Asymptotics of Knot and Link Diagrams

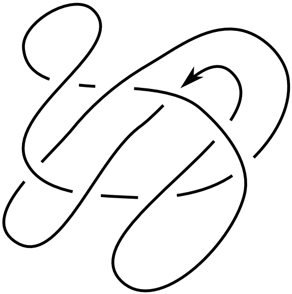
Harrison Chapman

Mock AMS Conference, July 29, 2015

Natural questions about knot diagrams

Question

What fraction of 8-crossing diagrams are trefoils?



Natural questions about knot diagrams

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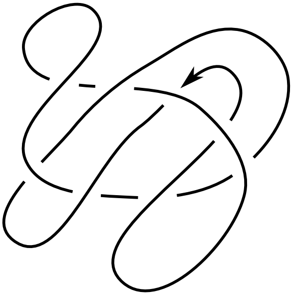
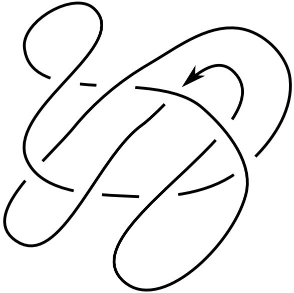
12.48%



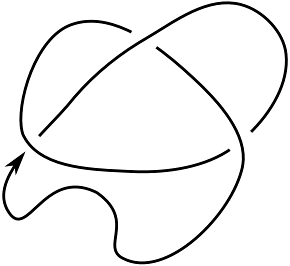
Natural questions about knot diagrams

Question

What is the average minimal crossing # of an 8-crossing diagram?



$\Rightarrow$

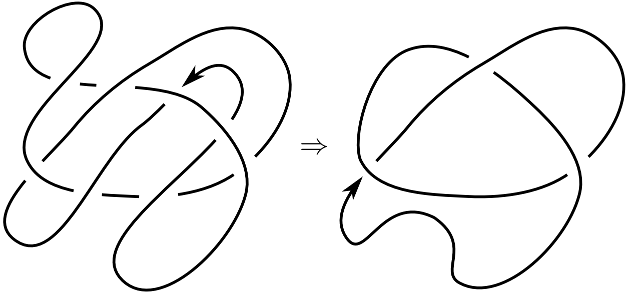


Natural questions about knot diagrams

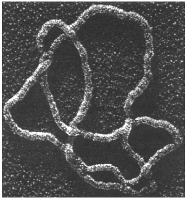
Question

What is the average minimal crossing # of an 8-crossing diagram?

0.52

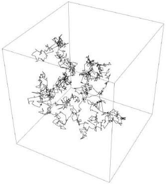


Polymers



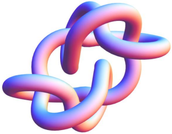
$\longrightarrow$

Random curves

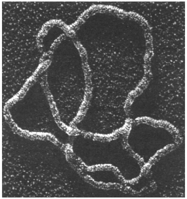


$\longrightarrow$

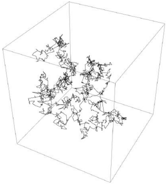
Knots



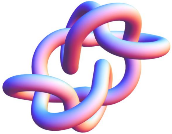
Polymers



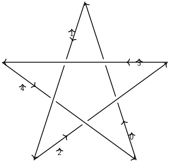
Random curves



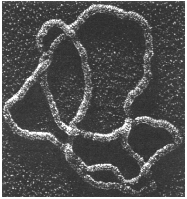
Knots



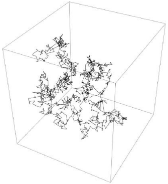
Random combinatorial



Polymers



Random curves



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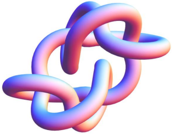
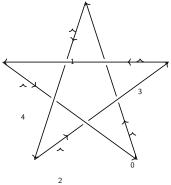
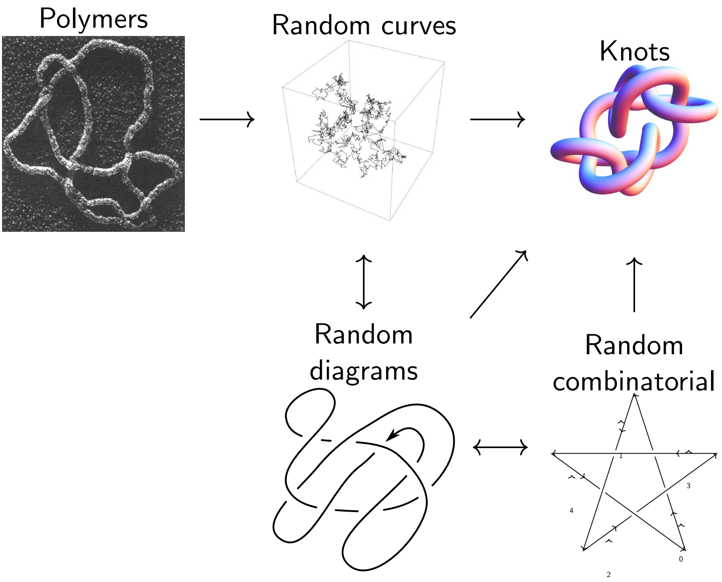


???

Random combinatorial

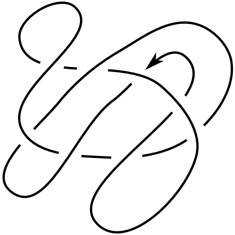
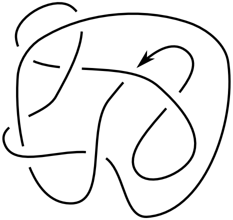
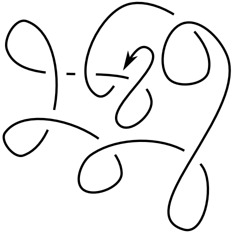



Polymers
Random curves
Knots
Random diagrams
Random combinatorial
1
3
4
0
2

Random diagrams

Definition

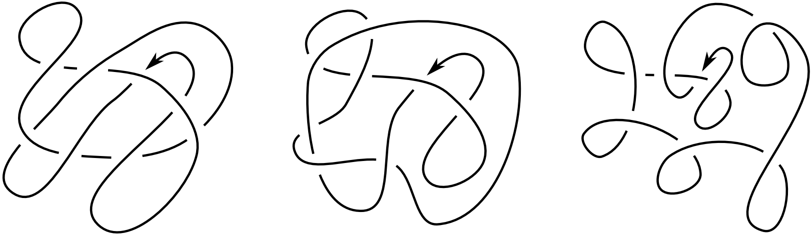
In the **random diagram model** of random knotting, a n-crossing diagram is drawn uniformly from the finite set of n-crossing knot diagrams.

Random diagrams

Definition

A **knot diagram** is a equivalence class of generic immersions of the oriented S^1 into the sphere S^2 together with over-under strand information at each double point up to diffeomorphism of S^2.

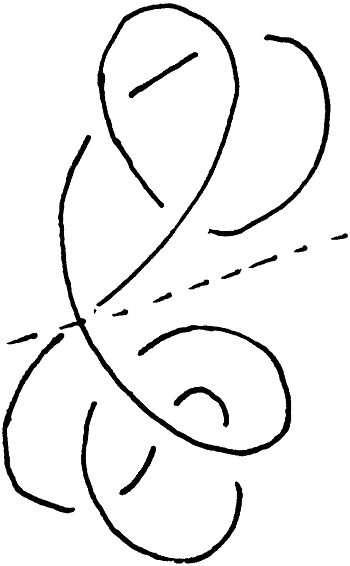


Different diagram classes of interest

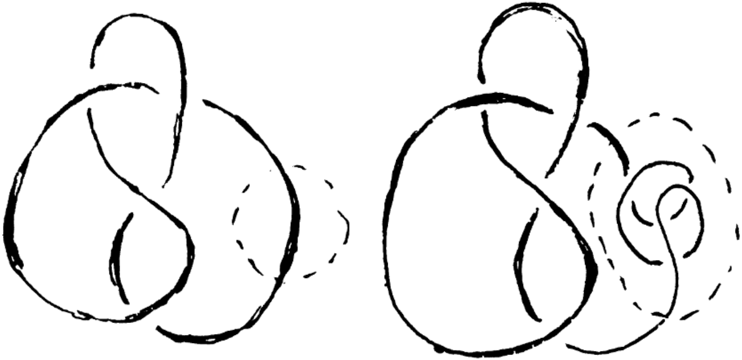
We can restrict to different classes of diagrams and prove results for them; some of those considered are,

1. **Knot diagrams**: diagrams with precisely one component
2. **Link diagrams**: diagrams with any number of components
3. **Reduced diagrams**: (knot or link) diagrams which have no disconnecting vertices (called **isthmi**)
4. **Prime diagrams**: (knot or link) diagrams which cannot be disconnected by removing any pair of edges

Diagram with an isthmus



Prime vs. composite diagrams



How to sample diagrams?

Idea

Tabulate all diagrams using software, then draw randomly from the list

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There are **a lot of** *diagrams! 10-crossing link diagrams take 1GB memory; 11-crossing ones take 10GB*

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Symmetry *makes it difficult to weight probabilities correctly*

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Efficiently sample **simpler objects** *which, in the limit, behave like diagrams*

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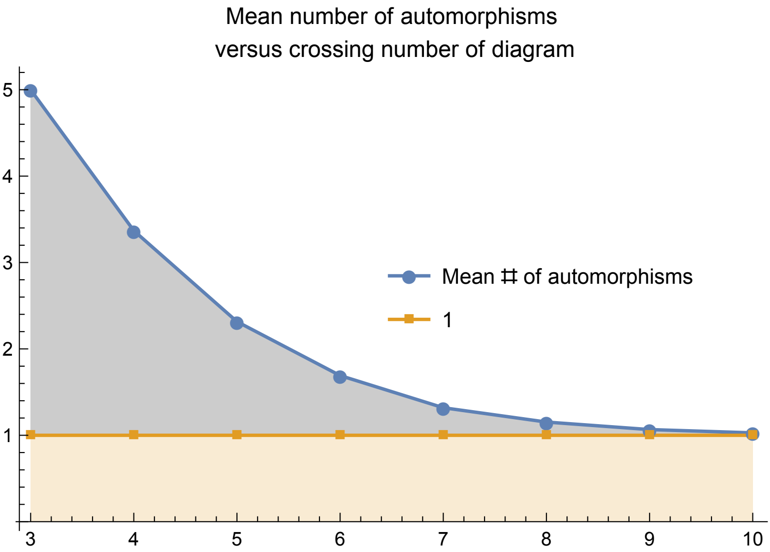
Definition

A **rooted diagram** is a diagram together with a choice of edge, called the **root edge** and a direction for that edge

Theorem (C.)

The ratio of diagrams with n crossings which have nontrivial automorphism group is exponentially small. Hence asymptotically, rooted diagrams behave like diagrams.

Size of the automorphism group of a random diagram



Substructure in diagrams

Asymmetry in diagrams results from their tendencies to contain specific substructures called tangles

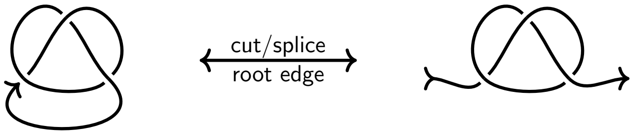
Definition

A $2k$-**tangle** is a diagram-like object having $2k$ half-edges which lie in the exterior face.

A tangle is contained in a diagram D if there exists some disk which, when intersected with D, produces the tangle.

Rooted diagrams and 2-tangles

Rooted (knot or link) diagrams are equivalently viewed as **2-tangles**



Specific tangles appear often

In fact, many kinds of tangles appear often almost surely in diagrams

Theorem (C.)

Consider a fixed class of diagrams $\mathcal{K}$ which grows "smoothly" and let P be a tangle which is appropriately "admissible" for the class. Then there exist constants $c > 0$ and $d < 1$ so that

$$\mathbb{P}(\textit{a diagram } D \textit{ in } \mathcal{K}_n \textit{ contains} \leq cn \textit{ copies of } P \textit{ as a subtangle}) < d^n.$$

Key requirement for proof.

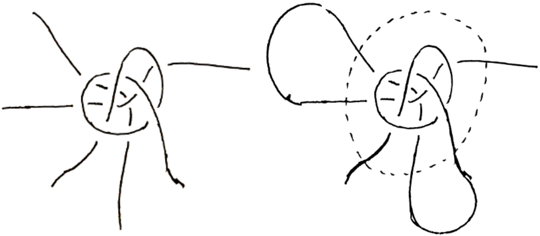
There is an "expansion" operation on diagrams which produces a new diagram containing P so that a diagram in n crossings has n/k valid expansion sites (k depends on expansion). $\square$

Admissible tangles

Knot diagrams and any 2-tangle of one component; expansion operation: connect summation

Admissible tangles

Knot diagrams and any $2k$-tangle of k components; expansion operation: connect summation (after placing into a 2-tangle)



Admissible tangles

Knot diagrams (prime, reduced, or otherwise) and any prime *abab* 4-tangle of two components; expansion operation: 4-tangle replacement

Smooth growth

Caveat

The prior theorem has a hypothesis that the class of diagrams $\mathcal{K}$ grow **smoothly**; *that*

$$\lim_{n\to\infty} |\mathcal{K}_n|^{1/n} = \limsup_{n\to\infty} |\mathcal{K}_n|^{1/n}.$$

- The class of rooted link diagrams and the class of rooted prime link diagrams are counted exactly and hence known to grow smoothly
- Explicit formulas for counts of different classes are still unknown

Smooth growth for diagrams

Theorem (C.)

The following additional classes of rooted diagrams grow smoothly;

1. *Rooted knot diagrams*
2. *Rooted reduced knot diagrams and rooted reduced link diagrams*
3. *Rooted prime knot diagrams*

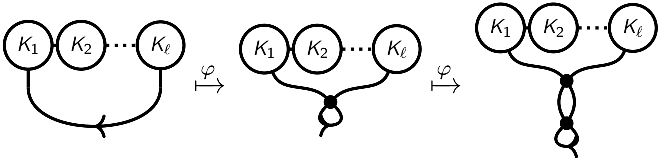
Smooth growth for diagrams

Idea of proof.

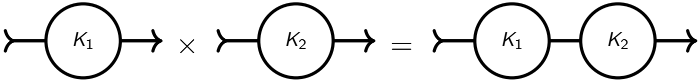
1. Let $\mathcal{K}$ be the class of interest
2. We know $\limsup_{n\to\infty} |\mathcal{K}_n|^{1/n}$ exists (Cauchy-Hadamard), so there exists n where there are "enough" diagrams in $\mathcal{K}_n$
3. Find $m > n$ and injections from $\mathcal{K}_n$ into $\mathcal{K}_m$ and $\mathcal{K}_{m+1}$
4. Define a "nice" composition on diagrams to get lower bounds on $|\mathcal{K}_N|$ for large N and, hence, $\liminf_{n\to\infty} |\mathcal{K}_n|^{1/n}$ □

Smooth growth for knot diagrams

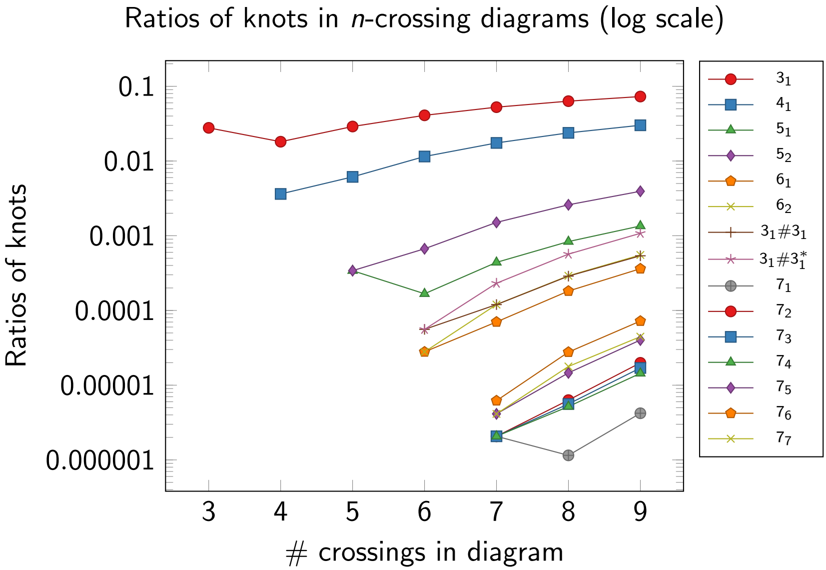
- The injection maps are the single and double twists of the root edge



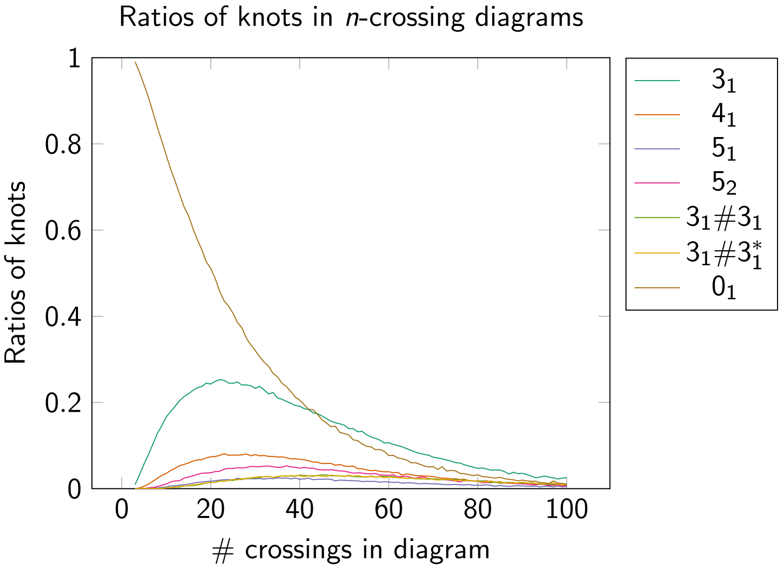
- Composition is end-to-end composition



Knotting in diagrams; tabulation data



Knotting in diagrams; random sampling



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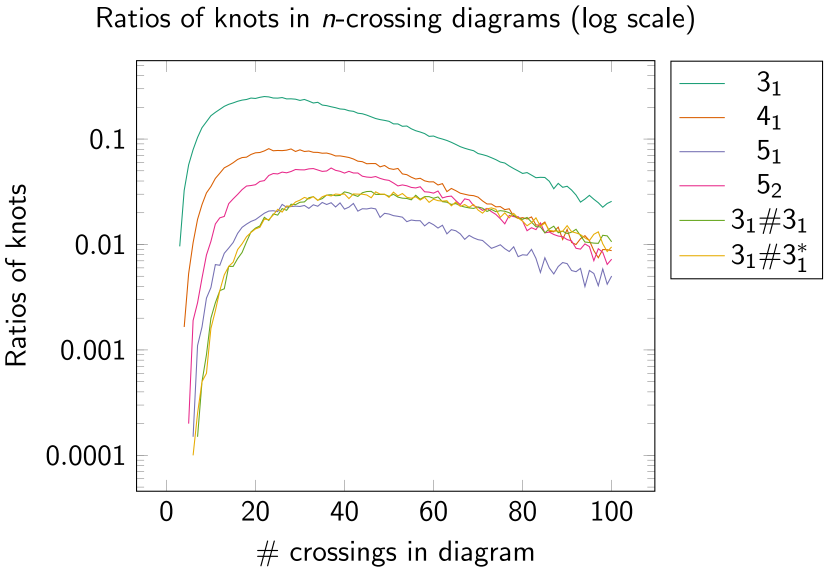
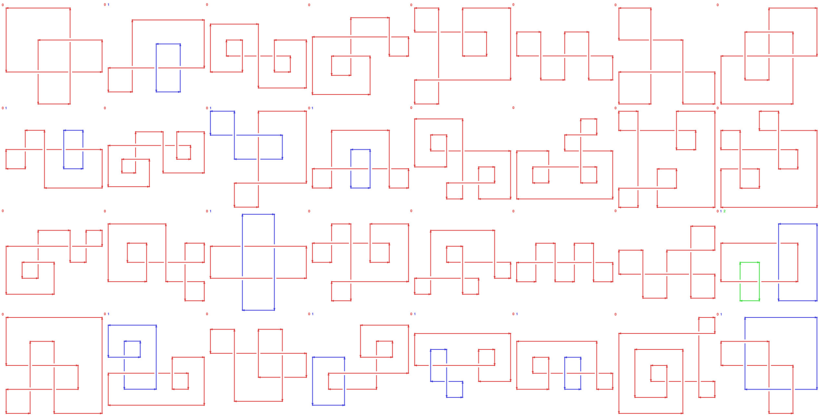
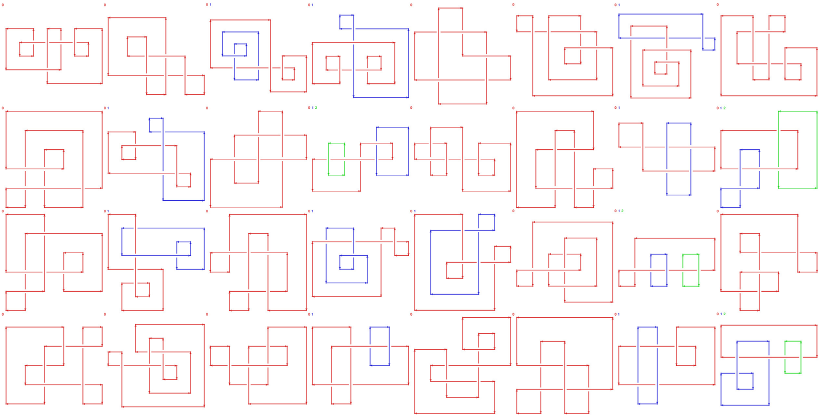


The space of shadows



Link shadows. Pictures generated by Eric Lybrand.

The space of shadows



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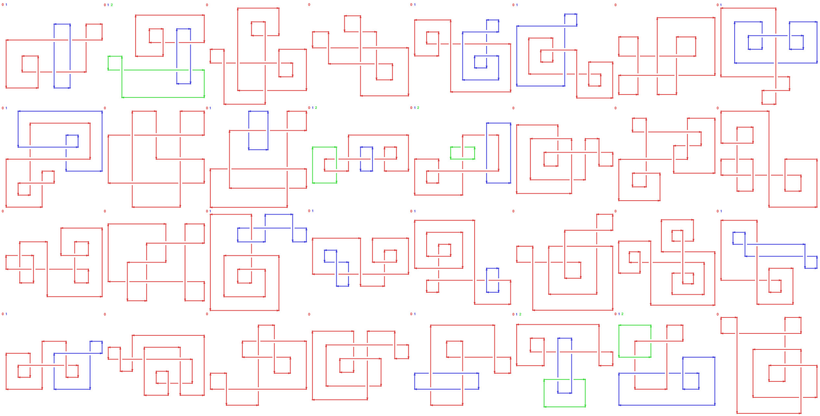

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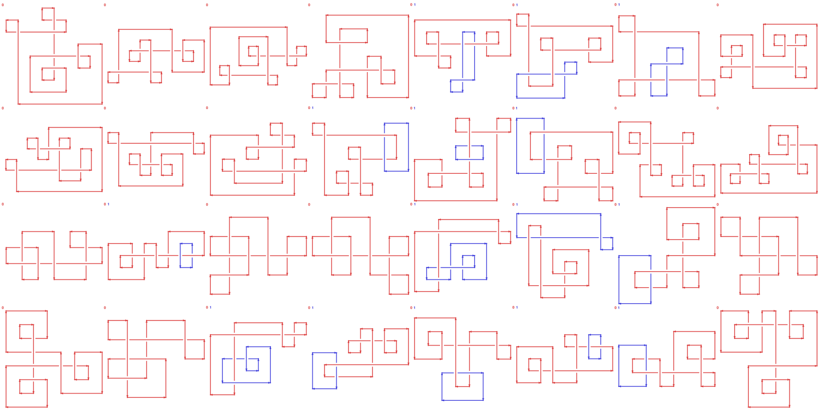

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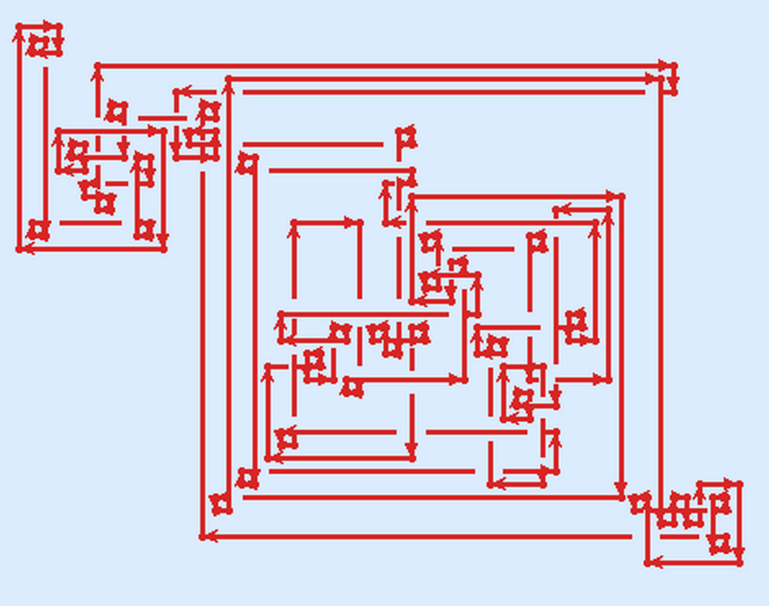
Large random knot diagrams



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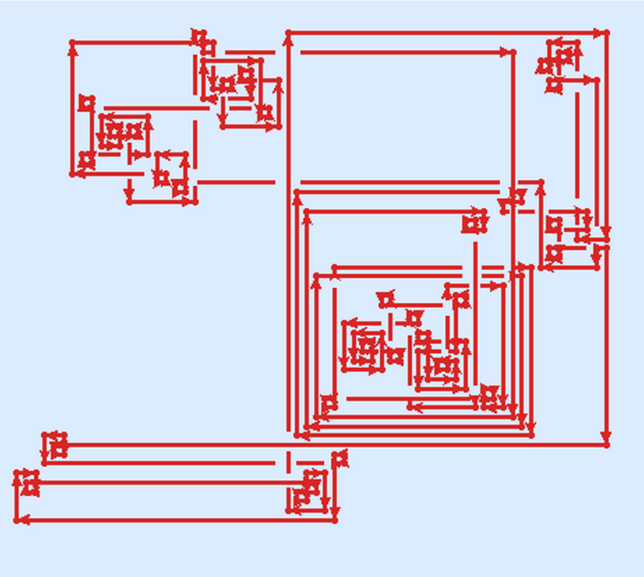

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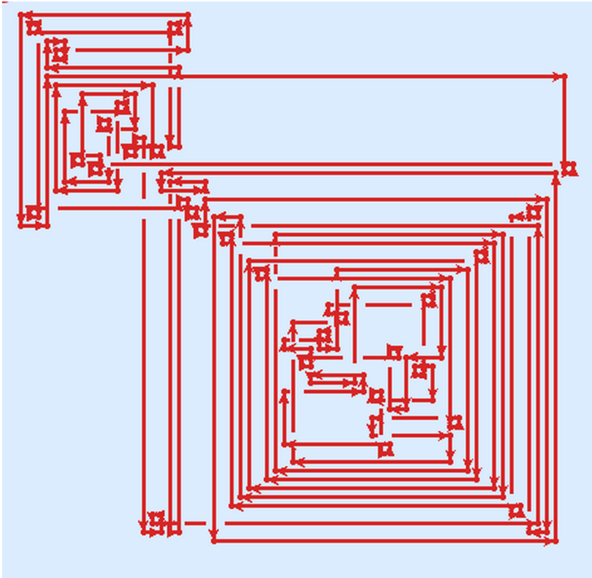

Thank you!