Knot Probabilities in Random Diagrams

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Keywords:

Suppose that one is given an n-crossing knot diagram chosen at random from the (finite) set of such diagrams. What is the probability that it is a diagram of the unknot? In this paper, we report on a computer experiment which gives precise answers to this and similar questions for $n \leq 12$ by direct enumeration and classification of knot diagrams. From the point of view of classical knot theory, this is a particularly simple model of random knotting. Part of our interest is to provide data which can be compared to results about more complicated distributions, such as the distribution of knots provided by selecting random closed equilateral n-gons, closed lattice walks, or in combinatorial models such as Even-Zohar et. al.ś *Petaluma* model.

1. DEFINITIONS

definition of diagram arnold's plane curve invariants equivalence relations

2. CONSTRUCTING THE DATABASE OF DIAGRAMS

Brinkman and McKay [**? ?**] gave an algorithm for producing isomorphism-free classes of embedded planar graphs, implemented in the software `plantri`.

In the spirit of Brinkmann and McKay, we now define two expansion moves which change graph type and embedding and a modification of the planar embedding:

Definition 1. The expansion move E_1 ("loop creation") adds a loop edge to an existing vertex. The expansion move E_2 ("bigon creation") doubles an existing edge. We'll call the reverse of these moves "reduction" moves "loop collapse" and "bigon collapse". We also define a move

Expansion moves graphic here

that we call F_1 ("flip"). If a single pair of parallel edges joins two vertices, the resulting loop can

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separate a portion of the graph from another portion. This is a pair of parallel edges which does not correspond to a bigon face. In this case, we can flip one portion of the graph to the other side of the loop at one of the two crossings, as in the move below:

Flipping out graphic here

We can now show

Proposition 2. *Every connected* 4*-regular embedded planar (multi)graph* G *can be obtained from a connected, embedded planar simple graph of vertex degree* ≤ 4 G_0 *by a series of* E_1 *and* E_2 *expansions. The graph isomorphism type of* G_0 *is canonical, but the embedding of* G_0 *depends on an arbitrary choice of exterior face in the embedding of* G*.*

Proof. We proceed by descent, using the reverse of the two expansion moves. Suppose we have a connected 4-regular embedded planar multigraph G. Denote (arbitrarily) a single face of the embedding which is not a monogon face as the "exterior" face f_0.

If G is simple, there is nothing to prove. If not, G has a loop edge, which can be removed by an E_1 loop collapse, or a pair of edges joining the same endpoints. If this pair defines a bigon face which is not the distinguished face f_0, it can be removed by an E_2 bigon collapse.

If the pair of edges does not bound a bigon face, or bounds only f_0 as a bigon face, we must perform a flip move. We do so with a little care to avoid the possibility of an infinite number of flips. The cycle generated by the pair of edges separates the sphere on which the diagram is embedded. One side is exterior (contains f_0) and the other is interior, and there is some portion of the graph contained in the interior. We can perform an F_1 move at one (or both) vertices (as below) to flip whichever portion of the graph is interior to the outside. This creates a bigon face, which we can then collapse with an E_2 move.

Flipping out graphic here.

All of these three options reduce the number of edges by one, so there are finitely many steps. When the process terminates, we have removed all loop edges and replaced all sets of parallel edges with single edges, yielding a simple graph G_0. The isomorphism type of G_0 is determined, since F_1 does not change graph isomorphism type, and E_2 and E_1 commute (with respect to graph isomorphism type).

We can maintain an embedding during the process to arrive at an embedding for the simple graph G_0. Since we don't destroy the exterior face during the descent process, the embedding has a marked exterior face, as well.

NOTE: The question remains— does a different choice of exterior face lead to the same embedding of the graph G_0 (with a corresponding different choice of exterior face)? Or can different choices of exterior face lead to different embeddings of the G_0 after the process is complete? Commutativity of the operations is the issue here. It is clear that loop edges must occur at different vertices by the vertex degree bound (excluding the trivial case of the figure-8 graph), so loop collapses commute with each other. Similarly, bigon collapses with disjoint vertex sets commute with each other, and with loop collapses as long as the loop isn't based at either endpoint. However, there are some cases where the same vertex is the site of more than one operation, and here it may be the case that order matters.

We do know that G_0 is connected, because none of our three operations can disconnect the graph. And it has vertex degree ≤ 4 because each operation reduces vertex degree. $\square$

To construct all connected, embedded 4-regular graphs from the embedded planar simple graphs with vertex degree ≤ 4 requires us to invert this process.

Construct the adjacency matrix of the graph A. At every vertex with degree 2 or 1, add a "ghost" loop edge with a weight of 0, At each edge

plantri and the plantri theorem reduction theorem for shadows reduction for diagrams

3. CLASSIFYING KNOT TYPES

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4. RESULTS

giant pictures, compared with tait's classification our distributions monogon and bigon fractions degree of alternatingness universal properties? comparison with distribution from ERPs, lattice walks, and petaluma.

5. FUTURE DIRECTIONS

transitions, unknotting number and so forth.