Discrete Fourier transform and Jacobi θ function identities

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Discrete Fourier transform and Jacobi θ function identities

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Classical theta function identities are derived using properties of eigenvectors corresponding to the discrete Fourier transform $\Phi(2)$. In particular, an extended Watson addition formula is obtained corresponding to $\Phi(2)$, and the corresponding result for the discrete Fourier transform $\Phi(3)$ is obtained. © *2010 American Institute of Physics.* [doi:10.1063/1.3272005]

I. INTRODUCTION

The discrete Fourier transform (DFT) is an important tool for applications in engineering and physics, and it is also a source of interesting mathematical problems. Mehta[1] has given constructions of eigenfunctions of the DFT in terms of the discrete analog of the Hermite functions that arise in the continuous cases. Various identities among these eigenvectors are also investigated. The derivatives of Jacobi theta functions as eigenfunctions of the DFT first appeared in Ref. 2. The properties of Jacobi theta functions and their derivatives under DFT are further investigated in Ref. 3. Matveev[4] has proven beautiful consequences of the fact that the DFT Φ is a fourth root of unity, i.e., $\Phi^4=I$. Given any absolutely summable series g_n, he has constructed eigenfunctions of the DFT from the series g_n. This is then applied for the case when the series arises as the summands of a ν theta function with characteristic (a,b), namely, $\theta_{a,b}(x,\tau,\nu)$. This reduces to usual theta function when $\nu=1$. We extend the result of Matveev to derive identities of theta functions and, in particular, obtain an extended Watson addition formula corresponding to $\Phi(2)$ and $\Phi(3)$. The applications of Watson's addition formula have appeared in Ref. 5. In Ref. 6 another addition formula for theta function is given which, in particular, implies the Winquist identity as well as other identities. The method we use is conceptually simple and does not depend on the properties of zeros of theta functions and their infinite product representation.

The basic notations adopted in this paper and some preliminary results are presented in Sec. II. The identities of theta functions corresponding to the DFT $\Phi(2)$ are discussed in Sec. III. Section IV contains identities corresponding to $\Phi(3)$.

II. PRELIMINARY RESULTS

The matrix $\Phi(n)$ corresponding to the DFT of size n is given by

$$\Phi_{jk}(n)=\frac{1}{\sqrt{n}}q^{jk}, \quad j,k=0,\ldots,n-1. \quad q=\frac{2\pi i}{n}. \tag{1}$$

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Definition: For $f=(f_0,\ldots,f_{n-1})^t \in C^n$ we define the DFT $\tilde{f}\in C^n$ by $\tilde{f}=\Phi f=(\tilde{f_0},\tilde{f_1},\ldots,\widetilde{f_{n-1}})$, where

$$\tilde{f}_k=\frac{1}{\sqrt{n}}\sum_{j=0}^{n-1} f_j e^{2\pi ijk/n}.$$

It is clear that $\Phi^4=I$. The multiplicities corresponding to eigenvalues of the DFT $\Phi(n)$ are given by (see Ref. 4)

$$n=4m+2 \Rightarrow m_1=m, \quad m_2=m+1, \quad m_3=m, \quad m_0=m+1,$$

$$n=4m \Rightarrow m_1=m, \quad m_2=m, \quad m_3=m-1, \quad m_0=m+1,$$

$$n=4m+1 \Rightarrow m_1=m, \quad m_2=m, \quad m_3=m, \quad m_0=m+1,$$

$$n=4m+3 \Rightarrow m_1=m+1, \quad m_2=m+1, \quad m_3=m, \quad m_0=m+1,$$

where m_k is the multiplicity of i^k.

Let τ be a complex number with a positive imaginary part. The theta function $\theta_{a,b}(x,\tau)$ with characteristics (a,b) is defined by (see Ref. 7)

$$\theta_{a,b}(x,\tau)=\sum_{m=-\infty}^{m=\infty}\exp[\pi i\tau(m+a)^2+2\pi i(m+a)(x+b)].$$

$\theta_{a,b}(x,\tau)$ is connected with $\theta_{0,0}(x,\tau)=\theta(x,\tau)$ by

$$\theta_{a,b}(x,\tau)=\theta(x+a\tau+b,\tau)exp[\pi ia^2\tau+2\pi a(x+b)].$$

The four classical Jacobi theta functions may be represented as

$$\theta_{0,0}(x,\tau)=\sum_{m=-\infty}^{m=\infty}\exp(\pi im^2\tau+2\pi imx),$$

$$\theta_{1/2,1/2}(x,\tau)=(-1)\sum_{m\in Z}\exp[\pi i\tau(m+1/2)^2+2\pi i(m+1/2)(x+1/2)],$$

$$\theta_{1/2,0}(x,\tau)=\sum_{m\in Z}\exp[\pi i\tau(m+1/2)^2+2\pi i(m+1/2)x],$$

$$\theta_{0,1/2}(x,\tau)=\sum_{m\in Z}\exp[\pi i\tau m^2+2\pi im(x+1/2)].$$

The Jacobi theta functions which are related to eigenfunctions of the DFT $\Phi(n)$ are given by

$$\theta_{j/n,0}(x,\tau)=\sum_{m\in Z}\exp[\pi i\tau(m+j/n)^2+2\pi i(m+j/n)x] \quad \text{for } j=0,1,2,\ldots,n-1.$$

Matveev[4] has proven the following theorem which will be crucial to the following sections.

Theorem 2.1: *(Matveev) For any τ with positive imaginary part the vector $v(x,\tau,k)$ with components $v_j(x,\tau,k)$, $j=0,1,2,\ldots,n-1$ given by*

$$v_j(x,\tau,k) = \theta_{j/n,0}(x,\tau) + (-1)^k\theta_{-j/n,0}(x,\tau) + \frac{1}{\sqrt{n}}\left[(-i)^k\theta\left(\frac{j+x}{n},\frac{\tau}{n^2}\right) + (-i)^{3k}\theta\left(\frac{x-j}{n},\frac{\tau}{n^2},\right)\right], \tag{2}$$

is an eigenvector of the DFT with an eigenvalue i^k ,

$$\Phi(n)v(x,\tau,k) = i^k v(x,\tau,k).$$

■

The proof of the above theorem follows from the fact that $\Phi^4=I$. It has been suggested in Ref. 4 that this may have applications to determine the identities of theta functions. For a given value of n, take eigenvectors of the form (2) corresponding to the eigenvalue i^k,with different values of x and τ. At the most m_k of these are linearly independent. Thus minors of the matrix consisting of the eigenvectors $v(x,\tau,k)$ of order greater than m_k vanish. This may lead to new identities among theta functions. This idea is explored for the DFT $\Phi(2)$ in the following section to obtain an extended Watson addition formula. In Sec. IV, the corresponding result for the DFT $\Phi(3)$ is derived.

III. IDENTITIES CORRESPONDING TO THE DFT $\Phi(2)$

The DFT $\Phi(2)$ has only two eigenvalues +1 and -1. The eigenvector corresponding to eigenvalue +1 is given by

$$v(x,\tau,0) = \begin{bmatrix} 2\theta(x,\tau) + \sqrt{2}\theta\left(\frac{x}{2},\frac{\tau}{4}\right) \\ 2\theta_{1/2,0}(x,\tau) + \sqrt{2}\theta\left(\frac{x+1}{2},\frac{\tau}{4}\right) \end{bmatrix}. \tag{3}$$

The eigenvector corresponding to eigenvalue of -1 is given by

$$v(x,\tau,2) = \begin{bmatrix} 2\theta(x,\tau) - \sqrt{2}\theta\left(\frac{x}{2},\frac{\tau}{4}\right) \\ 2\theta_{1/2,0}(x,\tau) - \sqrt{2}\theta\left(\frac{x+1}{2},\frac{\tau}{4}\right) \end{bmatrix}. \tag{4}$$

We have

$$\Phi(2)[v(x,\tau,0) + v(x,\tau,2)] = v(x,\tau,0) - v(x,\tau,2).$$

This gives the following two identities:

$$\theta(x,\tau) + \theta_{1/2,0}(x,\tau) = \theta\left(\frac{x}{2},\frac{\tau}{2^2}\right), \tag{5}$$

$$\theta(x,\tau) - \theta_{1/2,0}(x,\tau) = \theta\left(\frac{x+1}{2},\frac{\tau}{2^2}\right). \tag{6}$$

(5) and (6) are equivalent to

$$\theta(2x,4\tau) + \theta_{1/2,0}(2x,4\tau) = \theta(x,\tau), \tag{7}$$

$$\theta(2x,4\tau) - \theta_{1/2,0}(2x,4\tau) = \theta_{0,1/2}(x,\tau). \tag{8}$$

Using (3),

$$v\left(x+\frac{\tau}{2},\tau,0\right) = \begin{bmatrix} 2a\theta_{1/2,0}(x,\tau) + a\sqrt{2}\theta\left(\frac{x}{2},\frac{\tau}{4}\right) \\ 2a\theta(x,\tau) - a\sqrt{2}\theta\left(\frac{x+1}{2},\frac{\tau}{4}\right) \end{bmatrix}, \tag{9}$$

where $a=\exp(-\pi i\tau/4-\pi ix)$,

$$v(x+1,\tau,0) = \begin{bmatrix} 2\theta(x,\tau) + \sqrt{2}\theta\left(\frac{x+1}{2},\frac{\tau}{4}\right) \\ -2\theta_{1/2,0}(x,\tau) + \sqrt{2}\theta\left(\frac{x}{2},\frac{\tau}{4}\right) \end{bmatrix}. \tag{10}$$

Since $v(x+1,\tau,0), v(x+\tau/2,\tau,0)$ are eigenvectors corresponding to the same eigenvalue of 1, which has multiplicity of 1 we have $\det(v(x+1,\tau,0),v(x+\tau/2,\tau,0))=0$. This gives

$$2\theta^2(x,\tau) + 2\theta_{1/2,0}^2(x,\tau) = \theta^2\left(\frac{x+1}{2},\frac{\tau}{4}\right) + \theta^2\left(\frac{x}{2},\frac{\tau}{4}\right). \tag{11}$$

(11) is equivalent to

$$2\theta^2(2x,4\tau) + 2\theta_{1/2,0}^2(2x,4\tau) = \theta_{0,1/2}^2(x,\tau) + \theta^2(x,\tau). \tag{12}$$

Now consider

$$\theta(2x,2\tau)\theta(0,2\tau) = \sum_{m,n}\exp(\pi i(m^2+n^2)2\tau + 2\pi im2x). \tag{13}$$

Let $m+n=n_1$ and $m-n=n_2$, so that n_1 and n_2 are of the same parity. Rewriting (13) in terms of n_1 and n_2 we have

$$\theta(2x,2\tau)\theta(0,2\tau) = \sum_{n_1\equiv n_2 (\mathrm{mod}\ 2)}\exp(\pi i(n_1^2+n_2^2)\tau + 2\pi i(n_1+n_2)x)$$

$$= \sum_{n_1,n_2 \text{ are even}}\exp(\pi i(n_1^2+n_2^2)\tau + 2\pi i(n_1+n_2)x) + \sum_{n_1,n_2 \text{ are odd}}\exp(\pi i(n_1^2+n_2^2)\tau + 2\pi i(n_1+n_2)x)$$

$$= \theta^2(2x,4\tau) + \theta_{1/2,0}^2(2x,4\tau).$$

From (12) we have

$$\theta^2(x,\tau) + \theta_{0,1/2}^2(x,\tau) = 2\theta(2x,2\tau)\theta(0,2\tau). \tag{14}$$

This is a well known transformation of the Landen type. All other transformations of the Landen type can be derived using a similar method.

Now the $\det(v(x,\tau,0),v(x+1,\tau,0)=0$. This gives

$$-4\theta(x,\tau)\theta_{1/2,0}(x,\tau) + 2\sqrt{2}\theta(x,\tau)\theta\left(\frac{x}{2},\frac{\tau}{4}\right) - 2\sqrt{2}\theta_{1/2,0}(x,\tau)\theta\left(\frac{x}{2},\frac{\tau}{4}\right) + 2\theta^2\left(\frac{x}{2},\frac{\tau}{4}\right) - 4\theta(x,\tau)\theta_{1/2,0}(x,\tau)$$

$$-2\sqrt{2}\theta_{1/2,0}(x,\tau)\theta\left(\frac{x+1}{2},\frac{\tau}{4}\right) - 2\sqrt{2}\theta(x,\tau)\theta\left(\frac{x+1}{2},\frac{\tau}{4}\right) - 2\theta^2\left(\frac{x+1}{2},\frac{\tau}{4}\right) = 0.$$

Using Eqs. (5) and (6) the terms with coefficients of $2\sqrt{2}$ cancel each other out. This gives

$$\theta^2\left(\frac{x}{2},\frac{\tau}{4}\right) - \theta^2\left(\frac{x+1}{2},\frac{\tau}{4}\right) = 4\theta(x,\tau)\theta_{1/2,0}(x,\tau). \tag{15}$$

(15) is equivalent to

$$\theta^2(x,\tau) - \theta^2_{0,1/2}(x,\tau) = 4\theta(2x,4\tau)\theta_{1/2,0}(2x,4\tau). \tag{16}$$

Applying the same argument as in the derivation of (14) we have

$$2\theta(2x,4\tau)\theta_{1/2,0}(2x,4\tau) = \theta_{1/2,0}(2x,2\tau)\theta_{1/2,0}(0,2\tau). \tag{17}$$

At $x=0$ we have

$$2\theta(0,4\tau)\theta_{1/2,0}(0,4\tau) = \theta^2_{1/2,0}(0,2\tau). \tag{18}$$

From (16) and (17) we have

$$\theta^2(x,\tau) - \theta^2_{0,1/2}(x,\tau) = 2\theta_{1/2,0}(2x,2\tau)\theta_{1/2,0}(0,2\tau). \tag{19}$$

This is again an identity of the Landen type. From (14) at $x=0$ we have

$$\theta^2(0,\tau) + \theta^2_{0,1/2}(0,\tau) = 2\theta^2(0,2\tau). \tag{20}$$

Similarly from (19) at $x=0$,

$$\theta^2(0,\tau) - \theta^2_{0,1/2}(0,\tau) = 2\theta^2_{1/2,0}(0,2\tau). \tag{21}$$

From (20) and (21) we have

$$\theta^4(0,\tau) - \theta^4_{0,1/2}(0,\tau) = 4\theta^2(0,2\tau)\theta^2_{1/2,0}(0,2\tau). \tag{22}$$

Using (18) we have

$$\theta^4(0,\tau) - \theta^4_{0,1/2}(0,\tau) = \theta^4_{1/2,0}(0,\tau). \tag{23}$$

This is a well known Jacobi identity between the null values of theta functions. Many of the classical identities involving the squares of theta functions can be obtained by the method illustrated above. In the next theorem we illustrate the technique to extend the classical Watson's addition formula for theta functions (see Ref. 8).

Theorem 3.1: *(Extended Watson's addition formula)*

$$\begin{aligned}\theta_{0,1/2}(x_1+x_2,\tau)\theta(x_1-x_2,\tau) - \theta_{0,1/2}(x_1-x_2,\tau)\theta(x_1+x_2,\tau) &= 2\theta(2x_1+2x_2,4\tau)\theta_{1/2,0}(2x_1-2x_2,4\tau)\\ &\quad - 2\theta(2x_1-2x_2,4\tau)\theta_{1/2,0}(2x_1+2x_2,4\tau).\end{aligned} \tag{24}$$

Proof:

Using (5) and (6)

$$\theta\left(\frac{x_1 \pm x_2}{2},\frac{\tau}{2}\right) = \theta(x_1 \pm x_2,2\tau) + \theta_{1/2,0}(x_1 \pm x_2,2\tau), \tag{25}$$

$$\theta\left(\frac{x_1 \pm x_2+1}{2},\frac{\tau}{2}\right) = \theta(x_1 \pm x_2,2\tau) - \theta_{1/2,0}(x_1 \pm x_2,2\tau). \tag{26}$$

From (3) and (4), we have

$$v(x_1+x_2,2\tau,0)=\begin{bmatrix} 2\theta(x_1+x_2,2\tau)+\sqrt{2}\theta\left(\dfrac{x_1+x_2}{2},\dfrac{\tau}{2}\right) \\ 2\theta_{1/2,0}(x_1+x_2,2\tau)+\sqrt{2}\theta\left(\dfrac{x_1+x_2+1}{2},\dfrac{\tau}{2}\right)\end{bmatrix},$$

$$v(x_1-x_2,2\tau,0)=\begin{bmatrix} 2\theta(x_1-x_2,2\tau)+\sqrt{2}\theta\left(\dfrac{x_1-x_2}{2},\dfrac{\tau}{2}\right) \\ 2\theta_{1/2,0}(x_1-x_2,2\tau)+\sqrt{2}\theta\left(\dfrac{x_1-x_2+1}{2},\dfrac{\tau}{2}\right)\end{bmatrix}$$

are eigenvectors of $\Phi(2)$ corresponding to eigenvalues +1 which has multiplicity of 1. Therefore,

$$\det(v(x_1+x_2,2\tau,0),v(x_1-x_2,2\tau,0))=0.$$

This gives

$$\begin{aligned}&4\theta(x_1+x_2,2\tau)\theta_{1/2,0}(x_1-x_2,2\tau)+2\sqrt{2}\theta(x_1+x_2,2\tau)\theta\left(\frac{x_1-x_2+1}{2},\frac{\tau}{2}\right)+2\sqrt{2}\theta_{1/2,0}(x_1\\&\quad -x_2,2\tau)\theta\left(\frac{x_1+x_2}{2},\frac{\tau}{2}\right)+2\theta\left(\frac{x_1+x_2}{2},\frac{\tau}{2}\right)\theta\left(\frac{x_1-x_2+1}{2},\frac{\tau}{2}\right)-4\theta_{1/2,0}(x_1+x_2,2\tau)\theta(x_1-x_2,2\tau)\\&\quad -2\sqrt{2}\theta_{1/2,0}(x_1+x_2,2\tau)\theta\left(\frac{x_1-x_2}{2},\frac{\tau}{2}\right)-2\sqrt{2}\theta(x_1-x_2,2\tau)\theta\left(\frac{x_1+x_2+1}{2},\frac{\tau}{2}\right)\\&\quad -2\theta\left(\frac{x_1+x_2+1}{2},\frac{\tau}{2}\right)\theta\left(\frac{x_1-x_2}{2},\frac{\tau}{2}\right)=0.\end{aligned}\tag{27}$$

In this expression consider the terms with $2\sqrt{2}$ as coefficient. Using formulas (25) and (26), we have

$$\begin{aligned}A&=\theta(x_1+x_2,2\tau)\theta\left(\frac{x_1-x_2+1}{2},\frac{\tau}{2}\right)\\&=\theta(x_1+x_2,2\tau)\theta(x_1-x_2,2\tau)-\theta(x_1+x_2,2\tau)\theta_{1/2,0}(x_1-x_2,2\tau),\end{aligned}$$

$$\begin{aligned}B&=\theta_{1/2,0}(x_1-x_2,2\tau)\theta\left(\frac{x_1+x_2}{2},\frac{\tau}{2}\right)\\&=\theta_{1/2,0}(x_1-x_2,2\tau)\theta(x_1+x_2,2\tau)+\theta_{1/2,0}(x_1+x_2,2\tau)\theta_{1/2,0}(x_1-x_2,2\tau),\end{aligned}$$

$$\begin{aligned}C&=\theta_{1/2,0}(x_1+x_2,2\tau)\theta\left(\frac{x_1-x_2}{2},\frac{\tau}{2}\right)\\&=\theta_{1/2,0}(x_1+x_2,2\tau)\theta(x_1-x_2,2\tau)+\theta_{1/2,0}(x_1+x_2,2\tau)\theta_{1/2,0}(x_1-x_2,2\tau),\end{aligned}$$

$$\begin{aligned}D&=\theta(x_1-x_2,2\tau)\theta\left(\frac{x_1+x_2+1}{2},\frac{\tau}{2}\right)\\&=\theta(x_1-x_2,2\tau)\theta(x_1+x_2,2\tau)-\theta(x_1-x_2,2\tau)\theta_{1/2,0}(x_1+x_2,2\tau).\end{aligned}$$

It is clear that $A+B-C-D=0$. Hence all the terms with coefficients $2\sqrt{2}$ cancel each other out. Equation (27) becomes

$$4\theta(x_1+x_2,2\tau)\theta_{1/2,0}(x_1-x_2,2\tau)-4\theta_{1/2,0}(x_1+x_2,2\tau)\theta(x_1-x_2,2\tau)$$
$$=2\theta\left(\frac{x_1+x_2+1}{2},\frac{\tau}{2}\right)\theta\left(\frac{x_1-x_2}{2},\frac{\tau}{2}\right)-2\theta\left(\frac{x_1-x_2+1}{2},\frac{\tau}{2}\right)\theta\left(\frac{x_1+x_2}{2},\frac{\tau}{2}\right).$$

Replacing x_1, x_2, τ by $2x_1, 2x_2, 2\tau$, respectively, we obtain

$$2\theta(2x_1+2x_2,4\tau)\theta_{1/2,0}(2x_1-2x_2,4\tau)-2\theta_{1/2,0}(2x_1+2x_2,4\tau)\theta(2x_1-2x_2,4\tau)=\theta_{0,1/2}(x_1+x_2,\tau)\theta(x_1-x_2,\tau)-\theta_{0,1/2}(x_1-x_2,\tau)\theta(x_1+x_2,\tau).$$

This proves (24). ■

Theorem 3.2: *(Watson addition formula)*

$$\theta_{1/2,1/2}(x_1,\tau)\theta_{1/2,1/2}(x_2,\tau)=\theta(x_1+x_2,2\tau)\theta_{1/2,0}(x_1-x_2,2\tau)-\theta(x_1-x_2,2\tau)\theta_{1/2,0}(x_1+x_2,2\tau). \tag{28}$$

Proof: Consider first term of the left hand side of (24),

$$\theta_{0,1/2}(x_1+x_2,\tau)\theta(x_1-x_2,\tau)=\sum\exp\left[\pi i(m^2+n^2)\tau+2\pi i\left((m+n)x_1+(m-n)x_2+\frac{m}{2}\right)\right]. \tag{29}$$

Let $m+n=n_1$, $m-n=n_2$, where n_1 and n_2 are of same parity. Rewriting (29) in terms of n_1 and n_2,

$$=\sum_{n_1\equiv n_2 (\mathrm{mod}\ 2)}\exp\left[\pi i\left(\frac{n_1^2+n_2^2}{2}\right)\tau+2\pi i\left(n_1x_1+n_2x_2+\frac{n_1+n_2}{4}\right)\right]=\sum_{n_1,n_2\ \text{are even}}\exp\left[\pi i\left(\frac{n_1^2+n_2^2}{2}\right)\tau+2\pi i\left(n_1x_1+n_2x_2+\frac{n_1+n_2}{4}\right)\right]+\sum_{n_1,n_2\ \text{are odd}}\exp\left[\pi i\left(\frac{n_1^2+n_2^2}{2}\right)\tau+2\pi i\left(n_1x_1+n_2x_2+\frac{n_1+n_2}{4}\right)\right]$$
$$=\theta_{0,1/2}(2x_1,2\tau)\theta_{0,1/2}(2x_2,2\tau)+\theta_{1/2,1/2}(2x_1,2\tau)\theta_{1/2,1/2}(2x_2,2\tau).$$

Similarly, using the same argument it is easy to show that

$$\theta_{0,1/2}(x_1-x_2,\tau)\theta(x_1+x_2,\tau)=\theta_{0,1/2}(2x_1,2\tau)\theta_{0,1/2}(2x_2,2\tau)-\theta_{1/2,1/2}(2x_1,2\tau)\theta_{1/2,1/2}(2x_2,2\tau).$$

By using the above two results in (24), we get

$$\theta_{1/2,1/2}(2x_1,2\tau)\theta_{1/2,1/2}(2x_2,2\tau)=\theta(2x_1+2x_2,4\tau)\theta_{1/2,0}(2x_1-2x_2,4\tau)-\theta(2x_1-2x_2,4\tau)\theta_{1/2,0}(2x_1+2x_2,4\tau).$$

Watson's addition formula (28) is obtained by replacing x_1, x_2, τ by $x_1/2, x_2/2, \tau/2$, respectively, in the above equation. ■

IV. IDENTITIES CORRESPONDING TO THE DFT $\Phi(3)$

The DFT $\Phi(3)$ has three eigenvalues 1, i, and -1 each of multiplicity of 1, (see Ref. 4). The eigenvector corresponding to eigenvalue +1 is given by using Theorem 2.1,

$$v(x,\tau,0)=\begin{bmatrix} 2\theta(x,\tau)+\frac{2}{\sqrt{3}}\theta\left(\frac{x}{3},\frac{\tau}{3^2}\right) \\ \theta_{1/3,0}(x,\tau)+\theta_{2/3,0}(x,\tau)+\frac{1}{\sqrt{3}}\left[\theta\left(\frac{x+1}{3},\frac{\tau}{3^2}\right)+\theta\left(\frac{x-1}{3},\frac{\tau}{3^2}\right)\right] \\ \theta_{1/3,0}(x,\tau)+\theta_{2/3,0}(x,\tau)+\frac{1}{\sqrt{3}}\left[\theta\left(\frac{x+2}{3},\frac{\tau}{3^2}\right)+\theta\left(\frac{x-2}{3},\frac{\tau}{3^2}\right)\right]\end{bmatrix}.$$

The eigenvector corresponding to eigenvalue of -1 is given by

$$v(x,\tau,2) = \begin{bmatrix} 2\theta(x,\tau) - \frac{2}{\sqrt{3}}\theta\left(\frac{x}{3},\frac{\tau}{3^2}\right) \\ \theta_{1/3,0}(x,\tau) + \theta_{2/3,0}(x,\tau) - \frac{1}{\sqrt{3}}\left[\theta\left(\frac{x+1}{3},\frac{\tau}{3^2}\right) + \theta\left(\frac{x-1}{3},\frac{\tau}{3^2}\right)\right] \\ \theta_{1/3,0}(x,\tau) + \theta_{2/3,0}(x,\tau) - \frac{1}{\sqrt{3}}\left[\theta\left(\frac{x+2}{3},\frac{\tau}{3^2}\right) + \theta\left(\frac{x-2}{3},\frac{\tau}{3^2}\right)\right] \end{bmatrix}.$$

Similarly the eigenvector corresponding to eigenvalue i is given by

$$v(x,\tau,1) = \begin{bmatrix} 0 \\ \theta_{1/3,0}(x,\tau) - \theta_{2/3,0}(x,\tau) - \frac{i}{\sqrt{3}}\left[\theta\left(\frac{x+1}{3},\frac{\tau}{3^2}\right) - \theta\left(\frac{x-1}{3},\frac{\tau}{3^2}\right)\right] \\ \theta_{2/3,0}(x,\tau) - \theta_{1/3,0}(x,\tau) - \frac{i}{\sqrt{3}}\left[\theta\left[\frac{x+2}{3},\frac{\tau}{3^2}\right) - \theta\left(\frac{x-2}{3},\frac{\tau}{3^2}\right)\right] \end{bmatrix}.$$

We have

$$\theta(x+1,\tau) = \theta(x,\tau), \quad \theta_{1/3,0}(x+1,\tau) = \omega\theta_{1/3,0}(x,\tau), \quad \theta_{2/3,0}(x+1,\tau) = \omega^2\theta_{2/3,0}(x,\tau),$$

where $\omega = e^{2\pi i/3}$ is the cube root of unity. We have

$$\Phi(3)[v(x,\tau,0) + v(x,\tau,1) + v(x,\tau,2)] = v(x,\tau,0) + iv(x,\tau,1) - v(x,\tau,2).$$

By equating the first component we obtain

$$\theta(x,\tau) + \theta_{1/3,0}(x,\tau) + \theta_{2/3,0}(x,\tau) = \theta\left(\frac{x}{3},\frac{\tau}{3^2}\right). \tag{30}$$

Replacing x by $x+1$ and $x+2$ in (30) we get the following identities:

$$\theta(x,\tau) + \omega\theta_{1/3,0}(x,\tau) + \omega^2\theta_{2/3,0}(x,\tau) = \theta\left(\frac{x+1}{3},\frac{\tau}{3^2}\right), \tag{31}$$

$$\theta(x,\tau) + \omega^2\theta_{1/3,0}(x,\tau) + \omega\theta_{2/3,0}(x,\tau) = \theta\left(\frac{x+2}{3},\frac{\tau}{3^2}\right). \tag{32}$$

The above identities are equivalent to

$$\theta(x,\tau) = \theta(3x,9\tau) + \theta_{1/3,0}(3x,9\tau) + \theta_{2/3,0}(3x,9\tau), \tag{33}$$

$$\theta_{0,1/3}(x,\tau) = \theta(3x,9\tau) + \omega\theta_{1/3,0}(3x,9\tau) + \omega^2\theta_{2/3,0}(3x,9\tau), \tag{34}$$

$$\theta_{0,2/3}(x,\tau) = \theta(3x,9\tau) + \omega^2\theta_{1/3,0}(3x,9\tau) + \omega\theta_{2/3,0}(3x,9\tau). \tag{35}$$

Identities (30)–(35) are extensions of the identities (5)–(8).

Theorem 4.1: *[Extended Watson addition formula corresponding to* $\Phi(3)$ *]*

$$\begin{aligned} &3\theta(3x+3y,9\tau)\theta_{1/3,0}(3x-3y,9\tau) + 3\theta(3x+3y,9\tau)\theta_{2/3,0}(3x-3y,9\tau) \\ &\quad - 3\theta(3x-3y,9\tau)\theta_{1/3,0}(3x+3y,9\tau) - 3\theta_{2/3,0}(3x+3y,9\tau)\theta(3x-3y,9\tau) \\ &= \theta_{0,1/3}(x+y,\tau)\theta(x-y,\tau) + \theta_{0,2/3}(x+y,\tau)\theta(x-y,\tau) \\ &\quad - \theta(x+y,\tau)\theta_{0,1/3}(x-y,\tau) - \theta(x+y,\tau)\theta_{0,2/3}(x-y,\tau). \end{aligned} \tag{36}$$

Proof: Consider the eigenvectors $v(x+y,3\tau,0)$, $v(x-y,3\tau,0)$,

$$v(x+y,3\tau,0)=\begin{bmatrix} 2\theta(x+y,3\tau)+\frac{2}{\sqrt{3}}\theta\left(\frac{x+y}{3},\frac{\tau}{3}\right) \\ \theta_{1/3,0}(x+y,3\tau)+\theta_{2/3,0}(x+y,3\tau)+\frac{1}{\sqrt{3}}\left[\theta\left(\frac{x+y+1}{3},\frac{\tau}{3}\right)+\theta\left(\frac{x+y-1}{3},\frac{\tau}{3}\right)\right] \\ \theta_{1/3,0}(x+y,3\tau)+\theta_{2/3,0}(x+y,3\tau)+\frac{1}{\sqrt{3}}\left[\theta\left(\frac{x+y+2}{3},\frac{\tau}{3}\right)+\theta\left(\frac{x+y-2}{3},\frac{\tau}{3}\right)\right] \end{bmatrix},$$

$$v(x-y,3\tau,0)=\begin{bmatrix} 2\theta(x-y,3\tau)+\frac{2}{\sqrt{3}}\theta\left(\frac{x-y}{3},\frac{\tau}{3}\right) \\ \theta_{1/3,0}(x-y,3\tau)+\theta_{2/3,0}(x-y,3\tau)+\frac{1}{\sqrt{3}}\left[\theta\left(\frac{x-y+1}{3},\frac{\tau}{3}\right)+\theta\left(\frac{x-y-1}{3},\frac{\tau}{3}\right)\right] \\ \theta_{1/3,0}(x-y,3\tau)+\theta_{2/3,0}(x-y,3\tau)+\frac{1}{\sqrt{3}}\left[\theta\left(\frac{x-y+2}{3},\frac{\tau}{3}\right)+\theta\left(\frac{x-y-2}{3},\frac{\tau}{3}\right)\right] \end{bmatrix}.$$

These are eigenvectors corresponding to the same eigenvalue of 1 which has multiplicity of 1. Therefore, any 2×2 minor of $[v(x+y,3\tau,0),\ v(x-y,3\tau,0)]$ is zero. We consider the minor formed by the first two components in the eigenvectors $v(x+y,3\tau,0)$, $v(x-y,3\tau,0)$, i.e.,

$$\begin{vmatrix} v_0(x+y,3\tau,0) & v_0(x-y,3\tau,0) \\ v_1(x+y,3\tau,0) & v_1(x-y,3\tau,0) \end{vmatrix}=0.$$

In the following expansion we have used the temporary shorthand notation $\theta_{j/3,0}(x\pm y,3\tau)=\theta_{j/3,0}(x\pm y)$ and $\theta((x\pm y\pm j)/3,\tau/3)=\theta((x\pm y\pm j)/3)$ for j=0,1,2,

$$\begin{aligned} &2\theta(x+y)\theta_{1/3,0}(x-y)+2\theta(x+y)\theta_{2/3,0}(x-y)+\frac{2}{\sqrt{3}}\theta(x+y)\theta\left(\frac{x-y+1}{3}\right)+\frac{2}{\sqrt{3}}\theta(x+y)\theta\left(\frac{x-y-1}{3}\right) \\ &\quad+\frac{2}{\sqrt{3}}\theta\left(\frac{x+y}{3}\right)\theta_{1/3,0}(x-y)+\frac{2}{\sqrt{3}}\theta\left(\frac{x+y}{3}\right)\theta_{2/3,0}(x-y)+\frac{2}{3}\theta\left(\frac{x+y}{3}\right)\theta\left(\frac{x-y+1}{3}\right) \\ &\quad+\frac{2}{3}\theta\left(\frac{x+y}{3}\right)\theta\left(\frac{x-y-1}{3}\right)-2\theta(x-y)\theta_{1/3,0}(x+y)-2\theta(x-y)\theta_{2/3,0}(x+y) \\ &\quad-\frac{2}{\sqrt{3}}\theta(x-y)\theta\left(\frac{x+y+1}{3}\right)-\frac{2}{\sqrt{3}}\theta(x-y)\theta\left(\frac{x+y-1}{3}\right)-\frac{2}{\sqrt{3}}\theta\left(\frac{x-y}{3}\right)\theta_{1/3,0}(x+y) \\ &\quad-\frac{2}{\sqrt{3}}\theta\left(\frac{x-y}{3}\right)\theta_{2/3,0}(x+y)-\frac{2}{3}\theta\left(\frac{x-y}{3}\right)\theta\left(\frac{x+y+1}{3}\right)-\frac{2}{3}\theta\left(\frac{x-y}{3}\right)\theta\left(\frac{x+y-1}{3}\right)=0 \end{aligned} \tag{37}$$

Label the summands on the left hand side of (37) successively as $A_1,A_2,A_3,\ldots,A_{16}$. Then $\sum_{k=1}^{16}A_k=0$. By using formulas (30)–(32) we have $A_3+A_4+A_5+A_6+A_{11}+A_{12}+A_{13}+A_{14}=0$. (37) becomes

$$\begin{aligned} &3\theta(x+y)\theta_{1/3,0}(x-y)+3\theta(x+y)\theta_{2/3,0}(x-y)-3\theta(x-y)\theta_{1/3,0}(x+y)-3\theta(x-y)\theta_{2/3,0}(x+y) \\ &\quad=\theta\left(\frac{x-y}{3}\right)\theta\left(\frac{x+y+1}{3}\right)+\theta\left(\frac{x-y}{3}\right)\theta\left(\frac{x+y+2}{3}\right)-\theta\left(\frac{x+y}{3}\right)\theta\left(\frac{x-y+1}{3}\right) \\ &\qquad-\theta\left(\frac{x+y}{3}\right)\theta\left(\frac{x-y+2}{3}\right), \end{aligned} \tag{38}$$

replacing x, y, τ by $3x, 3y, 3\tau$ we get the required formula in the theorem.

Corollary 4.1.1:

$$3\theta(3x,9\tau)\theta_{1/3,0}(0,9\tau) + 3\theta(3x,9\tau)\theta_{2/3,0}(0,9\tau) - 3\theta(0,9\tau)\theta_{1/3,0}(3x,9\tau) - 3\theta_{2/3,0}(3x,9\tau)\theta(0,9\tau)$$
$$= \theta_{0,1/3}(x,\tau)\theta(0,\tau) + \theta_{0,2/3}(x,\tau)\theta(0,\tau) - \theta(x,\tau)\theta_{0,1/3}(0,\tau) - \theta(x,\tau)\theta_{0,2/3}(0,\tau). \quad (39)$$

Proof: Put $y=x$ in the Theorem 4.1 and replace x by $x/2$ to get (39). ∎

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