11/16/99 (T.F. Weiss)

Lecture #18: Continuous time periodic signals

Motivation:

- Representation of continuous time, periodic signals in the frequency domain
- Periodic signals occur frequently — motion of planets and their satellites, vibration of oscillators, electric power distribution, beating of the heart, vibration of vocal chords, etc.

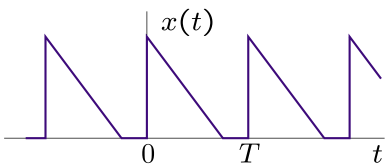
Outline:

- Fourier series of periodic functions
- Examples of Fourier series — periodic impulse train
- Fourier transforms of periodic functions — relation to Fourier series
- Conclusions

Fourier series of a periodic function

Periodic time function

$x(t)$ is a periodic time function with period T.



Such a periodic function can be expanded in an infinite series of exponential time functions called the *Fourier series*,

$$x(t) = \sum_{n=-\infty}^{\infty} X[n] e^{j2\pi nt/T}.$$

Fourier series coefficients

The coefficients of the Fourier series can be found as follows.

$$\frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-j2\pi nt/T}\,dt = \frac{1}{T}\int_{-T/2}^{T/2}\left(\sum_{k=-\infty}^{\infty} X[k]e^{j2\pi kt/T}\right)e^{-j2\pi nt/T}\,dt,$$

$$\frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-j2\pi nt/T}\,dt = \sum_{k=-\infty}^{\infty} X[k]\left(\frac{1}{T}\int_{-T/2}^{T/2} e^{j2\pi(k-n)t/T}\,dt\right).$$

The integral can be evaluated as follows.

$$\frac{1}{T}\int_{-T/2}^{T/2} e^{j2\pi(k-n)t/T}\,dt = \begin{cases} 1 & \text{if } k = n, \\ 0 & \text{if } k \neq n. \end{cases}$$

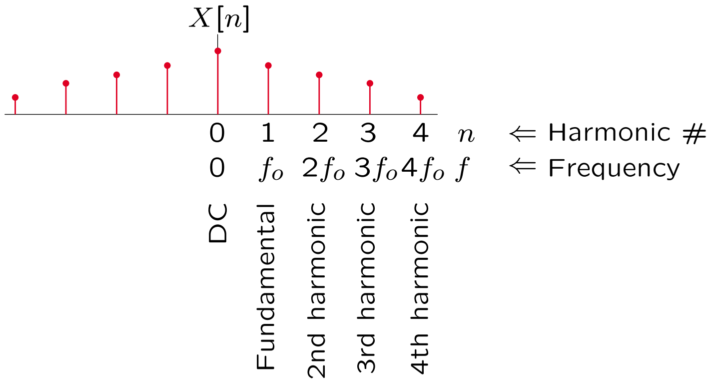
The set of exponential time functions are said to be an *orthonormal basis*.

The coefficients are

$$X[n] = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-j2\pi nt/T}\,dt.$$

Definition of line spectra, harmonics

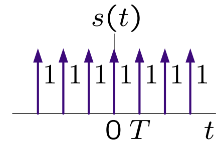
The fundamental frequency $f_o = 1/T$. The Fourier series coefficients plotted as a function of n or nf_o is called a *Fourier spectrum*.



Examples of Fourier series of periodic time functions

Periodic impulse train

The periodic impulse train is an important periodic time function and we derive its Fourier series coefficients.



$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT).$$

The Fourier series coefficients are found as follows

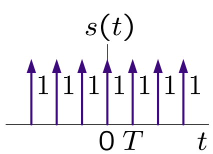
$$\begin{aligned} S[n] &= \frac{1}{T}\int_{-T/2}^{T/2} s(t)e^{-j2\pi nt/T}\,dt, \\ &= \frac{1}{T}\int_{-T/2}^{T/2} \sum_{n=-\infty}^{\infty} \delta(t-nT)e^{-j2\pi nt/T}\,dt, \\ &= \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)e^{-j2\pi nt/T}\,dt = \frac{1}{T}. \end{aligned}$$

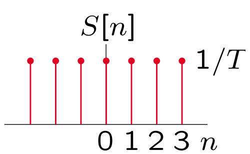
Periodic impulse train, cont'd

The Fourier series coefficients are

$$S[n] = \frac{1}{T}.$$

The time function and spectrum are shown below.





To summarize, the periodic impulse train can be represented by its Fourier series,

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{j2\pi nt/T}.$$

Periodic impulse train, cont'd

The Fourier series of the periodic impulse train is

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{j2\pi nt/T}.$$

It is not obvious that the two expressions are equal. To investigate this, we define the partial sum of the Fourier series, $s_N(t)$,

$$s_N(t) = \frac{1}{T}\sum_{n=-N}^{N} e^{j2\pi nt/T},$$

and investigate its behavior as $N \to \infty$.

Periodic impulse train, cont'd

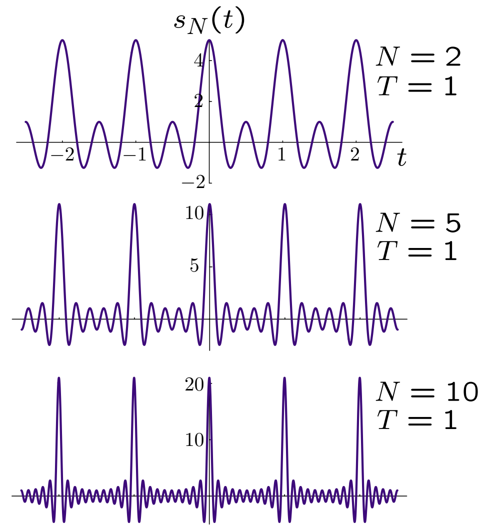
The partial sum of the Fourier series is

$$s_N(t) = \frac{1}{T}\sum_{n=-N}^{N} e^{j2\pi nt/T} = \frac{1}{T}\sum_{n=-N}^{N}\left(e^{j2\pi t/T}\right)^n.$$

We can use the summation formula for a finite geometric series (Lecture 10) to sum this series,

$$\begin{aligned} s_N(t) &= \frac{1}{T}\frac{\left(e^{j2\pi t/T}\right)^{-N} - \left(e^{j2\pi t/T}\right)^{N+1}}{1 - e^{j2\pi t/T}}, \\ s_N(t) &= \frac{1}{T}\frac{e^{j(2N+1)\pi t/T} - e^{-j(2N+1)\pi t/T}}{e^{j\pi t/T} - e^{-j\pi t/T}}, \\ s_N(t) &= \frac{1}{T}\left(\frac{\sin(2N+1)\pi t/T}{\sin \pi t/T}\right). \end{aligned}$$

Periodic impulse train, cont'd



$$s_N(t) = \frac{1}{T}\left(\frac{\sin(2N+1)\pi t/T}{\sin \pi t/T}\right).$$

Note that this function is periodic with period T, and

$$s_N(nT) = \frac{2N+1}{T}.$$

The first zero of $s_N(t)$ is at

$$t = \frac{T}{2N+1}.$$

Thus, as $N \to \infty$, each lobe gets larger and narrower. To determine if each lobe acts as an impulse, we need to find its area.

Periodic impulse train, cont'd

The area of each period of $s_N(t)$ is simply

$$\begin{aligned} \text{Area} &= \int_{-T/2}^{T/2} s_N(t)\,dt = \int_{-T/2}^{T/2}\frac{1}{T}\sum_{n=-N}^{N} e^{j2\pi nt/T}\,dt \\ &= \sum_{n=-N}^{N}\frac{1}{T}\int_{-T/2}^{T/2} e^{j2\pi nt/T}\,dt \end{aligned}$$

The integral is zero except when $n = 0$ when it equals T so that Area $= 1$. Thus, each lobe of $s_N(t)$ becomes: tall, height is $s_N(nT) = (2N+1)/T$; narrow, width is $2T/(2N+1)$; and its area is 1. Thus, the partial sum approaches an infinite impulse train of unit area,

$$s(t) = \sum_{n=-\infty}^{\infty}\delta(t - nT) = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{j2\pi nt/T}.$$

Fourier transform of a periodic function

Fourier transform of a periodic impulse train

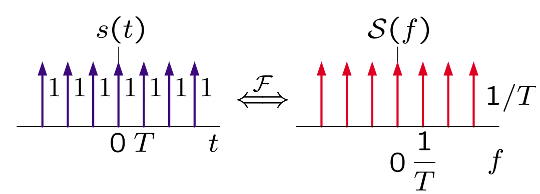
We have two expressions for a periodic impulse train,

$$s(t) = \sum_{n=-\infty}^{\infty}\delta(t - nT) = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{j2\pi nt/T}.$$

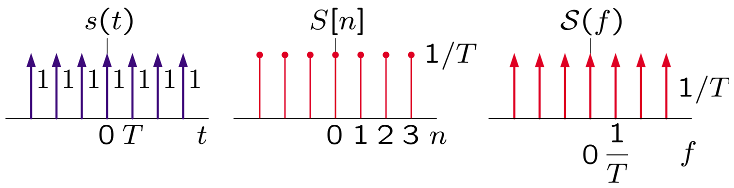
The Fourier transform of each expression is

$$\mathcal{S}(f) = \sum_{n=-\infty}^{\infty} e^{-j2\pi nTf} = \frac{1}{T}\sum_{n=-\infty}^{\infty}\delta\left(f - \frac{n}{T}\right).$$

Therefore, the Fourier transform of a periodic impulse train in time is a periodic impulse train in frequency.



Relation of Fourier series spectrum to Fourier transform of a periodic impulse train



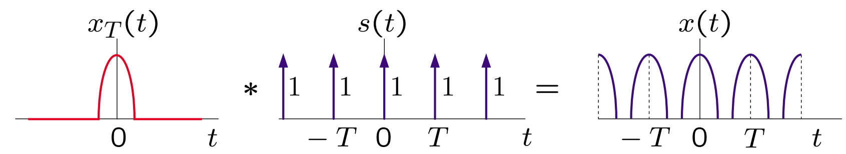
Therefore, the Fourier transform of the periodic impulse train has an impulse at the frequency of each Fourier series component and the area of the impulse equals the Fourier series coefficient.

Fourier transform of an arbitrary periodic function

Representation of a periodic function

An arbitrary periodic function can be generated by convolving a pulse, $x_T(t)$, that represents one period of the periodic function with a periodic impulse train, $s(t) = \sum_n \delta(t - nT)$,

$$x(t) = x_T(t) * s(t) = x_T(t) * \sum_{n=-\infty}^{\infty} \delta(t - nT) = \sum_{n=-\infty}^{\infty} x_T(t - nT).$$

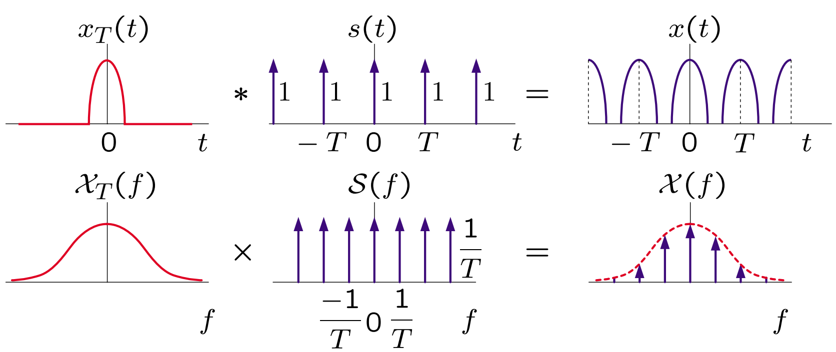


Fourier transform of a periodic function

The Fourier transform of the periodic function is

$$x(t) = x_T(t) * s(t) \overset{\mathcal{F}}{\Longleftrightarrow} \mathcal{X}(f) = \mathcal{X}_T(f) \times \mathcal{S}(f).$$

$$\mathcal{X}(f) = \mathcal{X}_T(f) \times \frac{1}{T} \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T}\right) = \sum_{n=-\infty}^{\infty} \left(\frac{\mathcal{X}_T\left(\frac{n}{T}\right)}{T}\right) \delta\left(f - \frac{n}{T}\right).$$



Fourier transform of a periodic function, cont'd

An important conclusion is that the Fourier transform of a periodic function consists of impulses in frequency at multiples of the fundamental frequency. Thus, periodic continuous time functions can be represented by a countably infinite number of complex exponentials.

Fourier series coefficients

The Fourier transform of the periodic function is

$$\mathcal{X}(f) = \sum_{n=-\infty}^{\infty} \left(\frac{\mathcal{X}_T\left(\frac{n}{T}\right)}{T} \right) \delta\left(f - \frac{n}{T}\right).$$

Recall that

$$\mathcal{X}_T(f) = \int_{-T/2}^{T/2} x_T(t) e^{-j2\pi f t}\, dt$$

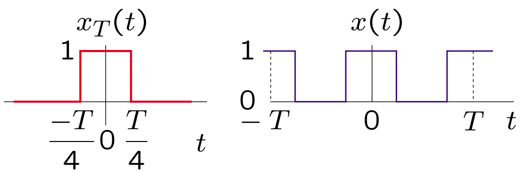
Therefore,

$$\frac{\mathcal{X}_T\left(\frac{n}{T}\right)}{T} = \frac{1}{T} \int_{-T/2}^{T/2} x_T(t) e^{-j2\pi n t/T}\, dt = X[n].$$

Therefore, for an arbitrary periodic continuous time function, the Fourier transform consists of impulses (located at the harmonic frequencies) whose areas are the Fourier series coefficients.

Fourier series of a square wave — generation of the square wave

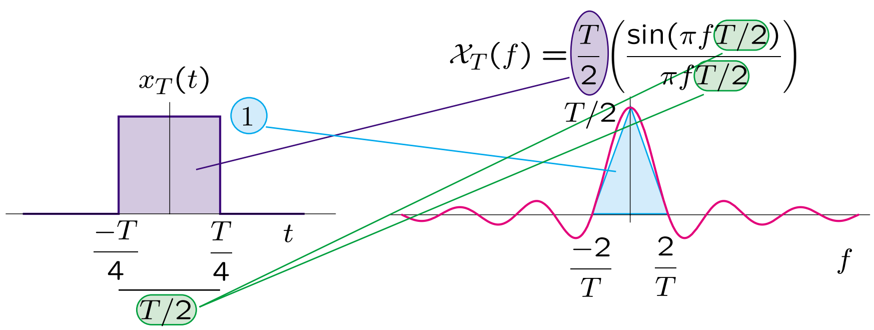
We will find the Fourier series of a square wave by finding the Fourier transform of one period.



$$x(t) = x_T(t) * s(t)$$

Fourier series of a square wave — Fourier transform of one period of the square wave

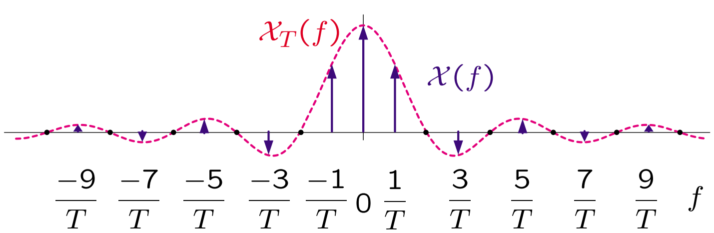
$$x_T(t) \overset{\mathcal{F}}{\Longleftrightarrow} \mathcal{X}_T(f)$$



Fourier series of a square wave — Fourier transform of square wave

To obtain the Fourier transform of the square wave, we take the Fourier transform of one period of the square wave and multiply it by the Fourier transform of the periodic impulse train.

$$\begin{aligned} \mathcal{X}(f) &= \mathcal{X}_T(f) \times \mathcal{S}(f) = \frac{T}{2} \left(\frac{\sin(\pi f T/2)}{\pi f T/2} \right) \times \frac{1}{T} \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T}\right), \\ &= \sum_{n=-\infty}^{\infty} \frac{1}{2} \left(\frac{\sin(n\pi/2)}{n\pi/2} \right) \delta\left(f - \frac{n}{T}\right). \end{aligned}$$



Two-minute miniquiz problem

Problem 18-1 — Fourier series of the square wave

a) Determine the Fourier series coefficients of the square wave.

b) From the Fourier series coefficients determine the average value of the square wave.

Two-minute miniquiz solution

Problem 18-1 — Fourier series of the square wave

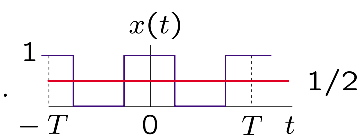
a) Since the Fourier transform of the square wave is

$$\mathcal{X}(f) = \sum_{n=-\infty}^{\infty} \frac{1}{2}\left(\frac{\sin(n\pi/2)}{n\pi/2}\right)\delta\left(f-\frac{n}{T}\right),$$

the Fourier series coefficients are

$$X[n] = \frac{1}{2}\left(\frac{\sin(n\pi/2)}{n\pi/2}\right)$$

b) $X[0] = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,dt = 1/2.$



Fourier series of a square wave — synthesis of a square wave

We can synthesize the square wave by adding complex exponentials weighted by their Fourier series coefficients,

$$x(t) = \sum_{n=-\infty}^{\infty} X[n]e^{j2\pi nt/T} = \sum_{n=-\infty}^{\infty} \frac{1}{2}\left(\frac{\sin(n\pi/2)}{n\pi/2}\right)e^{j2\pi nt/T}.$$

Note that the coefficients are even functions of n. Hence, we can rewrite the series as follows

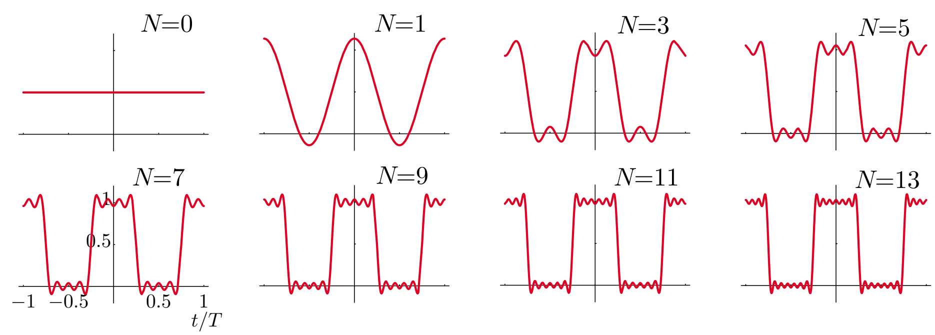
$$\begin{aligned} x(t) &= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{\sin(n\pi/2)}{n\pi/2}\right)\left(e^{j2\pi nt/T} + e^{-j2\pi nt/T}\right), \\ x(t) &= \frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{\sin(n\pi/2)}{n\pi/2}\right)\cos(2\pi nt/T). \end{aligned}$$

Note that all the even harmonics of $x(t)$ are zero except for the term for $n = 0$.

Fourier series of a square wave — synthesis of a square wave, cont'd

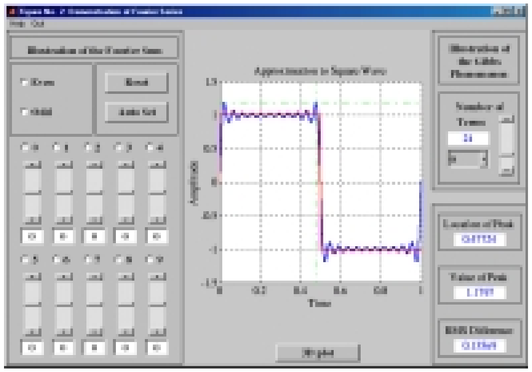
To investigate the synthesis of the square wave, we consider the partial sum of the Fourier series,

$$x_N(t) = \frac{1}{2} + \sum_{n=1}^{N} \left(\frac{\sin(n\pi/2)}{n\pi/2}\right)\cos(2\pi nt/T).$$



Fourier series of a square wave — synthesis of a square wave, cont'd

Demo illustrating the Gibbs' phenomenon.

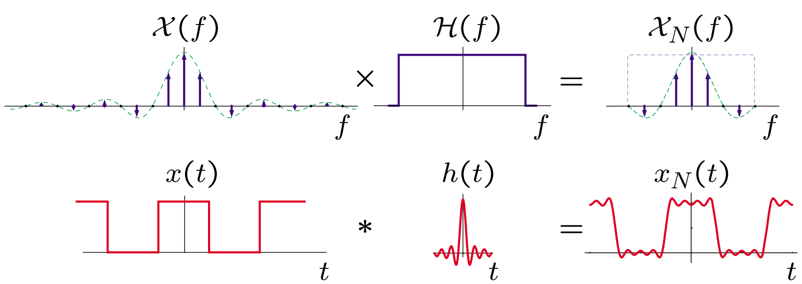


Fourier series of a square wave — Gibbs' phenomenon

As harmonics are added to synthesize the square wave, the partial sum of the Fourier series converges to the square wave everywhere except near the discontinuity where the partial sum takes on the value of 1/2. There are oscillations on either side of the discontinuity whose maximum over and undershoot approach 9% of the discontinuity independent of N. We can interpret the Gibbs' phenomenon by examining the partial sum of the Fourier series as a filtering problem.

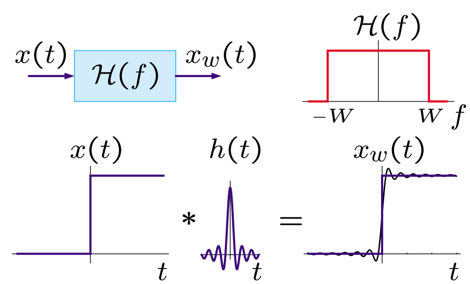
Gibbs' phenomenon interpretation as an ideal LPF of a square wave

Truncation of the Fourier series of a square wave can be interpreted in the frequency domain as passing the square wave through an ideal lowpass filter that truncates the spectrum. That can be interpreted in the time domain as the convolution of the square wave with a sinc function. The response of the ideal lowpass filter to the discontinuity of the square wave gives rise to the oscillations.



Gibbs' phenomenon — step response of an ideal LPF

The Gibbs' phenomenon can be investigated further by examining the step response of an ideal lowpass.



Hence,

$$x_W(t) = u(t) * h(t) = \int_{-\infty}^{t} h(\tau)\,d\tau \text{ where } h(t) = 2W\left(\frac{\sin(2\pi Wt)}{2\pi Wt}\right).$$

Gibbs' phenomenon — step response of an ideal LPF, cont'd

We evaluate the integral

$$x_w(t) = \int_{-\infty}^{t} 2W \left(\frac{\sin(2\pi W\tau)}{2\pi W\tau} \right) d\tau,$$

by changing variables $y = 2\pi Wt$ which yields

$$x_w(t) = \int_{-\infty}^{2\pi Wt} \left(\frac{\sin y}{y} \right) dy.$$

This is closely related to a tabulated function called the sine integral function $\mathrm{Si}(t)$ defined as

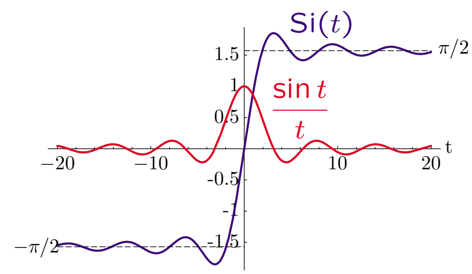
$$\mathrm{Si}(t) = \int_{0}^{t} \frac{\sin y}{y}\, dy.$$

Gibbs' phenomenon — step response of an ideal LPF, cont'd

The sine integral function,

$$\mathrm{Si}(t) = \int_{0}^{t} \frac{\sin y}{y}\, dy,$$

is plotted below.

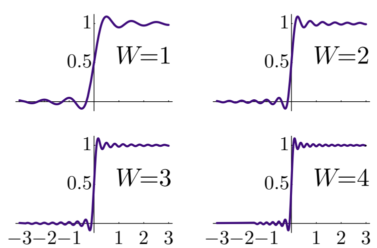


Gibbs' phenomenon — step response of an ideal LPF, cont'd

We express the step with the truncated spectrum in terms of the sine integral function as

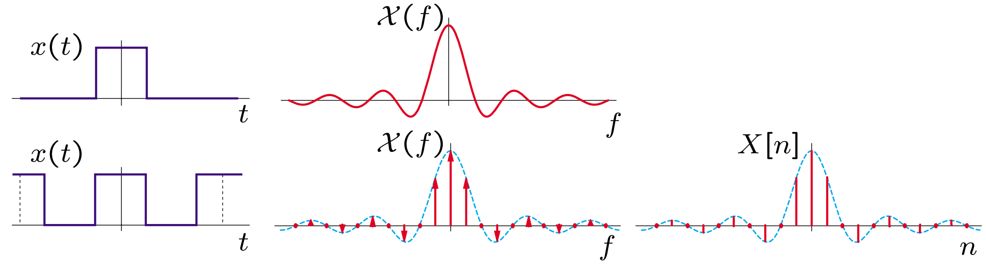
$$x_w(t) = \frac{1}{2} + \frac{1}{\pi}\mathrm{Si}(2\pi Wt),$$

which is shown plotted below for several values of W.



Conclusions

Time Function | Fourier Transform | Fourier Series



- Aperiodic, continuous functions of time $\overset{\mathcal{F}}{\Longleftrightarrow}$ aperiodic, continuous functions of frequency.
- Periodic, continuous functions of time $\overset{\mathcal{F}}{\Longleftrightarrow}$ impulses whose areas are the Fourier series coefficients and located at discrete frequencies.

Historical perspective

Jean Baptiste Joseph Fourier (1768-1830)

Joseph Fourier was born in Auxerre, France on March 21, 1768 and died in Paris on May 4, 1830. This is one of two portraits that has survived.

Joseph Fourier, continued

- Born of humble origins — his father was the town tailor, he was orphaned at age 9, and raised by a neighbor.
- Went to military school and discovered mathematics at age 13.
- Taught mathematics, rhetoric, philosophy, and history in a Benedictine school in his home town.
- Became active in local politics and social causes. His political honesty got him in hot water — he was arrested first by the Robespierre regime and later by the enemy post-Robespierre regime. He was just barely saved from the guillotine twice.

Joseph Fourier, continued

- Taught at the Ecole Polytechnique where he developed an excellent reputation as a lecturer and began publishing mathematical research. He taught with Lagrange and Monge — two eminent mathematicians.
- When Napoleon Bonaparte was put in charge of a French expedition to Egypt, Fourier was chosen as a scientific member. He held political and administrative positions in Egypt and proved to be an able administrator. When the French occupation of Egypt ended, Fourier returned to teaching mathematics in France.

Joseph Fourier, continued

- In 1802, Napoleon appointed Fourier as the prefect of Isère, a French department whose center was Grenoble. This was an important administrative/political position somewhat akin to the governor of a US state. Fourier always hoped to complete his tasks and return to a scholarly life, but he continued to take on important administrative posts throughout his life. His mathematical and scientific studies were largely part-time efforts.

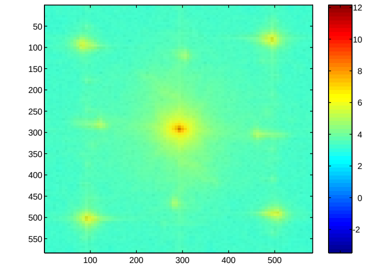
Joseph Fourier, continued

- In 1807, Fourier submitted a paper on the use of trigonometric series to solve problems in heat conduction to the Institute of France. It was reviewed by four famous French mathematicians — Lagrange, Laplace, Lacroix, and Monge. Three of the four referees voted to accept the paper, Lagrange was opposed. Lagrange had been on one side of a raging controversy on the representation of functions by trigonometric series. Lagrange did not believe that arbitrary functions could be expanded in trigonometric series as Fourier's paper claimed. Fourier's original paper was never published, but in 1822 he published his work in a book *The Analytical Theory of Heat* in 1822. Although he did not originate trigonometric series, nor did he determine the precise conditions for their validity, he did use them to solve problems in heat conduction thus illustrating their utility.

Joseph Fourier, continued

Fourier's work and that of those that followed him have had a profound effect on science and mathematics. Fourier transforms are common tools in many different fields of science.

Fourier



Fourier transformed

The image of Fourier (left) is 582 by 582 pixels. The logarithm of the magnitude of the Fourier transform of this image is plotted on the right.