Some Results for Discrete Population Models

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1 2.4.6

Consider the second-iterate map f^2 of the logistic map $f(x) = rx(1-x)$.

(a)

We have

$$\begin{aligned} f^2(x) &= f(f(x)) \\ &= r\left(rx(1-x)\right)\left(1-rx(1-x)\right) \\ &= -r^2\left(rx^4 - 2rx^3 + (r+1)\,x^2 - x\right). \end{aligned}$$

(b)

To find the fixed points of f^2, we must solve $f^2(x) - x = 0$. We already know that $x_0 = 0$ and $x_1 = \frac{r-1}{r}$ are solutions; factoring out the corresponding linear factors and assuming $r \neq 0$, we see that the remaining fixed points are the solutions to

$$r^2x^2 - (r^2+r)x + r + 1 = 0,$$

and thus

$$x_2, x_3 = \frac{r+1 \pm \sqrt{r^2-2r-3}}{2r}. \quad \text{(respectively)}$$

These are the points of a two-cycle in f, and they are real and distinct only when $r^2 - 2r - 3 > 0$: for our purposes, when $r > 3$.

(c)

The derivative of f^2 is easily computed from the expanded polynomial expression:

$$\frac{\mathrm{d}}{\mathrm{d}x}f^2(x) = -r^2\left(4rx^3 - 6rx^2 + 2\,(r+1)\,x - 1\right).$$

We also note that by the chain rule, $\frac{\mathrm{d}}{\mathrm{d}x}f^2(x) = f'(f(x))f'(x)$.

(d)

The two-cycle we found in (b) is stable if $|\frac{\mathrm{d}}{\mathrm{d}x}f^2(x_2)|$ and $|\frac{\mathrm{d}}{\mathrm{d}x}f^2(x_3)|$ are less than one. In fact, either inequality implies the other, for (WLOG) if the sequence $\{y_{2n} = f^{2n}(y_0)\}$ converges to x_2, then $y_{2n+1} \to f(x_2) = x_3$. So we compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}f^2(x_2) &= f'(f(x_2))f'(x_2)\\ &= f'(x_3)f'(x_2)\\ &= (r-2rx_3)(r-2rx_2)\\ &= r^2(1-2(x_2+x_3)+4x_2x_3).\end{aligned}$$

From part (c) we have $x_2 + x_3 = \frac{r+1}{r}$ and

$$\begin{aligned}x_2x_3 &= \frac{(r+1)^2-(r^2-2r-3)}{4r^2}\\ &= \frac{r+1}{r^2}.\end{aligned}$$

So the derivative becomes

$$\begin{aligned}r^2(1-2(x_2+x_3)+4x_2x_3) &= r^2-2r(r+1)+4(r+1)\\ &= -r^2+2r+4.\end{aligned}$$

This is less than 1 for $r > 3$ and greater than -1 for $r < 1+\sqrt{6}$; accordingly, the two-cycle is stable when r is in that range.

2 2.4.7

Consider the fourth-iterate map f^4 of the logistic map $f(x) = rx(1-x)$.

(b)

(see attachment)

As f^4 is a 16^{th} degree polynomial, it does not bear copying out here, nor is it easily manipulated by hand. Using the computer algebra system SymPy, I constructed the symbolic polynomial expressions for f^4 and $g = (f^4-x)/(f^2-x)$. The roots of the latter polynomial are fixed points of f^4 but not of f^2 or f; that is, they are the points of four-cycles in f.

I then made a function minval(r) to approximate the minimum value taken by $g(x)$ on our interval of interest, $0 < x < 1$, for a given value of r. This function is rather a blunt instrument: I could have used the symbolic expression for g to compute its derivative and find extrema in the usual fashion, but even the derivate of g has degree 11 and was beyond the capabilities of any symbolic rootfinding algorithms I could find. Numeric rootfinding on g' did not seem to offer any immediate advantage over numeric minimum-hunting on g.

Based on the discussion in class and the observation that the two-cycle becomes unstable when $r > 1 + \sqrt{6}$, I conjectured that the nondegenerate four-cycle would appear when r exceeded this value. Indeed, my computations confirm that

$$\mathrm{minval}(1 + \sqrt{6} - \epsilon) < \mathrm{minval}(1 + \sqrt{6}) \approx 0 < \mathrm{minval}(1 + \sqrt{6} + \epsilon), \quad (\epsilon > 0)$$

so that g (and thus $f^4 - x$) has real roots when r exceeds $1 + \sqrt{6}$, so far as I was able to test.

I also performed a bisection search for other solutions to minval(r) = 0 with $0 < r < 4$; as expected, all tests converged to $r \approx 1 + \sqrt{6}$.

(c)

The four-cycle x_1, x_2, x_3, x_4 is stable when $|\frac{\mathrm{d}}{\mathrm{d}x} f^4(x_i)| < 1$ for all i. As in (2.4.6(d)), we need only check one of the points. The attached SymPy script uses a bisection search to find a numerical solution $1 + \sqrt{6} < r < 4$ to $|\frac{\mathrm{d}}{\mathrm{d}x} f^4(x^*(r))| - 1 = 0$, where $x^*(r)$ is the first (i.e., least) root of the polynomial $g = (f^4 - x)/(f^2 - x)$.

It turns out that $r \approx r_0 = 3.54409094646132 15$ is the only such solution. Thus $|\frac{\mathrm{d}}{\mathrm{d}x} f^4(x_i)| < 1$, and the four-cycle is stable, when $1 + \sqrt{6} < r < r_0$.

3 2.4.8

The Breverton-Holt map is given by

$$\begin{aligned} f(x_{n+1}) &= \frac{r x_n}{1 + \frac{r-1}{K} x_n} \quad (r, K > 0) \\ &= \frac{r K x_n}{K + (r-1)\, x_n}. \end{aligned}$$

Claim. The closed-form solution $x_n = g(n)$ is given by

$$\begin{aligned} g(n) &= \frac{r^n x_0}{1 + \frac{r^n - 1}{K} x_0} \\ &= \frac{r^n K x_0}{K + (r^n - 1)\, x_0}. \end{aligned}$$

Proof (1). Substitute $u_n = 1/x_n$ and $T = \frac{r-1}{rK}$. We then have

$$\begin{aligned}
u_{n+1} &= \frac{K + (r-1)\,x_n}{rKx_n} \\
&= \frac{u_n}{r} + \frac{r-1}{rK} \\
&= \frac{u_n}{r} + T \\
&= \frac{\frac{u_{n-1}}{r} + T}{r} + T \\
&\vdots \\
&= \frac{u_{n+1-k}}{r^k} + T\left(\left(\frac{1}{r}\right)^{k-1} + \left(\frac{1}{r}\right)^{k-2} + \cdots + 1\right) \\
&\vdots \\
&= \frac{u_0}{r^{n+1}} + T\left(\left(\frac{1}{r}\right)^{n} + \left(\frac{1}{r}\right)^{n-1} + \cdots + 1\right) \\
&= \frac{u_0}{r^{n+1}} + T\left(\frac{1 - r^{-n-1}}{1 - r^{-1}}\right) \\
&= \frac{u_0}{r^{n+1}} + \left(\frac{r-1}{rK}\right)\left(\frac{r^{n+1}-1}{(r-1)\,r^n}\right) \\
&= \frac{Ku_0 + r^{n+1} - 1}{r^{n+1}K} \\
&= \frac{K + \left(r^{n+1}-1\right)x_0}{r^{n+1}Kx_0} \\
&= \frac{1}{g(n+1)},
\end{aligned}$$

so that $x_{n+1} = g(n+1)$. ■

Proof (2). More formally, we induct on n. When $n = 0$, we have

$$g(n) = \frac{r^0 K x_0}{K + (r^0 - 1)\,x_0} = x_0,$$

so $g(n) = x_n$ in the base case.

Now let $n \geq 0$ and suppose that $g(n) = x_n$. We then have

$$\begin{aligned}
x_{n+1} &= \frac{rKx_n}{K + (r-1)\,x_n} \\
&= \frac{rKx_n}{K + (r-1)\,g(n)} \\
&= \frac{rx_n}{1 + (r-1)\,\frac{r^n x_0}{K+(r^n-1)x_0}} \\
&= \frac{rx_n\,(K + (r^n - 1)\,x_0)}{K + (r^n - 1 + (r-1)\,r^n)\,x_0} \\
&= \frac{rx_n\,(K + (r^n - 1)\,x_0)}{K + (r^{n+1} - 1)\,x_0} \\
&= \frac{r^{n+1}Kx_0}{K + (r^{n+1} - 1)\,x_0} \\
&= g(n+1).
\end{aligned}$$

By induction, $x_n = g(n)$ for all n. ■

4 2.4.10

Consider the tent map, given by

$$x_{n+1} = f(x_n) = \begin{cases} \mu x & \text{for } 0 \leq x \leq 0.5, \\ \mu(1-x) & \text{for } 0.5 < x \leq 1. \end{cases}$$

(a)

(See attachment)

(b)

On the left-hand side of the domain, we solve $x = \mu x$ to find the fixed point $x = 0$ (stable when $|\mu| < 1$). If $\mu = 1$, then *every* point $x \leq 0.5$ is a fixed point. Each point in this class is technically unstable, for if $x_0 = x + \epsilon$ then the sequence (x_n) does not converge to x; but of course the sequence does converge to $x + \epsilon$, so the usual concept of stability is not very meaningful in this case.

On the right, we solve $x = \mu(1-x)$ and find the fixed point $x = \frac{\mu}{\mu+1}$. This value is always less than one, but is greater than 0.5 only when $\mu > 1$. It follows that this fixed point is never stable.

(c)

The orbits of period two correspond to pairs of fixed points of $f^2(x)$. Every nondegenerate 2-cycle has a point on the right side of the domain (for if $x_0, x_1, x_2 \leq 0.5$, then $x_2 = \mu^2 x_0 = x_0$ only if $x_0 = 0$ or $\mu = 1$), so it suffices to find those points. We have

$$f^2(x) = \begin{cases} \mu - \mu^2(1-x) & \text{for } 0.5 < x < \min\{1, 1 - \frac{1}{2\mu}\}, \\ \mu^2(1-x) & \text{for } 1 - \frac{1}{2\mu} \leq x \leq 1. \end{cases}$$

For the first "piece", the only fixed point is $x = \frac{\mu}{\mu+1}$; this is the fixed point of the first-iterate map and represents a degenerate 2-cycle.

In the remaining part of the domain, $\mu^2(1-x) - x = 0$ has the solution $x = \frac{\mu^2}{\mu^2+1}$ when this value is at least $1 - \frac{1}{2\mu}$. Solving for μ, we find

$$\begin{aligned} \frac{\mu^2}{\mu^2+1} &\geq \frac{2\mu - 1}{2\mu} \\ \implies 2\mu^3 &\geq 2\mu^3 - \mu^2 + 2\mu - 1 \\ \implies \mu^2 - 2\mu + 1 &\geq 0 \\ \implies \mu &\geq 1. \end{aligned}$$

(For example, with $\mu = 2$, we get the orbit $\frac{4}{5}, \frac{2}{5}, \frac{4}{5}, \ldots$.) Notice that $\frac{\mathrm{d}}{\mathrm{d}x} f^2(x) = -\mu^2$ in this region, so the two-cycle is never stable.

(d)

In a search for a 3-cycle, the following lemma will simplify things enormously:

Lemma. The n^{th} iterate of the tent map f with $\mu = 2$ is given by

$$f^n(x) = f(2^{n-1}x \mod 1)$$

for all $n > 0$.

Proof. We induct on n. Clearly,

$$f^1(x) = f(x) = f(2^0 x \mod 1).$$

Furthermore, for $x \leq 0.5$ we have

$$f^2(x) = f(f(x)) = f(2x) = f(2x \mod 1),$$

while for $x > 0.5$

$$\begin{aligned} f^2(x) &= f(2(1-x)) \\ &= f(1 - 2(1-x)) \\ &= f(2x - 1) \\ &= f(2x \mod 1). \end{aligned}$$

So the claim holds for $n \leq 2$.

Now suppose that $n > 2$ and $f^k(x) = f(2^{k-1}x \mod 1)$ for all $1 \leq k < n$. We then have, for all x,

$$\begin{aligned} f^n(x) &= f(f^{n-1}(x)) \\ &= f(f(2^{n-2}x \mod 1)) \\ &= f^2(2^{n-2}x \mod 1)) \\ &= f(2^{n-1}x \mod 1)). \end{aligned}$$

So the claim holds for all n. ■

Armed with this lemma, we see that any point on a 3-cycle will be a solution to $g(x) = f(4x \mod 1) = x$. With $x < 1/8$, we have $g(x) = 8x$; no good. With $1/8 < x < 1/4$, we have $g(x) = 2(1 - 4x)$ and we find the fixed point $x = 2/9$, which gives us the cycle $\frac{2}{9}, \frac{4}{9}, \frac{8}{9}, \frac{2}{9}, \ldots$.

5 2.4.14

Consider the model for infection spread given by

$$I_0 = 1, \quad I_{n+1} = I_n + kI_n(N - I_n).$$

(a)

This is the (unsimplified) logistic model with nontrivial equilibrium $I^* = N$. For $0 < kN < 2$, we have

$$-1 < \frac{\mathrm{d}}{\mathrm{d}I}\left(I + kI(N - I)\right) = kN + 1 - 2kI^* = 1 - kN < 1,$$

so the equilibrium is stable: *everyone* eventually gets sick.

(b)

To account for recovery, assume each individual is sick for exactly d days, can infect others during this period, and is immune after recovery. For each day n, let I_n be the number of sick people on that day, let J_n be the number of *new* cases (that is, the number of people whose first day sick is day n), and let P_n be the remaining susceptible population. The new model is then

$$\begin{aligned} J_0 = 1, &\quad J_{n+1} = kI_nP_n; \\ I_0 = J_0, &\quad I_{n+1} = I_n + J_{n+1} - J_{n+1-d}; \\ P_0 = N - J_0, &\quad P_{n+1} = P_n - J_{n+1}. \end{aligned}$$

Thus, the rate of new infections is unchanged from the old model; each new case decrements the susceptible population; and each disease victim is removed from the count of sick people after d days sick.

6

(a)

(see attachment)

(b)

The Riker model is given by $x_{n+1} = f(x_n) = x \exp\left[r(1 - \frac{x}{k})\right]$. Denote $y = f(x)$ and $h(x) = r(1 - \frac{x}{k})$. Then $y = xe^{h(x)}$ and $f^2(x) = f(y) = xe^{h(x)+h(y)}$.

If x^* and $y^* = f(x^*)$ are the points of a two-cycle, then $x^* = x^* e^{h(x^*)+h(y^*)}$, so that $h(x^*) + h(y^*) = 0$. We then have

$$\begin{aligned} 0 &= h(x^*) + h(y^*) \\ &= r\left(2 - \frac{1}{k}(x+y)\right) \\ \implies \quad x + y &= 2k. \end{aligned}$$

The derivative of f^2 at x^* is given by

$$\begin{aligned} \left(\frac{\mathrm{d}}{\mathrm{d}x} f^2\right)(x^*) &= (1 + xh'(x^*) + xh'(y^*)f'(x^*)) \exp\left[h(x^*) + h(y^*)\right] \\ &= 1 - \frac{r}{k}x^* - \frac{r}{k}x^*\left(1 - \frac{r}{k}x^*\right)e^{h(x)} \\ &= 1 - \frac{r}{k}\left(x^* + y^* - \frac{r}{k}x^* y^*\right) \\ &= 1 - \frac{r}{k}\left(2k - \frac{r}{k}x^* y^*\right) \\ &= 1 - 2r + \frac{r^2}{k^2}x^* y^* \\ &\leq 1 - 2r + \frac{r^2}{k^2}k^2. \end{aligned}$$

So the derivative at x^* is bounded by the derivative at the fixed point $x = k$, which by the above quadratic is less than 1 while $r < 2$. This of course is the value at which a two-cycle appears.