w(D) = \sum_{\sigma \in S_n} w(T_\sigma).$$

Comparing this with (22), we obtain $\det V = \sum_{\substack{D \in \mathcal{T}; \\ D \text{ has no 3-cycles}}} w(D)$. This proves Proposition 1.6.6. $\square$

1.7. The numbers $w_0, w_1, w_2, \ldots$

Our goal is to prove that the left hand sides in Proposition 1.6.6 and in Proposition 1.5.11 are equal. To that purpose, we shall show that the right hand sides are equal. These right hand sides are already very similar:

$$\sum_{\substack{D\in\mathcal{T};\\ D \text{ has no 3-cycles}}} w(D) \qquad \text{versus} \qquad \sum_{D\in\mathcal{T}} w(D).$$

Yet, they at least look different: The latter is a sum containing a lot of addends that the former does not.

We shall reconcile this difference by showing that all these addends (i.e., all the addends corresponding to tournaments $D \in \mathcal{T}$ that have at least one 3-cycle) cancel each other out. This will be achieved by reversing the arcs of a cycle; Proposition 1.1.5 will come rather handy here.

First, we introduce some notations:

Convention 1.7.1. For each $k \in \mathbb{N}$, we let

$$w_k := \sum_{\substack{D\in\mathcal{T};\\ D \text{ has } k \text{ many 3-cycles}}} w(D).$$

(Here, "k many" means "exactly k many".)

Thus,

$$\sum_{D\in\mathcal{T}} w(D) = w_0 + w_1 + w_2 + \cdots \tag{23}$$

(this infinite sum is well-defined, since all sufficiently large $k \in \mathbb{N}$ satisfy $w_k = 0$). [7] Proposition 1.6.6 says that

$$\begin{aligned}\det V &= \sum_{\substack{D\in\mathcal{T};\\ D \text{ has no 3-cycles}}} w(D) = \sum_{\substack{D\in\mathcal{T};\\ D \text{ has 0 many 3-cycles}}} w(D) \\ &= w_0 \end{aligned}\tag{24}$$

(by the definition of w_0). If we can now show that all integers $k > 0$ satisfy $w_k = 0$, then we will see that the right hand sides of (24) and (23) are equal, and thus we will obtain

$$\det V = \sum_{D\in\mathcal{T}} w(D) = \prod_{1\leq i<j\leq n} \left(x_j - x_i\right) \qquad \text{(by Proposition 1.5.11)};$$

this will prove Theorem 1.4.1.

[7] Note that, because of the way we defined 3-cycles, the # of 3-cycles in a tournament D is always a multiple of 3, since each 3-cycle (u, v, w) leads to two other 3-cycles (v, w, u) and (w, u, v). So we have $w_k = 0$ for all $k \in \mathbb{N}$ that are not multiples of 3.

1.8. The great cancelling

So how do we prove that all $k > 0$ satisfy $w_k = 0$? We begin with lemmas:

Lemma 1.8.1. Let $D \in \mathcal{T}$ be a tournament. Let (u, v, w) be a 3-cycle of D. Let D' be the digraph obtained from D by reversing the arcs uv, vw and wu (this means replacing them by vu, wv and uw). Then, D' is again a tournament in $\mathcal{T}$, and satisfies

$$\operatorname{sign}(D') = -\operatorname{sign} D.$$

Proof. Clearly, D' is again a tournament (since reversing an arc in a tournament yields a tournament). It remains to prove that $\operatorname{sign}(D') = -\operatorname{sign} D$.

The pairs uv, vw and wu are arcs of D (since (u, v, w) is a 3-cycle of D). Hence, none of the pairs vu, wv and uw is an arc of D (by the tournament axiom, since D is a tournament).

We know that the digraph D' is obtained from D by reversing the arcs uv, vw and wu. Let us refer to all the other arcs of D as *inert*.

Thus, the arcs of D are uv, vw, wu and all the inert arcs of D. Therefore, the arcs of D' are vu, wv, uw and all the inert arcs of D (since D' is obtained from D by reversing the arcs uv, vw and wu). Note that none of the arcs vu, wv and uw appears among the inert arcs of D, since none of the pairs vu, wv and uw is an arc of D.

Now, the definition of $\operatorname{sign} D$ yields

$$\operatorname{sign} D = \prod_{\substack{(i,j) \text{ is an} \\ \text{arc of } D}} (-1)^{[i>j]} = (-1)^{[u>v]} \cdot (-1)^{[v>w]} \cdot (-1)^{[w>u]} \cdot \prod_{\substack{(i,j) \text{ is an} \\ \text{inert arc of } D}} (-1)^{[i>j]}$$

(since the arcs of D are uv, vw, wu and all the inert arcs). The definition of $\operatorname{sign}(D')$ yields

$$\operatorname{sign}(D') = \prod_{\substack{(i,j) \text{ is an} \\ \text{arc of } D'}} (-1)^{[i>j]} = (-1)^{[v>u]} \cdot (-1)^{[w>v]} \cdot (-1)^{[u>w]} \cdot \prod_{\substack{(i,j) \text{ is an} \\ \text{inert arc of } D}} (-1)^{[i>j]}$$

(since the arcs of D' are vu, wv, uw and all the inert arcs of D).

However, the vertices u and v are distinct (since (u, v, w) is a 3-cycle). Thus, exactly one of the two inequalities $u > v$ and $v > u$ holds. In other words, exactly one of the two truth values $[u > v]$ and $[v > u]$ equals 1, while the other equals 0. Hence, exactly one of the two numbers $(-1)^{[u>v]}$ and $(-1)^{[v>u]}$ equals -1, while the other equals 1. Therefore, these two numbers differ in sign but have the same magnitude. Consequently,

$$(-1)^{[u>v]} = -(-1)^{[v>u]}.$$

Similarly, $(-1)^{[v>w]} = -(-1)^{[w>v]}$ and $(-1)^{[w>u]} = -(-1)^{[u>w]}$. Now, our above formula for $\operatorname{sign} D$ becomes

$$\begin{aligned}\operatorname{sign} D &= \underbrace{(-1)^{[u>v]}}_{=-(-1)^{[v>u]}} \cdot \underbrace{(-1)^{[v>w]}}_{=-(-1)^{[w>v]}} \cdot \underbrace{(-1)^{[w>u]}}_{=-(-1)^{[u>w]}} \cdot \prod_{\substack{(i,j) \text{ is an} \\ \text{inert arc of } D}} (-1)^{[i>j]} \\ &= -\underbrace{(-1)^{[v>u]} \cdot (-1)^{[w>v]} \cdot (-1)^{[u>w]} \cdot \prod_{\substack{(i,j) \text{ is an} \\ \text{inert arc of } D}} (-1)^{[i>j]}}_{\substack{=\operatorname{sign}(D') \\ \text{(by our above formula for } \operatorname{sign}(D'))}} = -\operatorname{sign}(D').\end{aligned}$$

In other words, $\operatorname{sign}(D') = -\operatorname{sign} D$. This completes the proof of Lemma 1.8.1. $\square$

> **Lemma 1.8.2.** Let k be a positive integer. Let $(d_1, d_2, \ldots, d_n) \in \mathbb{N}^n$ be any n-tuple of nonnegative integers. Then,
>
> $$\sum_{\substack{D \in \mathcal{T}; \\ D \text{ has } k \text{ many 3-cycles}; \\ \deg^- i = d_i \text{ for each } i}} \operatorname{sign} D = 0.$$
>
> (Here, $\deg^- i$ means the indegree of the vertex i in the digraph D. Also, "for each i" means "for each $i \in \{1, 2, \ldots, n\}$".)

Proof. A *flippy pair* shall mean a pair (D, α), where

- $D \in \mathcal{T}$ is a tournament having k many 3-cycles and satisfying $\deg^- i = d_i$ for each i;
- α is a 3-cycle of D.

If (D, α) is a flippy pair, then we define a new flippy pair $\operatorname{flip}(D, \alpha)$ as follows:

- Let (u, v, w) be the 3-cycle α.
- We obtain a new digraph D' from D by reversing the arcs uv, vw and wu (this means replacing them by vu, wv and uw). Note that this digraph D' is again a tournament in $\mathcal{T}$, and again has k many 3-cycles (because Proposition 1.1.5 shows that (# of 3-cycles of D') = (# of 3-cycles of D) $= k$). This tournament D' furthermore satisfies the equalities $\deg^- i = d_i$ for each i (since D satisfies these equalities, and since the indegrees of the vertices have not changed from D to D' [8]).
- We let α' be the 3-cycle (u, w, v) of D'. (This is indeed a 3-cycle of D', due to the construction of D'.)

[8] because each of the three vertices u, v and w lost one incoming arc and gained another when we reversed the arcs uv, vw and wu

- We let $\operatorname{flip}(D, \alpha)$ be the flippy pair (D', α'). (This is indeed a flippy pair, because we have seen that D' is a tournament in $\mathcal{T}$ having k many 3-cycles and satisfying $\deg^- i = d_i$ for each i, and that α' is a 3-cycle of D'.)

Thus, we have defined a map flip that sends flippy pairs to flippy pairs. It is easy to see that this map is its own inverse: That is, if (D, α) is a flippy pair, and if $(D', \alpha') = \operatorname{flip}(D, \alpha)$, then $(D, \alpha) = \operatorname{flip}(D', \alpha')$ (because D' is obtained from D by reversing the arcs uv, vw and wu, and thus D can be recovered from D' by reversing the arcs uw, wv and vu).

Furthermore, the map flip changes the sign of a tournament: That is, if (D, α) is a flippy pair, and if $(D', \alpha') = \operatorname{flip}(D, \alpha)$, then

$$\operatorname{sign}(D') = -\operatorname{sign} D. \tag{25}$$

[*Proof of (25):* Let (D, α) be a flippy pair. Let $(D', \alpha') = \operatorname{flip}(D, \alpha)$. By the definition of the map flip, we know that the digraph D' is obtained from D by reversing the arcs uv, vw and wu, where (u, v, w) is the 3-cycle α. Thus, Lemma 1.8.1 yields $\operatorname{sign}(D') = -\operatorname{sign} D$. This proves (25).]

Thus, if (D, α) is a flippy pair satisfying $\operatorname{sign} D = 1$, and if $(D', \alpha') = \operatorname{flip}(D, \alpha)$, then (D', α') is a flippy pair satisfying $\operatorname{sign}(D') = -1$ (because (25) yields $\operatorname{sign}(D') = -\underbrace{\operatorname{sign} D}_{=1} = -1$). Hence, we obtain a map

$$\begin{aligned} &\text{from the set } \{\text{flippy pairs } (D, \alpha) \text{ satisfying } \operatorname{sign} D = 1\} \\ &\text{to the set } \{\text{flippy pairs } (D, \alpha) \text{ satisfying } \operatorname{sign} D = -1\}, \end{aligned}$$

which sends each flippy pair (D, α) to $\operatorname{flip}(D, \alpha)$. Similarly, we obtain a map

$$\begin{aligned} &\text{from the set } \{\text{flippy pairs } (D, \alpha) \text{ satisfying } \operatorname{sign} D = -1\} \\ &\text{to the set } \{\text{flippy pairs } (D, \alpha) \text{ satisfying } \operatorname{sign} D = 1\}, \end{aligned}$$

which sends each flippy pair (D, α) to $\operatorname{flip}(D, \alpha)$. These two maps are mutually inverse (since the map flip is its own inverse), and thus are bijections. Hence, the bijection principle yields that

$$\begin{aligned} &|\{\text{flippy pairs } (D, \alpha) \text{ satisfying } \operatorname{sign} D = 1\}| \\ &= |\{\text{flippy pairs } (D, \alpha) \text{ satisfying } \operatorname{sign} D = -1\}|. \end{aligned}$$

In other words, there are as many flippy pairs (D, α) satisfying $\operatorname{sign} D = 1$ as there are flippy pairs (D, α) satisfying $\operatorname{sign} D = -1$. Hence, in the sum

$$\sum_{(D,\alpha) \text{ is a flippy pair}} \operatorname{sign} D,$$

the addends equal to 1 and the addends equal to -1 are equinumerous, and consequently these addends cancel each other out. The sum therefore equals 0 (since each addend of this sum is either a 1 or a -1). In other words,

$$\sum_{(D,\alpha) \text{ is a flippy pair}} \operatorname{sign} D = 0. \tag{26}$$

However, recall that the D in a flippy pair (D, α) has to be a tournament in $\mathcal{T}$ having k many 3-cycles and satisfying $\deg^- i = d_i$ for each i, whereas the α has to be a 3-cycle of D. Thus, the summation sign $\sum\limits_{(D,\alpha) \text{ is a flippy pair}}$ can be rewritten as follows:

$$\sum_{(D,\alpha) \text{ is a flippy pair}} = \sum_{\substack{D \in \mathcal{T}; \\ D \text{ has } k \text{ many 3-cycles;} \\ \deg^- i = d_i \text{ for each } i}} \sum_{\alpha \text{ is a 3-cycle of } D}.$$

Thus,

$$\begin{aligned}
\sum_{(D,\alpha) \text{ is a flippy pair}} \operatorname{sign} D &= \sum_{\substack{D \in \mathcal{T}; \\ D \text{ has } k \text{ many 3-cycles;} \\ \deg^- i = d_i \text{ for each } i}} \underbrace{\sum_{\alpha \text{ is a 3-cycle of } D} \operatorname{sign} D}_{= (\# \text{ of 3-cycles of } D) \cdot \operatorname{sign} D} \\
&= \sum_{\substack{D \in \mathcal{T}; \\ D \text{ has } k \text{ many 3-cycles;} \\ \deg^- i = d_i \text{ for each } i}} \underbrace{(\# \text{ of 3-cycles of } D)}_{\substack{= k \\ (\text{since } D \text{ has } k \text{ many 3-cycles})}} \cdot \operatorname{sign} D \\
&= \sum_{\substack{D \in \mathcal{T}; \\ D \text{ has } k \text{ many 3-cycles;} \\ \deg^- i = d_i \text{ for each } i}} k \cdot \operatorname{sign} D = k \sum_{\substack{D \in \mathcal{T}; \\ D \text{ has } k \text{ many 3-cycles;} \\ \deg^- i = d_i \text{ for each } i}} \operatorname{sign} D.
\end{aligned}$$

Therefore, (26) can be rewritten as

$$k \sum_{\substack{D \in \mathcal{T}; \\ D \text{ has } k \text{ many 3-cycles;} \\ \deg^- i = d_i \text{ for each } i}} \operatorname{sign} D = 0.$$

We can divide this equality by k (since k is positive), and obtain

$$\sum_{\substack{D \in \mathcal{T}; \\ D \text{ has } k \text{ many 3-cycles;} \\ \deg^- i = d_i \text{ for each } i}} \operatorname{sign} D = 0.$$

This proves Lemma 1.8.2. □

Lemma 1.8.3. Let k be a positive integer. Let $(d_1, d_2, \ldots, d_n) \in \mathbb{N}^n$ be any n-tuple of nonnegative integers. Then,

$$\sum_{\substack{D \in \mathcal{T}; \\ D \text{ has } k \text{ many 3-cycles;} \\ \deg^- i = d_i \text{ for each } i}} w(D) = 0.$$

(Here, $\deg^- i$ means the indegree of the vertex i in the digraph D. Also, "for each i" means "for each $i \in \{1, 2, \ldots, n\}$".)

Proof. We have

$$\begin{aligned}
&\sum_{\substack{D\in\mathcal{T};\\ D \text{ has } k \text{ many 3-cycles;}\\ \deg^- i=d_i \text{ for each } i}} \underbrace{w(D)}_{\substack{=(\operatorname{sign} D)\cdot \prod_{j=1}^{n} x_j^{\deg^- j}\\ \text{(by Proposition 1.5.10)}}} \\
&= \sum_{\substack{D\in\mathcal{T};\\ D \text{ has } k \text{ many 3-cycles;}\\ \deg^- i=d_i \text{ for each } i}} (\operatorname{sign} D)\cdot \prod_{j=1}^{n} \underbrace{x_j^{\deg^- j}}_{\substack{=x_j^{d_j}\\ \text{(since } \deg^- j=d_j\\ \text{(because } \deg^- i=d_i \text{ for each } i\text{))}}} \\
&= \sum_{\substack{D\in\mathcal{T};\\ D \text{ has } k \text{ many 3-cycles;}\\ \deg^- i=d_i \text{ for each } i}} (\operatorname{sign} D)\cdot \prod_{j=1}^{n} x_j^{d_j} = \left(\prod_{j=1}^{n} x_j^{d_j}\right) \underbrace{\sum_{\substack{D\in\mathcal{T};\\ D \text{ has } k \text{ many 3-cycles;}\\ \deg^- i=d_i \text{ for each } i}} \operatorname{sign} D}_{\substack{=0\\ \text{(by Lemma 1.8.2)}}} \\
&= 0.
\end{aligned}$$

This proves Lemma 1.8.3. ☐

Lemma 1.8.4. Let k be a positive integer. Then, $w_k = 0$.

Proof. The definition of w_k yields

$$\begin{aligned}
w_k &= \sum_{\substack{D\in\mathcal{T};\\ D \text{ has } k \text{ many 3-cycles}}} w(D) \\
&= \sum_{(d_1,d_2,\ldots,d_n)\in\mathbb{N}^n} \underbrace{\sum_{\substack{D\in\mathcal{T};\\ D \text{ has } k \text{ many 3-cycles;}\\ \deg^- i=d_i \text{ for each } i}} w(D)}_{\substack{=0\\ \text{(by Lemma 1.8.3)}}} \\
&\qquad \left(\begin{array}{c}\text{here, we have split up the sum according}\\ \text{to the } n\text{-tuple } \left(\deg^- 1, \deg^- 2, \ldots, \deg^- n\right)\end{array}\right) \\
&= 0.
\end{aligned}$$

This proves Lemma 1.8.4. ☐

1.9. The finish line

Proving Theorem 1.4.1 is now a matter of combining what we know:

Proof of Theorem 1.4.1. Proposition 1.5.11 yields

$$\begin{aligned}\prod_{1\leq i<j\leq n}\left(x_j-x_i\right) &= \sum_{D\in\mathcal{T}} w\left(D\right) = w_0+w_1+w_2+\cdots \qquad (\text{by (23)})\\ &= \sum_{k\in\mathbb{N}} w_k = w_0 + \sum_{k>0} \underbrace{w_k}_{\substack{=0\\ (\text{by Lemma 1.8.4})}} = w_0 + \underbrace{\sum_{k>0} 0}_{=0} = w_0 = \det V\end{aligned}$$

(by (24)). Thus follows Theorem 1.4.1. □

Here ends our scenic route to the Vandermonde determinant. A different combinatorial proof – also using tournaments – is sketched in [Bresso99, Exercises 2.4.1–2.4.6]. Yet another (not using tournaments) appears in [BenDre07]. Moreover, several variants of the Vandermonde determinant (type-B, type-C and type-D versions, for those who know the lingo of Coxeter groups) have been proved using tournaments by Bressoud [Bresso87].

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