The Aharonov-Bohm Effect: Electromagnetic Potentials and the Geometric Phase

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Abstract

The discovery of the Aharonov-Bohm effect in 1959 demonstrated that electron interference fringes could be shifted by a localised magnetic field, despite the field being zero at the location of the electron. There are two interpretations of this effect: the first implies that the vector potential must be considered a real quantity in quantum theory, if the principle of locality is to be upheld; secondly, the effect can be considered a nonlocal interaction between the electron and magnetic flux, retaining the classical idea of a non-physical vector potential. This review begins with an introduction to the Aharonov-Bohm effect and a discussion of the gauge freedom of the vector potential, before exploring its connection to the geometric phase. Experimental confirmations are reviewed and applications to topological insulators and the anomalous hall effect are presented.

Contents

1 Introduction

The Aharonov-Bohm (AB) effect demonstrated a remarkable prediction of quantum mechanics - electron interference fringes could be shifted by changes in a distant, localised magnetic flux despite the magnetic field remaining zero at the location of the electron. The profound and somewhat controversial implications of the effect can be realised by a thought experiment based on the double-slit diffraction of an electron beam. A modification is an infinitely thin solenoid positioned behind the slits giving a highly localised delta function flux line that is never intercepted by the electron along its path. In classical electrodynamics, the Lorentz force on a moving electron is zero in the region of zero magnetic field, and hence no measurable effect can be observed. However, the AB effect implied that the effects observed in field-free regions are due to the magnetic vector potential - a classically immeasurable vector field. The evolution of an electron is described by its Hamiltonian which inherently relies on potentials, specifically the vector potential. In 1959, Yakir Aharonov and David Bohm proposed that the vector potential is a real entity [1] with physical significance in quantum theory. This preserves the principle of locality in quantum mechanics - the electron acquires a phase shift in its wavefunction by local interaction with the vector potential itself rather than the magnetic field; an effect with no classical analog.

The gauge freedom of the vector potential led to an interpretation of the effect in terms of nonlocality - a concept that had been proposed by Einstein, Podolsky and Rosen in 1935 [2]. Since the vector potential is not considered to be physical due to gauge freedom, the electron interacts nonlocally with the magnetic flux - a concept that many scientists, including Aharonov and Bohm, were initially reluctant to accept. This is further supported by the fact that the vector potential can be eliminated from the electron phase shift derived by Aharonov and Bohm by use of Stokes' theorem, reinforcing the purely mathematical role of the vector potential.

Initial confirmation of the AB effect was made by Robert G. Chambers in 1960 at the University of Bristol, England [3]. This was received with much scepticism, in particular regarding the possibility of magnetic flux leakage. Improvements to Chambers' work were made by Akira Tonomura *et al.* by use of a toroidal magnet confined by a superconductor, exploiting the Meissner effect to eliminate flux leakage [4].

Despite its controversies over the past 60 years, the AB effect raises two fundamental concepts. The first is that the vector potential is a physical field in quantum mechanics, the second being the inherent nonlocality of quantum interactions, and the fact that a definite explanation of the AB effect remains unclear indicates that we are still lacking an intuitive understanding of quantum theory.

We review the AB effect in chapters 2, 3 and 4. In chapter 2, we discuss the gauge freedom of the vector potential alongside Maxwell's equations, and its implications in classical theory. Another form of the effect - the electric Aharonov-Bohm effect - is discussed in chapter 3 and experimental confirmations are reviewed in chapter 4. Chapter 5 examines Berry phase as a unifying concept to interpret the AB effect by consideration of cyclic adiabatic evolution of a quantum state. In chapter 6, we analyse practical applications, in the contexts of a topological insulator and the anomalous Hall effect.

2 The Magnetic Aharonov-Bohm Effect

As a graduate student at the University of Bristol, England, Yakir Aharonov worked on topological aspects of quantum mechanics, under the supervision of the eminent theorist David Bohm. Inspired by attending lectures on Bloch waves in spatially periodic potentials, Aharonov became interested scalar potentials which were periodic not in space, but in time. By changing this symmetry, the allowed wavefunctions now had a gap, not in energy, but in wavevector.

In his consideration of time varying potentials, Aharonov realised that one could measure a relative phase acquired by an electron in a changing potential, even if it is not in an electric field. With the help of Bohm, he extended this idea to the simpler case of a magnetic vector potential, in a paper which they coauthored in 1959. [1].

2.1 An Introductory Thought Experiment

In the Aharonov-Bohm effect, the wavefunction of an electron acquires a complex phase factor when it travels through a field-free region. Since a single phase cannot be measured physically, the effect can only be observed by performing an interference experiment, measuring the relative phases of two electron paths. The thought experiment devised by Aharonov is as follows: a solenoid is placed in the region between the slits and the detector; the electron beam will travel around both sides of the solenoid, once again forming an interference pattern on the other side as a result of the path difference from different paths taken. For an infinitely long solenoid, there will be a uniform magnetic field inside the solenoid and zero field outside. Any observer outside the solenoid should not be able to tell if there is a $\vec{B}$ field inside the solenoid or not. If there is a non-zero $\vec{B}$ field inside the solenoid, there exists a magnetic vector potential $\vec{A}$ outside of the solenoid.

In order to describe the effects on an electron in the presence of this solenoid, the two electromagnetic potentials must be considered in the region the electrons penetrate. There is no charge in the solenoid, hence the electric field gives

$$\vec{E} = -\vec{\nabla}V = 0. \tag{1}$$

Therefore, setting $V = 0$ for the electric vector potential will satisfy (1). There are two conditions required for the magnetic vector. Firstly, the definition of $\vec{B}$ must be obeyed:

$$\vec{B} = \vec{\nabla} \times \vec{A}. \tag{2}$$

The second is expressed via Stoke's theorem:

$$\oint_C \vec{A} \cdot d\vec{r} = \int_S (\vec{\nabla} \times \vec{A}) \cdot d\vec{S} = \int_S \vec{B} \cdot d\vec{S} = \Phi_B, \tag{3}$$

where C is the path round the solenoid and Φ_B the total magnetic flux through the solenoid.
The ideal form of $\vec{B}$, assumed here, is a single flux at one point in the region (solenoid) and can therefore be expressed using delta functions:

$$\vec{B} = \Phi_B \delta(x - x_0)\delta(y - y_0)\hat{\vec{e_z}}, \tag{4}$$

where x_0, y_0 are the coordinates of the chosen origin. The magnetic vector potential is usually chosen to be

$$\vec{A} = \frac{\Phi_B}{2\pi} \frac{x\hat{\vec{e_y}} - y\hat{\vec{e_x}}}{r^2} \tag{5}$$

to satisfy the above conditions. The vector potential is therefore non-zero outside the solenoid if there is a $\vec{B}$ field inside.

The wavefunction of the electron is then expressed in the form

$$|\psi(\vec{r}, t)\rangle = e^{ig(\vec{r})} |\psi'(\vec{r}, t)\rangle . \tag{6}$$

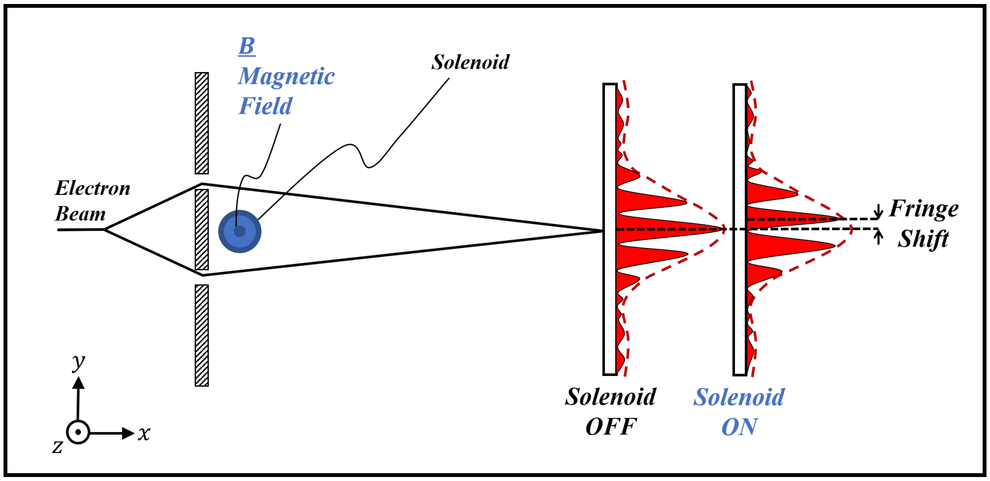

Figure 1: The apparatus for the AB experiment. The electrons originate from the source and pass through both slits, taking paths around the solenoid (shown by the circular $\vec{B}$ region). They reconvene on the detector to generate an interference pattern.

where $g(\vec{r})$ is the additional phase factor arising from the $\vec{B}$ field:

$$g(\vec{r}) = \frac{e}{\hbar}\int_0^{\vec{r}} \vec{A}(\vec{r'})\cdot d\vec{r'} \tag{7}$$

If the two paths (C_1, C_2) taken are expressed as

$$C_1 :\ |\psi_1(\vec{r},t)\rangle = e^{\int_{C_1} \vec{A}(\vec{r'})\cdot d\vec{r'}}\,|\psi_1'(\vec{r},t)\rangle \tag{8a}$$

$$C_2 :\ |\psi_2(\vec{r},t)\rangle = e^{\int_{C_2} \vec{A}(\vec{r'})\cdot d\vec{r'}}\,|\psi_2'(\vec{r},t)\rangle \tag{8b}$$

then the phase difference for the two paths is given by

$$\Delta g = g_1 - g_2 \tag{9a}$$

$$= \frac{e}{\hbar}\Big[\int_{C_1} \vec{A}(\vec{r'})\cdot d\vec{r'} - \int_{C_2} \vec{A}(\vec{r'})\cdot d\vec{r'}\Big] \tag{9b}$$

$$= \frac{e}{\hbar}\oint_C \vec{A}(\vec{r'})\cdot d\vec{r'} = \frac{e}{\hbar}\Phi_B, \tag{9c}$$

where the path difference forms a closed path round the solenoid, leading to an expression for the phase difference being proportional to the magnetic flux inside the solenoid.

The interference pattern of the electron will therefore be shifted by an amount proportional to the magnetic flux. This results in a shifting of the interference fringes, but without changing the envelope of the pattern. However, recall that the electron has no contact with the magnetic flux along its path; the flux is non-zero only *inside* the solenoid. It seems, therefore, that the vector potential is somehow inducing the phase shift; it is a mediator between Φ_B and the electron. However, the vector potential is classically regarded as a purely mathematical construct and does not represent anything physical due its *gauge freedom*.

2.2 Gauge Transformations of Potentials

Gauge Transformation in General

Many readers will have first encountered gauge transformations even when considering the gravitational potential energy of an object. The potential energy of a mass m at a height h is given by

$$U = mgh. \tag{10}$$

In practice, the height is usually measured with respect to the ground potential, $U = 0$. However, it is completely plausible to measure this height with respect to another reference point, such as the center of the Earth. In this respect, we have freedom of choice for the origin of our defined coordinate system: more formally, a *gauge freedom*. Furthermore, let us say that we would like to measure the potential energy of a mass with respect to the Earth's core, when we know that it is a height h above the ground. We could change the origin of our coordinate system, from the ground to the centre of the Earth, using a gauge transformation

$$U = mgh + C, \tag{11}$$

recalling that the gradient of the potential is related to the force on the mass by

$$\vec{F} = -\vec{\nabla} U. \tag{12}$$

It can be shown trivially that an additive constant in the potential introduces no change in the force, and therefore, the equations of motion of an object. The freedom to transform the coordinate system without changing the equations of motion is called *gauge invariance*.

Electromagnetic Gauge Transformations

Classical electromagnetism is governed by Maxwell's equations, in conjunction with the Lorentz force law. These equations describe how electric and magnetic fields are produced by charges and currents respectively. In addition, the equations show that an alternating magnetic field produces an electric field and *vice versa* [5]:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{13}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{14}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{15}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{16}$$

Electromagnetism was the first example of *gauge symmetry* within classical physics. The electric field $\vec{E}$ and magnetic field $\vec{B}$ are related to the scalar electric potential V and magnetic vector potential $\vec{A}$ by

$$\vec{B} = \vec{\nabla} \times \vec{A} \tag{17}$$

and

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}. \tag{18}$$

Suppose we have two sets of potentials $(V, \vec{A})$ and $(V', \vec{A}')$ which correspond to the same electric and magnetic field. How much can they differ and still satisfy this condition? Quantitatively,

$$\vec{A}' = \vec{A} + \vec{\theta}$$

and

$$V' = V + \chi. \tag{19}$$

If the two vector potentials correspond to the same magnetic field then we require

$$\vec{\nabla} \times \vec{\theta} = 0. \tag{20}$$

Noting that the curl of any continuously differentiable scalar function is equal to zero, we can write

$$\vec{\theta} = \vec{\nabla} \lambda. \tag{21}$$

Moreover, we need to get the same electric field, therefore

$$\vec{\nabla} \chi + \frac{\partial \vec{\Phi}_B}{\partial t} = \vec{\nabla} \Big(\chi + \frac{\partial \lambda}{\partial t} \Big) = 0. \tag{22}$$

The term in parentheses must not depend on position (since its gradient is identically zero everywhere), however it could depend on time, say $k(t)$:

$$\chi = -\frac{\partial \lambda}{\partial t} + k(t). \tag{23}$$

We can absorb $k(t)$ into λ without changing the gradient of λ. It then follows that

$$\vec{A}' = \vec{A} + \vec{\nabla} \lambda \tag{24}$$

and

$$V' = V - \frac{\partial \lambda}{\partial t}. \tag{25}$$

Thus, both the magnetic field and electric field are invariant under these new transformations, implying that $\vec{A}$ and V are not uniquely defined. This leads to the notion that the potentials can not symbolise anything physical [6].

We can further this argument by considering the Lorentz force which describes the force on a particle in the presence of an electric field or magnetic field or both:

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}). \tag{26}$$

From classical mechanics, if we know the equations of motions of a particle we can model everything about the system. In a region where $\vec{B} = 0$ even if $\vec{A}$ is non-zero, such as a solenoid, there is no identifiable effect of $\vec{A}$. Therefore, it was thought that $\vec{A}$ could not symbolise anything physical. The non-physical association with the vector potential was challenged in the advent of quantum mechanics.

In contrast to classical mechanics, quantum mechanics does not use Newtonian concepts such as force, focusing instead on energies and momenta. A particle is no longer described by a point, but a probability wave in space and time. The wave itself is comprised of wavelengths related to momenta and frequencies related to energies [7]. As a result, the energies and momenta are used to describe the evolution of a system meaning that potentials become a significant property within quantum mechanics, especially when considering AB effect. For this reason, the central equation of quantum mechanics, the Schrödinger equation, is conventionally written in terms of potentials. Quantum mechanically, only quantities which are gauge invariant can be measured. For this reason, the gauge invariant magnetic field was considered to be “real”, whereas the gauge dependent vector potential was considered to be non-physical, just a mathematical convenience. However, we have seen that a complication arises in the AB effect - it seems as though the vector potential is affecting the electrons because they have no contact with the magnetic flux. Aharonov and Bohm interpreted this consequence, previously believed to be immeasurable, as indicative of the “real" significance of potentials in quantum theory. This interpretation preserves the classical notion of *locality*, whereby objects can only interact if they are in the same place as the field, or via a mediator. Nevertheless, there is another interpretation in which this is not the case - nonlocality.

2.3 Nonlocal Interpretations

We have seen from the AB effect that the vector potential $\vec{A}$ can be regarded as something physical, on par with the field itself. However, this is troublesome given the gauge symmetry of Maxwell’s equations. Another way of viewing this is to abandon the intuitive principle of locality and treat the system as nonlocal. In this case, the electron interacts directly with the magnetic field, and the vector potential plays no part (and therefore is still a purely mathematical construct). The fact that the electron and magnetic flux are separated by a significant distance and do not enter into any common region does not matter. A way to picture this is that the electron and magnetic flux are instantaneously exchanging information, independent on their separation. This nonlocal interpretation has been studied in depth [8], [9].

Aharonov reviewed new methods [10], [11] which enable the AB effect to be determined at a precise time (previously one could only observe the interference pattern on the screen and have no knowledge of where and when this interaction happened). These methods measure the state of the electron (the modular drift velocity is measured) at sometime. It was found that the diffracting electron seems to change at the instant the electron wave packets and solenoid are in a straight line. This is explained [8] in terms of an instantaneous interaction between the solenoid and the electron wave packet. Such an interpretation is encouraged more by the violation of Bell’s inequality [12] and seems to be the more favoured way of interpreting the effect.

3 The Electric Aharonov-Bohm Effect

Although the magnetic AB effect has been verified experimentally and studied comprehensively, the electric AB effect is far more difficult to observe. In fact, up until now it has not been confirmed experimentally [13], despite multiple attempts [14], [15], [16], [17]. A more recent experiment was devised [18] in which the moving electrons are replaced by hydrogen ions - a higher mass-charge ratio - in order to lower the velocity. Among these experiments, none of them could completely eliminate the "real" force acting on the moving charge due to the influence of the "real" magnetic or electric field. The effect of the potential has not been reported quantitatively as well.

To understand why, we will first look at how such an effect can be observed. The following argument is similar to the original paper by Aharonov and Bohm [19] and a more recent paper by Rui-Feng Wang [20]. They proposed a possible experiment to observe the electric AB effect. Shown in Fig. 2, a coherent charged beam is split into two and each passes down a long cylindrical tube, which has to be metal to generate an electric field in, and then re-combine and interfere at the screen. The charged beam, divided into wave packets, is long compared to the wavelength of the charge but short compared to the tube. The charge enters the tube at t_0 and leaves at t_3. During $t_1 \leq t \leq t_2$, an electric potential difference V_0 is applied to the tubes. In this example, tube 1 is always connected to zero potential (earthed) while tube 2 is connected to an external voltage generator. The external voltage generator $U(t)$ is zero for $t < t_1$, V_0 for $t_1 \leq t \leq t_2$, and zero for $t > t_2$. Let us consider the case in which the voltage is switched off - meaning that $U(t)$ is zero for all t. If $\psi_1^0(x,t)$ and $\psi_2^0(x,t)$ represent the wavefunctions in tube 1 and tube 2 respectively, then their time evolutions are given by Schrödinger's equation (a position representation of the wavefunction will be used in this section):

$$i\hbar\frac{\partial\psi_1^0(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi_1^0(x,t)}{\partial x^2} \tag{27}$$

and

$$i\hbar\frac{\partial\psi_2^0(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi_2^0(x,t)}{\partial x^2}, \tag{28}$$

and the total wavefunction $\psi^0(x,t)$ is just $\psi_1^0(x,t) + \psi_2^0(x,t)$.

Now $U(t)$ is switched on and defined as above. The corresponding Schrödinger's equation is:

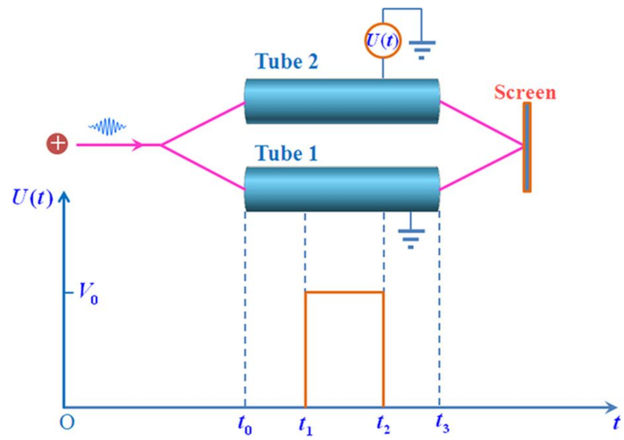

Figure 2: Figure showing basic setup of a possible experiment to observe the electric AB effect. $U(t)$ is the external voltage. Charge enters the tube at t_0. $U(t)$ is switched on at t_1 and off at t_2. Charge leaves at t_3. Credit: Rui-Feng Wang
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$$i\hbar\frac{\partial\psi_1(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi_1(x,t)}{\partial x^2} \tag{29}$$

$$i\hbar\frac{\partial\psi_2(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi_2(x,t)}{\partial x^2} + qU(t)\psi_2(x,t), \tag{30}$$

where $\psi_1(x,t)$ and $\psi_2(x,t)$ represent the wavefunctions perturbed by the electric potential in tube 1 and tube 2. Again, the total wavefunction $\psi(x,t)$ is just the sum of these two.

Comparing (27) & (28) and (29) & (30), we can see that

$$\psi_1(x,t) = \psi_1^0(x,t) \tag{31}$$

and

$$\psi_2(x,t) = \psi_2^0(x,t)e^{-\frac{iq}{\hbar}\int_0^t U(t')dt'}. \tag{32}$$

Therefore, when $t > t_2$,the wavefunction becomes:

$$\psi(x,t) = \psi_1^0(x,t) + \psi_2^0(x,t)e^{-\frac{iqV_0(t_2-t_1)}{\hbar}}. \tag{33}$$

Comparing this to the wavefunction of a switched-off voltage generator, a phase is acquired:

$$\varphi = \frac{iqV_0(t_2-t_1)}{\hbar}. \tag{34}$$

So in theory, the interference pattern should vary with the value $iqV_0(t_2 - t_1)$.

Rui-Feng Wang, in his paper [20], showed that a critical factor was neglected by Aharonov and Bohm in the original paper - the induced charge of the tube was not taken into account. This implies that the electric field exists in all space, meaning that the electric field is in fact non-zero in the interior of the metal tube. Furthermore, this field will penetrate into regions outside the metal tubes and thus cause overlapping between the electric fields of the two tubes and the one generated by the external voltage generator. Hence, the discussion above is unreliable since the induced charge is not taken into account.

If we include the induced charge q' which is equal to $-q$ and do the same analysis as before, we find that the induced charge cancels the effect of the potential:

$$\psi(x,t) = \psi_1(x,t_0)e^{\frac{iE_1 t}{\hbar}} + \psi_2(x,t_0)e^{\frac{iE_2 t}{\hbar}} \tag{35}$$

and

$$\psi(x,t) = \psi_1(x,t_0)e^{\frac{iE_1 t}{\hbar}} + \psi_2(x,t_0)e^{\frac{iE_2 t}{\hbar}}. \tag{36}$$

$$E_1 = E_K + E_{K'} + q'\alpha(y) + \frac{qq'}{4\pi\epsilon_0|x-y|} \tag{37}$$

and

$$E_2 = E_K + E_{K'} + q'\alpha(y) + \frac{qq'}{4\pi\epsilon_0|x-y|} + qU(t) + q'U(t) \tag{38}$$

E_K is the kinetic energy of the charge q and $E_{K'}$ is the kinetic energy of the induced charge.The parameter y is the position of the induced charge. The function $\alpha(t)$ is the potential function experienced by q' in the metal tube which ensures it does not leave the surface of the tube, independent of $U(t)$. The term $\frac{qq'}{4\pi\epsilon_0|x-y|}$ is the Coulomb term which arises due to the interaction between the charge and the induced charge. Putting q'=-q, we see that the contribution of the external potential is cancelled (no phase shift due to $U(t)$), yielding E_1=E_2, meaning that the wavefunctions evolves in the same manner, therefore no interference pattern can be observed.

Since in practice a charge always induces a charge in a conductor, it obvious to see why all the experiments so far cannot conclude with certainty that the interference pattern is solely due to the scalar potential $U(t)$, though one can state that the phase change is due to the potential energy of the whole system, including the induced charge.

4 Experimental Confirmations

4.1 First Discoveries of the Effect

Although Aharonov and Bohm are credited with recognizing the significance of the effect, they were not in fact the first to discover it. In 1949, ten years previously, Ehrenberg and Siday had published a paper in which they had highlighted the "significance of the arbitrariness of the magnetic vector potential" [21]. Unfortunately, due to the opaque title and heavy use of esoteric optical language, their work had largely gone unnoticed, despite reaching the same conclusions. Additionally, some claim that Walter Franz discovered the effect, based on a series of lectures given in Danzig in 1939. It seems that, although Franz may have had the idea, he did not pursue it further, lacking in the resources to do so [22]. Aharonov and Bohm had not been aware of Ehrenberg and Siday's results in 1959 when they wrote their original paper, however they were acknowledged in the followup paper on what Bohm now referred to as the ES-AB effect [23].

Among various other claims of scientists who already knew about the effect, in 1956 Möllenstedt and Dürker had performed the experiment but it had not been widely recognised, as either the conclusion was not clear or it was only ever published in German. Some claim that the credit goes to Aharonov and Bohm because of the latter's high profile in physics. Others simply state that Aharonov and Bohm were responsible for the clearest recognition of the significance and the highest quality work in the field.

4.2 Initial Confirmation - R. G. Chambers

The first positive experimental confirmation of the AB effect is accredited to R. G. Chambers, who first performed the thought experiment described in the original 1959 paper [24], [19]. Chambers, who was also working in Bristol, discussed with Aharonov and colleagues how one may overcome the experimental difficulties in verifying the effect.

The difficulties in demonstrating the simpler, magnetic AB effect were twofold: an object is needed to take the place of an infinitely long, infinitely thin solenoid; and the path separation must enclose a relatively large object, and interfere again.

The first issue was resolved by Sir Charles Frank, who suggested that an *iron whisker* would act approximately like an infinite solenoid. An iron whisker is formed of a single crystal of iron, typically a micron in diameter, and half a millimetre long, which acts like a single line of flux. Secondly, the issue of enclosing a finitely sized whisker was resolved by Möllenstedt and Dürker, who had recently performed interference experiments with separations of a few microns, using their newly invented "electron biprism".

An interferometer was built from an electron microscope modified to produce a beam which was diffracted by a biprism formed of two earthed metal plates and a 1.5 micron aluminised quartz fibre. A positive potential applied to the quartz fibre alters the effective angle of the biprism. With the interferometer working, as shown in Fig. 3, the interference patterns was measured in two scenarios, with fringes of the order 0.6 micron across. In the first, a pair of Helmholtz oriented coils produced a uniform magnetic field, as is labelled a'. With a field of 0.3 Gauss, the envelope and the fringe pattern were translated by 30 fringe widths. A 2.5 fringe width diagram is shown in Fig. 4.

The more interesting scenario, using the enclosed magnetic whisker represented by Fig. 3. The additional phase acquired by the electrons, passing though the field free region resulted in a fringe shift in the diffraction pattern, but with no translation of the envelope. In this case, the biprism proved unnecessary, as fresnel diffraction around the quartz fibre

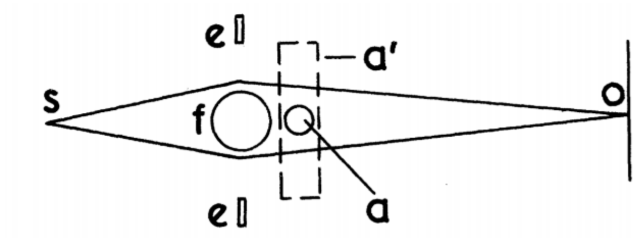

Figure 3: Schematic of Chambers' interferometer. The beam from a modified electron microscope source s is split by a biprism formed of earthed plates e, flanking an aluminesed quartz fibre f. Regions a and a' represent confined magnetic fields, Credit: R. G. Chambers.

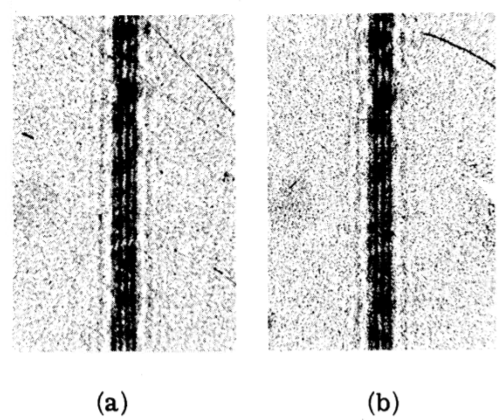

Figure 4: Interference pattern due to a biprism with (a) No field applied and (b) a field of type a'. The interference pattern in the second image is displaced 2.5 fringe widths from the first, credit: R. G. Chambers.

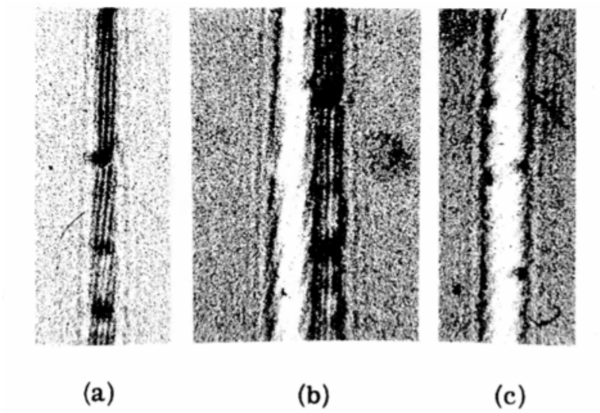

Figure 5: (a) Tilted fringes produced by the tapering of the whisker in shadow of the biprism fibre. (b) Fresnel fringes in the shadow of the whisker, just outside the shadow of the fibre. (c) Same as (b), but from a different part of the whisker.

was sufficient to observe a fringe shift, shown in Fig. 5. Chambers argued that the fibre entirely overshaddowed the whisker, so that no electrons may be in contact with it. Additionally, he calculated that any flux leakage was minimal, only enough to shift one fringe perhaps.

The phase shift is of the amount predicted by Aharonov and Bohm $\frac{e\Phi_B}{\hbar}$. Nevertheless, Chambers did acknowledge some form of flux leakage, as a result of the tapering whiskers, which results in the diagonal fringes in Fig. 5(a). For some critics this was enough ambiguity to cause problems.

Following Chamber's experiment on the AB effect in 1960, there were still disputes in the field over its existence. Particularly, Bocchieri *et al.* argued that the AB effect was a totally mathematical construct [25]. They explained that the gauge function χ from (19) could generate a multi-valued phase factor $e^{\frac{ie\chi}{\hbar}}$ which leads to the elimination of the vector potential.

There were also concerns about magnetic flux leakage from from the solenoidal magnet in previous experiments. A much more robust test using a toroidal magnet was performed in 1986 by Tonomura *et al.* at the Hitachi laboratory, Japan [25]:

4.3 Toroidal Magnet Experiment - A. Tonomura *et al.*

There will always be some flux leakage in a finite solenoid, since the two poles at the two ends have to be linked by closed magnetic lines. The longer the solenoid is, the less the magnetic field is outside, however the solenoid would need to be infinite in length to achieve precisely $\vec{B} = 0$ outside. A toroidal magnet is equally north and south at any point, since it is just a solenoid with the two ends joined. Every loop of wire around the toroid causes a magnetic field in the same direction inside the solenoid (recall that $\vec{B} = 0$ outside the "tube").

The scientists in the Hitachi lab coated the toroidal magnet with the superconductor niobium (Nb). Nb is advantageous in that its transition temperature (below which it is superconducting) is high: 9.2 K. The superconductor prevents the magnetic field from penetrating through via the Meissner effect.

The Meissner Effect

The Meissner effect was discovered in 1933 by W. Meissner and R. Ochsenfeld [26]. Shortly after its discovery, the effect was explained by brothers Fritz and Heinz London [27] via the London equation

$$\frac{4\pi\lambda}{c}\vec{\nabla} \times \vec{J} = -\vec{B}, \tag{39}$$

where λ is the penetration depth of a given material, c is the speed of light and $\vec{J}$ is the superconducting current density. Eqn. (39) has the following solution

$$\vec{B} = \vec{B}_0 e^{-z/\lambda_L}. \tag{40}$$

Here, z is the distance below the surface and λ_L is the London penetration depth which is dependent on the superconducting material. Eqn. (40) implies that $B \approx 0$ for distances appreciably beyond that of λ_L. Now consider a superconducting material above its superconducting temperature, T_c. An applied magnetic field will penetrate the material with no reduction its magnitude. Reducing the temperature below that of T_c will result in the field disappearing below the surface. This also implies that there is a surface current within the material which produces a magnetic field that exactly cancels that of the original field inside the volume of the material. Therefore, a magnetic field produced by a toroid coated in a superconducting material would be confined within the toroid itself, in other words, there is no magnetic flux leakage.

The field penetrates into the superconductor by a depth known as the London penetration depth - for Nb this is only 100 nm.

The schematic for the toroidal experiment is shown in Fig. 6. Part of the electron bean passes through

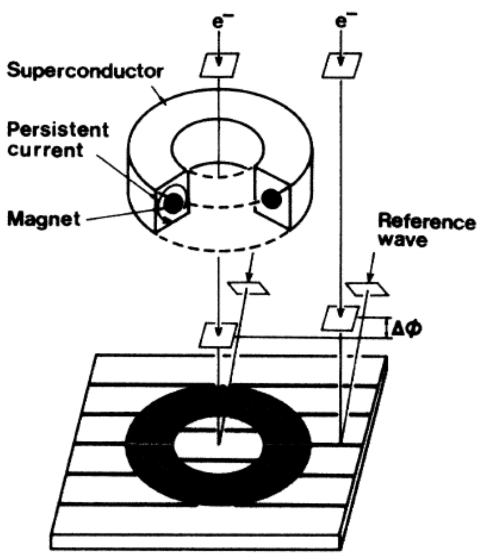

Figure 6: Diagram of the toroidal AB experiment [25], showing the two electron paths and a cross section of the inside of the toroid. The interference pattern is the lines drawn underneath.

the hole in the toroid. There is also a reference beam outside the toroid that acquires no extra phase shift - this acts as a reference point. The images obtained (Fig. 7) show that the interference pattern (the lines) has been shifted for the region in the hole of the toroid. It is important to note that this phase shift was performed at superconducting temperatures - $T = 5K$ - and therefore the magnetic field is precisely zero in the toroid hole.

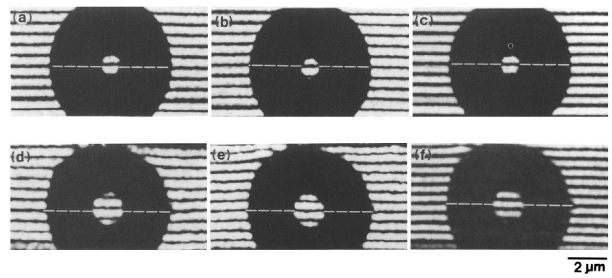

Figure 7: Interference patterns for six setups. (a), (b) and (c) have an even n quantisation, and (d), (e) and (f) have odd n. The first column is for $T = 15\,K$, the second and third columns are different amplifications for $T = 5\,K$ [25].

5 Berry Phase

Michael Berry, also of the University of Bristol, reformulated the AB effect in a geometric framework. In analogy to the classical notion of *parallel transport* [28], he showed that the phase acquired by the electrons in the AB effect can be interpreted as a special case of *Berry phase* [29]. This is a geometric phase which arises from the time evolution of a closed, cyclic, adiabatic quantum system.

5.1 Parallel Transport of a Vector

In geometry, providing that the surface of a topological space is locally Euclidean near each point, which allows nearby tangent spaces to connect, one can transport geometric data in the form of vectors along the smooth curvature of the space such that they stay tangent to the surface. Such an action is called the parallel transport. Consider a vector being parallel-transported on a sphere, as shown in Fig. 8. It can be seen from the figure that an originally-pointed south vector transported along a closed path of $1 \rightarrow 2 \rightarrow 3 \rightarrow 1$ will acquire an angle difference after its closed excursion on a sphere. At the end of its journey the vector is "rotated" despite it remaining in the tangent plane of the surface, without rotating about its normal axis to the surface during its transport. A system for which a parallel-transported vector does not return to its original state is said to

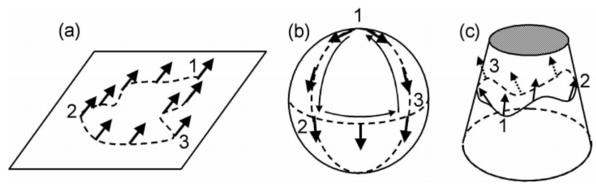

Figure 8: The parallel transport of a vector [30] on (a) a plane, (b) a sphere, (c) a cone.

be *nonholonomic* [31], and such an acquired rotation angle is known as the geometric phase. Note that the rotation angle is actually dependent on the "size" of the closed path. More formally, it can be shown that [28] for the parallel transport of a vector on a closed path C along a curve on a sphere, its geometric phase is equal to the solid angle of the area enclosed by C, with the centre of the sphere being its origin.

The underlying significance here is the fact that this angle of rotation is solely geometric. It relates purely to the intrinsic curvature of the manifold, in this case of a sphere. No such presence of a geometric phase would be found if a vector was parallel-transported along a manifold with flat curvature, like that of a plane or a cone, as seen again in Fig. 8. In the following section, it will be shown how Sir Michael Berry extended the classical parallel transport to study the quantum mechanics of nonholonomic adiabatic processes.

5.2 The Adiabatic Approximation

An adiabatic approximation can be cast on a quantum-mechanical system under the quantum adiabatic theorem[32]. It asserts that a system will be allowed to adapt its configuration providing that the set of external parameters that govern it undergo a small and gradual variation, such that the probability of an eigenstate $|\mathrm{n}(\vec{R})\rangle$ transitioning into another $|\mathrm{m}(\vec{R})\rangle$ becomes vanishingly small:

$$|\langle m| \frac{\partial}{\partial t} |n\rangle | \ll \frac{|E_\mathrm{m} - E_\mathrm{n}|}{\hbar}, \tag{41}$$

$\forall\, m \neq n$. Throughout this adiabatic process the probability density is modified and, as a result, a system that starts with an eigenstate $|\mathrm{n}(\vec{R}(0))\rangle$ of an initial Hamiltonian $\hat{H}_\mathrm{i}$ will continuously remain an instantaneous eigenstate $|\mathrm{n}(\vec{R}(t))\rangle$ up until the final Hamiltonian $\hat{H}_\mathrm{f}$.

In quantum mechanics, quantum states can be represented by vectors as wavefunctions $|\psi\rangle$ in a Hilbert space. Since $|\psi\rangle$ is defined up to a phase at each position in the Hilbert space, a case can be made on extending the classical parallel transport analogue, such that a state is said to have acquired a phase difference

$$|\psi_\mathrm{final}\rangle = e^{i\gamma(C)}\,|\psi_\mathrm{initial}\rangle \tag{42}$$

should it traverse a closed loop C. This phase difference is the quantum geometric phase.

5.3 The Geometric Phase of a Quantum State

In order to study the anholonomy of a quantum adiabatic process, consider a quantum mechanical system described by a Hamiltonian for which the set of parameters $\vec{R}$ it is dependent on are time varying:

$$\hat{H} = \hat{H}(\vec{R}), \quad \vec{R} = \vec{R}(t), \tag{43}$$

where the vector containing the set of n parameters denoted by $\vec{R} = (R_1, R_2, ..., R_\mathrm{n})$ spans the abstract parameter space. For example, an external vector field could be described by a three dimensional vector $\vec{R}$ where its parameters correspond to the three independent components of the vector field. At any instant, the orthonormal basis comprised by the eigenstates of the Hamiltonian satisfy the eigenvalue equation

$$\hat{H}(\vec{R}(t))\,|\mathrm{n}(\vec{R}(t))\rangle = E_\mathrm{n}(\vec{R}(t))\,|\mathrm{n}(\vec{R}(t))\rangle\,, \tag{44}$$

where $E_\mathrm{n}(\vec{R}(t))$ and $|\mathrm{n}(\vec{R}(t))\rangle$ are the instantaneous energy eigenvalues and eigenstates respectively. Though (44) does not consequently imply any phase relations between eigenstates with different $\vec{R}$, it does allow $|\mathrm{n}(\vec{R}(t))\rangle$ to be defined up to a $\vec{R}$ dependent phase factor. In this sense, the phase is arbitrary because it can be removed as long as one makes an appropriate choice of phase, otherwise known as a gauge, provided the phase is smooth and the basis function is single-valued in the parameter space of $\vec{R}$ along the path C.

Following the analogue of classical parallel transport, we can look for general solutions of the form

$$|\psi(t)\rangle = e^{i\Phi_{B\mathrm{n}}(t)} |\mathrm{n}(\vec{R}(t))\rangle \tag{45}$$

under the adiabatic approximation, such that it fulfills the time dependent Schrödinger equation

$$\hat{H}(\vec{R}(t)) |\psi(t)\rangle = i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle . \tag{46}$$

Inserting (45) into (46) and taking the overlap of $\langle \mathrm{n}(\vec{R}(t))|$ on its left hand side, we find a differential equation for $\Phi_B(t)$:

$$\frac{\partial}{\partial t}\Phi_{B\mathrm{n}}(t) = i \langle \mathrm{n}(\vec{R})| \frac{\partial}{\partial t} |\mathrm{n}(\vec{R})\rangle - \frac{E_\mathrm{n}(\vec{R})}{\hbar}. \tag{47}$$

It is then safe to assume the ansatz for (47) as

$$\Phi_{B\mathrm{n}}(t) = \gamma_\mathrm{n}(t) - \frac{1}{\hbar}\int_0^t E_\mathrm{n}(\vec{R}(t'))dt' \tag{48a}$$

$$= \gamma_\mathrm{n}(t) + \delta_\mathrm{n}(t), \tag{48b}$$

where $\gamma(t)$ is the geometric phase and $\delta(t)$ is the familiar dynamical phase factor. By applying the chain rule, this implies that the time derivative of the geometric phase can be written as

$$\frac{\partial}{\partial t}\gamma_\mathrm{n}(t) = i \langle \mathrm{n}(\vec{R}(t))| \vec{\nabla}_{\vec{R}} |\mathrm{n}(\vec{R}(t))\rangle \cdot \frac{\partial}{\partial t}\vec{R}(t), \tag{49}$$

and subsequently integrated

$$\gamma_\mathrm{n}(t) = \int_{\vec{R}(0)_C}^{\vec{R}(t)} \vec{A}_\mathrm{n}(\vec{R}) \cdot d\vec{R}(t'), \tag{50}$$

where is $\vec{A}_\mathrm{n}(\vec{R})$ a vector-valued function.

$$\vec{A}_\mathrm{n}(\vec{R}) = i \langle \mathrm{n}(\vec{R}(t'))| \vec{\nabla}_{\vec{R}} |\mathrm{n}(\vec{R}(t'))\rangle . \tag{51}$$

Under adiabatic approximation, it is shown by (50) that, on top of the usual dynamical phase obtained by the time evolution, the quantum state also acquires a geometric phase due to its adiabatic evolution around a circuit C.

5.4 Berry Phase, Berry Curvature and Berry Connection

Consider a gauge transformation with a differentiable phase factor $e^{i\xi(\vec{R})}$ where

$$|\mathrm{n}(\vec{R})\rangle \longrightarrow |\mathrm{n}'(\vec{R})\rangle = e^{i\xi(\vec{R})} |\mathrm{n}(\vec{R})\rangle , \tag{52}$$

hence $\vec{A}_\mathrm{n}(\vec{R})$ also transforms accordingly to

$$\vec{A}_\mathrm{n}(\vec{R}) \longrightarrow \vec{A}'_\mathrm{n}(\vec{R}) = \vec{A}_\mathrm{n}(\vec{R}) - \frac{\partial}{\partial \vec{R}}\xi(\vec{R}), \tag{53}$$

which shows that it is not gauge invariant. Consequently, the geometric phase is also transformed by

$$\gamma(t) \longrightarrow \gamma(t)' = \gamma(t) + \xi(\vec{R}(0)) - \xi(\vec{R}(t)) \tag{54}$$

where $\vec{R}(0)$ and $\vec{R}(t)$ are initial and final values of $\vec{R}(t)$ of the integral (50) performed on (53). (54) entails that so long as one chooses a suitable phase of $e^{i\xi(\vec{R})}$, also known as gauge fixing, one can always cancel out the contribution of $\gamma(t)$ on the circuit C. The fact that this geometric phase is gauge dependent means it does not correspond to any physical observables, leaving (45) with just

$$|\psi(t)\rangle = e^{i\delta_\mathrm{n}(t)} |\mathrm{n}(\vec{R}(t))\rangle . \tag{55}$$

So for nearly sixty years the significance of the geometric phase has been ignored, since many quantities of interest in quantum mechanics, such as A, revolve around taking the expectation values of the observables, namely

$$\langle A\rangle = \langle\psi| \hat{A} |\psi\rangle , \tag{56}$$

for which it is independent of the phase of the wavefunction.

In 1983, Sir Michael Berry pointed out that the geometric phase of the cyclic evolution of a quantum system under a closed circuit C is actually nontrivial. Providing that the state ends up at its initial point, such that

$$\vec{R}(T) = \vec{R}(0), \tag{57}$$

one can express (50) as a closed-loop integral

$$\gamma_{\mathrm{n}}(C) = \oint_C \vec{A}_{\mathrm{n}}(\vec{R}) \cdot d\vec{R}, \tag{58}$$

where $\gamma_{\mathrm{n}}(C)$ and $\vec{A}_{\mathrm{n}}(\vec{R})$ are now called the *Berry phase* and *Berry connection* respectively. Following Stokes' theorem, one can recast the line integral into a surface integral via

$$\gamma_{\mathrm{n}}(C) = \iint_S \vec{\nabla}_{\vec{R}} \times \vec{A}_{\mathrm{n}}(\vec{R}) \cdot d\vec{S} \tag{59a}$$

$$= \iint_S \vec{\Omega}_{\mathrm{n}}(\vec{R}) \cdot d\vec{S}. \tag{59b}$$

$\vec{\Omega}_{\mathrm{n}}$ is called the *Berry Curvature*, which is obviously a gauge invariant quantity due to the fact that

$$\vec{\nabla}_{\vec{R}} \times -\frac{\partial}{\partial R}\xi(\vec{R}) = 0. \tag{60}$$

Taking a closer look at the same gauge transformation from (52) under a closed loop C, $|n'(\vec{R})\rangle$ is required to be single-valued from the aforementioned precondition on the choice of gauge in section 5.3. Therefore, the Berry phase cannot be removed and can only be changed by an integer multiple of 2π under a gauge transformation, such that

$$\xi(\vec{R}(0)) - \xi(\vec{R}(T)) = 2n\pi, \tag{61}$$

where n is an integer. This makes the Berry phase a gauge invariant quantity and thus it is an observable.

5.5 Berry Phase in the Aharonov-Bohm Effect

In the same paper where Sir Berry addressed the Berry phase, he also showed how the resulting phase shift from the AB effect can be arrived from the geometric phase change interpretation. He considered a charged particle enclosed in a box by a potential $V(\vec{r} - \vec{R})$, as shown in Fig. 9. The box, centered at a point $\vec{R}$, is set to transport around a flux line such that it is not penetrated by the flux. The eigenvalue equation of the Hamiltonian reads

$$\vec{H}(\vec{p} - q\vec{A}(\vec{r}), \vec{r} - \vec{R})\,|n(\vec{R})\rangle = E_n\,|n(\vec{R})\rangle \tag{62}$$

which has the previously known solution

$$\psi_n(\vec{r}, \vec{R}) = \langle \vec{r}\,|n(\vec{R})\rangle \tag{63a}$$

$$= e^{ig}\psi_n^{'}(\vec{r} - \vec{R}), \tag{63b}$$

where

$$g = \frac{q}{\hbar}\int_{\vec{R}}^{\vec{r}} \vec{A}(\vec{r'}) \cdot d\vec{r'}. \tag{64}$$

Note that $\psi_n^{'}$, which is solely a function of the difference of $\vec{r}$ and $\vec{R}$, is also a solution for (62) but only acts locally within the system of the box, where the flux is absent, i.e.,

$$\left[-\frac{\hbar^2}{2m}\vec{\nabla}^2 + V(\vec{r} - \vec{R})\right]\psi_n^{'} = E_n\psi_n^{'}. \tag{65}$$

(51) and (63b) can then be used to calculate the Berry connection, as shown in (67). $\vec{A}(\vec{R})$ is not a function of $\vec{r}$, therefore it can simply be factorised in the first integral of (67a) and the orthogonalisation relations of the eigenstates can simply be applied; the second integral simply vanishes under the adiabatic evolution. Finally, (58) computes the geometric phase acquired by the initial state of the particle in the box after a closed circuit transport around the flux line as

$$\gamma_n(C) = \frac{q}{\hbar}\oint_C \vec{A}(\vec{R}) \cdot d\vec{R} \tag{66a}$$

$$= \frac{q}{\hbar}\Phi_B. \tag{66b}$$

This is equivalent to our result in (9c), which shows that Berry phase is a ubiquitous concept across physics.

$$\langle n(\vec{R})|\,\vec{\nabla}_{\vec{R}}\,|n(\vec{R})\rangle = \iiint_{\text{box}} [\psi_n^{'}(\vec{r}-\vec{R})]^* \left\{ -\frac{iq}{\hbar}\vec{A}(\vec{R})\psi_n^{'}(\vec{r}-\vec{R}) + \vec{\nabla}_{\vec{R}}\psi_n^{'}(\vec{r}-\vec{R}) \right\} d^3\vec{r} \tag{67a}$$

$$= -\frac{iq}{h}\vec{A}(\vec{R}) \tag{67b}$$

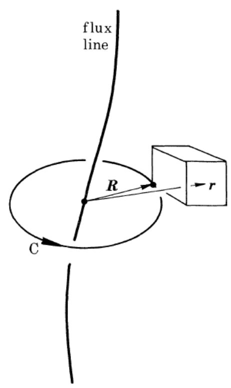

Figure 9: Ahanorov-Bohm effect in a box transported round a flux line. (Credit: Berry)

6 Applications

6.1 Topological Insulators

The Aharonov-Bohm effect has been observed in surface electron states of topological insulators- a new class of quantum matter with an insulating bulk gap supporting conduction at the surface [33]. It has been demonstrated by Hailin Peng *et al.* that such materials have the desired properties in the development of spin transport electronics, or spintronic devices. Spintronics exploit the spin degree of freedom of the electron in addition to its charge to increase data processing speed and in reduction of electrical power consumption in comparison to current semiconductor devices [34]. Peng *et al.* observed quantum AB oscillations when a layered nanoribbon of bismuth selenide, Bi_2Se_3, was subject to a magnetic field shown in Fig. 10.

Observed oscillations in the magnetoresistance associated with the 2D topological surface states under an applied field was attributed to the AB effect, where the interference trajectories of the electrons enclose a fixed area perpendicular to a magnetic field. The magnetoresistance is plotted as a function of applied field ranging $\pm 9\,T$ in Fig. 11. The primary resistance oscillation with period $0.62T$ was observed in the range $0.15T < B < 2T$ corresponding to the Aharonov-Bohm h/e flux quantisation. This is seen in the right inset of Fig.11 where the fast Fourier transform (FFT) of the quantity dR/dB is shown. The $h/2e$ flux quantisation is associated with the Altshuler-Aronov-Spivak (AAS) effect arising due to weak antilocalisation, associated with the spin-orbit coupling of the electron states [35].

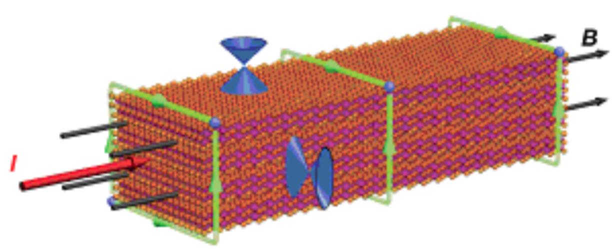

Figure 10: A layered Bi_2Se_3 nanoribbon with constant magnetic field applied along its length, shown by the black arrows. The green paths indicate the surface electron interference trajectories perpendicular to the field. Dirac surface states are shown by the blue cones on the top and side surfaces. Credit: Peng *et al.*

6.2 The Anomalous Hall Effect

Given the analogue between Berry phase and vector potentials, it is expected that such formalism has

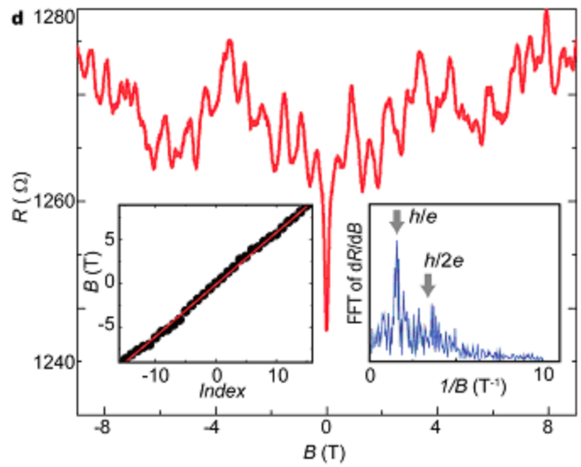

Figure 11: Aharonov-Bohm oscillations in the magnetoresistance as a function of applied field. Right inset: FFT of dR/dB demonstrating the primary h/e AB oscillation and secondary h/2e AAS oscillation. Credit: Peng *et al.*

a role in the anomalous Hall effect. The anomalous Hall effect (AHE) is similar to an ordinary Hall effect, except that it depends more on the magnetisation of the material than the external magnetic field, and is only observable in ferromagnetic/anti-ferromagnetic materials. The Hall effect is observed even in the absence of an external magnetic field. Such an effect arises from the spin-orbit coupling interaction and a broken symmetry called "time reversal symmetry" in the ferromagnet/anti-ferromagnet. The effect is a potential probe in electron-spin polarisation, particularly in nanoscale systems where direct measurement is not feasible [36].

Existing theories of this are divided into two categories, extrinsic and intrinsic. The extrinsic effect implies scattering of the electrons due to its spin, while the intrinsic is based on the Berry phase effect in the "crystal space" - a mathematical space where most calculations of the crystal are done [37], [38]. The intrinsic theory argues that when the electron Bloch wave moves adiabatically through the k-space in an electric field, the spin "rotates" due to an interaction between the spin and the potential in the path. This "rotation" acquires a phase (the Berry phase) which contributes to the momentum of the Bloch wave and hence its group velocity. Only in a ferromagnet/anti-ferromagnet does the Berry curvature have a non-zero contribution due to broken time reversal symmetry. It therefore seems as though there is a Hall effect, despite the external magnetic field being zero. The Berry curvature contributes by acting like an "external magnetic field" in the k-space [39].

7 Conclusions

The difference between potentials in quantum mechanics and those in classical mechanics, through the example of the Aharanov-Bohm effect, have been demonstrated. The consequences of the AB effect on the physical significance of vector potentials in quantum theory has been explored, and we have looked at interpreting this effect in terms of nonlocal behaviour of confined magnetic fluxes to an external particle.

In Michael Berry's paper on geometric phases, he reformulated the AB effect as a geometric problem. Through this analogy, the phase shift acquired by the electron is a specific case of Berry phase.

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