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OFDM Modulation Analysis & Simulations

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Contents

1 Project Purpose

The objective of this project is to study an OFDM system's operation.
The OFDM modulation (Orthogonal Frequency Division Multiplexing) makes equalization very easy with the use of the cyclic prefix. It allows to convert the transmission through a wideband frequency selective channel into multiple parallel and independent frequency flat subchannels.
This offers a lot of flexibility for per-subcarrier processing, we will focus on the following points in this report :

- Implementation of an OFDM chain on a frequency selective channel
- Adaptative modulation with bit and power allocation
- Channel estimation
- Coding across the frequencies and optimal decoding

For all the simulations performed in this project, an OFDM transmission with the following parameters is considered :

N	128	Number of Subcarriers
L_c	16	Cyclic prefix length
Δf	15 [kHz]	Subcarrier Spacing
f_c	2 [GHz]	Central Carrier frequency
	4-QAM	Mapping

The base-band ODFM chain is represented in figure 1.

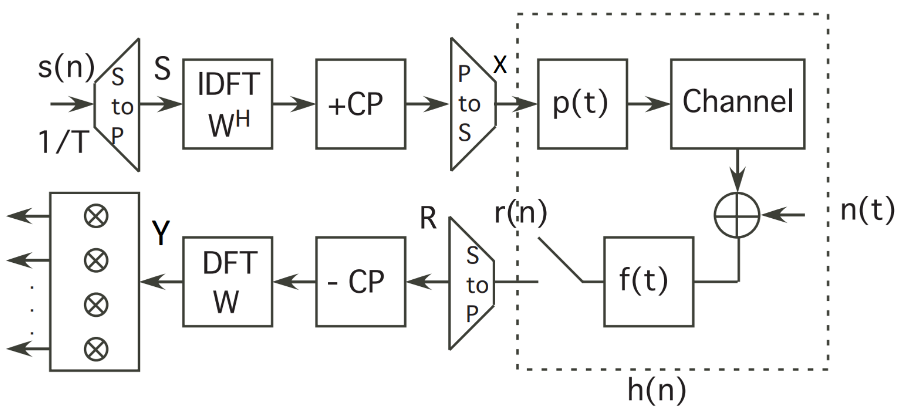

Figure 1: OFDM Base-Band Chain
from Reference (1)

2 Basic OFDM Chain Analysis

The first part of this project is dedicated to simulating a basic OFDM chain on an AWGN channel.
We will first derive the expression of the signals at different steps of the chain, then the quantities $\frac{E_s}{N0}$, the SNR and then apply it to the particular case of an AWGN channel.

The 4-QAM symbol s[n] are grouped by blocks, passed through an IDFT operator and then added to cyclic prefix to get

$$x[n] = \sum_{m=0}^{N-1} s[m] \frac{e^{j2\pi \frac{m(n-L_c)}{N}}}{\sqrt{N}} \tag{1}$$

$n \in [0, N + L_c - 1]$

At the receiver side, taking into account h[n] being the cascade of the pulse shaping filter (p(t), the channel and the adapted filter to p(t), we get

$$r[n] = \sum_{l=0}^{L_h-1} h[l]x[n-l] + n[n] \tag{2}$$

L_h being the length of the channel impulse response.
Removing the cyclic prefix at receiver :

$$\begin{aligned}
\overline{r}[n] &= r[n + L_c] && (3)\\
&= \sum_{l=0}^{L_h-1} h[l]x[n + L_c - l] + n[n + L_c] && (4)\\
&= \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} (\sum_{l=0}^{L_h-1} h[l]e^{j2\pi \frac{ml}{N}}) s[m] e^{j2\pi \frac{mn}{N}} + n[n + L_c] && (5)
\end{aligned}$$

Taking the DFT of blocks of length N we get

$$Y[k] = W[k]\, s[k] + V[k] \tag{6}$$

with the normalized operator

$$W[k] = \frac{1}{\sqrt{N}} \sum_{l=0}^{L_h-1} h[l] e^{-j2\pi \frac{kl}{N}} \tag{7}$$

Summarising in **matrix notations**, for an OFDM symbol with

- **Y**, **R** and **X** from figure 1
- $\hat{\mathbf{Y}}$ the obtained symbols after equalisation
- $\mathbf{\Lambda} = diag(\lambda)$ with $\lambda = \sqrt{N}\mathbf{Wh}$
- $\mathbf{H_c} = \mathbf{W}^H \mathbf{\Lambda W}$ the convolution matrix made circulant thanks to the add-CP step
- **N** the additive white Gaussian noise

$$\begin{aligned}
\mathbf{R} &= \mathbf{H_c X} + \mathbf{N} && (8)\\
&= \mathbf{W}^H \mathbf{\Lambda W W}^H \mathbf{S} + \mathbf{N} && (9)\\
&= \mathbf{W}^H \mathbf{\Lambda S} + \mathbf{N} && (10)
\end{aligned}$$

$$\mathbf{Y} = \mathbf{W}\mathbf{R} \tag{11}$$
$$= \mathbf{\Lambda}\mathbf{S} + \mathbf{W}\mathbf{N} \tag{12}$$

$$\hat{\mathbf{Y}} = \mathbf{\Lambda}^{-1}\mathbf{Y} \tag{13}$$
$$= \mathbf{S} + \underbrace{\mathbf{\Lambda}^{-1}\mathbf{W}\mathbf{N}}_{\mathcal{N}} \tag{14}$$

We see from equation 15 that each received symbol y_i is impacted by a flat channel characterised by a coefficient λ_i.

$$y_i = \lambda_i s_i + \nu_i \tag{15}$$

2.1 SNR

Calling $\sigma^2_{s,i}$ the energy on the symbol put on the channel i; the SNR associated to each channel can be deduced from the previous observation, we then get for the channel i:

$$\boxed{SNR_i = \frac{\sigma^2_{s,i}}{\frac{2\,N_0}{|\lambda_i|}}} \tag{16}$$

In the case of an AWGN channel h[n]=h[0]=1, then $\mathbf{\Lambda}$ is the identity matrix and we get the same SNR for each channel:

$$SNR = \frac{\sigma^2_{s,i}}{2N_0} \tag{17}$$

2.2 $\mathbf{Es/N_0}$

The Signal Energy as by convention to be taken at the receiver stage. It therefore corresponds to the energy of in the signal r[n] as :

$$E_s = \frac{\mathbf{E}\{r[n]r^*[n]\}}{2} \tag{18}$$
$$= \frac{\sigma^2_{OFDM}}{2N} \tag{19}$$

Defining σ^2_{OFDM} the energy in an OFDM symbol. The full computation is avalaible on Annexe 1, Section 5.1.1.

In the case of a AWGN channel we have that

$$\sigma^2_{OFDM} = \sigma^2_s(L_c + N)$$

We then obtain

$$\boxed{\frac{E_s}{N_0} = \frac{\sigma^2_s}{2N_0}\frac{L_c + N}{N}} \tag{20}$$

$$\rightarrow \frac{E_s}{N_0} = SNR\,\frac{L_c + N}{N} \tag{21}$$

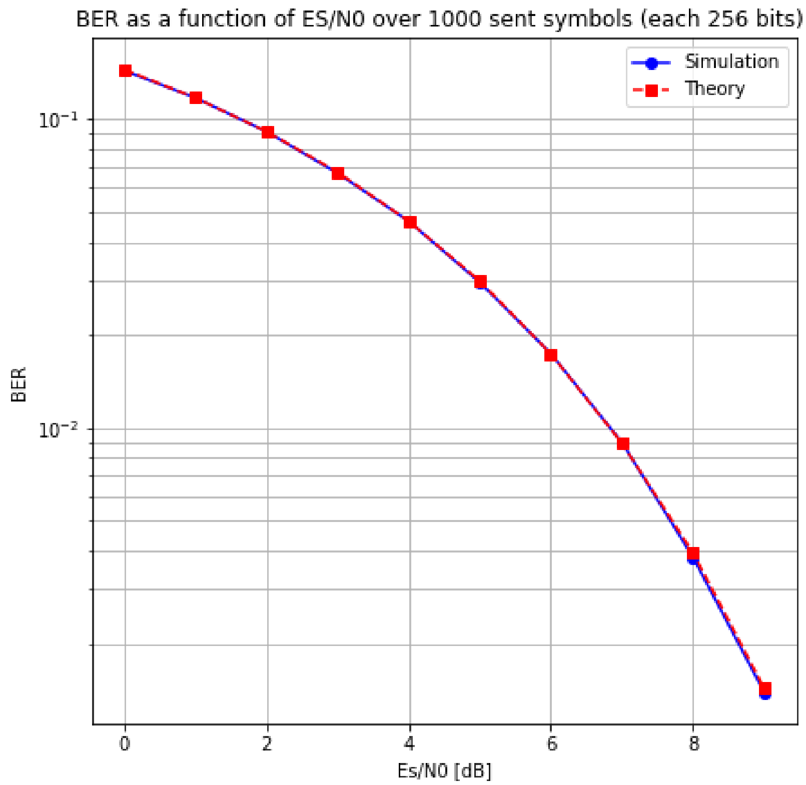

Figure 2: BER - $E_s/N0$ theoretical vs simulation

Now that the link between SNR and E_s/N_0 has been established, we can compare our simulation with theoretical results as we know that for 4-QAM constellations and grey coding, the BER can be expressed as

$$BER = \frac{1}{2} erfc\left(\sqrt{\frac{SNR}{2}}\right) \tag{22}$$

We get the curve on figure 2 for a simulation sending 1000 OFDM symbols.

The obtained results are consistent. Our simulation corresponds to the BER($\frac{E_s}{N0}$) of a 4-QAM constellation in AWGN conditions. Each sub-channel is independent from the others and is only affected by an AWGN noise (as the λ_i are equal to zero in this case.

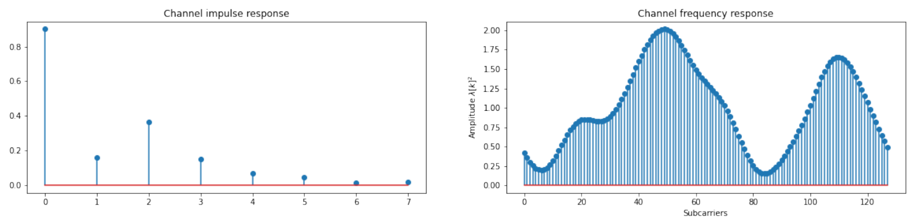

Figure 3: Channel of our simulation

3 Resource Allocation

3.1 Maximising bitrate

When we assume the channel is known at the transmitter and receiver side, it is possible for the transmitter to adapt the modulation and distribute powers differently in order to maximise the bits sent in one symbol. The input channel is given in figure 3

We first decide to maximise the bit rate for a given probability of error $p_e = 10^{-5}$ at the receiver.

3.1.1 Mathematical model

We first establish a mathematical model to maximise the bitrate for a given probability of error. We first allocate power on each channel. Once the power has been allocated, we can decide which constellation to use for the given probability of error at the receiving end. The probability of error is given by

$$p_e \approx 2\frac{\sqrt{M}-1}{\sqrt{M}} erfc(\frac{\sqrt{3SNR}}{2(M-1)}) = 10^{-5} \tag{23}$$

Using (23) and knowing $M = 2^{2b}$ with b the number of bits per subchannel

$$b_i \approx \frac{1}{2} log_2(1 + \frac{SNR_i}{\Gamma}) \tag{24}$$

This is the variable we want to maximise, more precisely, we want

$$\max_{p_k} \sum_k \frac{1}{2} log_2(1 + \frac{p_k}{\sigma_{n,k}^2 \Gamma}) \tag{25}$$

Equation (25) was achieved by replacing the SNR in (24)with the following equation. The power of our constellation $\sigma_{s,i}^2$ is fixed to 2 for our non-normalised 4-QAM constellation

$$SNR_i = \frac{\sigma_{s,i}^2}{\frac{2N_0}{\|\lambda_i\|^2}} \tag{26}$$

λ_i are the channel coefficients (DFT of the channel's impulse response) (see (15)). We can compute the SNR gap

$$\Gamma = \frac{2erfc^{-1}(\frac{p_{e,sym}}{2})^2}{3} = 8.42dB \tag{27}$$

Leaving the only unknown variable in (25) to be $2N_0$, the variance of the noise. This will be a simulation parameter. We want to compute the power (and bitrate) for different values of $\frac{E_s}{N_0}$. We will compare our results with the uniform case. So we assume the maximum power sent over our channel $P_{max} = \sigma_{s,i}^2 N$, the uniform allocation case.

$$P_{max} = \frac{E_s}{N_0} 2N_0 (\frac{N^2}{L_c + N}) \tag{28}$$

$P_{max} = \sigma_{s,i}^2 N = 256$, $\sigma_{s,i} = 2$, N and L_c are given in section 1

We state the maximisation (25) again:

$$\max_{p_k} \sum_k \frac{1}{2} log_2(1 + \frac{p_k}{\sigma_{n,k}^2 \Gamma}) \tag{29}$$

With constraints: $\sum p_k \leq P_{max}$ and $p_, \geq 0$

Using Lagrange maximisation under constraint and Karush-Kuhn-Tucker conditions, the solution to this problem is obtained with

$$p_k = [\mu - \sigma_{n,k}^2 \Gamma]^+ \tag{30}$$

Where $\sigma_{n,k}^2 = \frac{\sigma_n^2}{\|\lambda_k\|^2}$ and μ is adapted such that $\sum_k [\mu - \sigma_{n,k}^2 \Gamma] = P_{max}$

3.1.2 Simulation

In order to solve problem stated in (30), we use the water-filling algorithm. More precicely,
For $\frac{E_s}{N_0} \in$ [0dB, 10dB, 20dB]
Start with μ = 0

1. Compute p_k using (30)
2. Compute $\sum p_k = P_{tot}$
3. If $P_{tot} \leq P_{max}$, increase μ, else stop

Once p_k is computed, we can compute b_k using (24).

Figure 4 shows the simulated results. The horizontal bar represents the waterlevel μ. The power of the subcarrier is the waterlevel to which the blue curve is substracted. That substraction gives the power required on each channel to respect the given probability of error. If the level can't be reached for a given subchannel, the subchannel isn't used ((30) gives a negative value and is thus zeroed).

It is important to compare this method of power allocation with uniform allocation (where a power of 2 is sent on each subcarrier). An interesting metric is the ratio of bits sent with our allocation over bits sent with an uniform allocation. Figure 5 shows the power allocation and bits per symbol for each method.

$\frac{E_s}{N_0}$	Bits/symbol	Gain over uniform allocation	subcarriers used
0dB	16	1.396	57
10dB	77	1.081	104
20dB	227	1.002	128

3.2 Minimising MSE

A second way of allocating power given in the statement implies having a fixed bitrate and minimising errors on the receiving end. The transmitter takes the channel into account and allocates the power in order to minimise the sum of mean square error at the receiver.

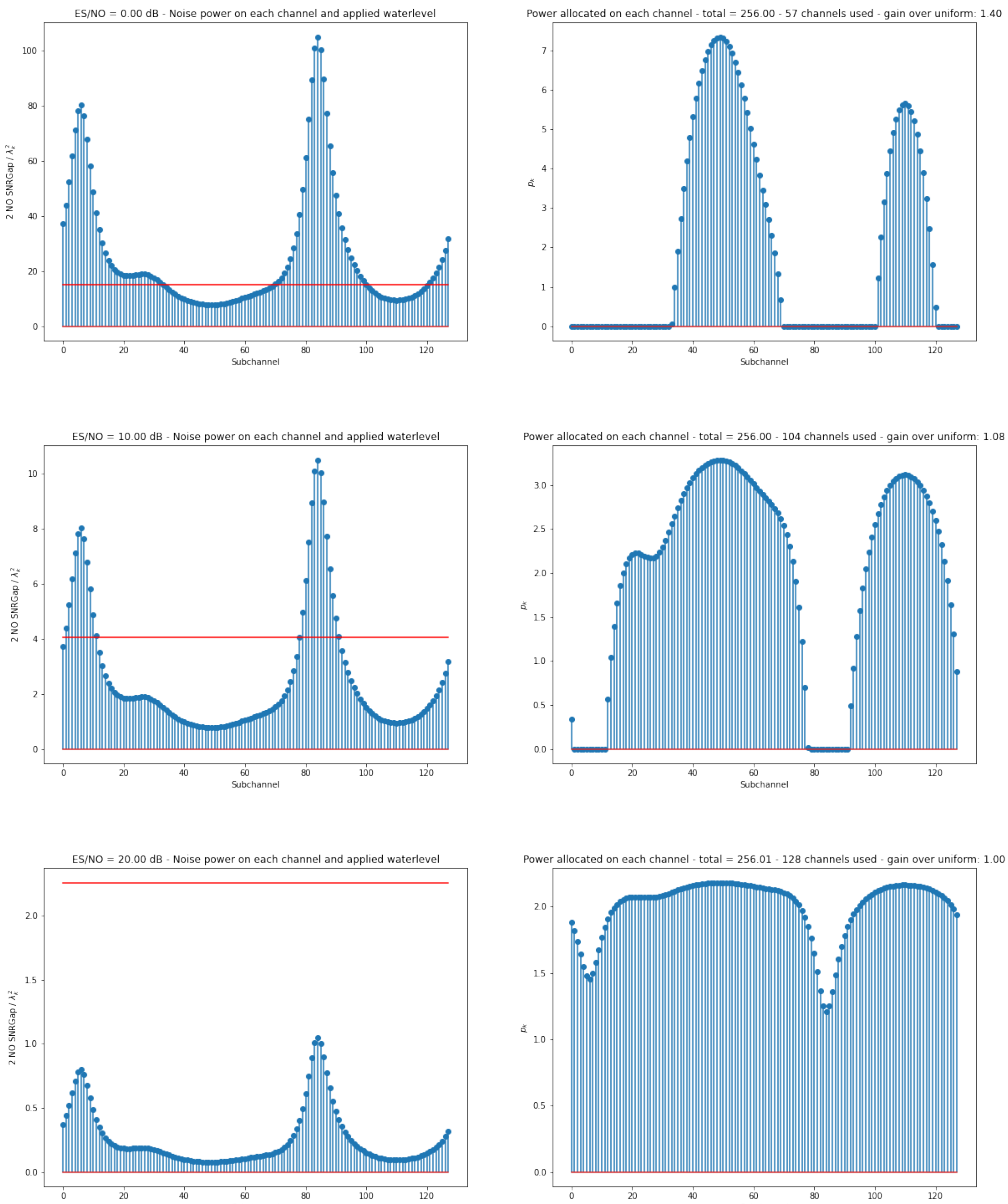

Figure 4: Power allocation for different values of ES/N0

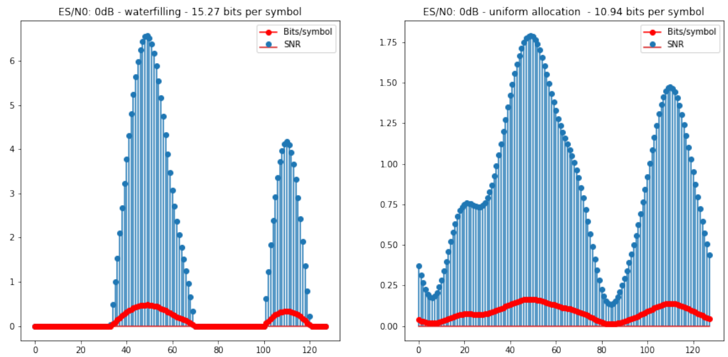

Figure 5: Uniform power allocation against waterfilling

3.2.1 Mathematical model

We model the received channel as follows

$$\hat{s_k} = s_k + \frac{\sigma_n^2}{\|\lambda_k\|^2} \tag{31}$$

We want to minimise the MSE ($\frac{\sigma_n^2}{\|\lambda_k\|^2}$) whilst taking into account the power allocated on each subchannel. If we were not to take into account for the ratio MSE/Power, we would allocate infinite power on all channels.

The problem is modelled as

$$\min_{p_k} \sum_k \frac{MSE}{p_k} = \min_{p_k} \sum_k \frac{\sigma_n^2}{p_k\|\lambda_k\|^2} \tag{32}$$

With constraints: $\sum p_k \leq P_{max}$ and $p_, \geq 0$

The solution, using Lagrange and Karush-Kuhn-Tucker yields:

$$p_k = \frac{P_{max}}{\|\lambda_k\| \sum_k \frac{1}{\|\lambda_k\|}} \tag{33}$$

Figure 6 shows how the power is allocated with this method compared to the previous one. From equation (33) we notice the power allocated doesn't depend on the ratio $\frac{E_s}{N_0}$. We also notice that the second method allocates a lot of power on "bad" subchannels, the ones with a lot of noise, instead of just disabling them. This increase in power yields to increase the signal to noise ratio on these channels.

Comparing the bitrate with both methods makes no sense since the second method has a fixed bitrate and a variable error rate whereas the first method fixes the error rate and varies the bitrate.

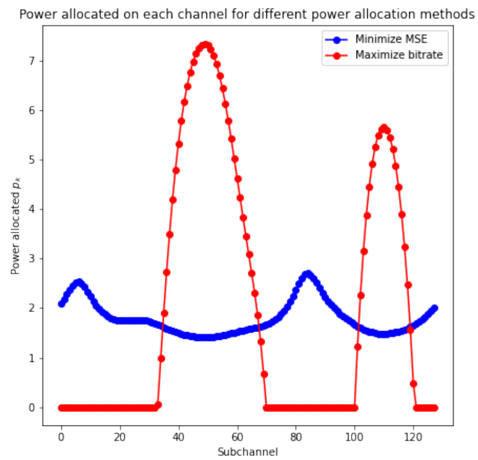

Figure 6: Power allocated when minimising MSE compared to maximising bitrate

4 Channel Estimation

The goal of this section is to estimate h[n], the impulse response of the channel that we will impose of length 8.

To perform it, a training sequence known at both sides is send by the transmitter. This training sequence is made of 2 identical OFDM symbols, named here as T.
The symbol received for one training sequence is given by :

$$\mathbf{Y_T} = \mathbf{\Lambda S_T} + \mathbf{WN} \tag{34}$$

Defining

$$\mathbf{\Sigma} = diag(S_T)$$

and knowing that

$$\lambda = \sqrt{N}\mathbf{Wh}$$

This equation can be rewritten as

$$\mathbf{Y_T} = \mathbf{\Sigma}\sqrt{N}\mathbf{Wh} + \underbrace{\mathbf{WN}}_{\nu} \tag{35}$$

Naming $\tilde{\mathbf{h}}$ the non-zero part of h (of length 8) as $\mathbf{h} = \begin{pmatrix} \tilde{\mathbf{h}} \\ 0 \end{pmatrix}$, and $\tilde{\mathbf{M}}$ this equation can again be rewritten as

$$\boxed{\mathbf{Y_T} = \tilde{\mathbf{M}}\tilde{\mathbf{h}} + \nu} \tag{36}$$

That is a linear regression where $\tilde{\mathbf{h}}$, the only unknown, is the weight vector to estimate.
We will first estimate it by **zero Forcing**, then with a **Maximum Likelihood Estimator** and finally compare this 2 techniques based on a **MSE** criterion.

4.1 Zero-Forcing Estimator

The Zero-Forcing estimator is built by neglecting the noise term ν. We then directly find

$$\hat{\tilde{\mathbf{h}}} = \tilde{\mathbf{M}}^{+}\mathbf{Y_T} \tag{37}$$

with $\tilde{\mathbf{M}}^{+}$ the pseudo-inverse matrix of $\tilde{\mathbf{M}}$.
As we send 2 training sequences, we can take the arithmetical mean of the 2 estimations to get a more accurate one.

The plots of the estimation of h[n] and $\lambda[n]$ as well as the actual impulse response are available on figure 7 for $\frac{E_s}{N0}$ equal to 0, 10 and 20 [dB].

4.2 Maximum Likelihood Estimator

As we are manipulating Gaussian quantities, the MLE is equivalent to a minimisation of the euclidean distance. The estimator can the be expressed as : sequence.

$$\begin{aligned}
\hat{\tilde{\mathbf{h}}} &= \min_{\tilde{h}} \sum_{i=1}^{2} (\mathbf{Y}_{\mathbf{T},\mathbf{i}} - \tilde{\mathbf{M}}\tilde{\mathbf{h}})^{H} (\mathbf{Y}_{\mathbf{T},\mathbf{i}} - \tilde{\mathbf{M}}\tilde{\mathbf{h}}) && (38)\\
&= \frac{\partial}{\partial \tilde{h}} \left(\sum_{i=1}^{2} (\mathbf{Y}_{\mathbf{T},\mathbf{i}} - \tilde{\mathbf{M}}\tilde{\mathbf{h}})^{H} (\mathbf{Y}_{\mathbf{T},\mathbf{i}} - \tilde{\mathbf{M}}\tilde{\mathbf{h}}) \right) && (39)\\
&= (2\tilde{\mathbf{M}}^{H}\tilde{\mathbf{M}})^{-1} (\tilde{\mathbf{M}}^{H}\mathbf{Y}_{\mathbf{T},\mathbf{1}} + \tilde{\mathbf{M}}^{H}\mathbf{Y}_{\mathbf{T},\mathbf{2}}) && (40)
\end{aligned}$$

Where $\mathbf{Y}_{\mathbf{T},\mathbf{i}}$ is the received sequence for the i^{th} training sequence. The full computation is avalaible on Annexe 1, Section 5.1.1.

The plots of the estimation of h[n] and $\lambda[n]$ as well as the actual impulse response are available on figure 8 for $\frac{E_s}{N0}$ equal to 0, 10 and 20 [dB].

4.3 MSE comparison

Computing the mean square error of the estimator in both cases for an average of 50 OFDM symbols we get the plots on figure 9.
Some comments about this figure:

- The MSE associated to the Zero forcing is approximately twice the value of the one associated to the MLE. This second method is therefore more specific but the zero-forcing technique is lighter in calculations.
- The MSE in logarithmic scale tends to be linear with an increasing number of OFDM symbols on which the average is made.
- The estimator is unbiased since the MSE tends to zero with increasing Es/N0.

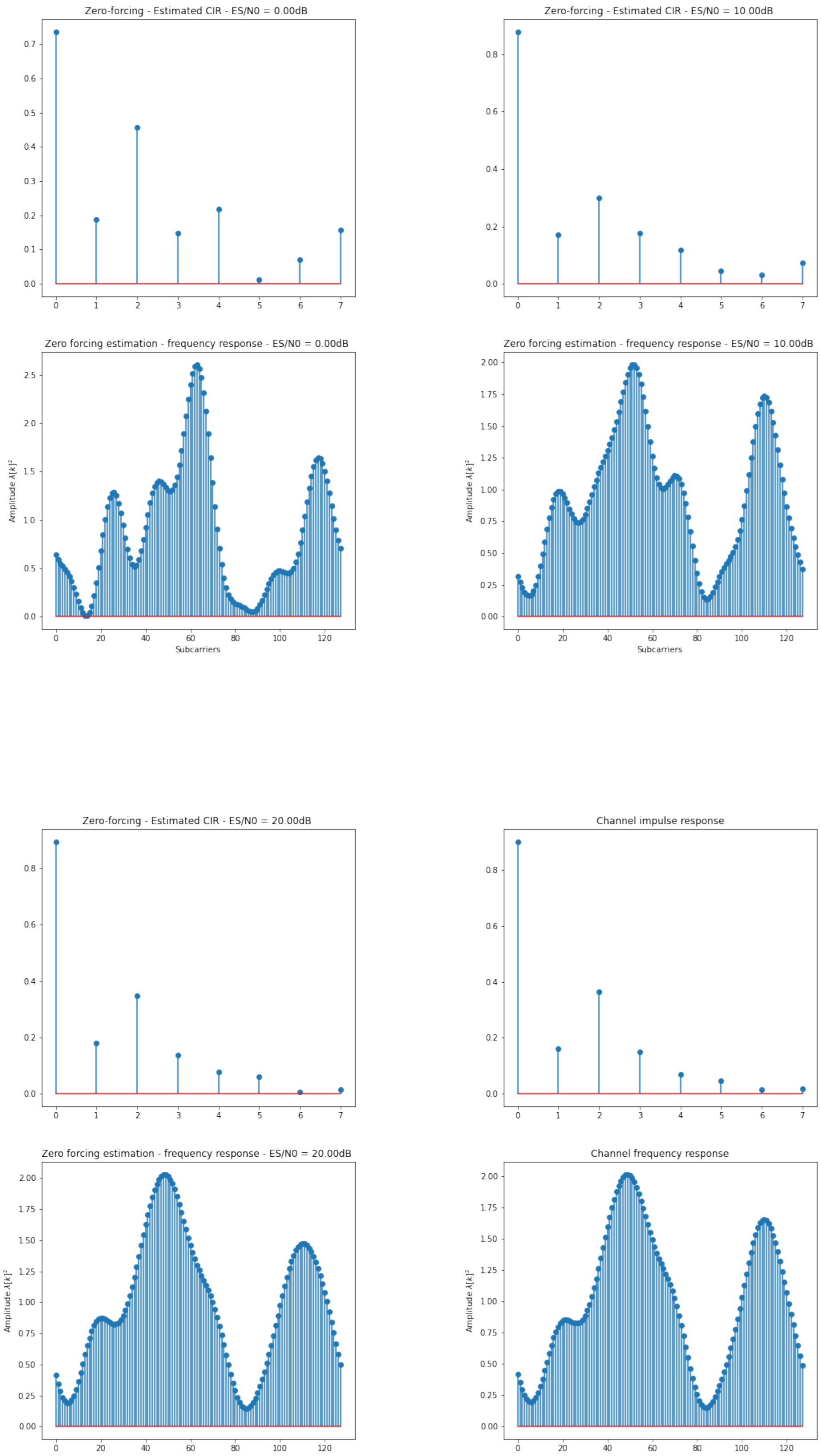

Figure 7: Channel estimation using zero-forcing

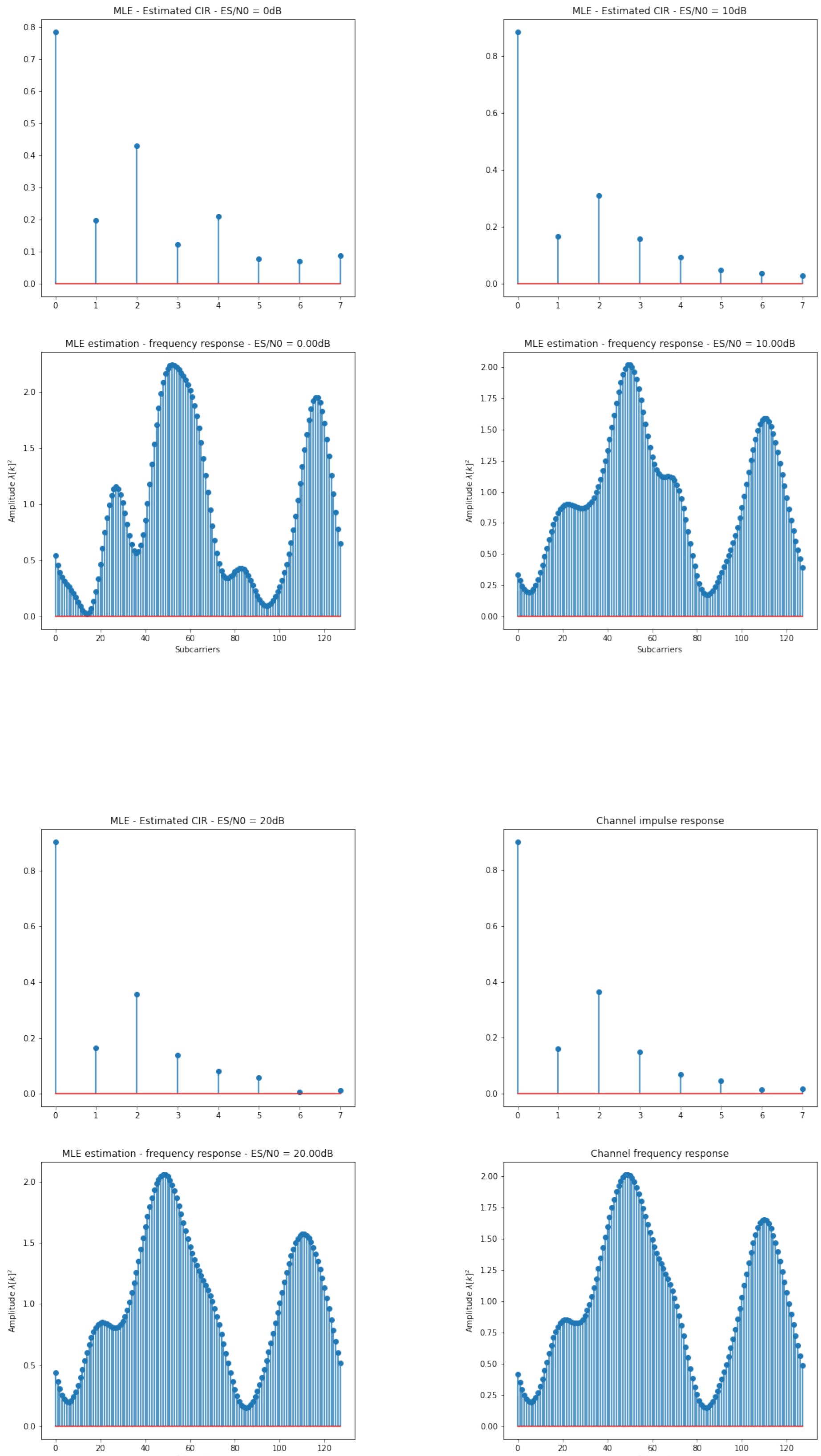

Figure 8: Channel estimation using MLE

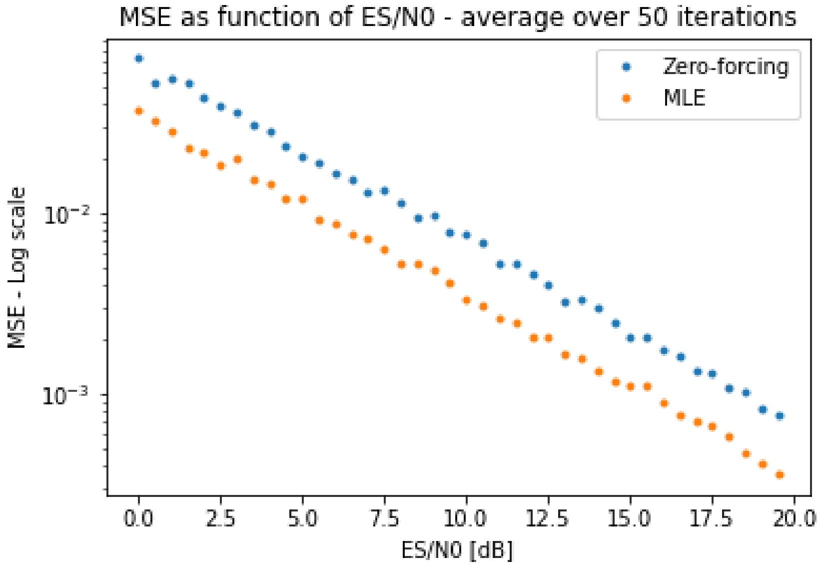

Figure 9: MSE of Channel Estimator

5 Optimal Viterbi decoding

In order to minimise the bit error rate in the received decoded signal, one might use convolutional codes. In this project, we analyse the benefits of viterbi decoding when having information about our channel.

5.1 Mathematical model

Viterbi decoding is based on maximum likelihood sequence decoding. We will consider a soft decision

$$L(s) = p(y_k \| s, \Lambda) = \prod_k p(y_k \| s_k, \lambda_k) \tag{41}$$

We want to maximise this expression which gives

$$\hat{s} = argmax(L(s)) \tag{42}$$

Using the fact that the received signal follows a normal distribution of mean s_k and variance σ_k^2 and using (16) we obtain

$$\hat{s} = argmin \sum_k (\|\lambda\|^2 \|y_k - s_k\|^2) \tag{43}$$

As we are using soft decisions, y_k is the received signal without hard threshold during decoding.

It is important to understand the intuition behind this mathematical model. Here we send one 4-QAM symbol over each subcarrier. Each QAM symbol contains bits x_1 and x_2 of the viterbi encoding. A classical viterbi decoding looks at the possible states and compares the received signal with the expected encoded signal. It then weights each vertex with a "cost" that's computed as the error between the expected encoded signal and the actual received signal. This is the hamming distance $\|y_k - s_k\|^2$. In our particular case, these hamming distances are weighted by $\|\lambda\|^2$. This means a symbol received on a good subcarrier (low SNR) will be trusted more (lower weight) than on a bad subcarrier.

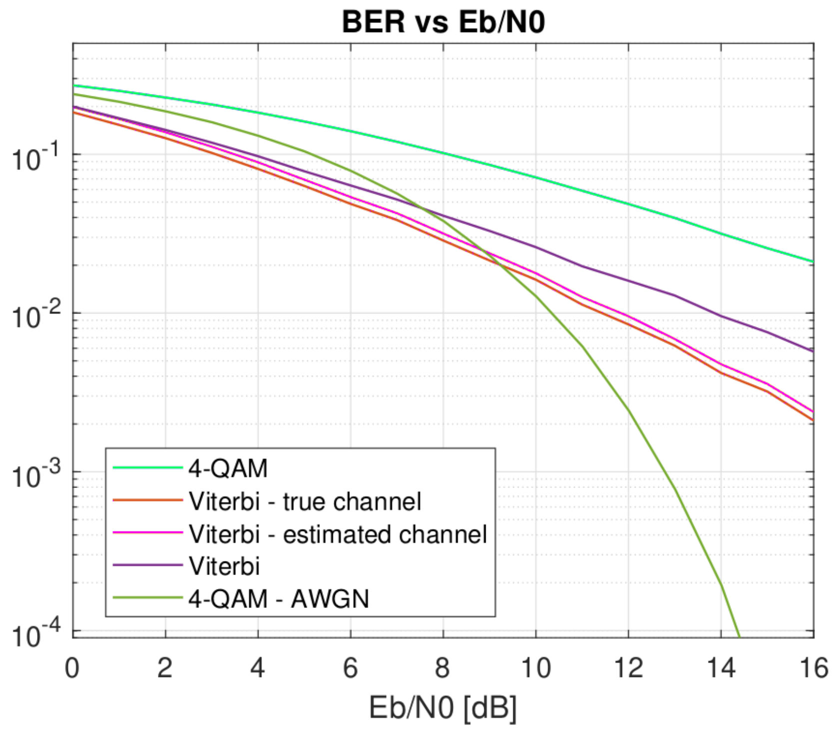

Figure 10: BER using viterbi

5.1.1 Simulation results

Figure 10 shows that the BER is always better when the channel is known. Viterbi as expected has clear advantages on the bit error rate, but the bitrate is divided by two in our case. This is a typical compromise one has to make in an OFDM chain.

Annexe

Es full Computation

$$E_s = \frac{E\left[s[n]\, s^*[n]\right]}{2}$$

Cos AWGN

$$= \frac{1}{2} E\left[x[n]\, x^*[n]\right]$$

$$= \frac{1}{2} E\left[\left(\sum_{m=0}^{N-1} s[m] \frac{e^{j\frac{2\pi m}{N}(n-L_c)}}{\sqrt{N}}\right)\left(\sum_{m'=0}^{N-1} s^*[m'] \frac{e^{-2\pi j \frac{m'(n-L_c)}{N}}}{\sqrt{N}}\right)\right]$$

$$= \frac{1}{2N} \sum_{m} \sum_{m'} \overbrace{E\left[s[m]\, s^*[m']\right]}^{\frac{\sigma^2_{OFDM}}{N}\delta(m-m')} e^{j\frac{2\pi (n-L_c)}{N}(m-m')}$$

$$= \frac{1}{2N} \sum_{m} \sum_{m'} \frac{\sigma^2_{OFDM}}{N} \delta(m-m') \underbrace{e^{j\frac{2\pi (n-L_c)}{N}(m-m)}}_{1}$$

$$= \sum_{m} \frac{\sigma^2_{OFDM}/N}{2N}$$

$$= \frac{\sigma^2_{OFDM}}{2N}$$

MLE full Computation

$$\hat{h} = \min_{\tilde{h}} \sum_{i=1}^{2} \left(\underline{y}_{T,i} - \underline{\underline{\tilde{M}}}_i \tilde{\underline{h}}\right)^H \left(\underline{y}_{T,i} - \underline{\underline{\tilde{M}}}_i \tilde{h}\right)$$

$$\downarrow$$

$$\frac{\partial}{\partial h}\left(\sum_i (y_i - M_i h)^H (y_i - M_i h)\right) = 0$$

$$\Leftrightarrow \sum_{i=1}^{2} -(y - Mh)^H M = 0$$

$$\Leftrightarrow -(y_1 - M_1 h)^H M_1 - (y_2 - M_2 h)^H M_2 = 0$$

$$\Leftrightarrow -y_1^H M_1 + h^H M_1^H M_1 - y_2^H M_2 + h^H M_2^H M_2 = 0$$

$$\Rightarrow h^H (M_1^H M_1 + M_2^H M_2) = y_1^H M_1 + y_2^H M_2$$

$$\Leftrightarrow \hat{h} = \left((y_1^H M_1 + y_2^H M_2)(M_1^H M_1 + M_2^H M_2)^{-1}\right)^H$$

$$\boxed{\hat{h} = \left(\underbrace{M_1^H M_1}_{\substack{(8\times N)(N\times 8) \\ 8\times 8}} + M_2^H M_2\right)^{-1} \left(M_1^H y_1 + M_2^H y_2\right)}$$

$\rightarrow 8\times 1$ OK $\leftarrow$ $\underbrace{(8\times N)(N\times 1)}_{8\times 1}$

References

[1] *LELEC2795 - Radiation and communication systems*

[2] *LELEC2880 - Modem Design*