Random Knot Diagrams

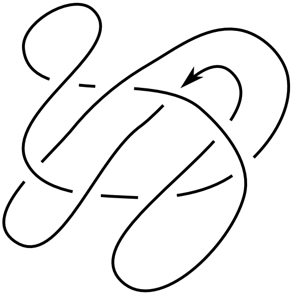
Harrison Chapman (UGA - Graduate student)
joint w/ Jason Cantarella (UGA), Matt Mastin (Wake Forest)

AMS Western Spring Sectionals 2015 (UNLV) – April 18, 2015

Natural questions about knot diagrams

Question

What fraction of 8-crossing diagrams are trefoils?

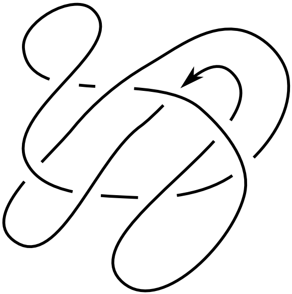


Natural questions about knot diagrams

Question

What fraction of 8-crossing diagrams are trefoils?

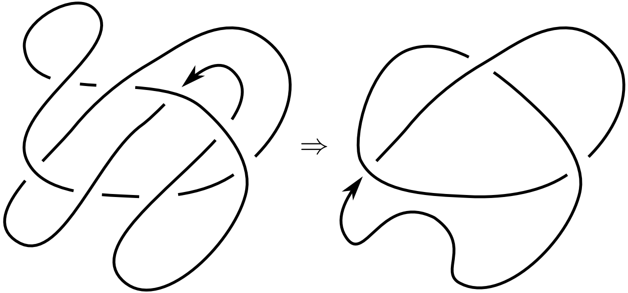
12.48%



Natural questions about knot diagrams

Question

What is the average minimal crossing # of an 8-crossing diagram?

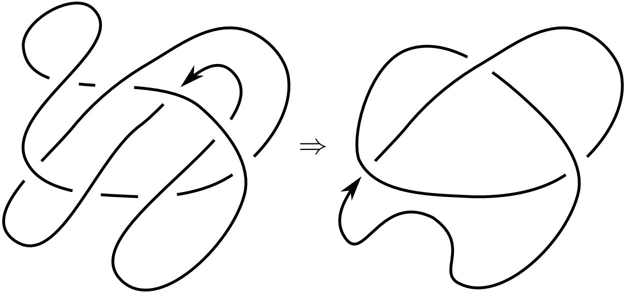


Natural questions about knot diagrams

Question

What is the average minimal crossing # of an 8-crossing diagram?

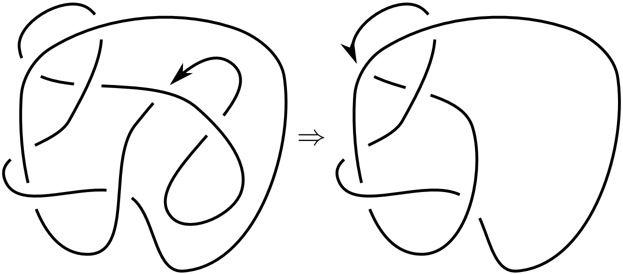
0.52



Natural questions about knot diagrams

Definition

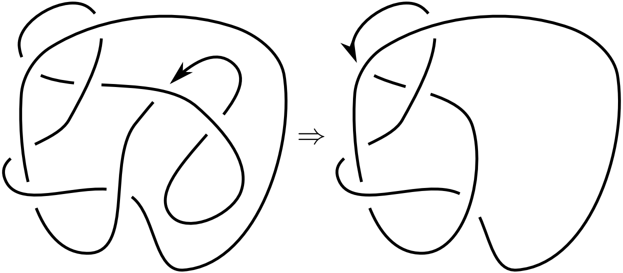
Let **untwisting** be the operation; recursively RI untwist loops in a diagram until there are no more.



Natural questions about knot diagrams

Question

What is the average crossing # of a untwisted 8-crossing diagram?

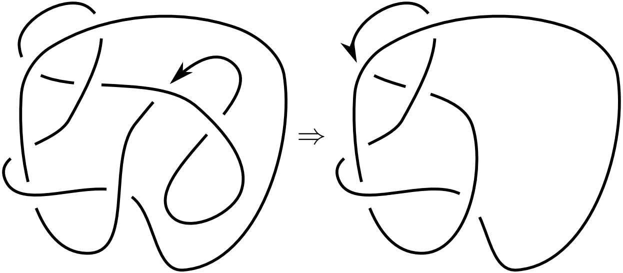


Natural questions about knot diagrams

Question

What is the average crossing # of a untwisted 8-crossing diagram?

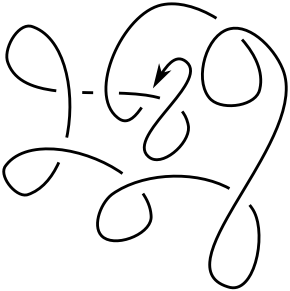
2.20



Natural questions about knot diagrams

Question

How many 8-crossing diagrams can be untwisted to the unknot?

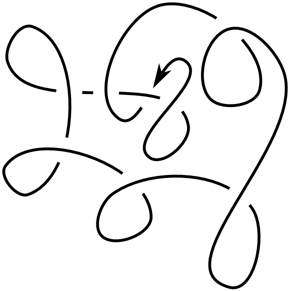


Natural questions about knot diagrams

Question

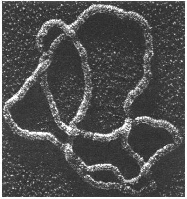
How many 8-crossing diagrams can be untwisted to the unknot?

42.05%



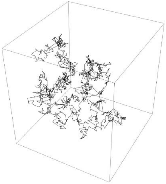
Ansatz

Polymers



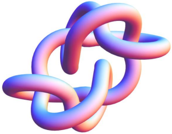
$\longrightarrow$

Random curves



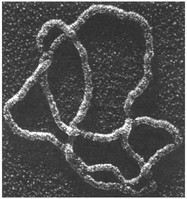
$\longrightarrow$

Knots

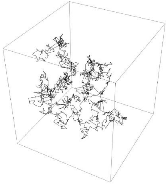


Combinatorial approaches

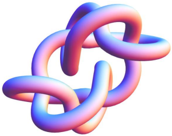
Polymers



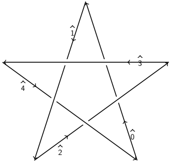
Random curves



Knots

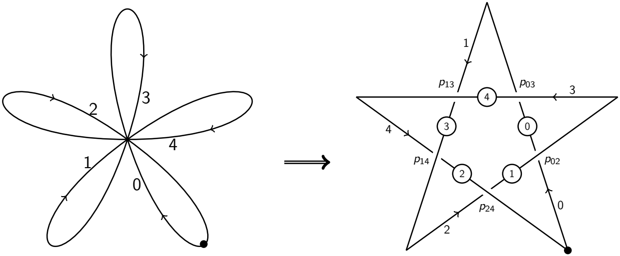


Random combinatorial



The Petaluma model

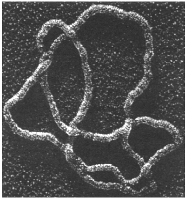
Satisfying theorems have been proven for the Petaluma model



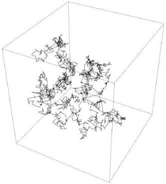
Petal diagram $\Longrightarrow$ Star diagram

(Diagram from Evan-Zohar, Hass, et al.)

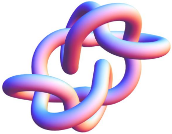
Polymers



Random curves

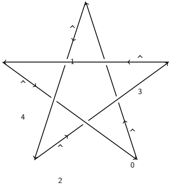


Knots

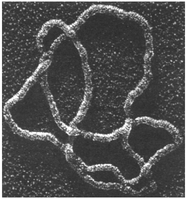


???

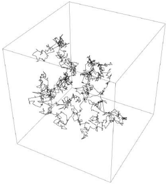
Random combinatorial



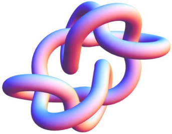
Polymers



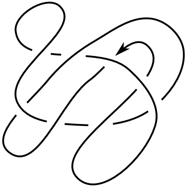
Random curves



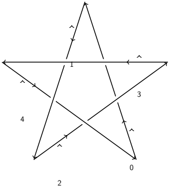
Knots



Random diagrams



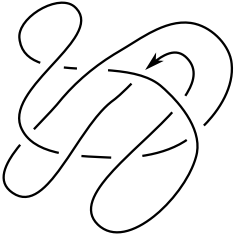
Random combinatorial

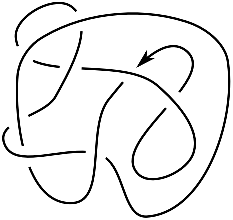


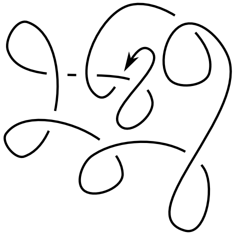
Random diagrams

Definition

In the **random diagram model** of random knotting, a n-crossing diagram is drawn uniformly from the finite set of n-crossing knot diagrams.



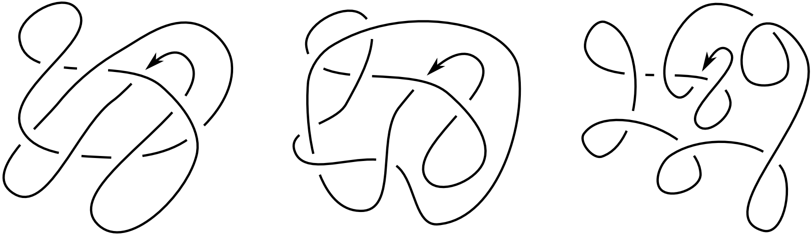




Random diagrams

Definition

A **knot diagram** is a equivalence class of generic immersions of the oriented S^1 into the sphere S^2 together with over-under strand information at each double point up to diffeomorphism of S^2.



How to enumerate knot diagrams

Definition

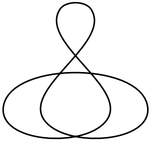
A **knot shadow** is a equivalence class of generic immersions of the unoriented S^1 into the sphere S^2 up to diffeomorphism of S^2.

1. Enumerate shadows
2. Assign crossing and orientation information and identify equivalent diagrams

Tabulating knot shadows

Proposition

Knot shadows $\leftrightarrow$ *1-component 4-valent embedded planar multigraphs up to embedded isomorphism*

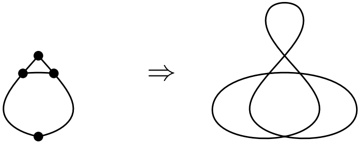


Tabulating knot shadows

Proposition

Knot shadows $\leftrightarrow$ 1-component 4-valent embedded planar multigraphs up to embedded isomorphism

1. Add loops and edges to planar simple graphs

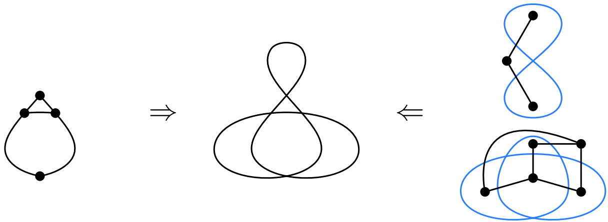


Tabulating knot shadows

Proposition

Knot shadows $\leftrightarrow$ 1-component 4-valent embedded planar multigraphs up to embedded isomorphism

1. Add loops and edges to planar simple graphs
2. Generate multiquadrangulations by connect sum, take dual graphs



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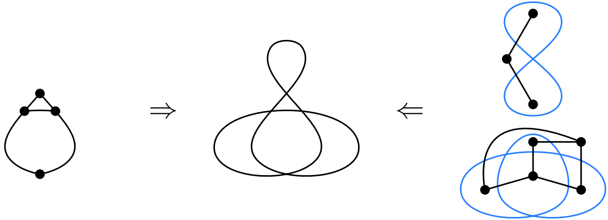


Actually generate all **link shadows**, then restrict to knot shadows

Assign crossings, orientation, identify

1. Orient each component. (2 choices)
2. Assign over-under information to each vertex. (2^n choices)

n	# knot shadows	2^{n+1} (# shadows)	# knot diagrams
3	6	96	36
4	19	608	276
5	76	4,864	2,936
6	376	48,128	35,872
7	2,194	561,664	484,088
8	14,614	7,482,368	6,967,942
9	106,421	108,975,104	105,555,336

Verifying against existing shadow counts

oriented	$n=0$	1	2	3	4	5
S^2, S^1	1	1	3	9	37	182
S^2	1	1	2	6	21	99
S^1	1	1	2	6	21	97
–	1	1	2	6	19	76

Curves on S^2. The number of types

V.I. Arnol'd. *Topological Invariants of Plane Curves*

A008989 Number of immersions of unoriented circle into unoriented sphere with n double points.

1, 1, 2, 6, 19, 76, 376, 2194 (list; graph; refs; listen; history; text; internal format)

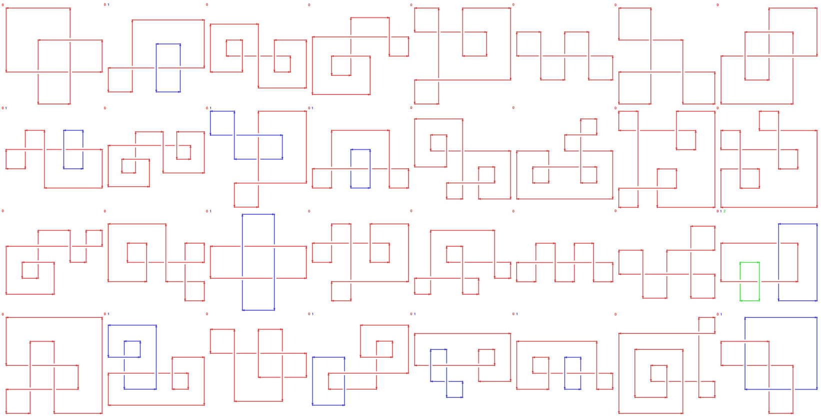
OFFSET	0,3
REFERENCES	V. I. Arnold, Topological Invariants of Plane Curves..., American Math.
LINKS	Table of n, a(n) for n=0..7.
CROSSREFS	Sequence in context: A150119 A181770 A138800 * A057240 A079564 A079453 Adjacent sequences: A008986 A008987 A008988 * A008990 A008991 A008992
KEYWORD	nonn
AUTHOR	N. J. A. Sloane.
EXTENSIONS	Two more terms from Guy H. Valette (guy.valette(AT)skynet.be), Feb 09 20
STATUS	approved

OEIS A008989

n	# knot shadows
0	1
1	1
2	2
3	6
4	19
5	76
6	376
7	2194
8	14614
9	106421

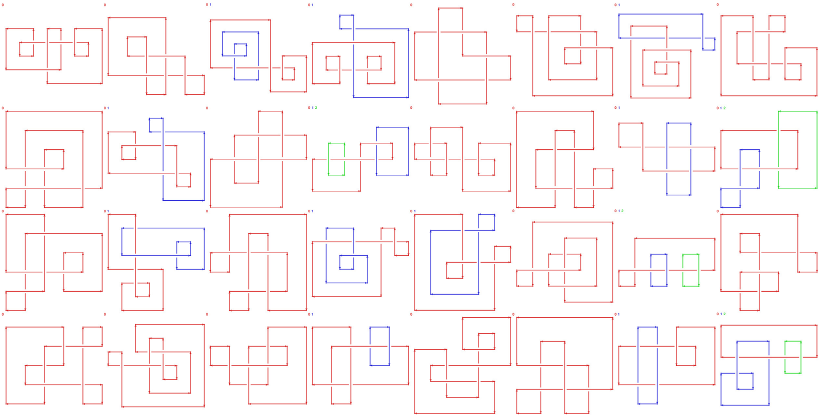
We have not found any existing counts of **diagrams**.

The space of shadows



Link shadows. Pictures generated by Eric Lybrand (UGA undergrad).

The space of shadows



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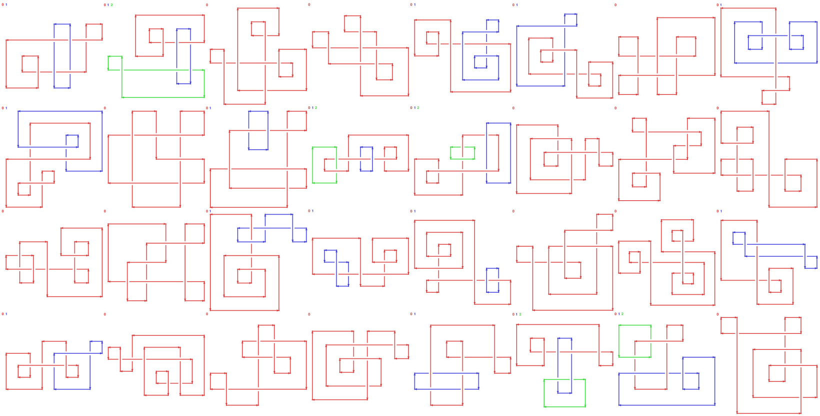

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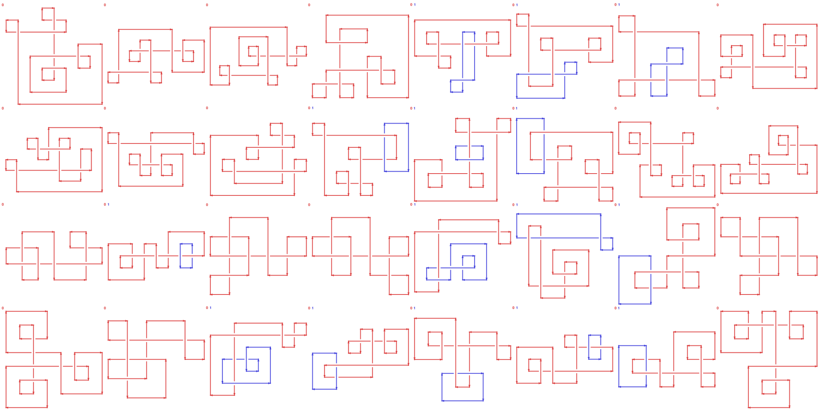
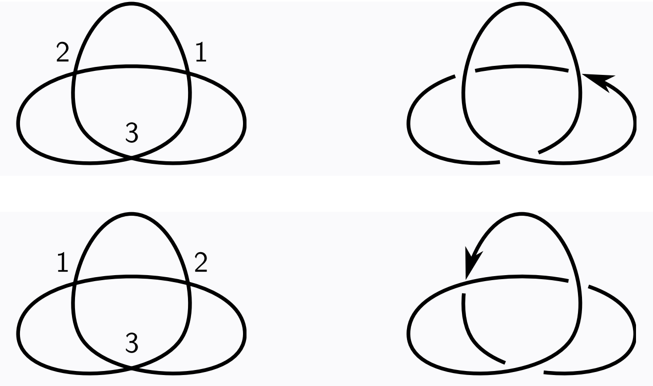
The space of shadows



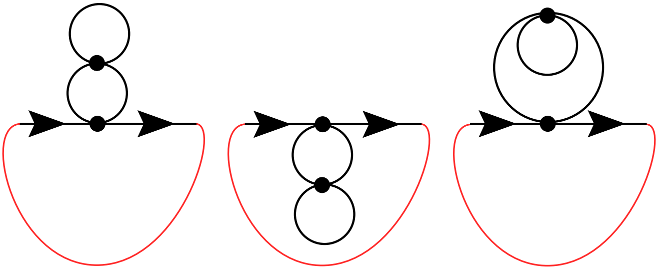
Link shadows. Pictures generated by Eric Lybrand (UGA undergrad).

Tabulation is difficult!

Accounting for symmetry is complicated.



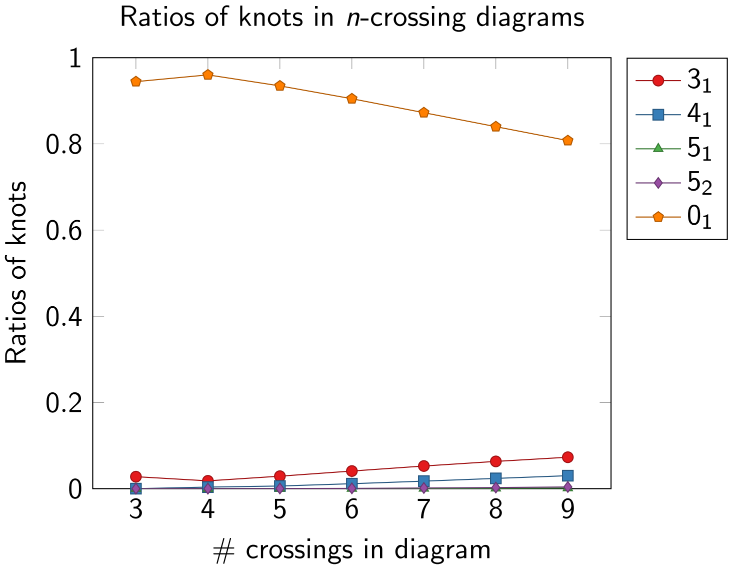
Breaking symmetries could make counting easier

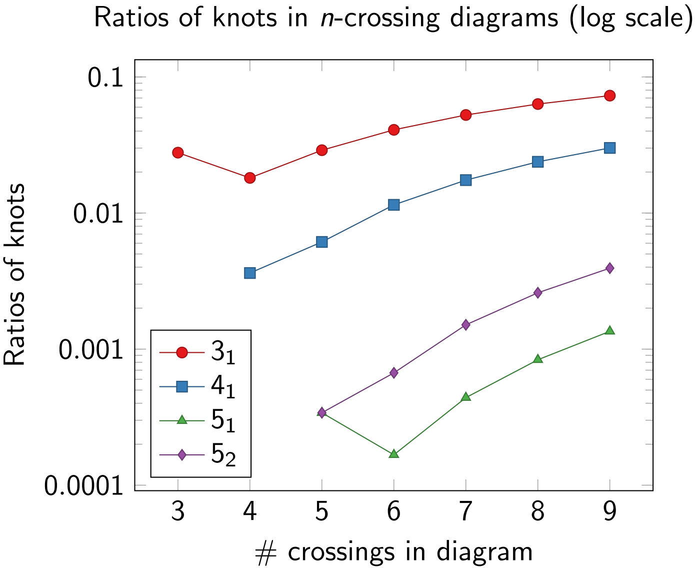


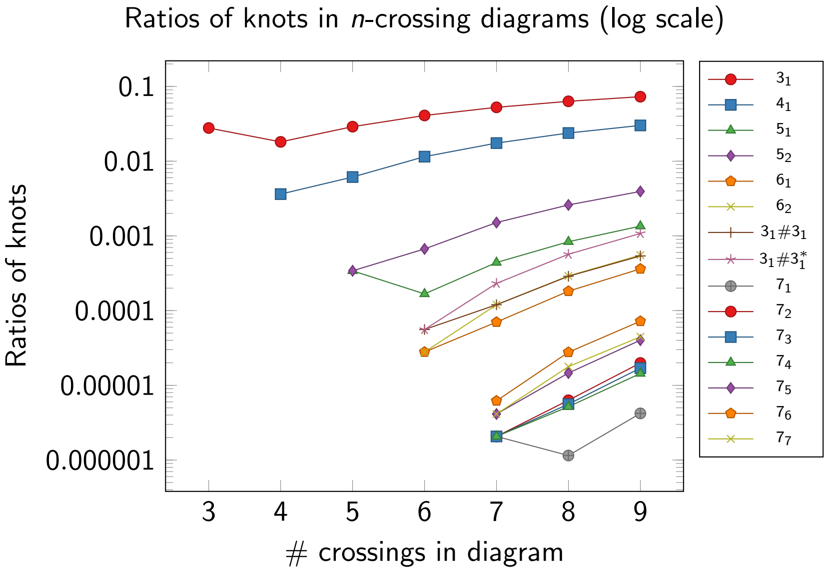
Two-leg diagrams counted by generating function (Bouttier, et al. 2003):

$$G_0 = \frac{24g - 1 + \sqrt{1-12g}}{9g(1+\sqrt{1-12g})} = 1 + 2g + 9g^2 + 54g^3 + 378g^4 + \cdots$$

Knotting in diagrams




Ratios of knots in n-crossing diagrams (log scale)
Ratios of knots
0.1
0.01
0.001
0.0001
3
4
5
6
7
8
9
crossings in diagram
3_1
4_1
5_1
5_2



Ratios of knots in n-crossing diagrams (log scale)
Ratios of knots
0.1
0.01
0.001
0.0001
0.00001
0.000001
3
4
5
6
7
8
9
crossings in diagram
3_1
4_1
5_1
5_2
6_1
6_2
$3_1\#3_1$
$3_1\#3_1^*$
7_1
7_2
7_3
7_4
7_5
7_6
7_7

A question on unknotting

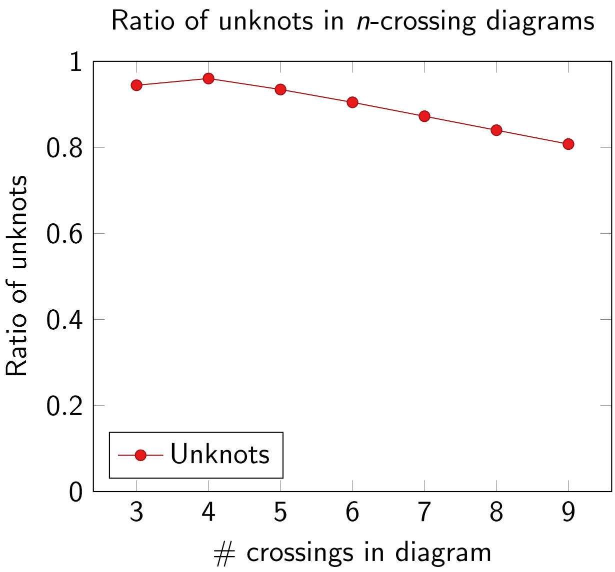
Theorem ([Frisch-Wassermann-Delbrück Conjecture] Sumners-Whittington 1988)

The ratio of unknots in random n-edge self-avoiding lattice polygons tends to zero exponentially with n.

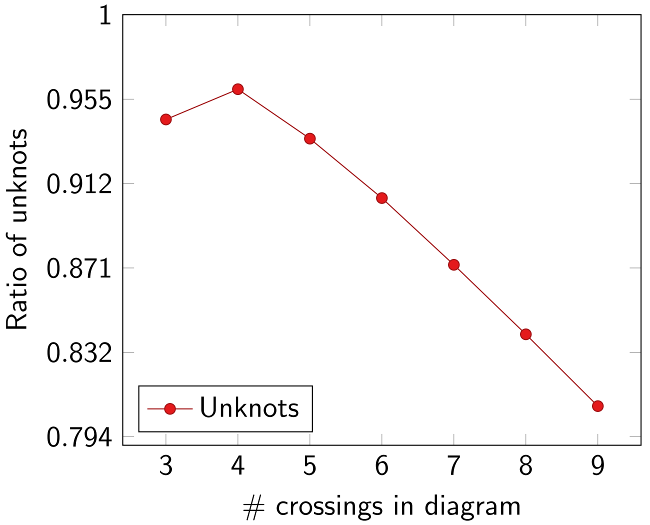
Conjecture

The ratio of unknots in diagrams tends to zero as n increases. (Exponentially?)

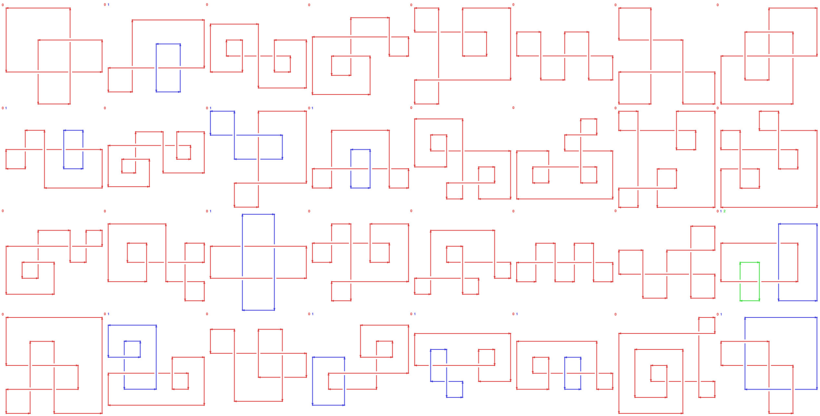
Dowden (2008) has recently proven a theorem for simple planar graphs similar to the Kesten pattern theorem (1963).


Ratio of unknots in n-crossing diagrams
Ratio of unknots
1
0.8
0.6
0.4
0.2
0
3
4
5
6
7
8
9
crossings in diagram
Unknots

Ratio of unknots in n-crossing diagrams (log scale)

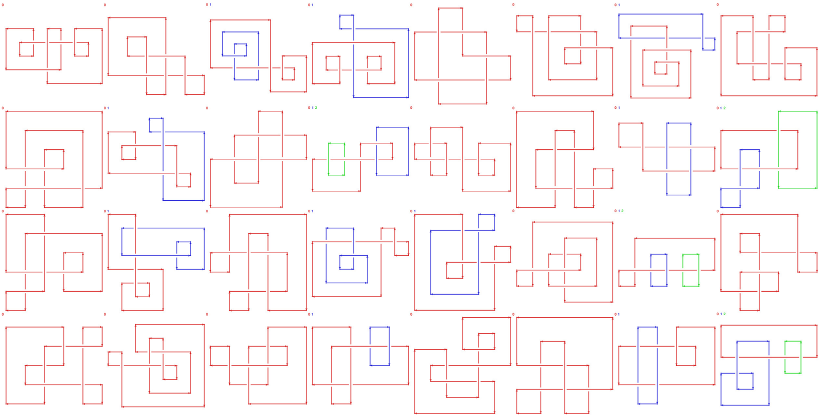


Why so many unknots?



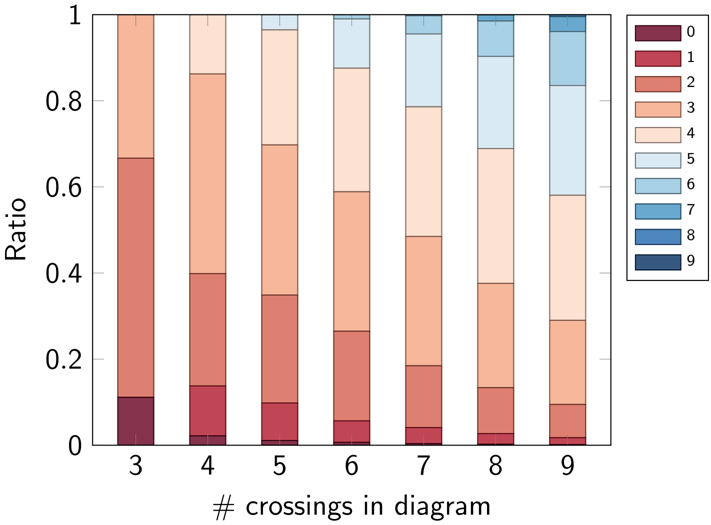
Link shadows. Pictures generated by Eric Lybrand (UGA undergrad).

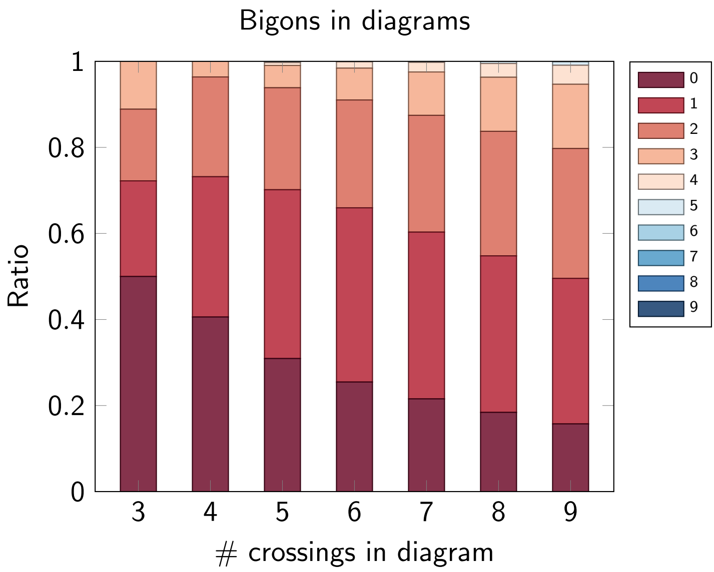
Why so many unknots?



Link shadows. Pictures generated by Eric Lybrand (UGA undergrad).

Reidemeister-I loops (monogons) in diagrams

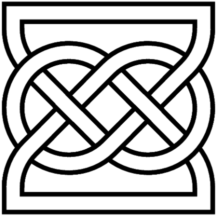


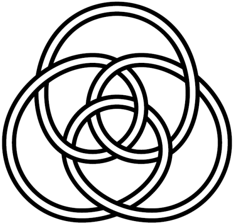

Bigons in diagrams
Ratio
1
0.8
0.6
0.4
0.2
0
3
4
5
6
7
8
9
crossings in diagram
0
1
2
3
4
5
6
7
8
9

Basic polyhedra 8^* and 9^*

Proposition

Conway's basic polyhedra 8^* and 9^* are the only shadows in ≤ 9 crossings with no monogons or bigons.

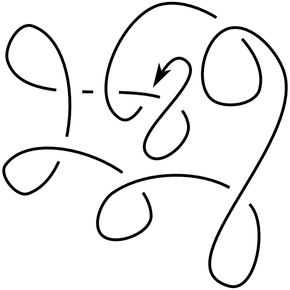




8_{18} (left), 9_{40} (right).

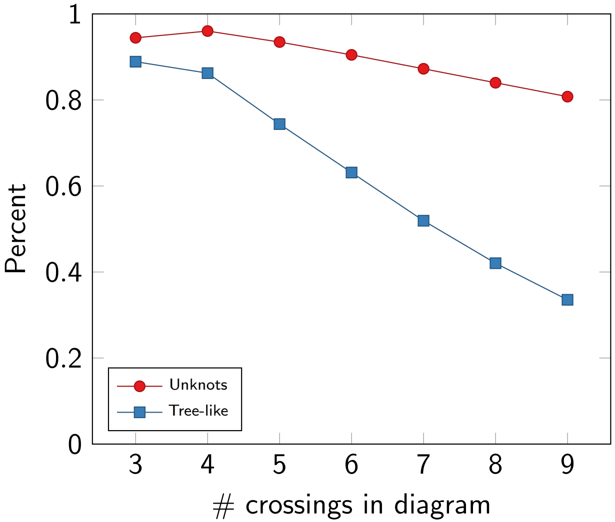
Some shadows are always unknots

A **tree-like curve** (Aicardi 1994) is a knot shadow which can be untwisted to the trivial shadow.

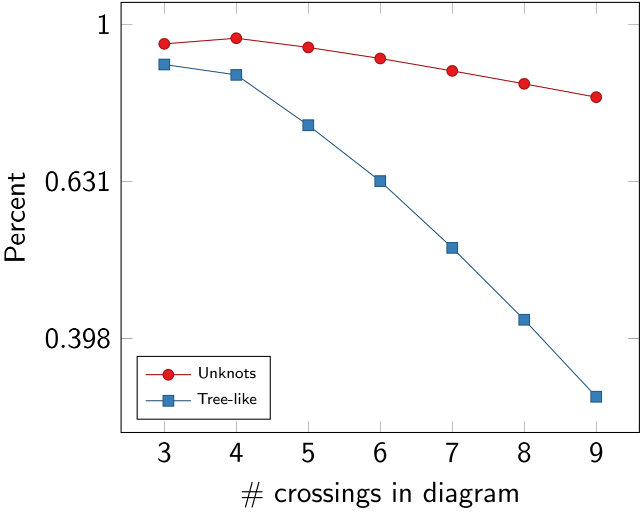


Tree-like curves ⇒ lower bound on unknottedness.

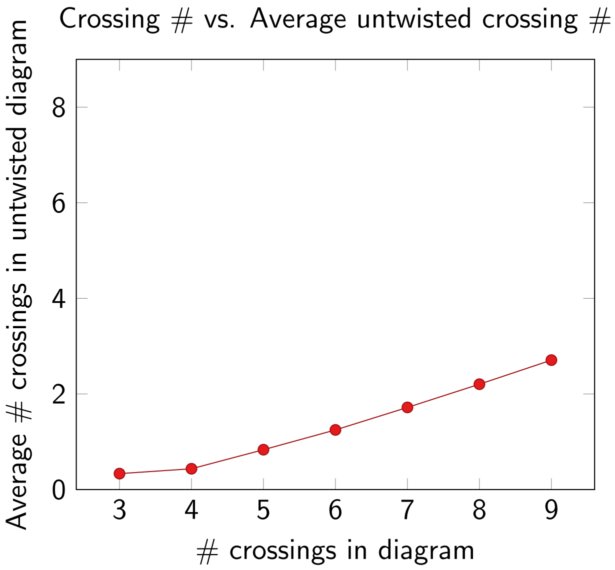
Ratio of unknots, tree-like curves in n-crossing diagrams



Ratio of unknots, tree-like curves in *n*-crossing diagrams (log scale)

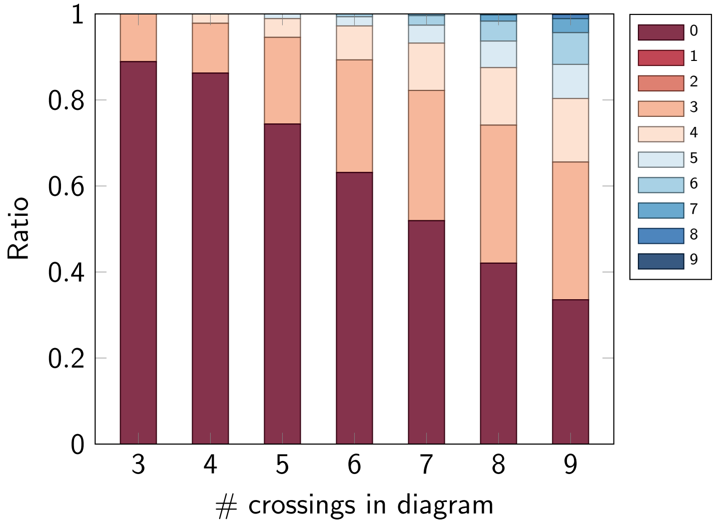


Tree-like curves alone explain only some of the unknot fraction



Crossing # vs. Average untwisted crossing #
Average # crossings in untwisted diagram
crossings in diagram
0
2
4
6
8
3
4
5
6
7
8
9

Untwisted crossing

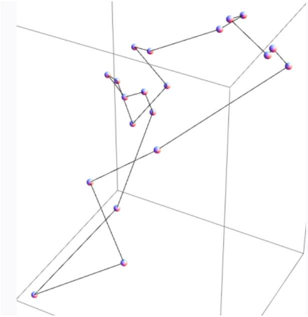


Questions to answer

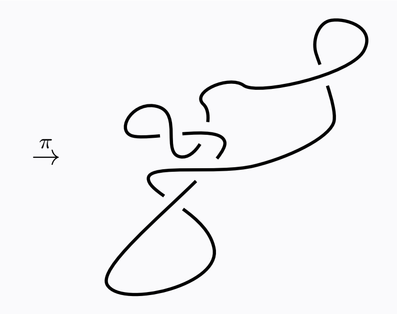
Random curves project to diagrams.

Question

How does the pushforward measure differ from uniform diagram sampling? (c.f. Hua, Nguyen, Raghavan, Arsuaga, Vasquez 2005)



$\xrightarrow{\pi}$



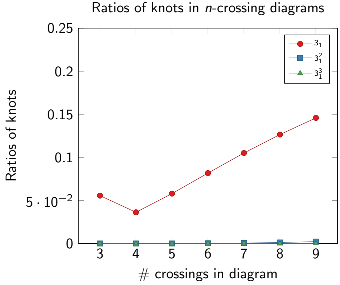
(from Shonkwiler)

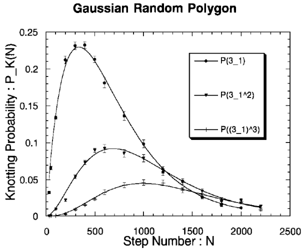
Questions to answer

Question

Let $PD(n, K)$ be the probability of knot type K in a random diagram of n crossings and $PC(n, K)$ the probability of knot type K in a random polygon of n edges.

If n, m are such that $PD(n, 0_1) = PC(m, 0_1)$, is there a relationship between $PD(n, K), PC(m, K)$ for other knot types K?





(from Deguchi, et. al.)

Questions to answer

Fact

No one will enumerate the 100-crossing knot diagrams.

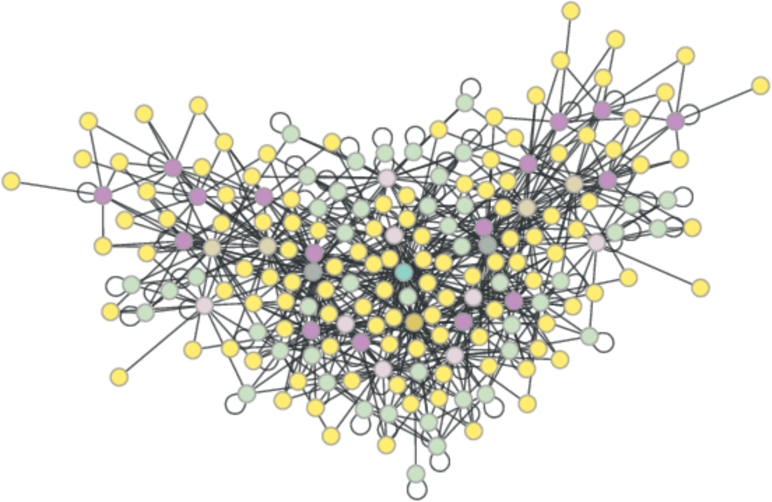
Question

Can we generate uniformly sampled random 100-crossing knot diagrams **another way**?

Future direction: Link diagrams

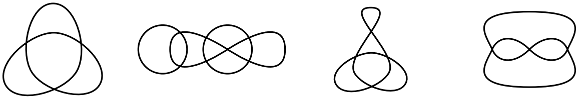
n	# link shadows	# knot shadows
0	1	1
1	1	1
2	3	2
3	7	6
4	30	19
5	124	76
6	733	376
7	4586	2194
8	33373	14614
9	259434	106421

Future direction: Knot distances



Thank you!

Coming soon: Cantarella, Chapman, Mastin. *Knot probabilities in random diagrams*.



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