On the asymptotics of uniformly random knot diagrams

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I. INTRODUCTION

There is a dearth of models for drawing random knots; self avoiding lattice walks [cite], random space polygons [cite], random braid words [cite], *Petaluma* [cite], et. al. In this paper we will discuss the *random diagram model* under which *knot diagrams* are drawn uniformly from the set of all diagrams with a given number of crossings. Alternating knot and link diagrams have been studied [cite] but little is published about the knottiness of arbitrary random diagrams of large size.

In this paper we begin by considering a slightly different object, *rooted diagrams*, which break symmetries (as opposed to in [1]). We are then able to prove that in the limit, knot diagrams behave similarly to rooted diagrams, so that these results carry over.

II. DEFINITIONS

II.1. Preliminaries

A *knot* is an isotopy class of embeddings of the circle into S^3. A *link* is an isotopy class of embeddings of one or more circles into S^3. Both of the prior are considered up to *ambient isotopy* of the embedded circles. The study of links and knots is well known to be equivalent to the study of *link diagrams* and *knot diagrams* (more formally defined below) up to the so-called *Reidemeister moves*. We wish to examine the underlying *planar map* structure of knot and link diagrams;

Definition 1. *A* planar map with n vertices P *is a graph* $\tilde{P}$ *embedded in the sphere. The planar map* P *is* 4-regular *or* quartic *if every vertex in the underlying graph* $\tilde{P}$ *has degree 4. A* rooted planar map *is a planar map together with a single edge marked with a direction.*

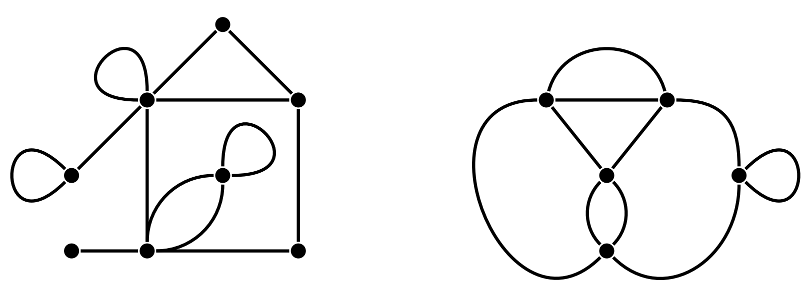

FIG. 1. Two planar maps. The map on the right is in the class of knot shadows.

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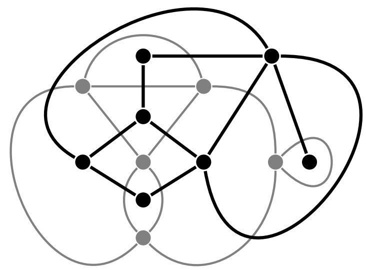

FIG. 2. Planar quadrangulation which is dual to a knot shadow.

Planar maps (indeed, maps on any surface) have a well defined notion of the *dual map*; a map $M = (V, E, F)$ has dual $M^* = (F, E^*, V)$, where there is an edge $(f_1, f_2) \in E^*$ if f_1 is adjacent to f_2 in M. The dual graph of a 4-regular planar map is a *planar quadrangulation*.

II.2. Diagrams and shadows

By breaking symmetries with a root, we may study certain classes of planar maps by way of the celebrated bijection with *blossom trees*[2]. We carry this idea to link and knot diagrams:

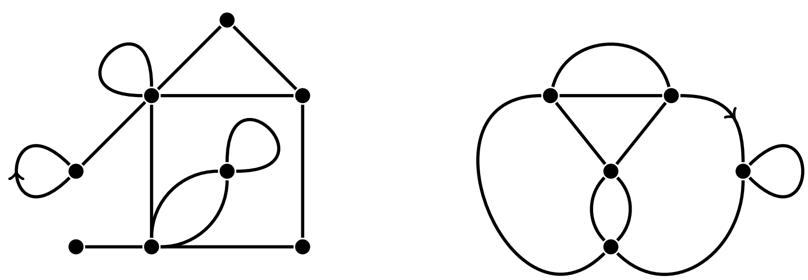

FIG. 3. Two rooted planar maps. The map on the right is in the class of rooted knot shadows.

Definition 2. *A* (rooted) link diagram with n crossings *is a 4-regular (rooted) planar map of n vertices together with a choice of over-under strand information at each vertex. The class of rooted link diagrams with n crossings is denoted $\mathcal{L}_n$.*

The class of maps which represent rooted link shadows *in n crossings, i.e. maps which can be found as the underlying map structure of a rooted link diagram are denoted $\mathscr{L}_n$.*

It is well understood that the class of rooted link shadows in n-vertices is identical to the class of 4-regular planar maps in n vertices is identical to $\mathscr{L}_n$; furthermore, the class of rooted planar quadrangulations is dual to $\mathscr{L}_n$.

Definition 3. *A* (rooted) knot diagram *is a (rooted) link diagram which consists of only one knot component. The class of rooted knot diagrams with n crossings is denoted by $\mathcal{K}_n$.*

The class of maps which represent rooted knot shadows *in n crossings are denoted $\mathscr{K}_n$.*

Knot shadows $\mathscr{K}_n$ represent a curious, small subclass of $\mathscr{L}_n$.

Rooted (knot or link) diagrams are equivalently viewed as *two-leg diagrams* or *2-tangles* as illustrated below.

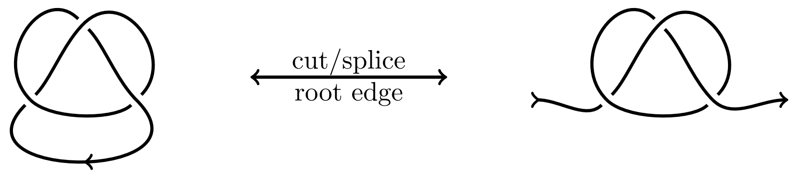

FIG. 4. Rooted diagrams of the trefoil and its mirror image

Additionally, rooted diagrams can be viewed as *four-leg diagrams*, or 4-*tangles* by deleting the root crossing. This identification is not injective for diagrams as it forgets the sign of the removed crossing.

A (rooted) shadow is *prime* if it cannot be disconnected by removing two edges (i.e. it is at least 3-connected); otherwise it is *composite*. A rooted shadow is *two-leg-prime* if it cannot be disconnected by removing two edges, one being the root edge. Diagrams are (two-leg-)prime if their underlying shadow structure is.

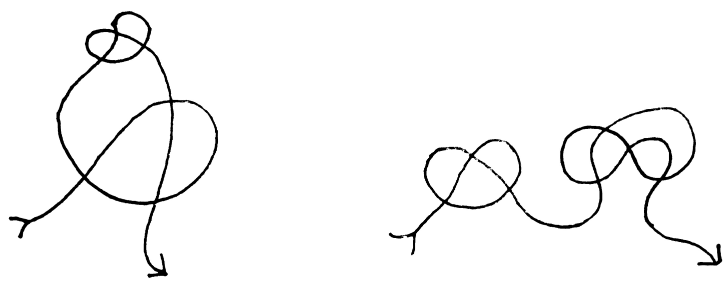

FIG. 5. A composite shadow which is two-leg-prime (left). A shadow which is not two-leg-prime (right).

There is a bijection between blossom trees and rooted link shadows.

Proposition 1. *There is a consistent way to order the components of a rooted link shadow. There is a consistent way to index the vertices of a rooted link shadow. There is a consistent way to index and orient the edges of a rooted link shadow so that the directed edges meet head-to-tail across the vertices of the shadow.*

Proof. Let L be a rooted link shadow. Begin by labelling the edges of L with the link component in which they lie. Index the root vertex and the root edge by 1. □

Corollary 2. *There is a consistent way to order and orient the components of a rooted link diagram. There is a consistent way to index the crossings of a rooted link diagram. There is a consistent way to index the edges of a rooted link diagram. These are the same as those for the underlying shadow.*

Proof. These are all induced on the diagram from its shadow. □

Corollary 3. *Rooted link diagrams are in bijection with rooted link shadows labelled with* $\{+, -\}$.

Proof. Given a rooted link diagram, there is a consistent orientation of its components. There is hence a labelling of the underlying rooted shadow with $\{+, -\}$ (they are the standard crossing signs). This process is reversible since the consistent component orientation for the diagram is identical to that of the shadow. □

III. ASYMPTOTIC STRUCTURE THEOREMS FOR DIAGRAMS

It is believed and numerically evident [3] that the number of link diagrams in a random link diagram grows exponentially, hence a random link diagram is almost certainly not a knot diagram.

III.1. A pattern theorem for classes of link diagrams

III.1.1. A pattern theorem

Theorem 2 in [4] provides a pattern theorem for knot and link shadows, provided a strategy of attaching a desired pattern. However, care is required in the case of knot or link *diagrams*, in which each vertex takes a value in the set $\{+,-\}$. In fact, we turn our attention to the dual case in which *faces* are labelled with an arbitrary set.

Definition 4. *Given a triple of label sets $S = (S_V, S_E, S_F)$, an* S-map *is a map $M = (V, E, F)$ together with a triple of maps (s_V, s_E, s_F), $s_* : * \to S_*$ which label each vertex, edge, and face of M with an element of S.*

Theorem 4. *Let S be a set and $\mathscr{M}[S]$ be some class of S-maps on a surface of type g and let P be a planar S-map with boundary that can be found as a submap of maps in $\mathscr{M}[S]$. Let $M(x)$ be the generating function by number of edges for $\mathscr{M}$. Let $H(x)$ be the generating function by number of edges for those maps M in $\mathscr{M}[S]$ that contain less than $ce(M)$ pairwise disjoint copies of P. Suppose that we can embed P in a possibly larger rooted planar S-map with boundary Q and attach copies of Q to each map K counted by $H(x)$ in such a way that*

1. *for some fixed positive integer k, at least $\lfloor e(K)/k \rfloor$ possible non-conflicting places of attachment exist,*
2. *only S-maps in $\mathscr{M}[S]$ are produced,*
3. *for any map produced as such we can identify the copies of Q that have been added and they are all pairwise disjoint, and*
4. *given the copies that have been added, the original map and associated places of attachment are uniquely determined.*

If $c > 0$ is sufficiently small, then $r(M) < r(H)$. The maps may be rooted or not.

The method of attachment is vague, but flexible. We will provide some examples which we use in our results for knot diagrams. The proof extends the proof of the original theorem for maps, and makes use of a lemma:

Lemma ([4], lemma 3)**.** *If*

1. *$F(z) \neq 0$ is a polynomial with non-negative coefficients and $F(0) = 0$,*
2. *$H(w)$ has a power series expansion with non-negative coefficients and $0 < r(H) < \infty$,*
3. *for some positive integer k the linear operator $\mathscr{L}$ is given by $\mathscr{L}(w^n) = z^n (F(z)/z)^{\lfloor n/k \rfloor}$, and*
4. *$G(z) = \mathscr{L}(H(w))$,*

then $r(H)^k = r(G)^{k-1}F(r(G))$.

The proof of the theorem then remains almost unchanged from the original theorem, although care will be necessary in defining attachment.

Proof of theorem 4. Let $G(z)$ be the generating function which counts S-maps $\mathscr{G}[S]$ which are the result of attaching some number between 0 and $\lfloor n/k \rfloor$ copies of Q to S-face maps $\mathscr{H}[S]$ counted by $H(x)$. The method of attachment leads to the relation $G(z) = \mathscr{L}(H(w))$, where $F(z) = z + z^q$ and q is the number of edges added when a copy of Q is attached, as

$$G(z) = \sum_{X \in \mathscr{G}[S]} z^{e(X)} = \sum_{Y \in \mathscr{H}[S]} z^{e(Y)} \left(1 + z^{q-1}\right)^{\lfloor n/k \rfloor} = \mathscr{L}(H(w)).$$

Let g_n be the coefficients of $G(z)$.

Suppose $M \in \mathscr{M}[S]$ contains m copies of Q. By property (3) of our attachment, $m \leq n$. If M had been produced from some S-map K in $\mathscr{H}[S]$ by our attachment process, we can find all possible K by removing at least $m - cn$ copies of Q from M. It is possible to bound from above the number of ways to do this by

$$\sum_{j \geq m-cn} \binom{m}{j} = \sum_{k<cn} \binom{m}{k} < \sum_{k<cn} \binom{n}{k} \leq n \binom{n}{cn} \leq \frac{n(ne)^{cn}}{cn^{cn}} = n \left(\frac{e}{c}\right)^{cn} =: t_n.$$

If $M(x) = \sum m_n x^n$, then $m_n \geq g_n$ and $t_n > 1$ for sufficiently large n, so $m_n \geq g_n/t_n$. Hence,

$$1/r(M) \geq \limsup_{n\to\infty} (g_n/t_n)^{1/n} = \lim_{n\to\infty} (t_n)^{-1/n} \limsup_{n\to\infty} (g_n)^{1/n} \geq (c/e)^c / r(G).$$

By the prior lemma, $r(H)^k = r(G)^k(1 + r(G)^{q-1})$ so that

$$r(H)/r(M) \ \geq (1 + r(G)^{q-1})^{1/k}(c/e)^c.$$

As $\lim_{c\to 0^+} (c/e)^c = 1$ and $r(G)^k(1 + r(G)^{q-1}) = r(H)^k \geq 1/12^k$, it follows that $r(H)/r(M) > 1$ for sufficiently small c, completing the proof of the theorem. □

The conclusion is about radii of convergence of two power series, and may appear an esoteric result. However, application of the Cauchy-Hadamard theorem, together with one additional hypothesis, gives a more familiar tune:

Corollary 5 ([4])**.** *Suppose all of the hypotheses of theorem 4 and additionally that $\mathscr{M}[S]$ grows smoothly, i.e. that $\lim_{n\to\infty} m_n^{1/n}$ exists. Then there exists constants $c > 0$ and $d < 1$ and $N > 0$ so that for all $n \geq N$,*

$$\frac{h_n}{m_n} < d^n.$$

I.e., the pattern P is ubiquitous.

Because of Euler's formula, the number of vertices, edges, or faces in a link shadow or planar quadrangulation is entirely determined by choosing any one cardinality. Hence, we can size shadows by the number of vertices and still keep the above results.

The crux of applying this theorem to link diagrams then falls upon determining an "attachment" operation which satisfies the hypotheses, along with patterns valid for a given class of shadows. We can generally define attachment operations for different kinds of tangles in the dual. By abuse of notation, let $S = (\{0\}, \{0\}, S)$ be a set of labels for the faces of the dual (for now, we are concerned about diagrams, which only have labelled vertices).

1. **Connect sum.** Let L and Q' be rooted S-quadrangulations. Orient the remaining edges of L canonically by proposition 1. Define the *connect sum* of Q' into L at an edge $e \in L$, $L\#_e Q'$, by

   (a) Cut and split the edge e, creating a map L' and leaving a distinguished, oriented bigon f. Denote the two edges formed by splitting e by e_1, e_2, so that the loop $e_1(-e_2)$ is a counterclockwise cycle around f. If e was the root of L, make e_1 the new root of L'.

   (b) Cut and split the root edge ϵ of Q', creating a map Q and leaving a distinguished, oriented bigon g. Denote the two edges formed by splitting ϵ by ϵ_1, ϵ_2, so that the loop $\epsilon_1(-\epsilon_2)$ is a counterclockwise cycle around g. Make ϵ_2 the new root of Q.

   (c) Glue the map Q into the map L''s distinguished face f along the boundary of the distinguished bigon g so that e_1 and ϵ_2 are mapped to the same edge and so that the orientations of the boundaries align.

   (d) Forget about all edge orientations except for the root edge of L.

   Notice that none of the original faces in L and Q' are changed; hence the result is a new rooted S-quadrangulation. Any given S-quadrangulation in $2n$ edges has precisely $2n$ different non-conflicting sites for connect summation (i.e. $k = 1$ for this attachment operation). This process is reversible, given a 2-cycle which bounds an instance of Q' (collapse the disk to a single edge). If L^* and $(Q')^*$ each consist of only one link component, then

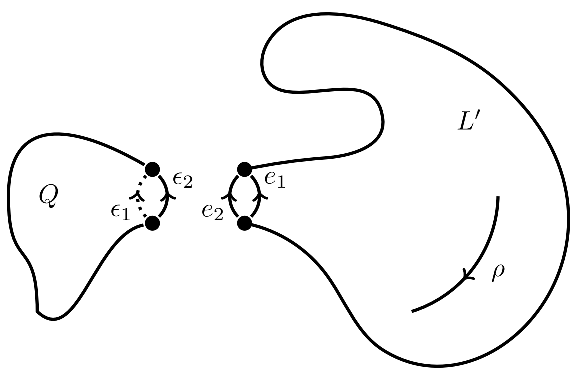

FIG. 6. The connect sum operation. Q' and L' are viewed as CW-complexes, and their boundaries are appropriately identified.

   $(L\#_e Q')^* =: L^*\#_e(Q')^*$ will as well. In fact, this attachment into the quadrangulation is precisely dual to the usual link connect sum from knot theory.

2. **4-tangle twist replacement.** Let L be a rooted S-quadrangulation, and Q a rooted S-quadrangulation with square boundary. Then given a face $f \in L$, we can define the *4-tangle twist replacement*, $L\&_f Q$ by,

   (a) Canonically index the vertices of L via proposition 1 and let the vertex of lowest index around f be v_1. Name the remaining edges and vertices from a walk around the face f clockwise from v_1 by $v_1, e_1, v_2, e_2, v_3, e_3, v_4, e_4$ (an edge or vertex may have more than one name under this scheme).

   (b) Divide face f into two new faces f_1, f_2 with induced S-labelling from f by inserting a new vertex o in the center of f and adding two new edges; a from o to v_1 between e_4

and e_1 and b from o to v_3 between e_2 and e_3. Let f_1 be the face adjacent to the vertex v_2 and f_2 be the face adjacent to v_4.

(c) Cut along edges a, b and open to create a new "exterior" face x. Let a_1, b_1, o_1 be the edges and vertex adjacent to f_1 and a_2, b_2, o_2 be the edges and vertex adjacent to f_2. Orient the edge a_1 to point from o_1 to v_1; the edge will point counter-clockwise around the face x.

(d) Glue Q into the face x by gluing the root of Q to a_1 with directions in agreement, and the rest of the edges to a_2, b_2, b_1.

The result will be a new S-quadrangulation. Any S-quadrangulation in $2n$ edges has n faces, and hence n attachment locations (i.e., $k = 2$). Observe that after this attachment, vertex v_1 still has the lowest index out of $\{v_1, v_2, v_3, v_4\}$ and for any face f in L, its lowest-index vertex v remains fixed. This process is reversible ?????; given a quadrangle $v_1e_1v_2e_2v_3e_3v_4e_4$ (where v_1 has the lowest canonical index) with interior diamond $v_1a_1o_1b_1v_3b_2o_2a_2$ bounding Q as a submap and faces $f_1 = v_1e_1v_2e_2v_3a_1o_1b_1$, $f_2 = v_1a_2o_2b_2v_3e_3v_4e_4$ with the same label s, remove the interior diamond and join f_1 and f_2 into a face f with label s;

We can then consider the following applications of this theorem to classes of link diagrams:

1. Let $\mathscr{K}$ be the class of all rooted knot shadows (so that $\mathscr{K}^*$ is the class of quadrangulations which are dual to knot shadows), and P be the dual of a prime 2-tangle shadow of only one component. Define the attachment operation of P into $\mathscr{K}^*$ as follows. Let Q be P with a rooted edge and embed P identically, and take the connect sum attachment operation. Since P was chosen to have one component, only maps in K^* are created (the attachment operation is just the connect sum).

2. Again consider $\mathscr{K}^*$, but let P be the dual of a prime 4-tangle shadow of one component with *alternating* loose strands and let attachment be 4-tangle replacement.

3. If we consider a tangle P^* which consists of more than one component, we obtain another proof that link diagrams almost surely have more than one component.

4. Consider again connect summation. Given a pattern P^* representing a $2k$-tangle, we can obtain a rooted two-leg-prime diagram Q^* as follows. Connect $2(k-1)$ of the loose tangle strands into pairs iteratively if the two strands correspond to different components of the tangle. It is possible that many new crossings are introduced in this step, but the resultant 2-tangle will have only one component. This proves the pattern theorem for arbitrary $2k$-tangles.

5. Consider the class of reduced knot shadows (i.e. those with no isthmi). Then gluing an alternating, reduced 4-tangle of one component keeps us in.

III.1.2. Strategy for proving smooth growth

The theorem in the prior section by itself does not sufficiently prove *ubiquity* as required to prove asymmetry. One may worry about bad cases; e.g., one in which . Indeed, we require that the class of maps *grow smoothly*, i.e. that (for $m_n = |\mathscr{M}_n|$) the limit

$$\lim_{n\to\infty} m_n^{1/n}$$

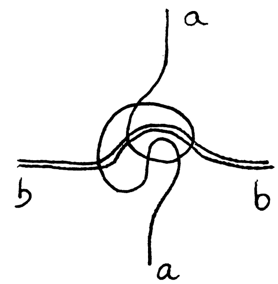

FIG. 7. A 4-tangle with alternating strands

exists.

Bender, et al. [4] give a powerful proof strategy for proving smooth growth of a sequence. We adapt that to prove the following theorem.

Theorem 6. *Let $\mathscr{C}$ be a class of combinatorial objects with generating function $\sum_{n=0}^{\infty} c_n\ z^n$; let the radius of convergence of the OGF be r, and $\mathscr{D}$ some other class with generating function $\sum_{n=0}^{\infty} d_n\ z^n$. Suppose that $0 > r \leq 1$ and let $C_i > 0$ and $1 - r > \delta > 0$ be arbitrary.*

Suppose there is a product operation $\times$ on elements $A, B \in \mathscr{C} \cup \mathscr{D}$ so that,

1. *$A \times B \in \mathscr{C}$,*
2. *there exists some fixed $k \in \mathbb{Z}_{\geq 0}$ so that $|A \times B| = |A| + |B| + k$, and*
3. *given any $C \in \mathscr{C}$, there is at most one maximal factorization $D_1 \times D_2 \times \cdots \times D_s = C$ with $D_i \in \mathscr{D}$.*

Suppose there exists $R \geq 0$ so that for $n \geq R$ there exists $\ell \in \mathbb{Z}_{\geq 0}$ and maps $\psi_0 : \mathscr{C}_n \hookrightarrow \mathscr{D}_{n+\ell}$ and $\psi_1 : \mathscr{C}_n \hookrightarrow \mathscr{D}_{n+\ell+1}$. Then the limit

$$\lim_{n\to\infty} c_n^{1/n}$$

exists.

It is known that there are at most 12^n planar maps, and so in our cases we will always have $r \geq 1/12$.

Proof. The proof breaks down into 3 steps;

1. *Show that there exists some $n \geq 0$ with $c_n > C_1(r+\delta)^{-n}$.* This step follows from the Cauchy-Hadamard theorem, which says that

$$\limsup_{n\to\infty} c_n^{1/n} = r^{-1}.$$

By the definition of lim sup, we have that if $a < r^{-1}$, then for any $M \geq R$ we have that there is some $n \geq M$ with $c_n^{1/n} > a$. For instance, we know that $(r + \delta/2)^{-1} < r^{-1}$, hence for any M we have some $n \geq M$ with $c_n > (r + \delta/2)^{-n}$. Notice now that as

$$\left(\frac{r+\delta}{r+\delta/2}\right) > 1,$$

there must be some $M \geq R$ so that for all $m \geq M$

$$\left(\frac{r+\delta}{r+\delta/2}\right) > C_1^{1/m}, \text{ implying that } (r+\delta/2)^{-m} > C_1(r+\delta)^{-m},$$

whence we then have (by lim sup) some $n \geq M$ with $c_n > (r+\delta/2)^{-n} > C_1(r+\delta)^{-n}$.

2. *Show that there exists some $m \geq 0$ with $d_m > C_2(r+\delta)^{-m}$ and $d_{m+1} > C_2(r+\delta)^{-(m+1)}$.* Notice that $(r+\delta) < 1$ and so for any $m \geq 0$, $(r+\delta)^{-m} < (r+\delta)^{-(m+1)}$. As there exist injections ψ_0, ψ_1 from $\mathscr{C}_n$ into $\mathscr{D}_{n+\ell}$ and $\mathscr{D}_{n+\ell+1}$, setting $m = n+\ell$ and $C_1 = C_2(r+\delta)^{-(m+n-1)}$ we have that

$$d_m \geq |\operatorname{im}\psi_0| = c_n > C_1(r+\delta)^{-n} = C_2(r+\delta)^{-(m+1)} > C_2(r+\delta)^{-m}$$

and

$$d_{m+1} \geq |\operatorname{im}\psi_1| = c_n > C_1(r+\delta)^{-n} = C_2(r+\delta)^{-(m+1)}.$$

3. *Show that there exists some N so that for any $n \geq N$, $c_n > (r+\delta)^{-n}$.* Consider k from the hypothesis. Let $C_2 = (r+\delta)^{-k}$. Let $N = (m+k)(m+k+1)$. Then if $n \geq N$, we can write n as a linear combination $a(m+k) + b(m+k+1) = am + b(m+1) + (a+b)k$, with $a, b \geq 0$. Observe that $c_n > d_m^a d_{m+1}^b$ as there exists a subset of objects $S \subset \mathscr{C}_n$ which can be expressed uniquely as a product of a elements of $\mathscr{D}_m$ and b elements of $\mathscr{D}_{m+1}$ (and $|S| > d_m^a d_{m+1}^b$). Then

$$c_n > d_m^a d_{m+1}^b > C_2^a(r+\delta)^{-am} C_2^b (r+\delta)^{-b(m+1)} = (r+\delta)^{-(am+b(m+1)+(a+b)k)} = (r+\delta)^{-n}.$$

To finish the proof we realize that this last step implies that the lim inf is r^{-1} and hence the limit result follows. Observe that $\liminf_{n\to\infty} c_n^{1/n} = r^{-1}$ if for any $\epsilon > 0$, there exists N so that for all $n \geq N$, $c_n^{1/n} > r^{-1} - \epsilon = \frac{1-r\epsilon}{r}$. We may assume that $\epsilon < 1$ since otherwise the inequality is clear since $c_n \geq 0$ always. So we are done if we can choose δ so that

$$\frac{r^2\epsilon}{1-r\epsilon} > \delta,$$

as then we have from our prior result that $c_n^{1/n} > (r+\delta)^{-1} > r^{-1} - \epsilon$. Indeed, we have $r^2\epsilon > 0$ and $1 > 1 - r\epsilon > 0$ so that the left hand side of the inequality is positive; but we may choose $\delta > 0$ as small as we desire. Hence for $N = (m+k)(m+k+1)$, the result that $\lim_{n\to\infty} c_n^{1/n} = r^{-1}$ follows. □

III.1.3. Smooth growth for knot and link diagrams

The class $\mathscr{L}$ of rooted link shadows has been counted exactly. Rooted link shadows are in bijection with rooted 4-regular planar maps and the coefficients of the generating function are known [5]. If $l_n = |\mathscr{L}_n|$, then:

$$l_n = \frac{2(3^n)}{(n+2)(n+1)}\binom{2n}{n}.$$

On the other hand, asymptotics of knot shadows are as of yet unknown. We are still however able to prove that they grow smoothly, as to prove our result of asymptotic asymmetry.

Theorem 7. *The class $\mathscr{K}$ of rooted knot shadows grows smoothly. I.e., the limit* $\lim_{n\to\infty} k_n^{1/n}$ *exists (and is equal to* $1/r(K)$*).*

Proof. As mentioned above, a very loose bound on $r = r(K)$ is $1/12 \leq r \leq 1$ as the number of planar maps in general is bounded by 12^n. Let $C_i > 0$ and $1 - r > \delta > 0$ be arbitrary. We need to define a composition $\times$ and subclass $\mathscr{D}$ of shadows which are prime under $\times$.

Define the concatenation $K = K_1 \times K_2$ on shadows in $\mathscr{K}$ by gluing the front leg of K_1 to the hind leg of K_2. Hence, we will take $\mathscr{D}$ to be the class of knot shadows which remain at least

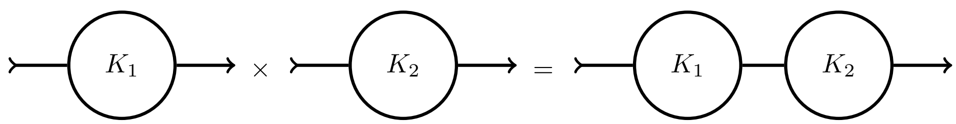

FIG. 8. The two-leg sum operation $\times$. If both A and B are two-leg-prime, then $A \times B$ has exactly one separating edge.

2-connected after removing the root edge.

Certainly $A \times B \in \mathscr{K}$ as we obtain a new 2-leg shadow. As $\times$ introduces no crossings, we have $k = 0$ and $|A \times B| = |A| + |B|$. Finally, a 2-leg shadow K either lies in $\mathscr{D}$ or has $\ell - 1$ disconnecting edges. Cutting these edges produces the disjoint union of ℓ well-ordered 2-leg shadows (well ordered from their position in the long curve K) which is the unique ordered $+$-decomposition of K into elements of $\mathscr{D}$.

Let φ be the map which twists the root edge, making the loop the new root (using the appropriate induced orientation). Then $\varphi : \mathscr{K}_* \hookrightarrow \mathscr{D}_{*+1}$, since deleting the root and smoothing the pointed edge produces a knot shadow, which must be at least 2-connected. Then we take $\psi_0 = \varphi$

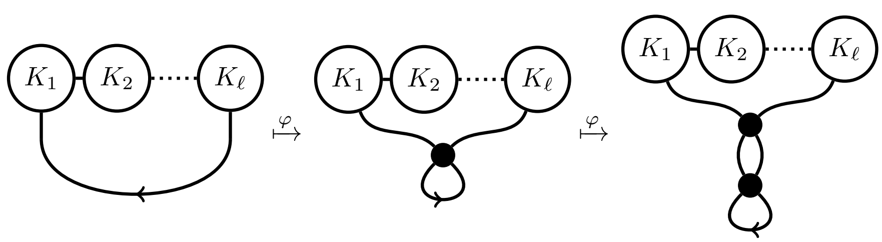

FIG. 9. The map φ adds a vertex and ensures that the new map is 2-leg-prime.

and $\psi_1 = \varphi^2$. This setup satisfies the hypotheses and hence proves the theorem. $\square$

Corollary 8. *There exists $N \geq 0$ and a constant $d < 1$ so that for $n \geq N$,*

$$\mathbb{P}(\textit{a knot diagram } K \textit{ is an unknot}) < d^n.$$

For any prime 2-tangle P, there exists $N \geq 0$ and constants $d < 1$, $c > 0$ so that for $n \geq N$,

$$\mathbb{P}(\textit{a knot diagram } K \textit{ contains} \leq cn \textit{ copies of } P \textit{ as connect summands}) < d^n.$$

Proof. The first statement will follow immediately from the second.

Let k_n count the number of knot shadows with n vertices and h_n count the number of knot shadows in n vertices which contain $\leq c'n$ copies of $\tilde{P}$, where $c' > 0, d < 1, N$ are chosen by the corollary so that for all $n \geq N$, $h_n/k_n < d^n$. Then the number of rooted knot diagrams in n crossings is 2^n, and the number of rooted knot diagrams that contain $\leq c'n$ copies of P is $2^{n-c'n}$ (there are two choices of orientation and 2^n choices of crossing sign, but $c'n$ of those are fixed if P) □

III.1.4. Smooth growth for prime knot and link diagrams

If, however, we are considering a class $\mathscr{P}$ of prime or reduced rooted diagrams, the method of proof for smoothness does not immediately carry over; it is possible that φ introduces numerous isthmi, in which case our diagrams created in the final step would not even be reduced. If $\mathscr{P}$ is the case of prime rooted link shadows exact counts are known from their bijection with simple quadrangulations [6];

$$s_n = \frac{4(3n)!}{n!(2n+2)!}.$$

In other cases again smoothness is more complicated to prove, although we can use a similar argument to that in the case of all knot shadows.

Proof. Step i is again immediate, so we begin with step ii. Let ψ, ψ' respectively be maps which take the root vertex to the two 4-tangle shadows:

Observe that neither ψ nor ψ' remove primeness or reducedness. Their images provide an injection into the spaces with 2 and 3 additional crossings, respectively. So take m appropriately.

Define the operation $+$ now by the detour-glom. Notice that primeness is preserved and the process is splittable; given the root edge we can identify the bendy edges and rebuild the old two shadows. Notice that $|A+B| = |A|+|B|+4$. Now let $C_3 \geq 1$ and $C_2 = C_3(r+\delta)^{-4}$. Then if there exist nonnegative integers a, b such that $n = am + b(m+1) + (a+b)4 = a(m+4) + b(m+5)$, i.e. if $n \geq (m+4)(m+5)$, then

$$p_n > p_{m+4}^a p_{m+5}^b > C_2^{a+b}(r+\delta)^{-(am+b(m+1))} > C_3^{a+b}(r+\delta)^{-(a(m+4)+b(m+5))} > C_3(r+\delta)^{-n}.$$

□

III.2. Asymmetry of diagrams and consequences

The following theorem of Richmond and Wormald [7] provides a sufficient set of criteria for almost all elements of $\mathscr{K}$ to have trivial automorphism group.

Theorem 9 (Richmond-Wormald 1996). *Let $\mathscr{C}$ be a class of rooted maps on a surface. Suppose that there is an outer-cyclic rooted planar map M_1 such that in all maps in $\mathscr{C}$, all copies of M_1 are pairwise disjoint, and such that*

1. *M_1 has no reflective symmetry in the plane preserving the unbounded face,*
2. *there exist constants $c > 0$ and $d < 1$ such that the proportion of n-vertex maps in $\mathscr{C}$ that do not contain at least cn pairwise disjoint copies of M_1 is at most d^n for n sufficiently large (M is "ubiquitous"), and*

3. *for any map M in $\mathscr{C}$ containing a copy of M_1, all maps obtainable by removing M_1 and gluing it back in to the same face are in $\mathscr{C}$ (M is "free").*

Then the proportion of n-vertex maps in $\mathscr{C}$ with nontrivial automorphisms is exponentially small.

We will prove this for $\mathscr{K}$ by proving it for its dual $\mathscr{K}^*$, a class of quadrangulations of the sphere. Specifically, we will take M_1 to be the dual of the underlying planar map of the following 2-tangle: Clearly M_1 has no reflective symmetry by inspection, and certainly any of the ways of

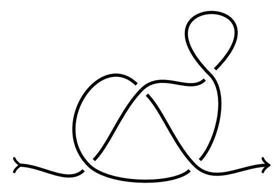

FIG. 10. The dual 2-tangle M_1, and its representation as a 2-tangle.

replacing M_1 keep the object in the class of quadrangulations dual to knot maps. Finally, the ubiquity condition is exactly the pattern theorem for 2-tangles proved in the prior section!

Application of the above theorems provides us with the following corollary which enables us to transfer any asymptotic results on rooted diagrams to unrooted diagrams.

Corollary 10. *Let L be a uniform random variable taking values in the space $\mathcal{K}_n$ or $\mathcal{L}_n$. Then there exist constants $C, \alpha > 0$ so that $\mathbb{P}(\operatorname{aut} L \neq 1) < Ce^{-\alpha n}$. Hence, rooted diagrams behave like unrooted diagrams.*

Indeed, link diagrams with n vertices are dual to quadrangulations with $n+2$ faces; there are $n+2$ ways of choosing the "exterior" root face and then 4 ways of rooting the edges around this chosen face. Hence if $\tilde{\ell}_n$, $\tilde{k}_n$ are the counts of unrooted link or knot diagrams we have that in the limit,

$$\tilde{\ell}_n \underset{n\to\infty}{\sim} \frac{\ell_n}{4(n+2)} \text{ and } \tilde{k}_n \underset{n\to\infty}{\sim} \frac{k_n}{4(n+2)}.$$

Corollary 11. *A random knot or link diagram has the pattern theorem. Namely, a random knot diagram is almost surely composite and almost surely knotted, and a random link diagram is almost surely not a knot diagram.*

IV. NUMERICAL RESULTS ON KNOTTING

One may be concerned that the "asymptotic" behavior proved in the prior section only applies to knot diagrams with an absurd number of crossings (in the sense that no physical knot should be expected to be so complicated). However, exact and numerical results show that this behavior is attained very quickly. For example, almost all 10-crossing knot diagrams have no nontrivial automorphisms!

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