Random Knot Diagrams

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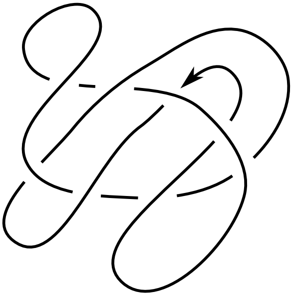
Harrison Chapman (UGA), Matt Mastin (Mailchimp, Inc.)
Crucial Assist: Eric Rawdon (St. Thomas)

AMS Spring Southeastern Section Meeting, 2016

Natural questions about knot diagrams

Question

What fraction of 8-crossing diagrams are trefoils?

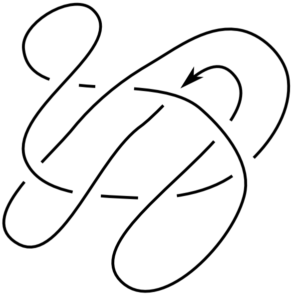


Natural questions about knot diagrams

Question

What fraction of 8-crossing diagrams are trefoils?

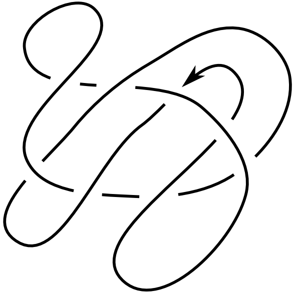
12.48%



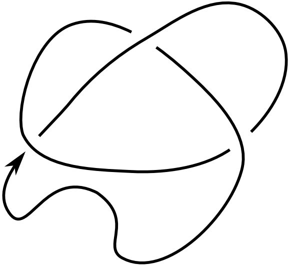
Natural questions about knot diagrams

Question

What is the average minimal crossing # of an 8-crossing diagram?



$\Rightarrow$

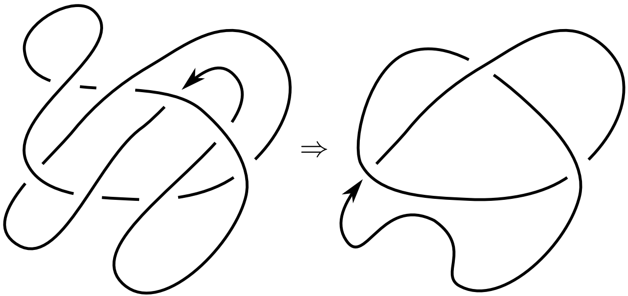


Natural questions about knot diagrams

Question

What is the average minimal crossing # of an 8-crossing diagram?

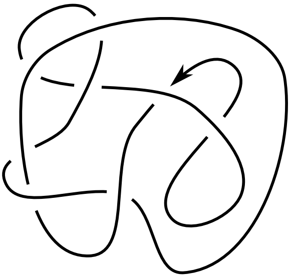
0.52



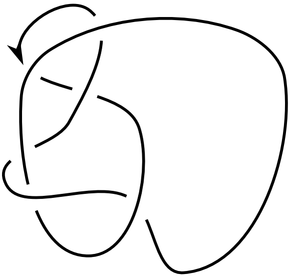
Natural questions about knot diagrams

Definition

The **untwisting** operator deletes all 1-crossing connect summands of a diagram. (Equivalently, performs all "available" Reidemeister I moves.)



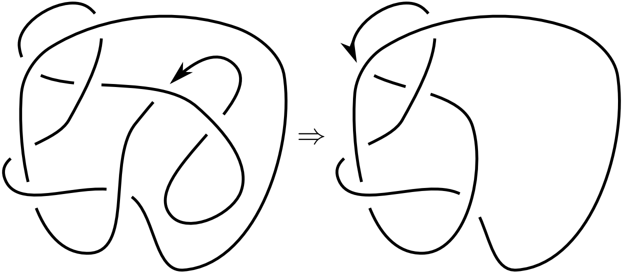
$\Rightarrow$



Natural questions about knot diagrams

Question

What is the average crossing # of a untwisted 8-crossing diagram?

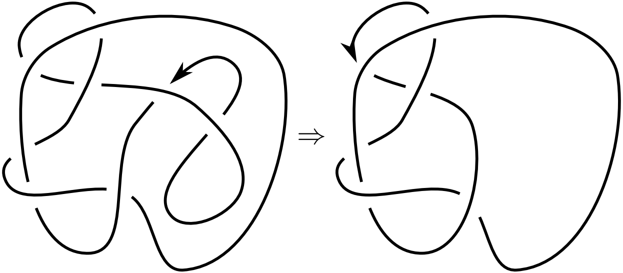


Natural questions about knot diagrams

Question

What is the average crossing # of a untwisted 8-crossing diagram?

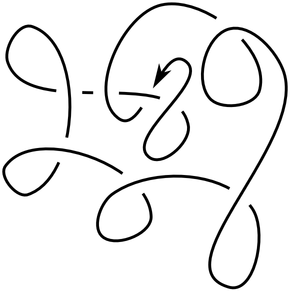
2.20



Natural questions about knot diagrams

Question

How many 8-crossing diagrams can be untwisted to the unknot?

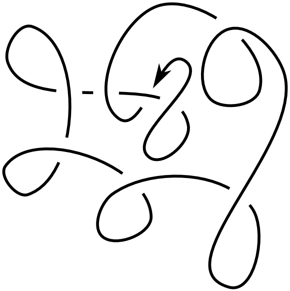


Natural questions about knot diagrams

Question

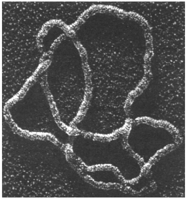
How many 8-crossing diagrams can be untwisted to the unknot?

42.05%

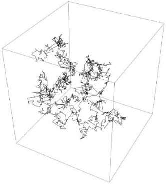


Ansatz

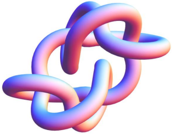
Polymers



Random curves

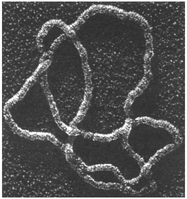


Knots

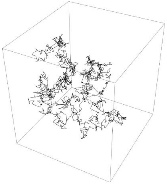


Combinatorial approaches

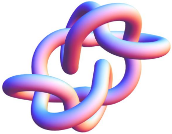
Polymers



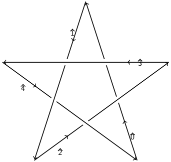
Random curves



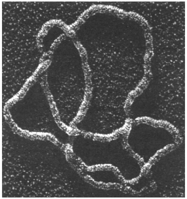
Knots



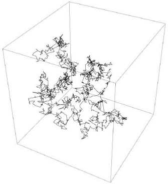
Random combinatorial



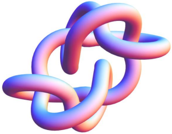
Polymers



Random curves

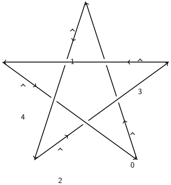


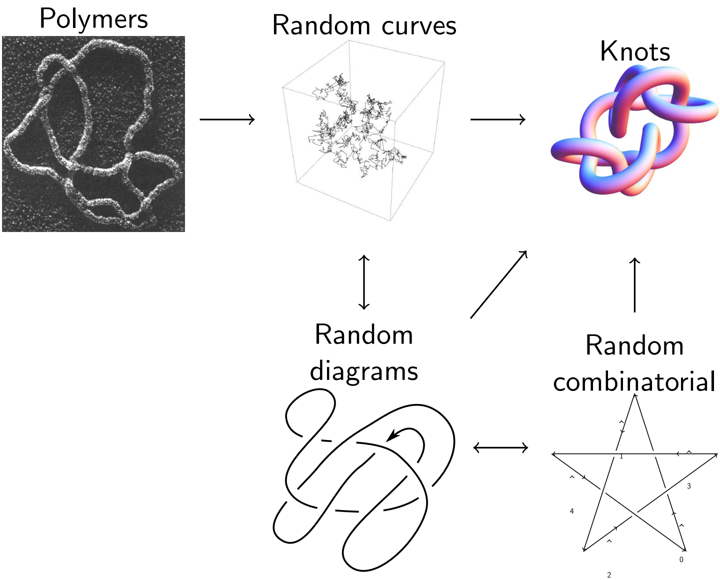
Knots



???

Random combinatorial

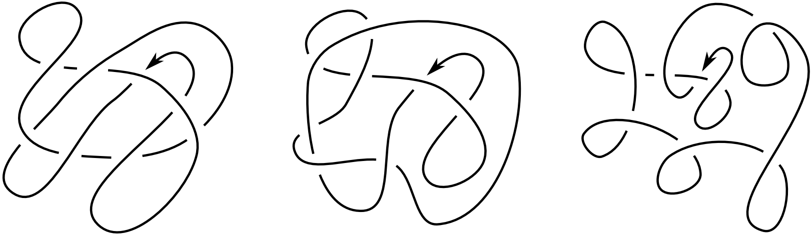



Polymers
Random curves
Knots
Random
diagrams
Random
combinatorial
1
3
4
0
2

Random diagrams

Definition

In the **random diagram model** of random knotting, a n-crossing diagram is drawn uniformly from the finite set of n-crossing knot diagrams.



How to enumerate knot diagrams (like a topologist)

Definition

A **knot shadow** is a equivalence class of generic immersions of the unoriented S^1 into the sphere S^2 up to diffeomorphism of S^2.

Plan to Enumerate Diagrams

1. *Enumerate shadows (and discard isomorphic shadows)*
2. *Assign crossing and orientation information (and discard crossing patterns related by an automorphism of the shadow)*

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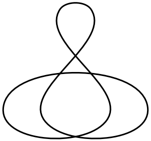
Observation (known to all combinatoricists, but new to me)

Symmetry stinks.

Tabulating knot shadows: `plantri`, two ways

Proposition

Knot shadows $\leftrightarrow$ 1-component 4-valent embedded planar multigraphs up to embedded isomorphism



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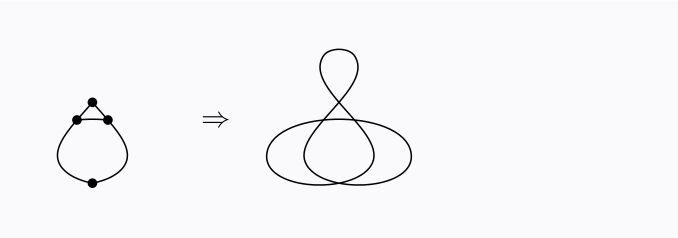
1. Add loops and edges to planar simple graphs (slow)



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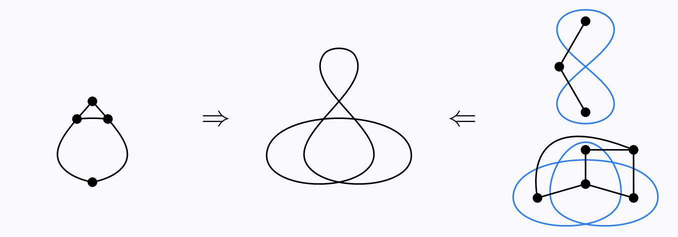
2. Generate multiquadrangulations of sphere by careful pattern of connect sums, take dual graphs (fast)



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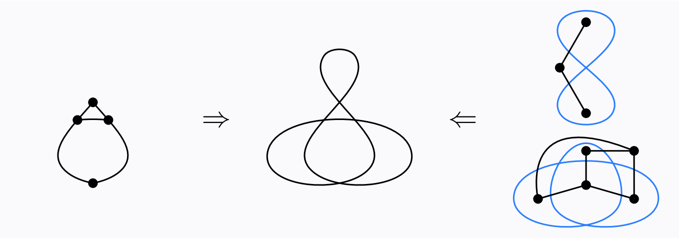

Actually generate all **link shadows**, then restrict to knot shadows

Verifying against existing shadow counts

oriented	$n=0$	1	2	3	4	5
S^2, S^1	1	1	3	9	37	182
S^2	1	1	2	6	21	99
S^1	1	1	2	6	21	97
–	1	1	2	6	19	76

Curves on S^2. The number of types

V.I. Arnol'd. *Topological Invariants of Plane Curves*

A008989 Number of immersions of unoriented circle into unoriented sphere with n double points.

1, 1, 2, 6, 19, 76, 376, 2194 (list; graph; refs; listen; history; text; internal format)

OFFSET	0,3
REFERENCES	V. I. Arnold, Topological Invariants of Plane Curves..., American Math.
LINKS	Table of n, a(n) for n=0..7.
CROSSREFS	Sequence in context: A150119 A181770 A138800 * A057240 A079564 A079453 Adjacent sequences: A008986 A008987 A008988 * A008990 A008991 A008992
KEYWORD	nonn
AUTHOR	N. J. A. Sloane.
EXTENSIONS	Two more terms from Guy H. Valette (guy.valette(AT)skynet.be), Feb 09 20
STATUS	approved

OEIS A008989

n	# knot shadows
0	1
1	1
2	2
3	6
4	19
5	76
6	376
7	2194
8	14614
9	106421
10	823832

We have not found any existing counts of **diagrams**.

Assign crossings, orientation, identify

1. Orient each component. (2 choices)
2. Assign over-under information to each vertex. (2^n choices)

n	# knot shadows	2^{n+1} (# shadows)	# knot diagrams
3	6	96	36
4	19	608	276
5	76	4,864	2,936
6	376	48,128	35,872
7	2,194	561,664	484,088
8	14,614	7,482,368	6,967,942
9	106,421	108,975,104	105,555,336
10	823,832	1,687,207,936	1,664,142,836

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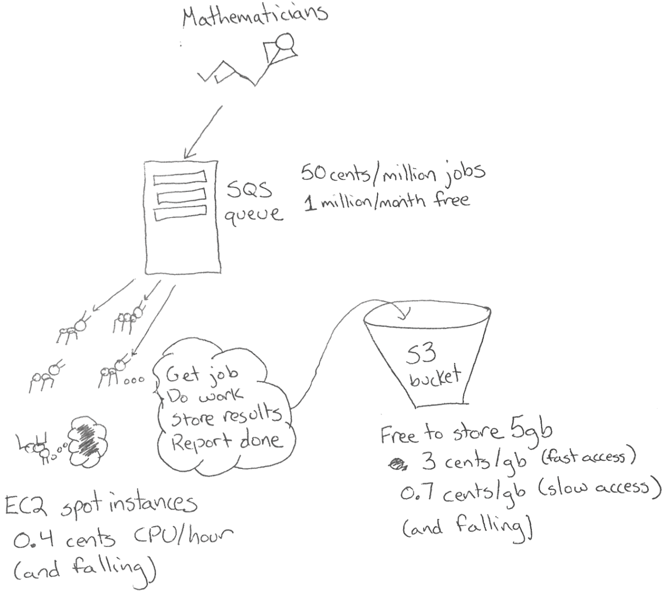
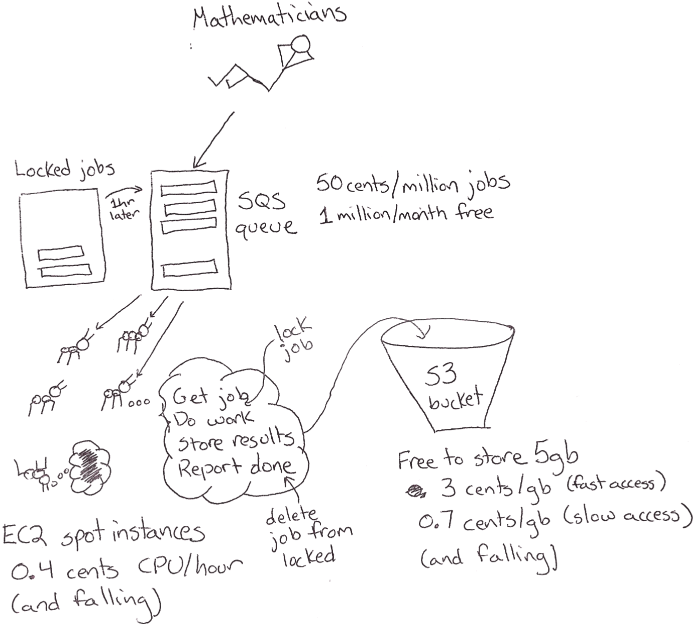
Observation

Symmetry becomes rare, quickly!

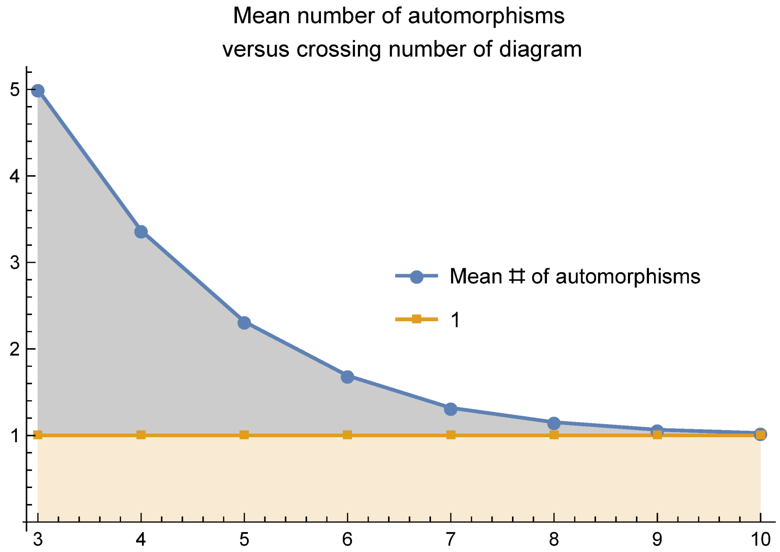
Methods


Mathematicians
SQS queue
50 cents/million jobs
1 million/month free
Get job
Do work
store results
Report done
S3 bucket
EC2 spot instances
0.4 cents CPU/hour
(and falling)
Free to store 5gb
3 cents/gb (fast access)
0.7 cents/gb (slow access)
(and falling)

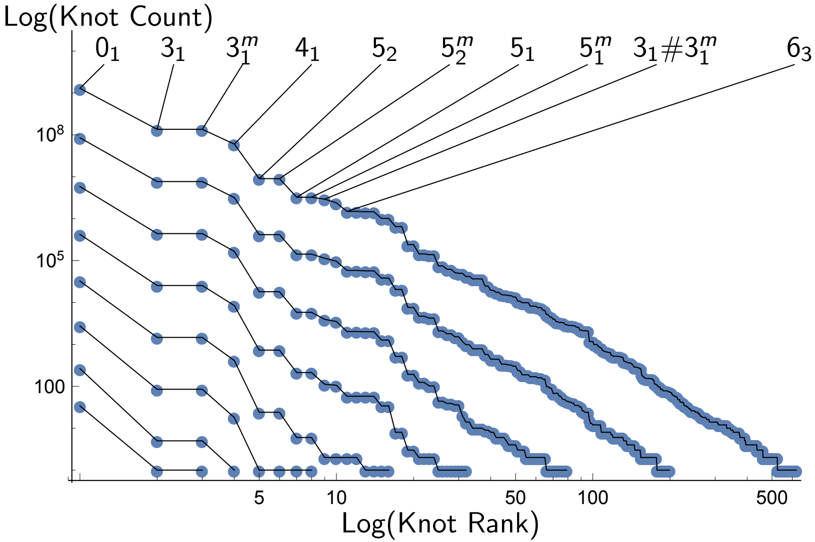
Methods


Mathematicians
Locked jobs
1hr later
SQS queue
50 cents/million jobs
1 million/month free
lock job
Get job
Do work
Store results
Report done
S3 bucket
EC2 spot instances
0.4 cents CPU/hour
(and falling)
delete job from locked
Free to store 5gb
3 cents/gb (fast access)
0.7 cents/gb (slow access)
(and falling)

Size of the automorphism group of a random diagram



Knotting in diagrams

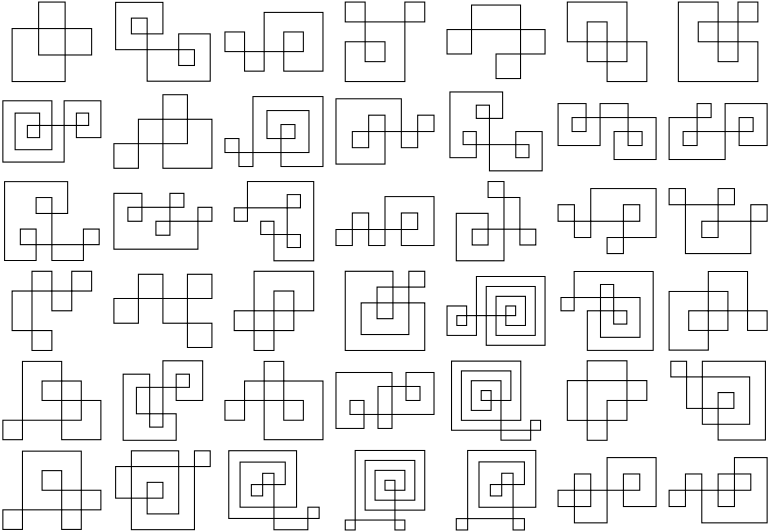


Unknot fraction

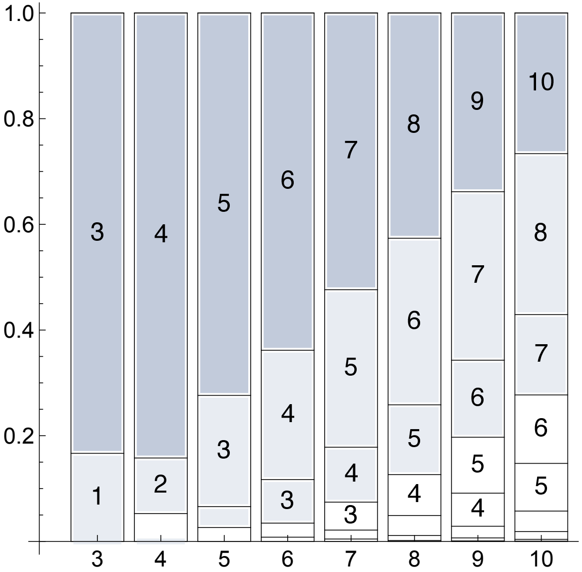
Cr	Unknots	(decimal)
3	$\frac{17}{18}$	0.94
4	$\frac{265}{276}$	0.96
5	$\frac{343}{367}$	0.93
6	$\frac{4057}{4484}$	0.90
7	$\frac{105583}{121022}$	0.87
8	$\frac{2926416}{3483971}$	0.84
9	$\frac{42626767}{52777668}$	0.81
10	$\frac{1291291155}{1664142836}$	0.78

Unknots are very common, even among 10 crossing diagrams. Why?

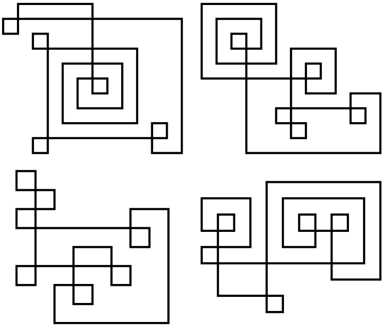
The space of shadows



Most diagrams are (very) composite



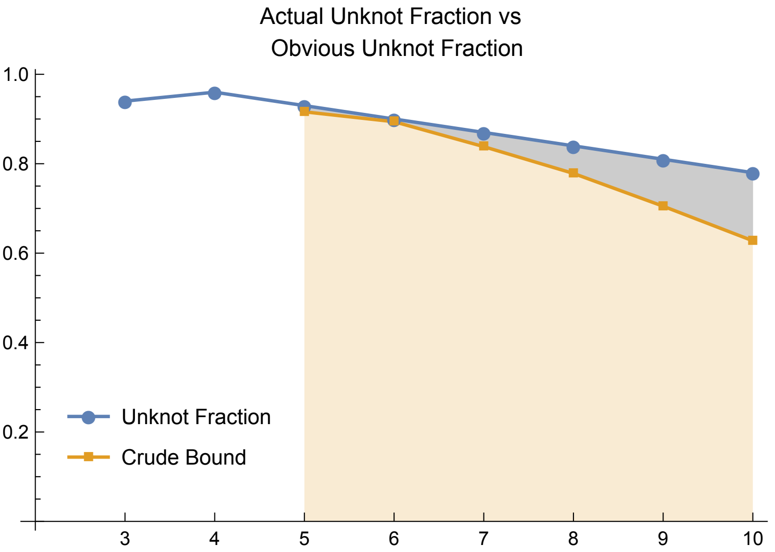
Maximally composite diagrams are "treelike"



Question

Treelike diagrams can't be knotted with any assignment of crossings. Does this (crude) bound explain the unknot fraction?

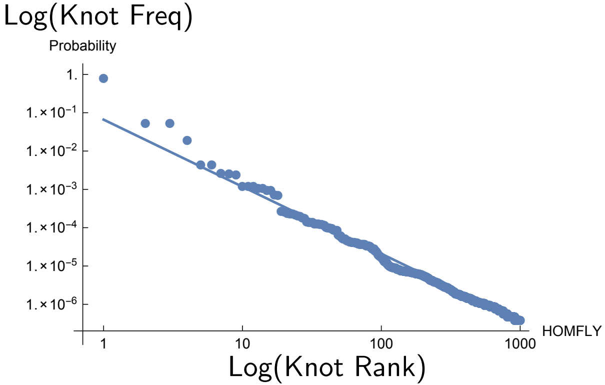
Pretty much.


Actual Unknot Fraction vs
Obvious Unknot Fraction
1.0
0.8
0.6
0.4
0.2
3
4
5
6
7
8
9
10
Unknot Fraction
Crude Bound

Future Direction: So what about those log-log plots?

Proposition (with Shonkwiler, 2015)

The symplectic structure on polygon space yields a fast direct sampling algorithm for closed equilateral polygons.

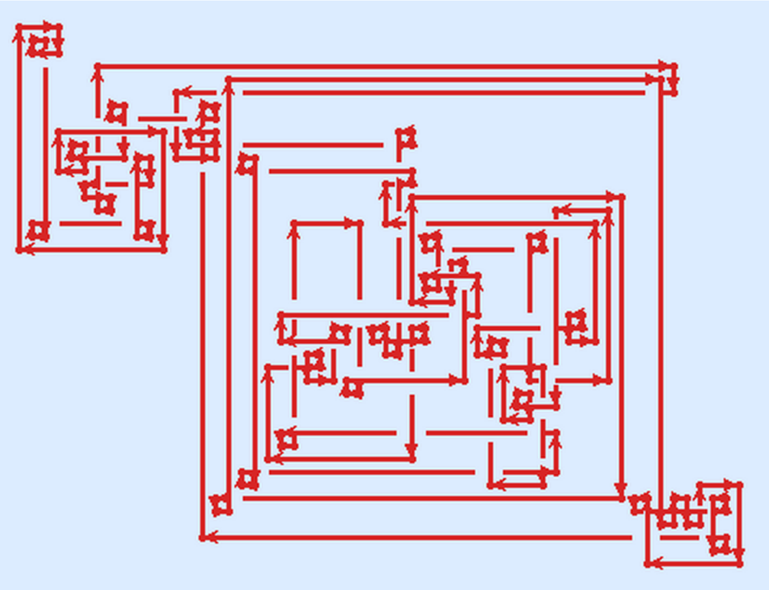


Future Direction: You can play, too!

- Knot Probabilities in Random Diagrams Cantarella, Chapman, Mastin. arXiv:1512.05749
- All data (and pictures for all the diagrams) available at www.jasoncantarella.com/wordpress/papers/
- A Fast Direct Sampling Algorithm for Random Equilateral Polygons Cantarella, Duplantier, Shonkwiler, Uehara. arXiv:1510.02466

Future Direction: Uniform sampling of large diagrams

Harrison Chapman has results on sampling large diagrams:



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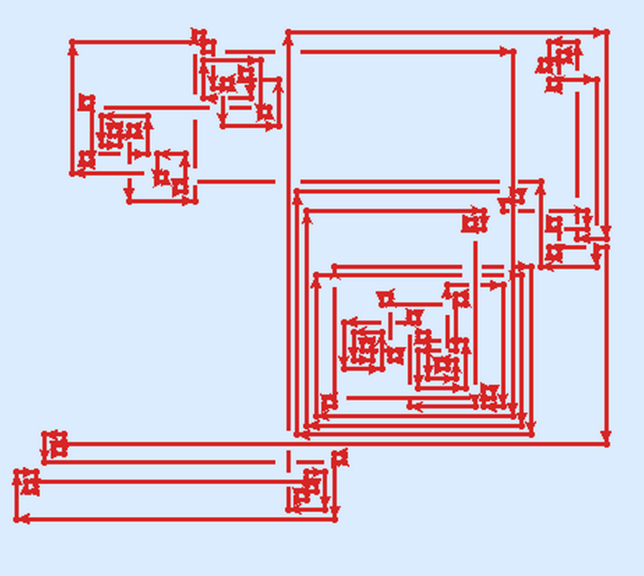

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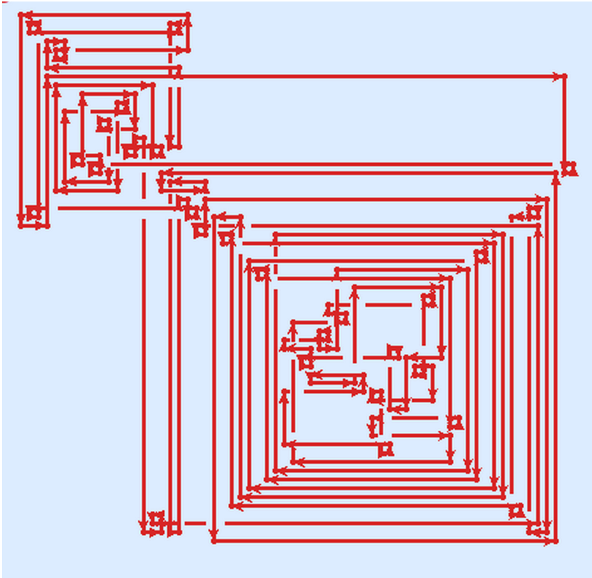
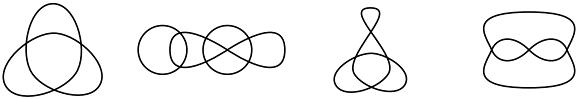


Thank you!



This research was supported in part by NSF grant DMS-1344994 (RTG in Algebra, Algebraic Geometry, and Number Theory, at the University of Georgia).