ON THE ASYMPTOTICS OF UNIFORMLY RANDOM KNOT DIAGRAMS

HARRISON CHAPMAN

Abstract. We study random knotting by considering knot and link diagrams as decorated, (rooted) combinatorial maps on spheres, and pulling them uniformly from among sets of a given number of vertices n. We prove some asymptotic results and examine how quickly this behavior occurs in practice. En route, we show how some asymptotic laws for unlabeled maps apply to decorated maps as well.

1. Introduction

1.1. **Random knotting.** There is a dearth of models for drawing random knots; self avoiding lattice walks [1], random space polygons [2], [3], random braid words [cite], *Petaluma* [4], et. al. In this paper we will discuss the *random diagram model* introduced in [5] under which *knot diagrams* are drawn uniformly from the set of all diagrams with a given number of crossings. There has been some work on sampling random diagrams[6], but the distributions are not precisely understood. As well, alternating knot and link diagrams have been studied [7] but little is published about the knottiness of arbitrary random diagrams of large size.

In this paper we begin by considering a slightly different object, *rooted diagrams*, which break symmetries (as opposed to in). We are then able to prove that in the limit, knot diagrams behave similarly to rooted diagrams, so that these results carry over.

1.2. **Definitions.**

1.2.1. *Knots, links, and tangles.* A *link* is an isotopy class of embeddings of one or more circles into S^3. A *knot* is an isotopy class of embeddings of exactly one circle into S^3. Both of the prior are considered up to *ambient isotopy* of the embedded circles. A *knot diagram* (resp. *link diagram*) is a generic immersion of a circle (resp. any number of circles) into the sphere S^2 (generic in that all intersection points are double points) together with over-under information at each double strand. The study of links and knots is well known to be equivalent to the study of link diagrams and knot diagrams up to the so-called *Reidemeister moves*, shown in figure 1 by a theorem of Reidemeister.

Date: October 29, 2015.

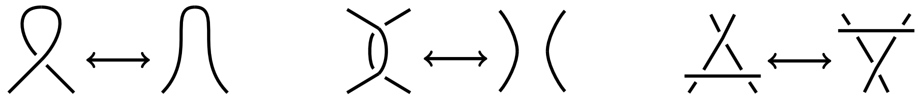

FIGURE 1. The three Reidemeister moves.

A $2k$-*tangle* is a generic immersion of k intervals and any number of (possibly no) closed circles into B^3 so that the $2k$ interval ends all lie in the boundary. A $2k$-*tangle diagram* is a generic immersion of k intervals and any number of closed circles into S^2 together with over-under information at each double point. In this paper, we will only discuss tangle diagrams in which all $2k$ ends of the intervals lie in the same face of the sphere, so that the $2k$-tangle diagram may be viewed as being an immersion into the disk D^2 with the $2k$ interval ends lying in the boundary circle.

FIGURE 2. A 6-tangle diagram with 3 strands, and a 6-tangle diagram with 4.

1.2.2. *Topological maps.* Diagrams are considered up to "embedded graph isomorphism." This precisely means that the viewpoint we should have is that of *topological maps* (in the cartographic sense) on surfaces.

If Σ is a surface, then its Euler characteristic $\chi(\Sigma)$ is a topological invariant. The type g of a surface is defined by $\chi(\Sigma) = 2 - 2g$ (this definition agrees precisely with the genus g of orientable surfaces).

Definition. A *map with n vertices* M is a graph $\Gamma(M)$ embedded on a surface Σ of type g so that every connected component of $\Sigma \setminus M$ is a topological disk. The connected components of $\Sigma \setminus M$ are called the *faces* of M.

A map M is *4-regular*, *4-valent*, or *quartic* if every vertex in the underlying graph $\Gamma(M)$ has degree 4.

If Σ is the sphere, then the map M is called *planar*.

The concerns of symmetry complicates the study of maps. A strategy to avoid this issue is to *root* the map by picking and directing a single edge.

Definition. A *rooted map* is a map together with a single edge marked with a direction, called a *root edge*.

An automorphism of a rooted map M would be required to fix the root edge and its direction; hence $\mathrm{aut}(M)$ is the trivial group.

In this paper we will only consider planar maps, although by considering maps on any oriented surface of arbitrary genus one can arrive at *virtual* diagrams. Furthermore, as each face must be a disk, maps' underlying graphs are necessarily connected.

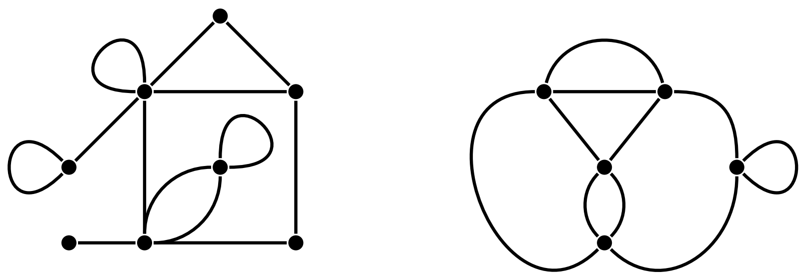

FIGURE 3. Two planar maps. The map on the right is in the class of knot shadows.

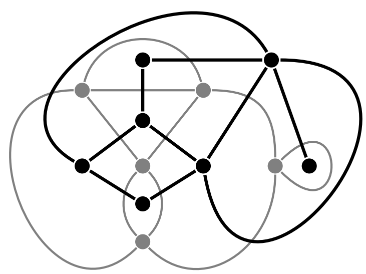

FIGURE 4. Planar quadrangulation which is dual to a knot shadow.

Maps have a well defined notion of *dual map*; a map $M = (V, E, F)$ has dual $M^* = (F, E^*, V)$, where there is an edge $(f_1, f_2) \in E^*$ if f_1 is adjacent to f_2 in M (faces are adjacent if they share an edge on their boundaries). The dual graph of a 4-regular map is a *quadrangulation*, i.e. a map for which every face has four bounding edges. A map is *simple* if it contains no parallel edges or self loops (its underlying graph is simple). Given a rooted map M, its dual is rooted as follows. Let ρ be the root edge of M pointing from v_1 to the root vertex v_2 be adjacent to the face f_1 and the root face f_2. Then (f_1, f_2) is the dual root edge and directed from f_1 to f_2, and f_2, v_2 are the dual root vertex and root face, respectively (and the dual of a dual rooted map is the original rooted map).

Maps have a notion of substructure,

Definition. A map P is a *submap* of a larger (possibly rooted) map M if there exists a cycle of k (possibly repeated) edges in M so that one of the two halves of M separated by the cycle is identical to P.

1.2.3. *Diagrams and shadows.* From here on, maps, shadows, and diagrams will be assumed rooted unless otherwise noted. Notice that we will use the word *crossings* to refer to the vertices of shadows and diagrams.

Definition. A *map decorated by a set* S, (M, s) is a (possibly unrooted) map M together with a mapping $s : V(M) \to S$ which associates to each vertex of M an element of S.

A *(unrooted) link shadow with* n *crossings* is a 4-regular (unrooted) planar map of n vertices. We will denote by $\mathscr{L}_n$ the set of all n-crossing link shadows.

A *(unrooted) link shadow with* n *crossings* is a 4-regular (unrooted) planar map decorated with $\{+, -\}$, i.e. a choice of over-under strand information at each vertex. We will denote the set of n-crossing link diagrams by $\mathcal{L}_n$.

Indeed, $\mathscr{L}_n$ is just another name for the class of 4-regular planar maps in n vertices; furthermore, the class of rooted planar quadrangulations is dual to $\mathscr{L}_n$. Hence, the class $\mathscr{L}$ of link shadows has been counted exactly [8]. If $\ell_n = |\mathscr{L}_n|$, then:

$$\ell_n = \frac{2(3^n)}{(n+2)(n+1)}\binom{2n}{n} \underset{n\to\infty}{\sim} \frac{2}{\sqrt{\pi}} 12^n n^{-5/2}.$$

From this the exact counts of link diagrams can be determined as well. If $\lambda_n = |\mathcal{L}_n|$, then

$$\lambda_n = \frac{2^{n+1}(3^n)}{(n+2)(n+1)}\binom{2n}{n} \underset{n\to\infty}{\sim} \frac{2}{\sqrt{\pi}} 24^n n^{-5/2}.$$

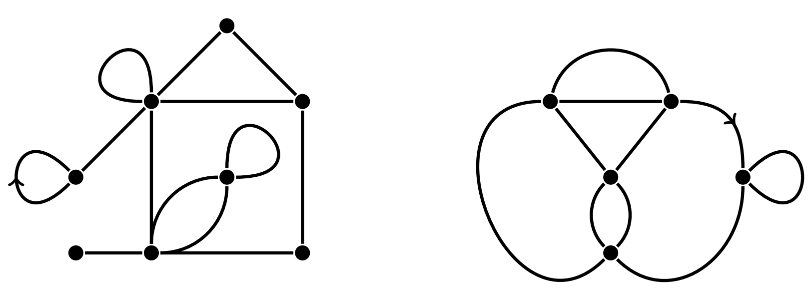

Figure 5. Two rooted planar maps. The map on the right is in the class of rooted knot shadows.

Restricting the number of "link components" complicates counting.

Definition. A *link component* of a (possibly unrooted) link shadow or diagram D is an equivalence class of edges modulo meeting across a vertex in D

A *(unrooted) knot shadow* is a (unrooted) link shadow which consists of precisely one link component. The class of knot shadows with n crossings is denoted by $\mathscr{K}_n$.

A *(unrooted) knot diagram* is a (unrooted) link diagram which consists of precisely one link component. The class of knot shadows with n crossings is denoted by $\mathcal{K}_n$.

Knot shadows $\mathscr{K}_n$ represent a curious, small subclass of $\mathscr{L}_n$. Indeed, exact counts for $k_n = |\mathscr{K}_n|$ and $\kappa_n = |\mathcal{K}_n|$ are not known except by experiments and conjectures[7]

Conjecture (Schaeffer-Zinn Justin 2004)**.** *There exist constants μ_K and c such that*

$$\frac{\kappa_n}{2^n} = k_n \underset{n\to\infty}{\sim} c\mu_K^n \cdot n^{\gamma-2},$$

where

$$\gamma = -\frac{1+\sqrt{13}}{6},$$

and $\mu_K \approx 11.4\ldots.$

Finally, we can define tangles using maps.

Definition. A *(unrooted) $2k$-tangle shadow* is a (unrooted) planar map with $2k$ degree 1 *leg vertices* and any number of 4-valent vertices. We will only consider tangle shadows in which all $2k$ leg vertices lie on the same "exterior" face. In this case, the shadow can be viewed as embedded in D^2 with leg vertices embedded in ∂D^2.

A *(unrooted) $2k$-tangle diagram* is a (unrooted) $2k$-tangle shadow decorated (at non-leg vertices) with signs $\{+,-\}$. We restrict ourselves to tangles which can be embedded into the disk.

A tangle shadow (resp. diagram) T is *contained* in a link shadow (diagram) D if there exists some disk B on the surface into which D is embedded such that

(1) The boundary ∂B intersects no vertices of D,
(2) The boundary ∂B intersects edges of D either transversally or not at all, and
(3) The interior of the intersection $B \cap D$ is isomorphic to the interior of T.

Rooted (knot or link) diagrams are equivalently viewed as *two-leg diagrams* or *2-tangle diagrams* as illustrated below.

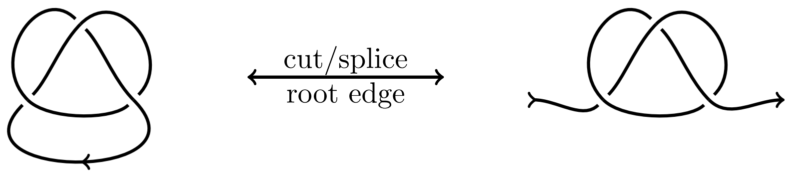

FIGURE 6. A rooted diagram of a trefoil, and its equivalent two-leg diagram

As a key portion of this paper, we will describe how the tools presented can be applied to other classes of diagram objects. To demonstrate this, we will prove the results for prime diagrams as well;

Definition. A (possibly unrooted) shadow D is *prime* if it has more than 1 vertex and is not 2-edge-connected, i.e. there is no way to disconnect $\Gamma(D)$ by removing 2 edges. A shadow which is not prime is *composite.*

A rooted shadow is *two-leg-prime* if it cannot be disconnected by removing two edges, *one being the root edge.*

Diagrams are (two-leg-)prime if their underlying shadow structure is.

We will denote by $\mathscr{PL}_n$ the set of prime link shadows, $\mathscr{PK}_n$ the set of prime knot shadows, $\mathcal{PL}_n$ the set of prime link diagrams, $\mathcal{PK}_n$ the set of prime knot diagrams, and $p\ell_n$, pk_n, $p\lambda_n$, and $p\kappa_n$ their respective cardinalities.

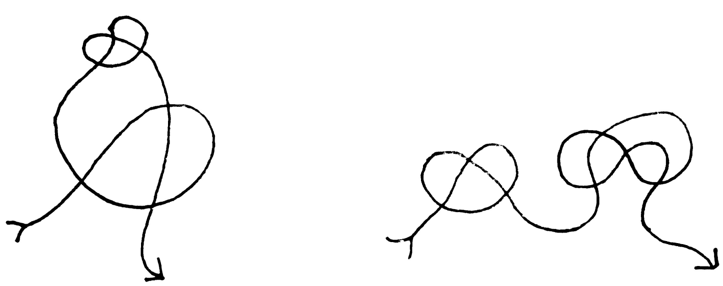

FIGURE 7. A composite shadow which is two-leg-prime (left). A shadow which is not two-leg-prime (right).

Again, the counts of prime link shadows and prime link diagrams are known precisely. Exact counts are known from their bijection with simple quadrangulations [9];

$$\frac{p\lambda_n}{2^n} = p\ell_n = \frac{4(3n)!}{n!(2n+2)!}.$$

1.3. **Result summary.** The primary goal of this paper is to prove the following result for unrooted knot diagrams:

Theorem 1. *Almost all unrooted knot diagrams are unknotted, i.e. they are in the same equivalence class as the unknoy* 0_1*.*

2. Asymptotic structure theorems for diagrams

2.1. **The Frisch-Wasserman-Delbrück conjecture.** On the topic of DNA topology, Frisch and Wasserman[10] and Delbrück[11] separately conjectured;

Conjecture (Frisch-Wasserman 1961, Delbrück 1962)**.** *As the size* n *of a circle embedded in space increases, the probability that the circle is knotted tends to 1.*

The conjecture was originally posed in the view of self-avoiding lattice polygons (SAPs), where size refers to the number of steps. We ask a similar question here for knot diagrams: *Is a knot diagram with* n *crossings almost certainly knotted as* n *tends to infinity*?

For SAPs on the lattice, the conjecture was proved in the affirmative decades later by Sumners and Whittington[1] who made use of Kesten's pattern theorem[12], [13] which states that patterns, (relatively) short walk configurations, appear linearly often in long self-avoiding walks.

We make use of a similar strategy: theorem 2 in [14] provides a pattern theorem for knot and link shadows, provided a strategy of attaching a desired pattern. However, care is required in the case of knot or link *diagrams*, in which each vertex takes a value in the set $\{+,-\}$. In fact, we turn our attention to the dual case in which *faces* are labelled with an arbitrary set.

2.2. **Tangles and the pattern theorem.**

Theorem 2. *Let* S *be a set and* $\mathscr{M}$ *be some class of decorated-maps on a surface of type* g *and let* P *be a planar decorated map with boundary that can be found as a submap of maps in* $\mathscr{M}$*. Let* $M(x)$ *be the generating function by number of edges for* $\mathscr{M}$*. Let* $H(x)$ *be the generating function by number of edges for those maps* M *in* $\mathscr{M}$ *that contain less than* $ce(M)$ *pairwise disjoint copies of* P*. Suppose that we can embed* P *in a possibly larger rooted planar labeled map with boundary* Q *and attach copies of* Q *to each map* K *counted by* $H(x)$ *in such a way that*

(1) for some fixed positive integer k*, at least* $\lfloor e(K)/k \rfloor$ *possible non-conflicting places of attachment exist,*

(2) only S-maps in $\mathscr{M}[S]$ are produced,

(3) for any map produced as such we can identify the copies of Q that have been added and they are all pairwise disjoint, and

(4) given the copies that have been added, the original map and associated places of attachment are uniquely determined.

If $1 > c > 0$ is sufficiently small, then $r(M) < r(H)$. The maps may be rooted or not.

The method of attachment is vague, but flexible. We will provide some examples which we use in our results for knot diagrams. The proof extends the proof of the original theorem for maps, and makes use of a lemma:

Lemma ([14], lemma 3)**.** *If*

(1) $F(z) \neq 0$ is a polynomial with non-negative coefficients and $F(0) = 0$,

(2) $H(w)$ has a power series expansion with non-negative coefficients and $0 < r(H) < \infty$,

(3) for some positive integer k the linear operator $\mathscr{L}$ is given by $\mathscr{L}(w^n) = z^n(F(z)/z)^{\lfloor n/k \rfloor}$, and

(4) $G(z) = \mathscr{L}(H(w))$,

then $r(H)^k = r(G)^{k-1}F(r(G))$.

The proof of the theorem then remains almost unchanged from the original theorem, although care will be necessary in defining attachment.

Proof of theorem 2. Let $G(z)$ be the generating function which counts S-maps $\mathscr{G}[S]$ which are the result of attaching some number between 0 and $\lfloor n/k \rfloor$ copies of Q to S-face maps $\mathscr{H}[S]$ counted by $H(x)$. The method of attachment leads to the relation $G(z) = \mathscr{L}(H(w))$, where $F(z) = z + z^q$ and q is the number of edges added when a copy of Q is attached, as

$$G(z) = \sum_{X \in \mathscr{G}[S]} z^{e(X)} = \sum_{Y \in \mathscr{H}[S]} z^{e(Y)} \left(1 + z^{q-1}\right)^{\lfloor n/k \rfloor} = \mathscr{L}(H(w)).$$

Let g_n be the coefficients of $G(z)$.

Suppose $M \in \mathscr{M}[S]$ contains m copies of Q. By property (3) of our attachment, $m \leq n$. If M had been produced from some S-map K in $\mathscr{H}[S]$ by our attachment process, we can find all possible K by removing at least $m - cn$ copies of Q from M. It is possible to bound from above the number of ways to do this by

$$\sum_{j\geq m-cn}\binom{m}{j}=\sum_{k<cn}\binom{m}{k}<\sum_{k<cn}\binom{n}{k}\leq n\binom{n}{cn}\leq\frac{n(ne)^{cn}}{cn^{cn}}=n\left(\frac{e}{c}\right)^{cn}=:t_n.$$

If $M(x)=\sum m_n x^n$, then $m_n\geq g_n$ and $t_n>1$ for sufficiently large n, so $m_n\geq g_n/t_n$. Hence,

$$1/r(M)\geq\limsup_{n\to\infty}(g_n/t_n)^{1/n}=\lim_{n\to\infty}(t_n)^{-1/n}\limsup_{n\to\infty}(g_n)^{1/n}\geq(c/e)^c/r(G).$$

By the prior lemma, $r(H)^k=r(G)^k(1+r(G)^{q-1})$ so that

$$r(H)/r(M)\ \geq(1+r(G)^{q-1})^{1/k}(c/e)^c.$$

As $\lim_{c\to 0^+}(c/e)^c=1$ and $r(G)^k(1+r(G)^{q-1})=r(H)^k\geq 1/12^k$, it follows that $r(H)/r(M)>1$ for sufficiently small c, completing the proof of the theorem. $\square$

The conclusion is about radii of convergence of two power series, and may appear an esoteric result. However, application of the Cauchy-Hadamard theorem, together with one additional hypothesis, gives a more familiar tune:

Corollary 3 ([14])**.** *Suppose all of the hypotheses of theorem 2 and additionally that $\mathscr{M}[S]$ grows smoothly, i.e. that $\lim_{n\to\infty}m_n^{1/n}$ exists. Then there exists constants $c>0$ and $d<1$ and $N>0$ so that for all $n\geq N$,*

$$\frac{h_n}{m_n}<d^n.$$

I.e., the pattern P is ubiquitous*.*

Because of Euler's formula, the number of vertices, edges, or faces in a link shadow or planar quadrangulation is entirely determined by choosing any one cardinality. Hence, we can size shadows by the number of vertices and still keep the above results.

2.2.1. *Smooth growth.* The theorem in the prior section by itself does not sufficiently prove *ubiquity* as required to prove asymmetry. One may worry about bad cases; e.g., one in which . Indeed, we require that the class of maps *grow smoothly*, i.e. that (for $m_n=|\mathscr{M}_n|$) the limit

$$\lim_{n\to\infty}m_n^{1/n}$$

exists.

Bender, et al. [14] give a powerful proof strategy for proving smooth growth of a sequence. We adapt that to prove the following theorem.

Theorem 4. *Let $\mathscr{C}$ be a class of combinatorial objects with generating function $\sum_{n=0}^{\infty} c_n\, z^n$; let the radius of convergence of the OGF be r, and $\mathscr{D}$ some other class with generating function $\sum_{n=0}^{\infty} d_n\, z^n$. Suppose that $0 > r \leq 1$ and let $C_i > 0$ and $1 - r > \delta > 0$ be arbitrary.*

Suppose there is a composition operation $\times$ on elements $A, B \in \mathscr{C} \cup \mathscr{D}$ so that,

(1) $A \times B \in \mathscr{C}$,
(2) there exists some fixed $k \in \mathbb{Z}_{\geq 0}$ so that $|A \times B| = |A| + |B| + k$, and
(3) given any $C \in \mathscr{C}$, there is at most one maximal factorization $D_1 \times D_2 \times \cdots \times D_s = C$ with $D_i \in \mathscr{D}$.

Suppose there exists $R \geq 0$ so that for $n \geq R$ there exists $\ell \in \mathbb{Z}_{\geq 0}$ and maps $\psi_0 : \mathscr{C}_n \hookrightarrow \mathscr{D}_{n+\ell}$ and $\psi_1 : \mathscr{C}_n \hookrightarrow \mathscr{D}_{n+\ell+1}$. Then the limit

$$\lim_{n\to\infty} c_n^{1/n}$$

exists.

It is known that there are at most 12^n planar maps, and so in our cases we will always have $r \geq 1/12$.

Proof. For some classes which can be shown to be subadditive, this follows by Fekete's lemma. However, there is a framework to show this result in more complicated cases.

The proof breaks down into 3 steps;

(1) *Show that there exists some $n \geq 0$ with $c_n > C_1(r + \delta)^{-n}$.* This step follows from the Cauchy-Hadamard theorem, which says that

$$\limsup_{n\to\infty} c_n^{1/n} = r^{-1}.$$

By the definition of lim sup, we have that if $a < r^{-1}$, then for any $M \geq R$ we have that there is some $n \geq M$ with $c_n^{1/n} > a$. For instance, we know that $(r + \delta/2)^{-1} < r^{-1}$, hence for any M we have some $n \geq M$ with $c_n > (r + \delta/2)^{-n}$. Notice now that as

$$\left(\frac{r+\delta}{r+\delta/2}\right) > 1,$$

there must be some $M \geq R$ so that for all $m \geq M$

$$\left(\frac{r+\delta}{r+\delta/2}\right) > C_1^{1/m}, \text{ implying that } (r+\delta/2)^{-m} > C_1(r+\delta)^{-m},$$

whence we then have (by lim sup) some $n \geq M$ with $c_n > (r+\delta/2)^{-n} > C_1(r+\delta)^{-n}$.

(2) *Show that there exists some $m \geq 0$ with $d_m > C_2(r+\delta)^{-m}$ and $d_{m+1} > C_2(r+\delta)^{-(m+1)}$.* Notice that $(r+\delta) < 1$ and so for any $m \geq 0$, $(r+\delta)^{-m} < (r+\delta)^{-(m+1)}$. As there exist injections ψ_0, ψ_1 from $\mathscr{C}_n$ into $\mathscr{D}_{n+\ell}$ and $\mathscr{D}_{n+\ell+1}$, setting $m = n+\ell$ and $C_1 = C_2(r+\delta)^{-(m+n-1)}$ we have that

$$d_m \geq |\operatorname{im}\psi_0| = c_n > C_1(r+\delta)^{-n} = C_2(r+\delta)^{-(m+1)} > C_2(r+\delta)^{-m}$$

and

$$d_{m+1} \geq |\operatorname{im}\psi_1| = c_n > C_1(r+\delta)^{-n} = C_2(r+\delta)^{-(m+1)}.$$

(3) *Show that there exists some N so that for any $n \geq N$, $c_n > (r+\delta)^{-n}$.* Consider k from the hypothesis. Let $C_2 = (r+\delta)^{-k}$. Let $N = (m+k)(m+k+1)$. Then if $n \geq N$, we can write n as a linear combination $a(m+k) + b(m+k+1) = am + b(m+1) + (a+b)k$, with $a, b \geq 0$. Observe that $c_n > d_m^a d_{m+1}^b$ as there exists a subset of objects $S \subset \mathscr{C}_n$ which can be expressed uniquely as a product of a elements of $\mathscr{D}_m$ and b elements of $\mathscr{D}_{m+1}$ (and $|S| > d_m^a d_{m+1}^b$). Then

$$c_n > d_m^a d_{m+1}^b > C_2^a(r+\delta)^{-am} C_2^b(r+\delta)^{-b(m+1)} = (r+\delta)^{-(am+b(m+1)+(a+b)k)} = (r+\delta)^{-n}.$$

To finish the proof we realize that this last step implies that the lim inf is r^{-1} and hence the limit result follows. Observe that $\liminf_{n\to\infty} c_n^{1/n} = r^{-1}$ if for any $\epsilon > 0$, there exists N so that for all $n \geq N$, $c_n^{1/n} > r^{-1} - \epsilon = \frac{1-r\epsilon}{r}$. We may assume that $\epsilon < 1$ since otherwise the inequality is clear since $c_n \geq 0$ always. So we are done if we can choose δ so that

$$\frac{r^2\epsilon}{1-r\epsilon} > \delta,$$

as then we have from our prior result that $c_n^{1/n} > (r+\delta)^{-1} > r^{-1} - \epsilon$. Indeed, we have $r^2\epsilon > 0$ and $1 > 1 - r\epsilon > 0$ so that the left hand side of the inequality is positive; but we may choose $\delta > 0$ as small as we desire. Hence for $N = (m+k)(m+k+1)$, the result that $\lim_{n\to\infty} c_n^{1/n} = r^{-1}$ follows. $\square$

2.2.2. *Smooth growth for knot and link diagrams.* The class $\mathscr{L}$ of rooted link shadows has been counted exactly. Rooted link shadows are in bijection with rooted 4-regular planar maps and the coefficients of the generating function are known [8]. If $l_n = |\mathscr{L}_n|$, then:

$$l_n = \frac{2(3^n)}{(n+2)(n+1)}\binom{2n}{n}.$$

On the other hand, asymptotics of knot shadows are as of yet unknown. We are still however able to prove that they grow smoothly, as to prove our result of asymptotic asymmetry.

Theorem 5. *The class $\mathscr{K}$ of rooted knot shadows grows smoothly. I.e., the limit $\lim_{n\to\infty} k_n^{1/n}$ exists (and is equal to $1/r(K)$).*

Proof. As mentioned above, a very loose bound on $r = r(K)$ is $1/12 \leq r \leq 1$ as the number of planar maps in general is bounded by 12^n. Let $C_i > 0$ and $1 - r > \delta > 0$ be arbitrary. We need to define a composition $\times$ and subclass $\mathscr{D}$ of shadows which are prime under $\times$.

Define the concatenation $K = K_1 \times K_2$ on shadows in $\mathscr{K}$ by gluing the front leg of K_1 to the hind leg of K_2. Hence, we will take $\mathscr{D}$ to be the class of knot shadows which remain at least 2-connected

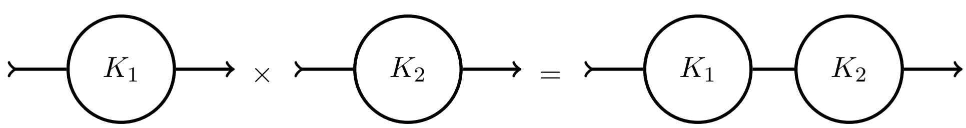

FIGURE 8. The two-leg sum operation $\times$. If both A and B are two-leg-prime, then $A \times B$ has exactly one separating edge.

after removing the root edge.

Certainly $A \times B \in \mathscr{K}$ as we obtain a new 2-leg shadow. As $\times$ introduces no crossings, we have $k = 0$ and $|A \times B| = |A| + |B|$. Finally, a 2-leg shadow K either lies in $\mathscr{D}$ or has $\ell - 1$ disconnecting edges. Cutting these edges produces the disjoint union of ℓ well-ordered 2-leg shadows (well ordered from their position in the long curve K) which is the unique ordered $+$-decomposition of K into elements of $\mathscr{D}$.

Let φ be the map which twists the root edge, making the loop the new root (using the appropriate induced orientation). Then $\varphi : \mathscr{K}_* \hookrightarrow \mathscr{D}_{*+1}$, since deleting the root and smoothing the pointed edge produces a knot shadow, which must be at least 2-connected. Then we take $\psi_0 = \varphi$ and $\psi_1 = \varphi^2$. This setup satisfies the hypotheses and hence proves the theorem. $\square$

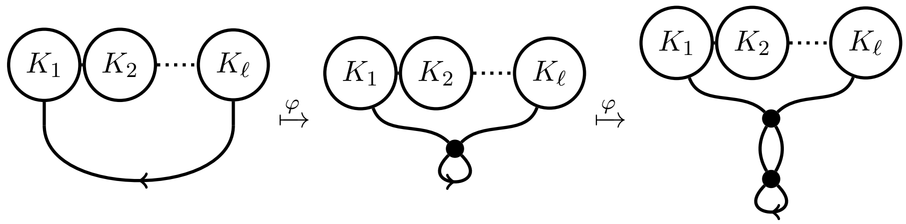

FIGURE 9. The map φ adds a vertex and ensures that the new map is 2-leg-prime.

Corollary 6. *There exists $N \geq 0$ and a constant $d < 1$ so that for $n \geq N$,*

$$\mathbb{P}(\textit{a knot diagram } K \textit{ is an unknot}) < d^n.$$

For any prime 2-tangle P, there exists $N \geq 0$ and constants $d < 1$, $c > 0$ so that for $n \geq N$,

$$\mathbb{P}(\textit{a knot diagram } K \textit{ contains} \leq cn \textit{ copies of } P \textit{ as connect summands}) < d^n.$$

Proof. The first statement will follow immediately from the second, given a prime 2-tangle corresponding to a prime knot diagram which is not an unknot. The second is a corollary of theorems 3 and 5: Let P be a prime 2-tangle which can be found as a connect summand of a knot diagram (i.e., it has one link component). If m_n is the number of knot diagrams, then there exists $c > 0$, $d > 1$, and $N > 0$ so that for all $n \geq N$, $\frac{h_n}{m_n} < d^n$, where h_n is the number of knot diagrams which contain at most cn copies of P as connect summands. This ratio is precisely the probability in the second statement. $\square$

2.2.3. *Smooth growth for prime knot and link diagrams.* If, however, we are considering a class $\mathscr{P}$ of prime or reduced rooted diagrams, the method of proof for smoothness does not immediately carry over; it is possible that φ introduces numerous isthmi, in which case our diagrams created in the final step would not even be reduced. In the case where $\mathscr{P}$ is the class of prime rooted link shadows, exact counts are known from their bijection with simple quadrangulations [9];

$$s_n = \frac{4(3n)!}{n!(2n+2)!}.$$

In other cases again smoothness is more complicated to prove, although we can use a similar argument to that in the case of all knot shadows.

Proposition 7. *Prime knots*

Proof. Step i is again immediate, so we begin with step ii. Let ψ, ψ' respectively be maps which take the root vertex to the two 4-tangle shadows:

Observe that neither ψ nor ψ' remove primeness or reducedness. Their images provide an injection into the spaces with 2 and 3 additional crossings, respectively. So take m appropriately.

Define the operation $+$ now by the detour-glom. Notice that primeness is preserved and the process is splittable; given the root edge we can identify the bendy edges and rebuild the old two shadows. Notice that $|A+B| = |A|+|B|+4$. Now let $C_3 \geq 1$ and $C_2 = C_3(r+\delta)^{-4}$. Then if there exist nonnegative integers a, b such that $n = am + b(m+1) + (a+b)4 = a(m+4) + b(m+5)$, i.e. if $n \geq (m+4)(m+5)$, then

$$p_n > p_{m+4}^a p_{m+5}^b > C_2^{a+b}(r+\delta)^{-(am+b(m+1))} > C_3^{a+b}(r+\delta)^{-(a(m+4)+b(m+5))} > C_3(r+\delta)^{-n}.$$

□

2.2.4. *Proof and constructions for the pattern theorem.* The crux of applying this theorem to link diagrams then falls upon determining an "attachment" operation which satisfies the hypotheses, along with patterns valid for a given class of shadows. We can generally define attachment operations for different kinds of tangles in the dual. By abuse of notation, let $S = (\{0\}, \{0\}, S)$ be a set of labels for the faces of the dual (for now, we are concerned about diagrams, which only have labeled vertices).

(1) **Connect sum.** Let L and Q' be rooted S-quadrangulations. Orient the remaining edges of L canonically by proposition **??**. Define the *connect sum* of Q' into L at an edge $e \in L$, $L\#_e Q'$, by
   (a) Cut and split the edge e, creating a map L' and leaving a distinguished, oriented bigon f. Denote the two edges formed by splitting e by e_1, e_2, so that the loop $e_1(-e_2)$ is a counterclockwise cycle around f. If e was the root of L, make e_1 the new root of L'.
   (b) Cut and split the root edge ϵ of Q', creating a map Q and leaving a distinguished, oriented bigon g. Denote the two edges formed by splitting ϵ by ϵ_1, ϵ_2, so that the loop $\epsilon_1(-\epsilon_2)$ is a counterclockwise cycle around g. Make ϵ_2 the new root of Q.

(c) Glue the map Q into the map L''s distinguished face f along the boundary of the distinguished bigon g so that e_1 and ϵ_2 are mapped to the same edge and so that the orientations of the boundaries align.

(d) Forget about all edge orientations except for the root edge of L.

Notice that none of the original faces in L and Q' are changed; hence the result is a new rooted S-quadrangulation. Any given S-quadrangulation in $2n$ edges has precisely $2n$ different non-conflicting sites for connect summation (i.e. $k = 1$ for this attachment operation). This process is reversible, given a 2-cycle which bounds an instance of Q' (collapse the disk to a single edge). If L^* and $(Q')^*$ each consist of only one link component,

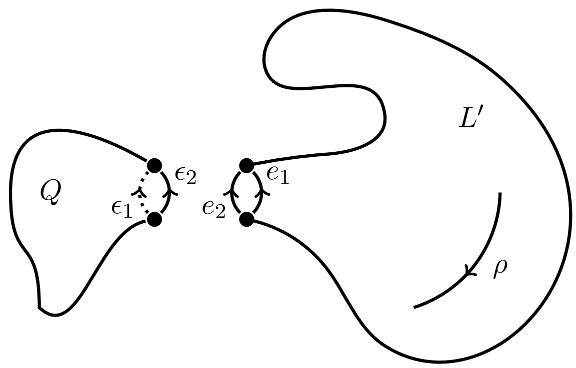

FIGURE 10. The connect sum operation. Q' and L' are viewed as CW-complexes, and their boundaries are appropriately identified.

then $(L\#_e Q')^* =: L^*\#_e(Q')^*$ will as well. In fact, this attachment into the quadrangulation is precisely dual to the usual link connect sum from knot theory. If Q is 2-leg prime, then no two copies of Q can intersect (as in this case there are precisely two paths of length 1 in Q between its two boundary vertices).

(2) **4-tangle replacement.** Let L be a rooted S-quadrangulation, and Q a rooted S-quadrangulation with square boundary. Then given a face $f \in L$, we can define a *4-tangle replacement* of f with Q by identifying the boundary of Q with the boundary of f in some manner. Precisely how to equate the boundaries will depend on the case at hand; what matters is that there be at least one valid way to glue Q into any face f and produce a new S-quadrangulation in the chosen class. Indeed, the result will always be a new S-quadrangulation. Any S-quadrangulation in $2n$ edges has n faces, and hence at least n non-conflicting attachment locations (i.e., $k = 2$).

To make the process reversible for all S (not just $S = \{0\}$), we must pick slightly different Q. Given a non-trivial S-map P, choose a planar rooted labeled map with boundary Q which cannot intersect with a copy of itself so that P is a unique copy of itself in Q, where all faces which are not in P are labeled with a special label $*$. Attachment of Q into the face f of an S-quadrangulation L then consists first of the actual attachment operation described above, and then relabeling all faces with label $*$ with the original label of f. The process is then reversed by

(a) Given a 4-cycle bounding a (relabeled) copy Q' of Q in L, tentatively delete the copy and replace it with a bare face f.
(b) Identify the unique copy of P inside of Q.
(c) If every face of $Q \setminus P$ does not have the same label x, then this would not in fact have been a place of attachment (hence reversal needn't be possible). If they do, then label the new face f with the label x.

Care must be taken in choice of Q in which to embed arbitrary P here; an important fact to remember is that submap insertion of any S-quadrangulation with boundary P will not introduce any shorter paths through Q, else there would be a 3-cycle in P (which is impossible as it is a quadrangulation).

If one is dealing with 4-tangle replacement within a class of knot S-shadow duals, the following lemma is helpful;

Lemma 8. *Given a link S-shadow L, a vertex v in L, and an 4-leg S-curve T with 2 link components, it is always possible to replace v by T in at least one way so that the result has the same number of components as L.*

Proof. TODO: This proof needs work

The vertex v in L and the tangle T are each either of crossing (abab) type or tangency (aabb) type. If they agree in type, replace v by T so that the strands agree. If they differ and $L \setminus v$ is of type abab, replace T in with type abba. If they differ and $L \setminus v$ is of type aabb, replace T with type abab. In all cases, the number of components is preserved. □

There are applications of this attachment in proving the weak pattern theorem for certain classes of maps:

i. Given arbitrary face labels S, a prime link dual S-shadow L and a prime 4-leg dual S-curve P, define Q as in figure 11. Observe that if there exist two copies Q' and Q'' of Q

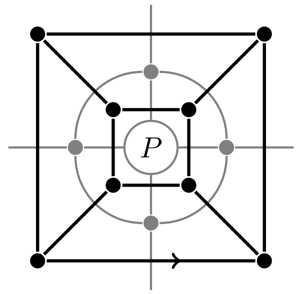

FIGURE 11. The labeled quadrangulation Q (in black) is dual to encircling the 4-leg curve P with a link component. The four faces bounding P have the distinguished label $*$.

in a prime link dual shadow L, then they cannot intersect: All paths between boundary vertices of Q are either of length 3, or greater. If there is an intersection between Q' and Q'', then without loss of generality one of two things happens: (1) There is a path along the root face of Q'' of 3 edges lies in Q' (since there are no paths of length 1 or 2) and runs between two adjacent vertices a and b of Q'; but then the remaining boundary edge of Q'' must run from a to b; but this gives the existence of a 2-cycle in L which we said was prime and hence has no 2-cycles. (2) The entire boundary cycle of Q'' lies within Q' and necessarily runs between two opposing vertices a and c of Q'. But then these two opposing vertices must be the same vertex in L; but then there is the 2-cycle a to b to $c = a$ in L, which again was chosen to have no 2-cycles.

ii. Given face labels S, a prime *knot* dual S-shadow K, and a nontrivial prime 4-leg dual S-curve with 2 components P, define Q as in figure 12, embedding P in such a way that Q has precisely 2 link components. Observe that, as in the case above, the shortest paths which both start and end on the boundary of Q are of length 3 between adjacent vertices, or length 4 between opposite vertices. As we are taking K to be prime, the same reasoning shows that copies of Q in K can not overlap. On the other hand, we must now be careful that the 4-tangle replacement into a knot dual S-shadow does not introduce new link components. Fortunately, there is always at least one way to insert a 4-leg dual S-curve while keeping constant the number of link components.

2.2.5. *Strategy for proving smooth growth.*

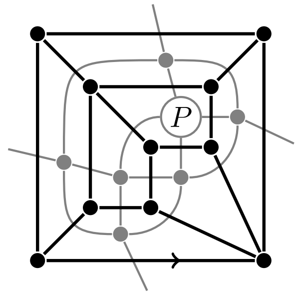

FIGURE 12. The labeled quadrangulation Q (in black) in which to embed P. The remaining faces bounding P have the distinguished label $*$.

2.3. **Asymmetry of diagrams and consequences.** The following theorem of Richmond and Wormald [15] provides a sufficient set of criteria for almost all elements of $\mathscr{K}$ to have trivial automorphism group.

Theorem 9 (Richmond-Wormald 1996)**.** *Let $\mathscr{C}$ be a class of rooted maps on a surface. Suppose that there is an outer-cyclic rooted planar map M_1 such that in all maps in $\mathscr{C}$, all copies of M_1 are pairwise disjoint, and such that*

(1) M_1 has no reflective symmetry in the plane preserving the unbounded face,
(2) there exist constants $c > 0$ and $d < 1$ such that the proportion of n-vertex maps in $\mathscr{C}$ that do not contain at least cn pairwise disjoint copies of M_1 is at most d^n for n sufficiently large (M is "ubiquitous"), and
(3) for any map M in $\mathscr{C}$ containing a copy of M_1, all maps obtainable by removing M_1 and gluing it back in to the same face are in $\mathscr{C}$ (M is "free").

Then the proportion of n-vertex maps in $\mathscr{C}$ with nontrivial automorphisms is exponentially small.

It has been suggested without proof in [16], [17] that classes of knot shadows are almost surely asymmetric. We will prove this for $\mathscr{K}$ by proving it for its dual $\mathscr{K}^*$, a class of quadrangulations of the sphere. Specifically, we will take M_1 to be the dual of the underlying planar map of the following 2-tangle: Clearly M_1 has no reflective symmetry by inspection, and certainly any of the ways of replacing M_1 keep the object in the class of quadrangulations dual to knot maps. Finally, the ubiquity condition is exactly the pattern theorem for 2-tangles proved in the prior section! The same pattern proves asymmetry for certain other classes of knots or links; for example, reduced diagrams.

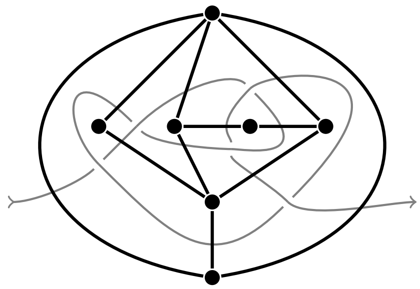

FIGURE 13. The dual 2-leg curve M_1 (black), and one of its representations as a 2-tangle (gray).

Additionally, we can use our 4-tangle replacement scheme to create a M_1 which shows asymmetry of prime knot (link) diagrams. Take M_1 as in figure 14. Then M_1 consists of exactly two link

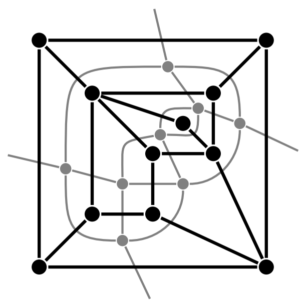

FIGURE 14. Choice of M_1 for proving that prime knot shadows are asymmetric

components and is of abab type; any way of replacing a vertex in a knot shadow with a 4-leg curve of abab type keeps the number of link components constant. Furthermore, M_1 is ubiquitous in prime (knot) diagrams as it is an application of corollary 6 to P, the square with an additional 2-path joining two of its opposite vertices.

Application of the above theorems provides us with the following corollary which enables us to transfer any asymptotic results on rooted diagrams to unrooted diagrams.

Corollary 10. *Let L be a uniform random variable taking values in the space $\mathcal{K}_n$ or $\mathcal{L}_n$. Then there exist constants $C, \alpha > 0$ so that $\mathbb{P}(\operatorname{aut} L \neq 1) < Ce^{-\alpha n}$. Hence, rooted diagrams behave like unrooted diagrams.*

Indeed, link diagrams with n vertices are dual to quadrangulations with $n+2$ faces; there are $n+2$ ways of choosing the "exterior" root face and then 4 ways of rooting the edges around this chosen face. Hence if $\tilde{\ell}_n$, $\tilde{k}_n$ are the counts of unrooted link or knot diagrams we have that in the limit,

$$\tilde{\ell}_n \underset{n\to\infty}{\sim} \frac{\ell_n}{4(n+2)} \text{ and } \tilde{k}_n \underset{n\to\infty}{\sim} \frac{k_n}{4(n+2)}.$$

Corollary 11. *A random knot or link diagram has the pattern theorem. Namely, a random knot diagram is almost surely composite and almost surely knotted, and a random link diagram is almost surely not a knot diagram.*

3. Some numerical results

[[Blurb about pattern appearances; counts of monogons in diagrams]]

One may be concerned that the "asymptotic" behavior proved in the prior section only applies to knot diagrams with an absurd number of crossings (in the sense that no physical knot should be expected to be so complicated). However, exact and numerical results show that this behavior is attained very quickly. For example, almost all 10-crossing knot diagrams have no nontrivial automorphisms!

References

[1] D. W. Sumners and S. G. Whittington, "Knots in self-avoiding walks," *J. Phys. A: Math. Gen.*, vol. 21, no. 7, pp. 1689–1694, Apr. 1988. DOI: 10.1088/0305-4470/21/7/030. [Online]. Available: http://dx.doi.org/10.1088/0305-4470/21/7/030.

[2] J. Cantarella, A. Y. Grosberg, R. Kusner, and C. Shonkwiler, "The expected total curvature of random polygons," *American Journal of Mathematics*, vol. 137, no. 2, pp. 411–438, 2015. DOI: 10.1353/ajm.2015.0015. [Online]. Available: http://dx.doi.org/10.1353/ajm.2015.0015.

[3] J. Cantarella, T. Deguchi, and C. Shonkwiler, "Probability theory of random polygons from the quaternionic viewpoint," *Communications on Pure and Applied Mathematics*, vol. 67, no. 10, pp. 1658–1699, Aug. 2013. DOI: 10.1002/cpa.21480. [Online]. Available: http://dx.doi.org/10.1002/cpa.21480.

[4] C. Even-Zohar, J. Hass, N. Linial, and T. Nowik, "Invariants of random knots and links," Dec. 2014. arXiv: 1411.3308v2 [math.GT]. [Online]. Available: http://arxiv.org/abs/1411.3308v2.

[5] J. Cantarella, H. Chapman, and M. Mastin, "An enumeration of knot diagrams," in preparation, 2015.

[6] Y. Diao, C. Ernst, and U. Ziegler, "Generating large random knot projections," in *Physical and Numerical Models in Knot Theory*. World Scientific, 2012, ch. 23, pp. 473–494. DOI: 10.1142/9789812703460_0023. [Online]. Available: http://www.worldscientific.com/doi/abs/10.1142/9789812703460_0023.

[7] G. Schaeffer and P. Zinn-Justin, "On the asymptotic number of plane curves and alternating knots," *Experiment. Math.*, vol. 13, no. 4, pp. 483–493, 2004, ISSN: 1058-6458.

[8] R. Chapman, *Surveys in Combinatorics 2011*, ser. London Mathematical Society Lecture Note Series. Cambridge University Press, 2011, ISBN: 9781139503686. [Online]. Available: https://books.google.com/books?id=98-l4J34SFYC.

[9] M. Albenque, É. Fusy, and D. Poulalhon, "On symmetric quadrangulations and triangulations," *Eur. J. Comb.*, vol. 35, pp. 13–31, Jan. 2014, ISSN: 0195-6698. DOI: 10.1016/j.ejc.2013.06.031. [Online]. Available: http://dx.doi.org/10.1016/j.ejc.2013.06.031.

[10] H. L. Frisch and E. Wasserman, "Chemical topology 1," *J. Am. Chem. Soc.*, vol. 83, no. 18, pp. 3789–3795, Sep. 1961. DOI: 10.1021/ja01479a015. [Online]. Available: http://dx.doi.org/10.1021/ja01479a015.

[11] M. Delbrück, "Knotting problems in biology," in *Mathematical Problems in the Biological Sciences*, R. Bellman, Ed. American Mathematical Society, 1962, ch. 5, pp. 55–63. DOI: 10.1090/psapm/014. [Online]. Available: http://dx.doi.org/10.1090/psapm/014.

[12] H. Kesten, "On the number of self-avoiding walks. II," *J. Math. Phys.*, vol. 5, no. 8, p. 1128, 1964. DOI: 10.1063/1.1704216. [Online]. Available: http://dx.doi.org/10.1063/1.1704216.

[13] ——, "On the number of self-avoiding walks," *J. Math. Phys.*, vol. 4, no. 7, p. 960, 1963. DOI: 10.1063/1.1704022. [Online]. Available: http://dx.doi.org/10.1063/1.1704022.

[14] E. A. Bender, Z.-C. Gao, and L. Richmond, "Submaps of maps. i. general 0-1 laws," *Journal of Combinatorial Theory, Series B*, vol. 55, no. 1, pp. 104–117, 1992, ISSN: 0095-8956. DOI: http://dx.doi.org/10.1016/0095-8956(92)90034-U. [Online]. Available: http://www.sciencedirect.com/science/article/pii/009589569290034U.

[15] L. Richmond and N. Wormald, "Almost all maps are asymmetric," *Journal of Combinatorial Theory, Series B*, vol. 63, no. 1, pp. 1–7, 1995, ISSN: 0095-8956. DOI: http://dx.doi.org/10.1006/jctb.1995.1001. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S0095895685710015.

[16] R. Coquereaux and J.-B. Zuber, "Maps, immersions and permutations," *ArXiv e-prints*, Jul. 2015. arXiv: `1507.03163 [math.CO]`.

[17] G. Schaeffer and P. Zinn-Justin, "On the asymptotic number of plane curves and alternating knots," *Experiment. Math.*, vol. 13, no. 4, pp. 483–493, 2004, ISSN: 1058-6458.

E-mail address: `hchapman@math.uga.edu`

Department of Mathematics, University of Georgia, Athens GA