Random Planar Diagrams

Harrison Chapman

University of Georgia

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Planar diagrams and knots

A planar diagram is the usual way to express a space curve or knot in two dimensions. We will only consider *oriented* planar diagrams, where components of links are given direction.

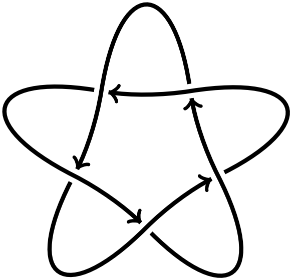


Figure : A typical planar diagram for the knot 5_1.

Planar diagrams as graphs

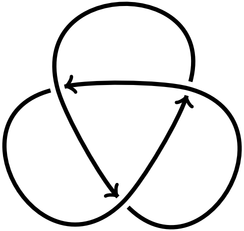
Definition

A **planar diagram** is an isotopy class of 4-valent embedded planar directed multigraphs

- together with information at each vertex as to which opposing pair of edges constitutes the over strand and under strand (**crossing sign**),
- and so that at every vertex, opposite edges do not both point towards or away from the vertex.

Two planar diagrams are equivalent if they differ only by **diagram isotopy**, wherein both the underlying planar digraph embeddings are isomorphic, and crossing signs agree.

A planar diagram



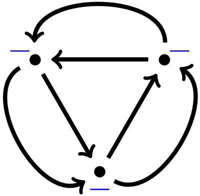


Figure : Planar diagram for a typical trefoil knot in a standard presentation, and as an annotated directed multigraph.

Crossing sign

Crossing signs are determined by the right hand rule.

Figure : Positive and negative crossings.

Diagram isotopy

Figure : The two diagrams are related by flipping the inside with the out and a rotation.

Planar diagram shadows

A **diagram shadow** is an isomorphism class of 4-valent embedded planar multigraphs, up to planar graph isomorphim.

Figure : A diagram shadow viewed both as a drawing and as a graph.

Counting planar diagrams

Finiteness

The spaces of planar diagrams or diagram shadows with [at most] *n* crossings is *finite*.

n	# knot shadows	# knot diagrams
3	6	< 96
4	16	< 512
5	63	< 4032
6	302	< 38656
7	1756	< 449536
8	11621	< 5949952
9	< 193903	$< 193903 * 2^{10}$

Table : Counts and bounds on knot diagrams and shadows. Numbers are large, but finite.

Exploring the space of diagrams

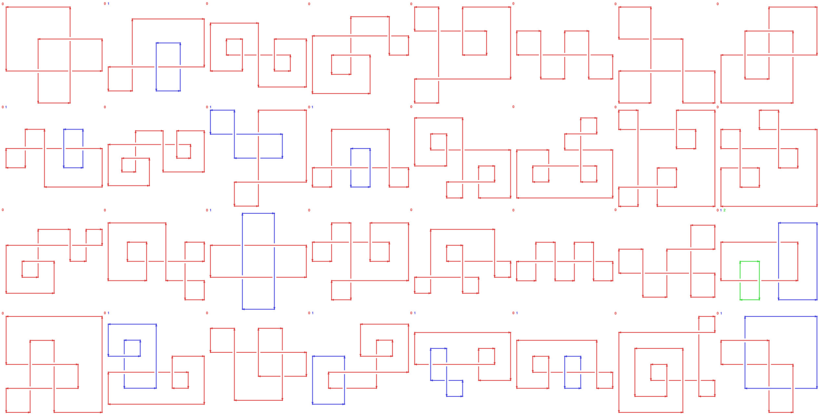


Figure : Planar diagrams

Exploring the space of diagrams

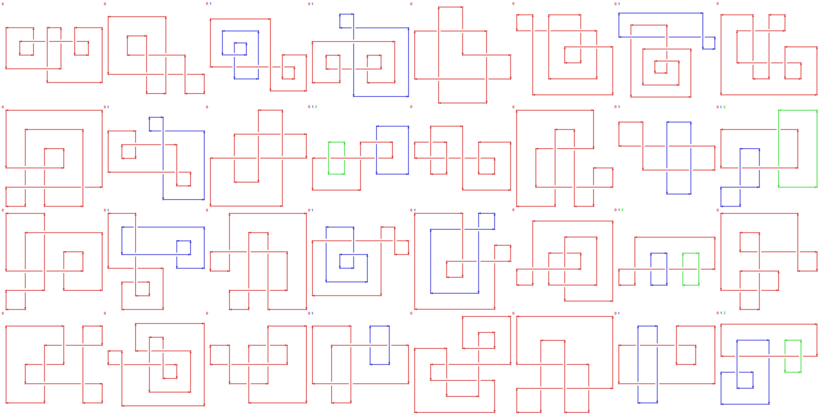


Figure : Planar diagrams

Exploring the space of diagrams

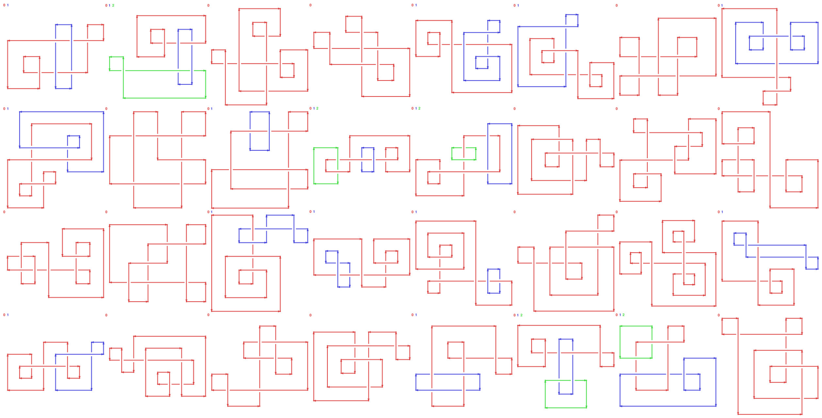


Figure : Planar diagrams

Exploring the space of diagrams

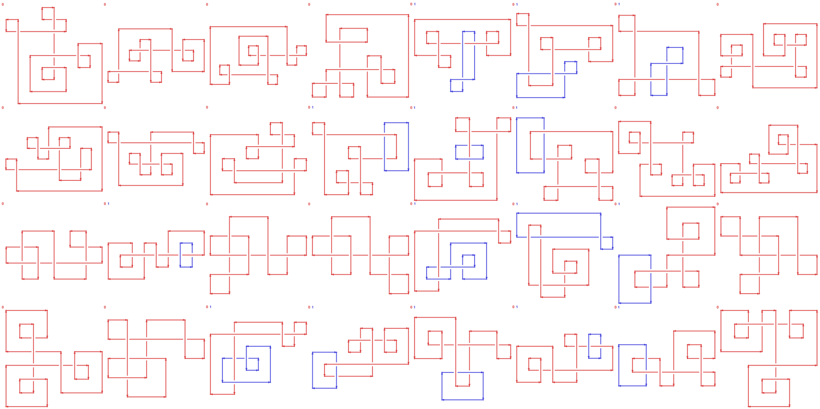


Figure : Planar diagrams. A map of all shadows with between 3 and 6 crossings is here.

Enumeration of appropriate planar graphs

The first step is to generate a complete list of diagram shadows.

plantri

A program `plantri` by McKay and Brinkmann is able to generate all planar embedded graphs (not multigraphs) with arbitrary numbers of vertices.

We then omit all such graphs with vertices of degree > 4.

Shadows from planar graphs

Given an appropriate graph G from `plantri` and a vertex $V \in G$, there are two different operations which are repeated until the vertex (and then all vertices) has degree 4:

1. Adding a self-loop to V inside one of the faces of the embedding. This increases the degree of the vertex by 2.
2. Doubling an edge adjacent to V in G. This increases the degrees of the two connected vertices each by 1.

This process produces a number of (possibly not all unique) diagram shadows.

Self loop addition

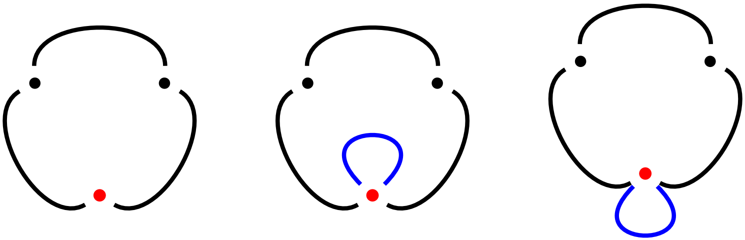


Figure : Addition of a self loop. From the graph on the left, with red selected vertex, the rightmost two graphs are produced.

Edge doubling

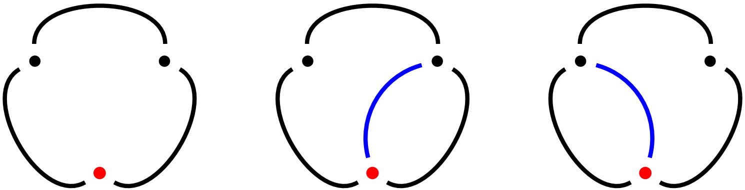


Figure : Doubling of an edge. From the graph on the left, with red selected vertex, the rightmost two graphs are produced.

Ordering shadows

The ultimate list of diagram shadows is ordered by how the algorithm produces them:

- The order in which `plantri` outputs its planar graphs is fixed.
- The algorithm which creates planar diagram shadows from the low-valence planar graph embeddings has order.

Exploding shadows

Given a diagram shadow, two sets of binary choices must be made to determine a planar diagram:

1. Edges must be given a direction.
2. Crossings must be given a sign.

Orienting components

One edge per component

The constraint on edge orientations in the definition of planar diagrams means that orienting all edges is equivalent to orienting one edge per component.

For knots, there is only one component to orient.

Assigning crossing signs

For any given crossing, there are two valid signs; $+$ and $-$.

Bound on diagrams for a shadow

For a k-crossing knot (1-component) diagram, there are 2 choices of orientation. There are then 2^k total choices for crossing signs. So there are at most 2^{k+1} planar diagram isotopy classes for each shadow.

The HOMFLY polynomial

HOMFLY polynomial

The **HOMFLY polynomial** is a knot-invariant polynomial of an oriented planar diagram defined by the following skein relation:

$$zP\left(\upuparrows\right) = aP\left(\nwarrow\!\!\!\!\nearrow\right) + a^{-1}P\left(\nwarrow\!\!\!\!\nearrow\right), \qquad P(0_1) = 1.$$

For a knot K,

$$P(K)(a, z) = P(K^*)(a^{-1}, z)$$

For two knots K, L,

$$P(K\#L) = P(K)P(L) \text{ and } P(K \cup L) = \left(\frac{a + a^{-1}}{z}\right)P(K)P(L)$$

Classifying knots with small crossing number

All knots (both prime and composite) with a crossing number of 7 or smaller are classified entirely by their HOMFLY polynomial.

Knot type	HOMFLY polynomial
0_1	1
3_1	$-2a^2 + a^2z^2 - a^4$
3_1^*	$-a^{-4} - 2a^{-2} + a^{-2}z^2$
4_1	$-a^{-2} - 1 + z^2 - a^2$
5_1	$3a^4 - 4a^4z^2 + a^4z^4 + 2a^6 - a^6z^2$
5_2	$-a^2 + a^2z^2 + a^4 - a^4z^2 + a^6$
$3_1\#3_1$	$4a^4 - 4a^4z^2 + a^4z^4 + 4a^6 - 2a^6z^2 + a^8$
$3_1\#3_1^*$	$2a^{-2} - a^{-2}z^2 + 5 - 4z^2 + z^4 + 2a^2 - a^2z^2$
6_1	$-a^{-2} + z^2 + a^2 - a^2z^2 + a^4$

Table : Some knot types and their HOMFLY polynomial

HOMFLY polynomial collisions

Negative amphichiral 8_{17}

In 8 crossings; the knots 8_{17} and 8_{17}^* have the same polynomial

$$-a^{-2} + 2a^{-2}z^2 - a^{-2}z^4 - 1 + 5z^2 - 4z^4 + z^6 - a^2 + 2a^2z^2 - a^2z^4$$

but are **different** as oriented knots.

Prime/composite collision

In 9-crossing knots; 9_{12} and $4_1\#5_2^*$ both have HOMFLY polynomial

$$-a^{-8} - 2a^{-6} + 2a^{-6}z^2 - a^{-4} + a^{-4}z^2 - a^{-4}z^4 - a^{-2}z^2 + a^{-2}z^4 + 1 - z^2.$$

The peculiar 8_{17}

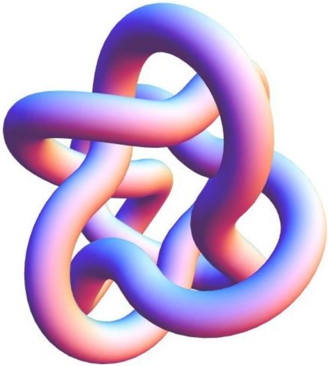


Figure : 8_{17} is not equal to its inverse if it is oriented.

Distinguishing 9_{12} and $4_1\#5_2^*$

The (prime) satellite knot of minimal crossing number has crossing number 13; hence, no complements of prime knots with crossing number < 13 have any incompressible non boundary-parallel tori.

Help from SnapPy

The software package `SnapPy` has a method `splitting_surfaces` which finds such surfaces; it finds none for 9_{12}, but two for $4_1\#5_2^*$.

The knots 9_{12} and $4_1\#5_2^*$ are also distinguished by their Kauffman polynomial (not bracket).

Experimental ratios

n	0_1	3_1	4_1	5_1	5_2	6_1	6_2
3	0.95833	0.02083	0.00000	0.00000	0.00000	0.00000	0.00000
4	0.95395	0.01974	0.00658	0.00000	0.00000	0.00000	0.00000
5	0.91897	0.03621	0.00690	0.00043	0.00043	0.00000	0.00000
6	0.88960	0.04675	0.01404	0.00032	0.00079	0.00005	0.00005
7	0.85244	0.06048	0.02009	0.00067	0.00181	0.00011	0.00015
8	0.81754	0.07132	0.02787	0.00105	0.00307	0.00024	0.00036

Table : Ratios of knots appearing in planar diagrams with $\leq n$ crossings.

Experimental ratios (unknots in diagrams)

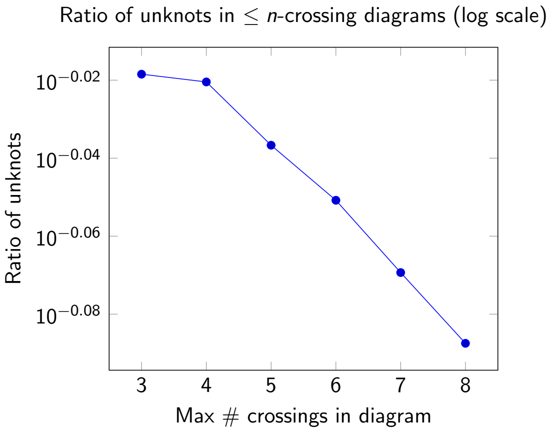


Figure : Unknot ratio decreases exponentially.

Experimental ratios (unknots in diagrams)

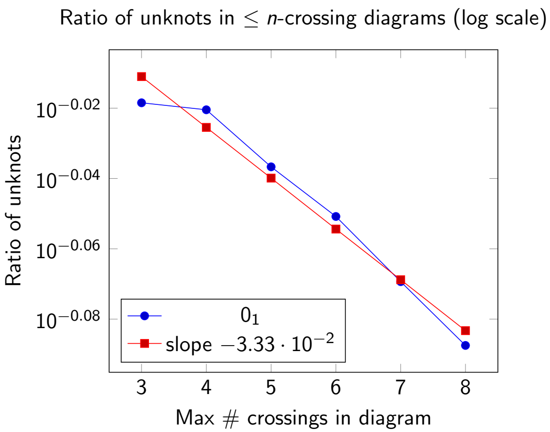


Figure : Unknot ratio decreases exponentially.

Experimental ratios (knotting)

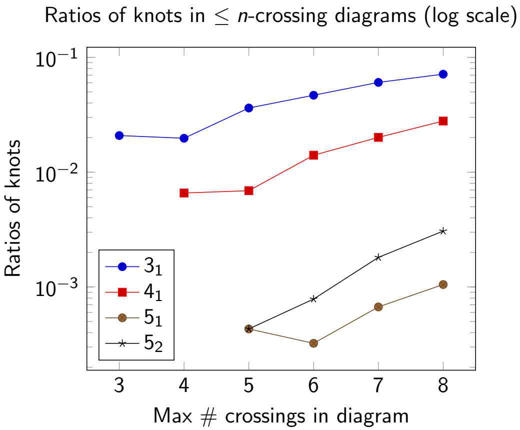


Figure : All ratios of knot types are still increasing.

Experimental ratios (knotting)

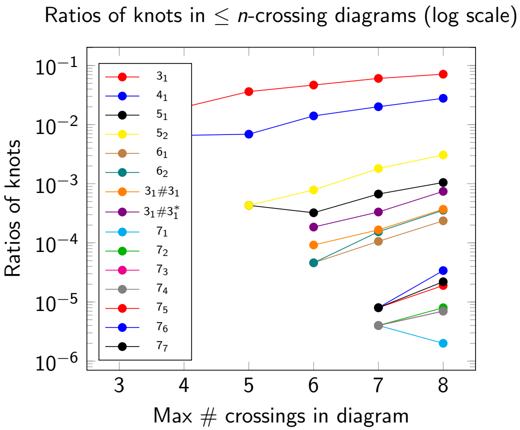


Figure : Ratios of knot types are still increasing.

Knot type and random polygons

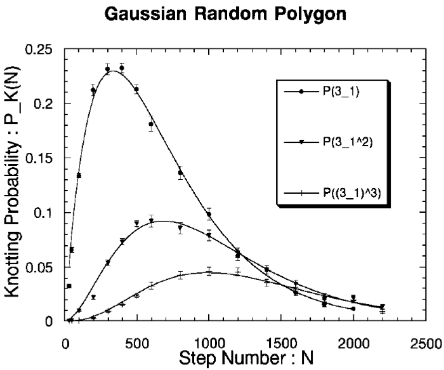


Figure : Random knotting probability for the knots 3_1, $3_1\#3_1$, and $3_1\#3_1\#3_1$ in Gaussian random polygons with N edges [Deguchi-Tsurusaki 1998]

Knotting in random planar diagrams

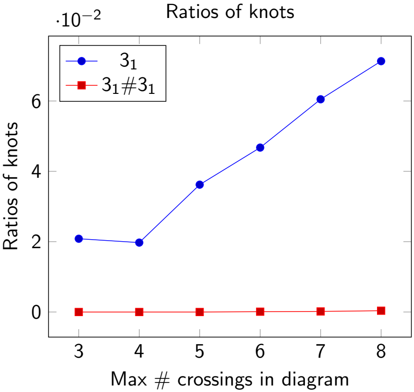


Figure : At $n = 8$, the ratio of 3_1 is 0.071; in Gaussian random polygons, the trefoil appears with probability approximately 0.07 when $N \approx 50$.

Unknotting in random confined polygons

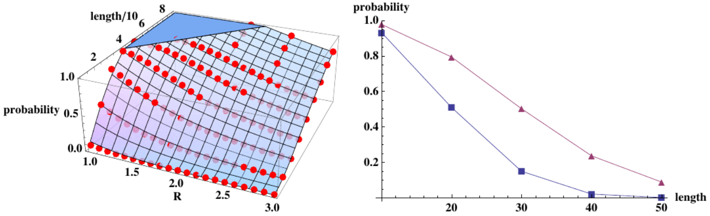


Figure : As a random polygon becomes more confined (i.e. the confinement radius R shrinks), unknotting probability decreases. Right: the blue data has $R = 1$, and the maroon has $R = 1.5$. [Diao-Ernst-Ziegler 2014]

Distance between knots

A notion of distance between knot types is the following.

Knot distance

The distance between two knots K and L is the minimum number of crossing sign toggles required to produce L from K.

The **unknotting number** of a knot K is the distance from K to the unknot 0_1.

Topoisomerase

An immediate application of knot distance is found in the enzyme *topoisomerase*, which aids in the replication of DNA by cutting a strand of a DNA double helix, pulling it through the other, and gluing it back.

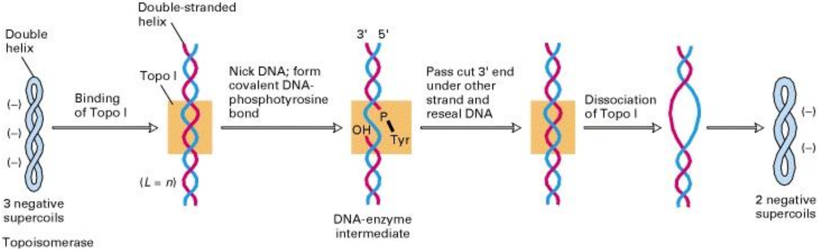


Figure : Topoisomerase brings strands of DNA closer to unlinking.

Known knot distances

Isabel Darcy has compiled tables of known ranges for the distances between knot types. Even in the first few rows there are unknowns.

	0_1	3_1	4_1	5_1	5_2	6_1	6_2	6_3	$3_1\#3_1$	$3_1\#3_1^*$
3_1	1	0	2	1	1	2	1	1	1	1
3_1^*	1	2	2	3	2	2	2	1	3	1
4_1	1	2	0	2-3	2	1	1	2	2-3	2-3
5_1	2	1	2-3	0	1	2-3	2	2	2	2
5_1^*	2	3	2-3	4	3	2-3	3	2	4	2
5_2	1	1	2	1	0	2	2	2	2	2
5_2^*	1	2	2	3	2	2	2	2	3	2
6_1	1	2	1	2-3	2	0	1	2	2-3	1-3
6_1^*	1	2	1	2-3	2	1	2	2	2-3	1-3

Table : A piece of the knot distance tabulation by Darcy.

Knot distance graph

As we've enumerated and classified every planar diagram with at most 9 crossings, we can create a graph of the knots based on their distance (for k up to 9):

Knot distance graph

Between any two vertices (knots), add an edge if there exists crossing switch on a planar diagram with k or fewer crossings which toggles between the two knot types.

Knot distance graph for 6-crossing diagrams

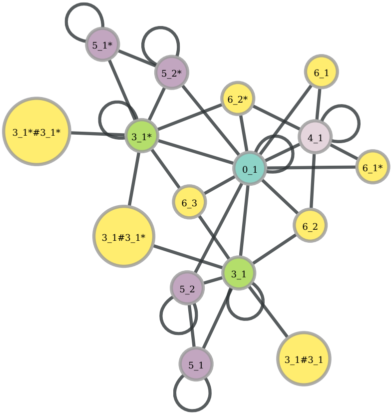


Figure : Knot distance graph up to and including diagrams of 6 crossings. Interactive version here.

Knot distance graph for 9-crossing diagrams

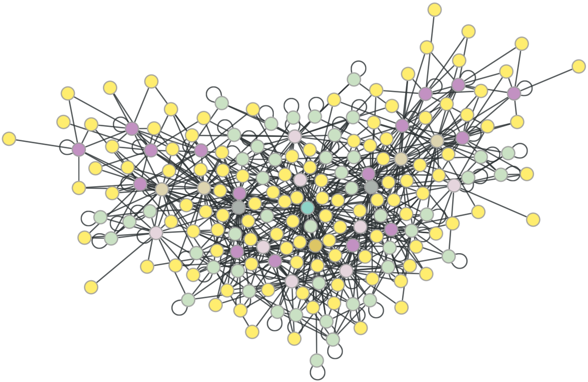


Figure : Knot distance graph up to and including diagrams of 9 crossings. Interactive version here.

Distance tabulations using the 9-crossing graph

	0_1	3_1	4_1	5_1	5_2	6_1	6_2	6_3	$3_1\#3_1$	$3_1\#3_1^*$
3_1	1	0	2	1	1	2	1	1	1	1
3_1^*	1	2	2	3	2	2	2	1	3	1
4_1	1	2	0	3	2	1	1	2	3	3
5_1	2	1	3	0	1	3	2	2	2	2
5_1^*	2	3	3	4	3	3	3	2	4	2
5_2	1	1	2	1	0	2	2	2	2	2
5_2^*	1	2	2	3	2	2	2	2	3	2
6_1	1	2	1	3	2	0	1	2	3	3
6_1^*	1	2	1	3	2	1	2	2	3	3

Table : Distances calculated from the 9-crossing distance graph. Cells in red are the upper bound for unknowns in Darcy's table

Transitions between knot types

We can additionally define the weight of any edge in the graph:

Edge weights

Let the weight of an edge (K, L) between two knot types be the number of pairs (D, x) of diagrams with knot type K and crossings $x \in D$ so that toggling the sign of x produces a diagram D' which is of knot type L.

Transition probabilities between knot types

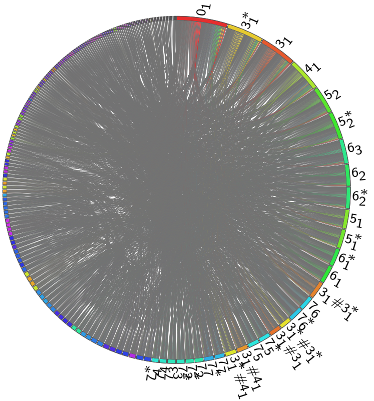


Figure : Chord diagram of the adjacency matrix of the knot distance graph. Interactive version here.