Random Knot Diagrams

Jason Cantarella (UGA)
Harrison Chapman (UGA), Matt Mastin (Mailchimp, Inc.)
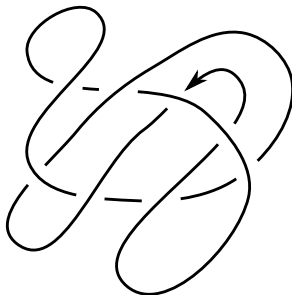Crucial Assist: Eric Rawdon (St. Thomas)

AMS Spring Southeastern Section Meeting, 2016

Natural questions about knot diagrams

Question
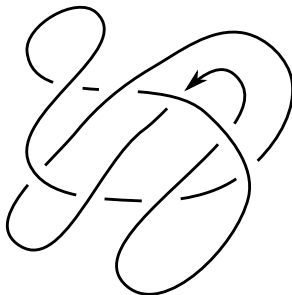
What fraction of 8-crossing diagrams are trefoils?

Natural questions about knot diagrams

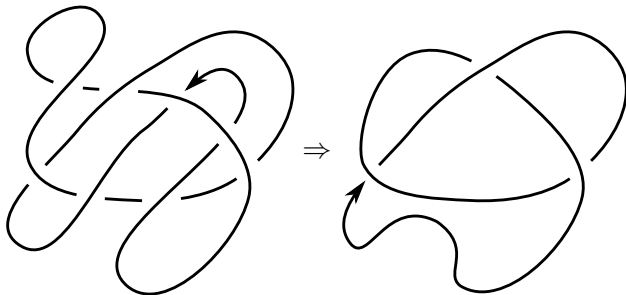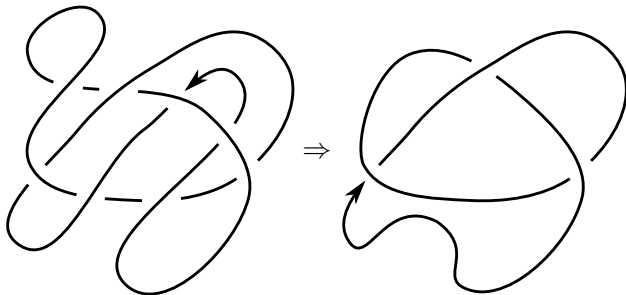### Question
What fraction of 8-crossing diagrams are trefoils?

12.48%

Natural questions about knot diagrams

Question
What is the average minimal crossing # of an 8-crossing diagram?

$\Rightarrow$

Natural questions about knot diagrams
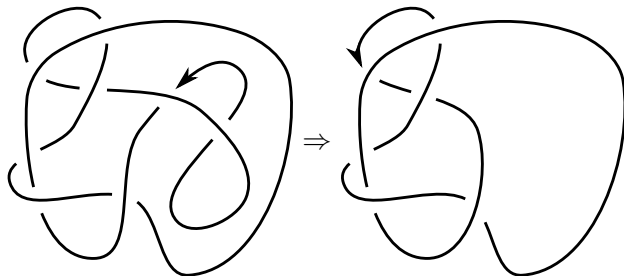
Question

What is the average minimal crossing # of an 8-crossing diagram?

0.52

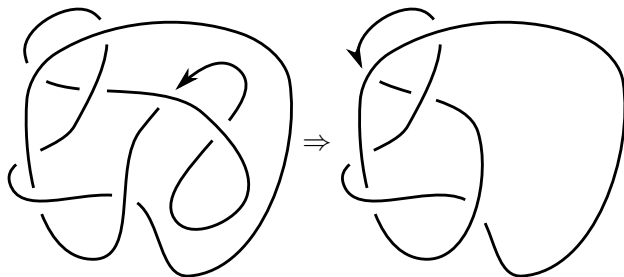# Natural questions about knot diagrams

Definition
The **untwisting** operator deletes all 1-crossing connect summands
of a diagram. (Equivalently, performs all "available" Reidemeister I
moves.)
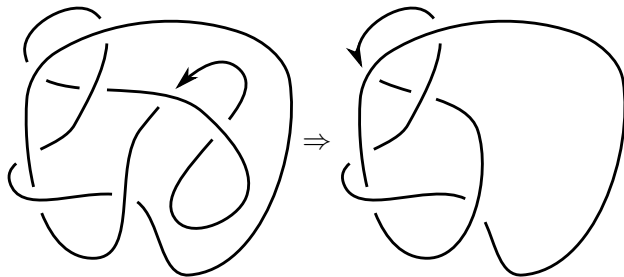
Natural questions about knot diagrams

Question

What is the average crossing # of a untwisted 8-crossing diagram?

Natural questions about knot diagrams
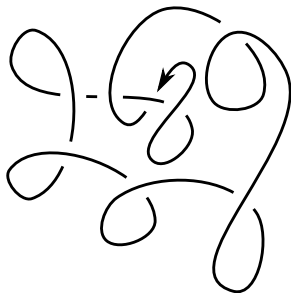
Question
What is the average crossing # of a untwisted 8-crossing diagram?

2.20

Natural questions about knot diagrams

Question

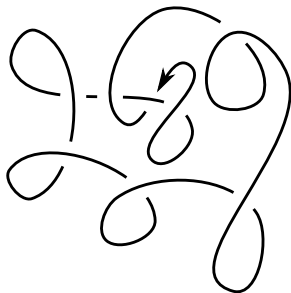How many 8-crossing diagrams can be untwisted to the unknot?
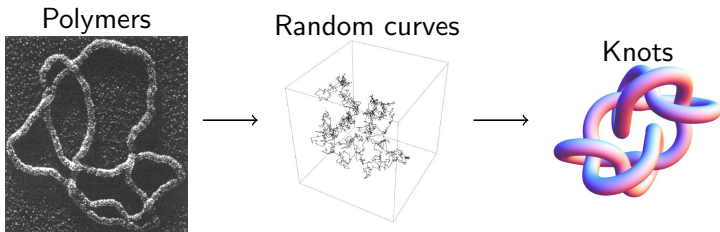
Natural questions about knot diagrams

Question
How many 8-crossing diagrams can be untwisted to the unknot?

42.05%

Ansatz

Polymers

Random curves

Knots

$\longrightarrow$

$\longrightarrow$
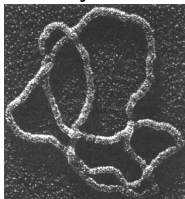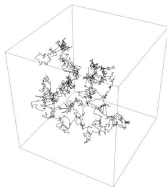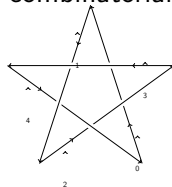
Combinatorial approaches

Polymers

Random curves
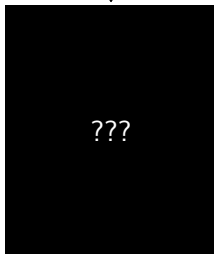
Knots

Random combinatorial

Polymers

Random curves

Knots

???
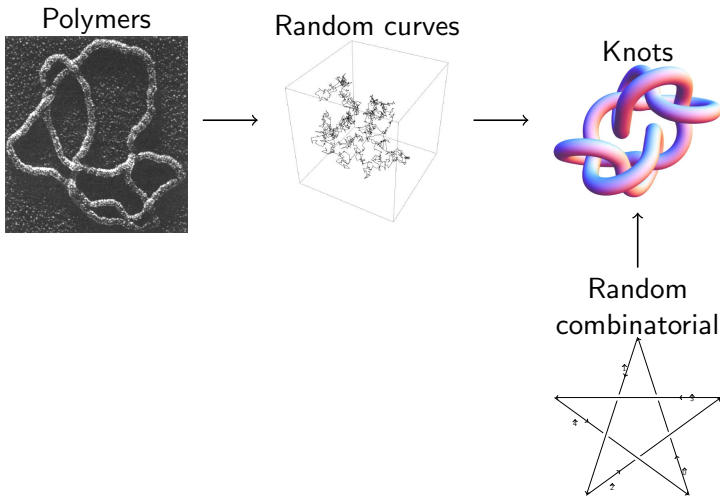
Random combinatorial

Polymers
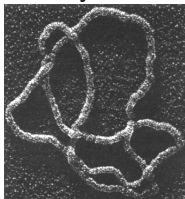
Random curves
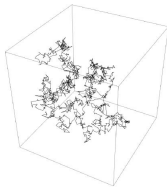
Knots

Random diagrams
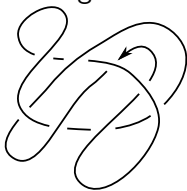
Random combinatorial
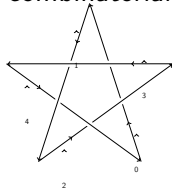
Random diagms

Definition
In the **random diagram model** of random knotting, a n-crossing diagram is drawn uniformly from the finite set of n-crossing knot diagrams.
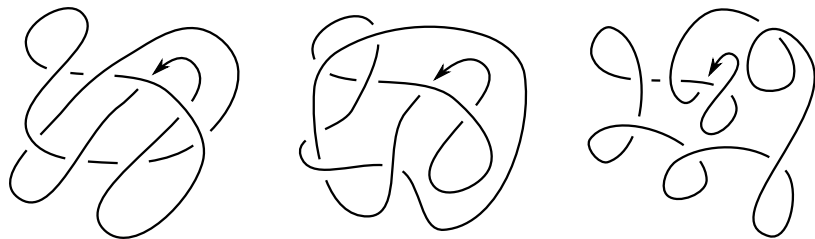
How to enumerate knot diagrams (like a topologist)

Definition
A **knot shadow** is a equivalence class of generic immersions of the unoriented S^1 into the sphere S^2 up to diffeomorphism of S^2.

Plan to Enumerate Diagrams

1. *Enumerate shadows (and discard isomorphic shadows)*
2. *Assign crossing and orientation information (and discard crossing patterns related by an automorphism of the shadow)*

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Observation (known to all combinatoricists, but new to me)
Symmetry stinks.

Proposition

Knot shadows ↔ 1-component 4-valent embedded planar multigraphs up to embedded isomorphism
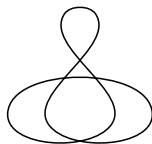
Tabulating knot shadows: `plantri`, two ways

Proposition

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Proposition

Knot shadows $\leftrightarrow$ 1-component 4-valent embedded planar multigraphs up to embedded isomorphism

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Tabulating knot shadows: `plantri`, two ways

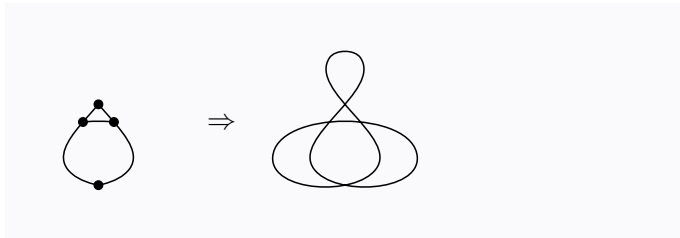## Proposition

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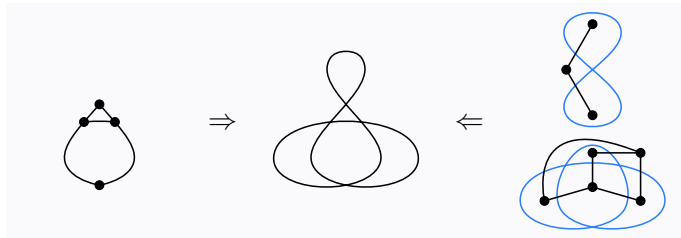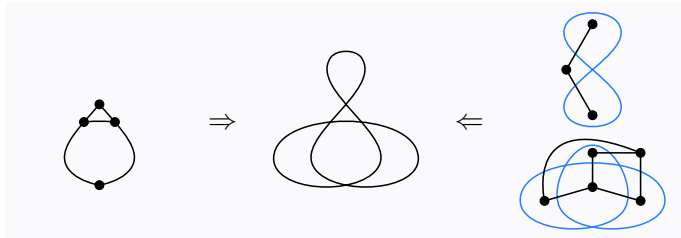2. Generate multiquadrangulations of sphere by careful pattern of connect sums, take dual graphs (fast)

Actually generate all **link shadows**, then restrict to knot shadows

Verifying against existing shadow counts

oriented	$n = 0$	1	2	3	4	5
S^2, S^1	1	1	3	9	37	182
S^2	1	1	2	6	21	99
S^1	1	1	2	6	21	97
–	1	1	2	6	19	76

Curves on S^2. The number of types

V.I. Arnol'd. *Topological Invariants of Plane Curves*

A008989   Number of immersions of unoriented circle into unoriented sphere with n double points.
1, 1, 2, 6, 19, 76, 376, 2194 (list; graph; refs; listen; history; text; internal format)
OFFSET   0,3
REFERENCES V. I. Arnold, Topological Invariants of Plane Curves..., American Math.
LINKS   Table of n, a(n) for n=0..7.
CROSSREFS   Sequence in context: A150119 A181770 A138800 * A057240 A079564 A079453
            Adjacent sequences: A008986 A008987 A008988 * A008990 A008991 A008992
KEYWORD   nonn
AUTHOR   N. J. A. Sloane.
EXTENSIONS Two more terms from Guy H. Valette (guy.valette(AT)skynet.be), Feb 09 20
STATUS   approved

OEIS A008989

n	# knot shadows
0	1
1	1
2	2
3	6
4	19
5	76
6	376
7	2194
8	14614
9	106421
10	823832

We have not found any existing counts of **diagrams**.

Assign crossings, orientation, identify

1. Orient each component. (2 choices)
2. Assign over-under information to each vertex. (2^n choices)

n	# knot shadows	2^{n+1} (# shadows)	# knot diagrams
3	6	96	36
4	19	608	276
5	76	4,864	2,936
6	376	48,128	35,872
7	2,194	561,664	484,088
8	14,614	7,482,368	6,967,942
9	106,421	108,975,104	105,555,336
10	823,832	1,687,207,936	1,664,142,836

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Observation
Symmetry becomes rare, quickly!

Size of the automorphism group of a random diagram

Mean number of automorphisms
versus crossing number of diagram

Mean ♯ of automorphisms
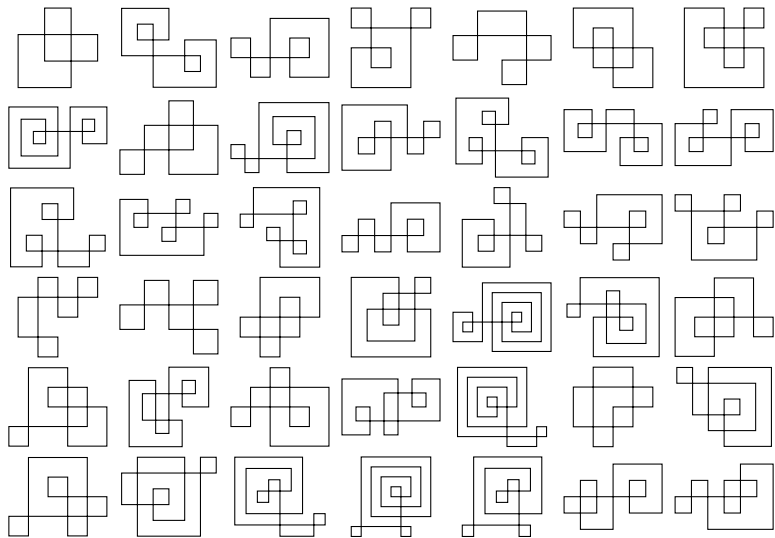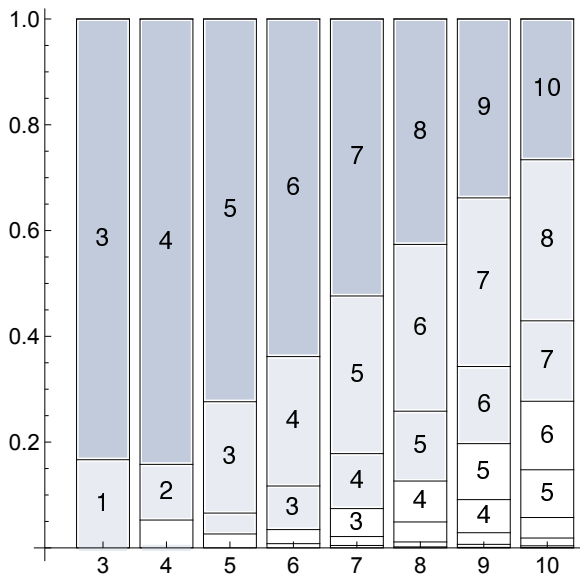
1
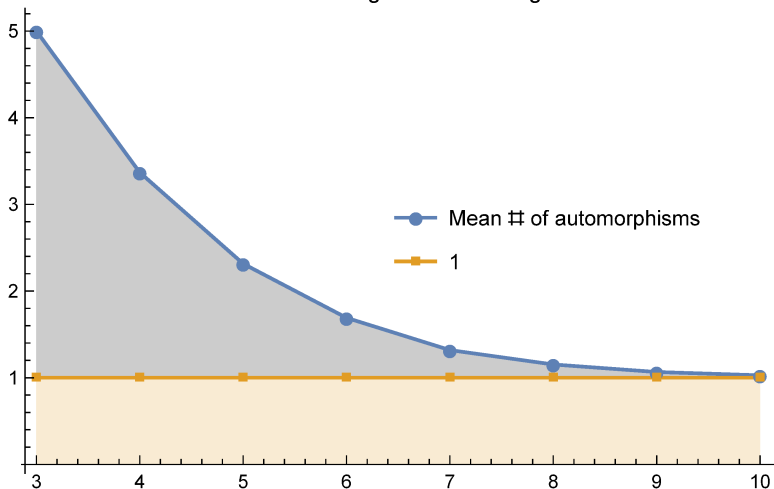
Knotting in diagrams
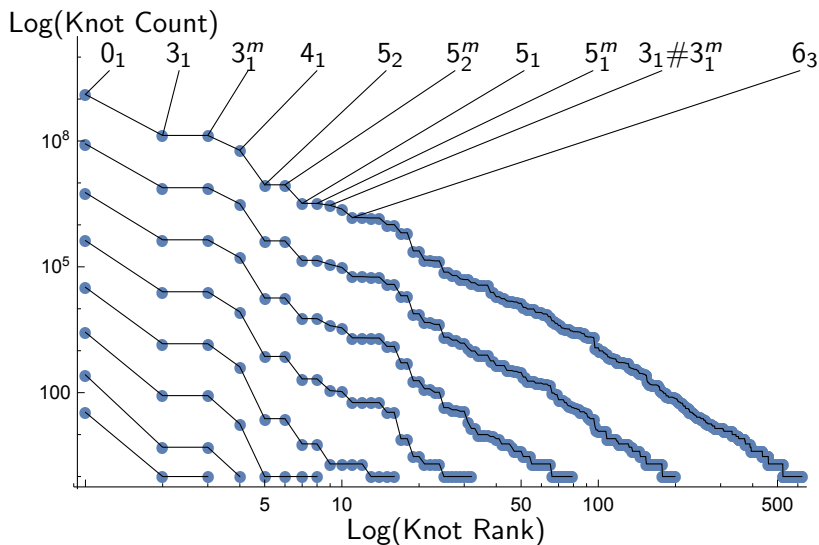
Unknot fraction

Cr	Unknots	(decimal)
3	$\frac{17}{18}$	0.94
4	$\frac{265}{276}$	0.96
5	$\frac{343}{367}$	0.93
6	$\frac{4057}{4484}$	0.90
7	$\frac{105583}{121022}$	0.87
8	$\frac{2926416}{3483971}$	0.84
9	$\frac{42626767}{52777668}$	0.81
10	$\frac{1291291155}{1664142836}$	0.78

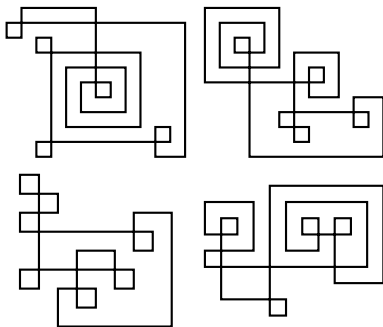Unknots are very common, even among 10 crossing diagrams.
Why?

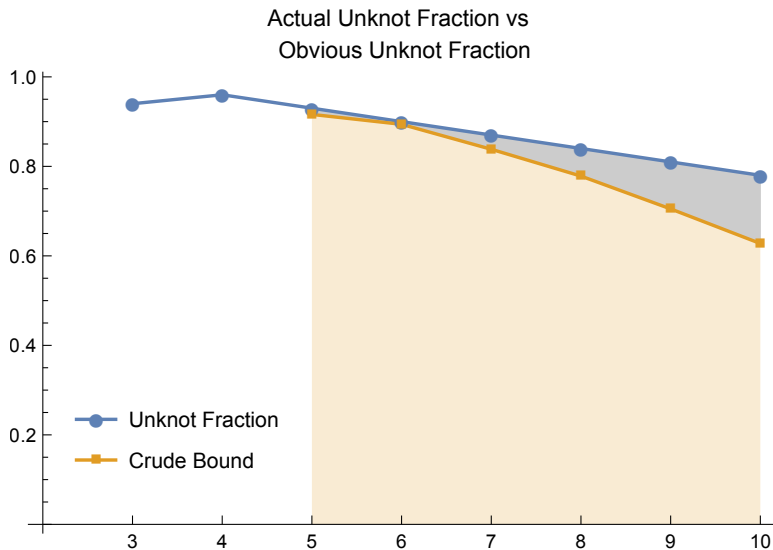# The space of shadows

Most diagrams are (very) composite

Maximally composite diagrams are "treelike"

Question

Treelike diagrams can't be knotted with any assignment of crossings. Does this (crude) bound explain the unknot fraction?

Pretty much.

Actual Unknot Fraction vs
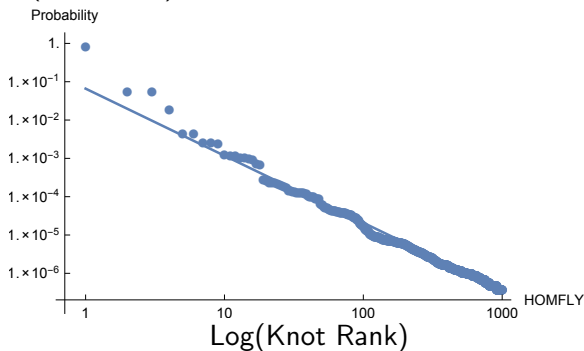Obvious Unknot Fraction

Future Direction: So what about those log-log plots?

Proposition (with Shonkwiler, 2015)

The symplectic structure on polygon space yields a fast direct sampling algorithm for closed equilateral polygons.


Log(Knot Freq)

Future Direction: You can play, too!

- Knot Probabilities in Random Diagrams Cantarella, Chapman, Mastin. arXiv:1512.05749
- All data (and pictures for all the diagrams) available at www.jasoncantarella.com/wordpress/papers/
- A Fast Direct Sampling Algorithm for Random Equilateral Polygons Cantarella, Duplantier, Shonkwiler, Uehara. arXiv:1510.02466
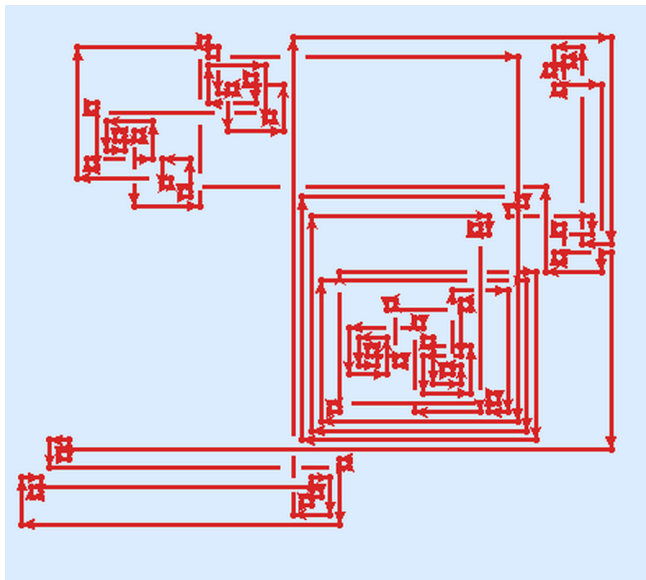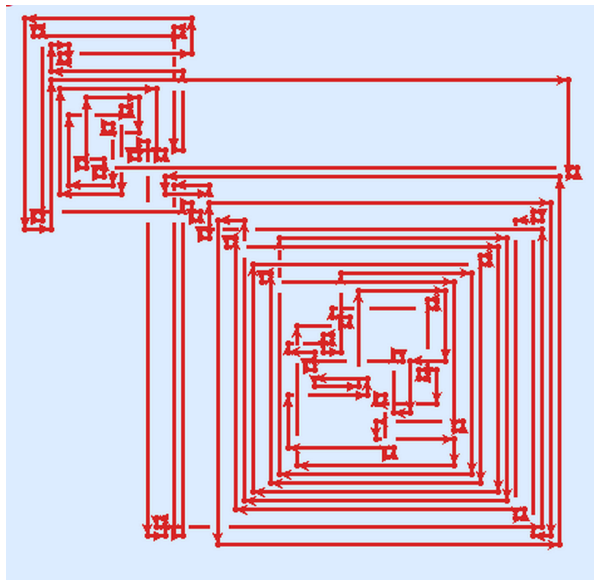
Future Direction: Uniform sampling of large diagrams

Harrison Chapman has results on sampling large diagrams:
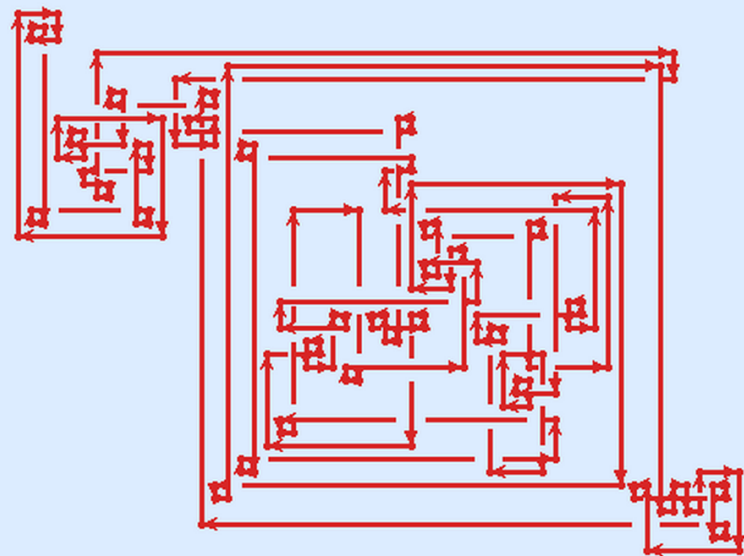
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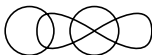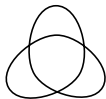# Future Direction: Uniform sampling of large diagrams

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Thank you!