Isomorphisms of Multigraphs in Terms of Isomorphisms of Colored Simple Graphs

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Keywords:

Definition 1. A **multigraph** G is a triple $G = (V, E, f)$ where V is a finite set of *vertices*, E is a finite set of *edges*, and f is a map from E to the power set of V, $\mathcal{P}(V)$, so that $f(e)$ is a subset of size either 1 or 2. The vertices in $f(e)$ are called the **endpoints** of e. Edges with $|f(e)| = 1$ are called **loops**.

An **isomorphism** from a graph G_1 to a graph G_2 is a pair of bijections (ϕ_V, ϕ_E) with $\phi_V : V_1 \to V_2$ and $\phi_E : E_1 \to E_2$ such that $(\phi_V \circ f_1)(e) = (f_2 \circ \phi_E)(e)$ (as sets) for all $e \in E_1$.

A **coloring on the vertices** of a graph is a map, C_V, from V to a set of colors X. An isomorphism (ϕ_V, ϕ_E) from a vertex colored graph G_1 to a vertex colored graph G_2 **respects the vertex coloring** if, and only if, $C_{V_1}(v) = (C_{V_2} \circ \phi_V)(v)$ for all $v \in V_1$.

A **coloring on the edges** of a graph is a map, C_E, from E to a set of colors X. An isomorphism (ϕ_V, ϕ_E) from an edge colored graph G_1 to an edge colored graph G_2 **respects the edge coloring** if, and only if, $C_{E_1}(e) = (C_{E_2} \circ \phi_E)(e)$ for all $e \in E_1$.

Definition 2. Given a multigraph $G = (V, E, f)$ we define an associated graph $\bar{G} = (\bar{V}, \bar{E}, \bar{f})$ by the following construction.

- The vertices of $\bar{G}$ are the vertices of G, in other words, $\bar{V} = V$.
- The edges of $\bar{G}$ come from collapsing non-loop edges that have the same endpoints. We do this by defining the edges of $\bar{G}$ to be a set of preimages of f. In particular, set

$$\bar{E} = \{f^{-1}(f(v)) : |f(v)| = 2\}.$$

- $\bar{f}(\bar{e}) = f(e)$ where e is any element of $\bar{e}$. Note that this is well-defined as all edges in the set $\bar{e}$ have the same endpoints by construction.

We also define a coloring on the vertices and edges of $\bar{G}$ as follows.

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- $C_{\bar{V}}(\bar{v}) =$ the number of edges $e \in E$ such that $f(e) = \{v\}$.
- $C_{\bar{E}}(\bar{e}) = |\bar{e}|$.

Lemma 3. *Given any graph G the associated graph $\bar{G}$ is simple.*

Proof. An edge in G is a loop precisely when $|f(v)| = 1$ which is excluded in the definition of $\bar{G}$. Similarly, duplicate edges are not allowed in $\bar{E}$ because it is defined to be the *set* of pre-images. Thus, $\bar{G}$ has no loops or multiedges and is therefore simple. □

Since $\bar{G} = (\bar{V}, \bar{E}, \bar{f})$ is always simple it can be described by only its set of vertices $\bar{V}$ and a collection of two element subsets of $\bar{V}$ giving the edges. In this case, a bijection $\bar{\phi}_V : \bar{V}_1 \to \bar{V}_2$ induces a map on the edge sets by $\bar{\phi}_E(\bar{e}) = \{\bar{\phi}_V(\bar{v}), \bar{\phi}_V(\bar{w})\}$ where $\bar{v}$ and $\bar{w}$ are the endpoints of $\bar{e}$. Thus, an isomorphism between simple graphs can be described by only a bijection on the vertex sets.

The following theorem says that there is a map from the set of isomorphisms $\bar{G}_1 \to \bar{G}_2$ to the power set of the isomorphisms $G_1 \to G_2$ whose image is a partition.

Theorem 4. *Let G_1 and G_2 be multigraphs and $\bar{G}_1$ and $\bar{G}_2$ the associated colored graphs given in Definition 2. Then, given an isomorphism $\bar{\phi}$ from $\bar{G}_1$ to $\bar{G}_2$ that respects the vertex and edge coloring, we can construct a set of isomorphisms $\Phi(\bar{\phi}) = \{(\phi_V^1, \phi_E^1), \ldots, (\phi_V^k, \phi_E^k)\}$ from G_1 to G_2. In addition, $\Phi(\bar{\phi}_1)$ and $\Phi(\bar{\phi}_2)$ are disjoint if, and only if, $\bar{\phi}_1$ and $\bar{\phi}_2$ are distinct isomorphisms. Moreover, if (ϕ_V, ϕ_E) is an isomorphism from G_1 to G_2, then there exists an isomorphism $\bar{\phi}$ from $\bar{G}_1$ to $\bar{G}_2$ such that (ϕ_V, ϕ_E) will be produced by this construction.*

Proof. We will first describe the construction and then prove that it satisfies the properties in the theorem.

IDEA: $\phi_V^i = \bar{\phi}$ for all i and the ϕ_E^i come from all choices of bijections between the *sets* labeling the edges of $\bar{G}_1$ and $\bar{G}_2$.

□