

# Random Knot Diagrams

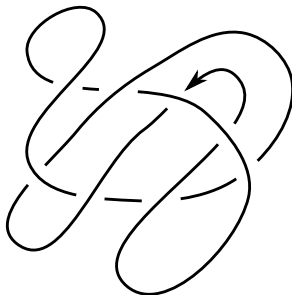
Harrison Chapman (UGA - Graduate student)  
joint w/ Jason Cantarella (UGA), Matt Mastin (Wake Forest)

AMS Western Spring Sectionals 2015 (UNLV) – April 18, 2015

# Natural questions about knot diagrams

## Question

What fraction of 8-crossing diagrams are trefoils?

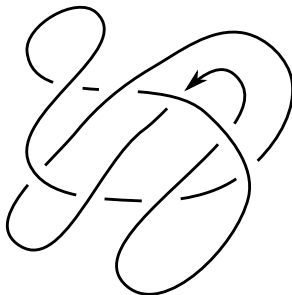


# Natural questions about knot diagrams

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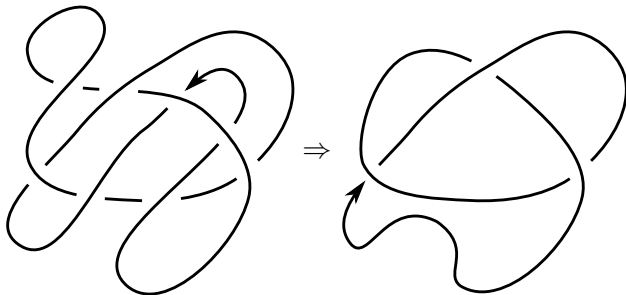
12.48%



# Natural questions about knot diagrams

## Question

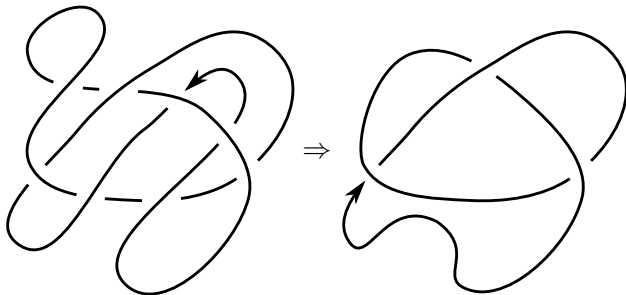
What is the average minimal crossing # of an 8-crossing diagram?



# Natural questions about knot diagrams

## Question

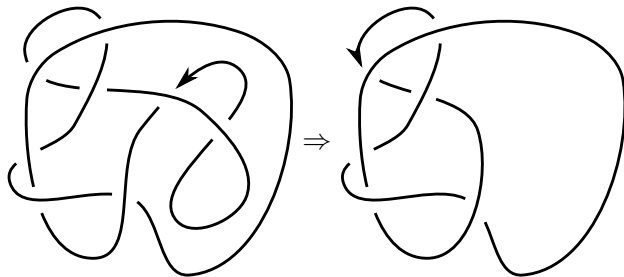
What is the average minimal crossing # of an 8-crossing diagram?  
0.52



# Natural questions about knot diagrams

## Definition

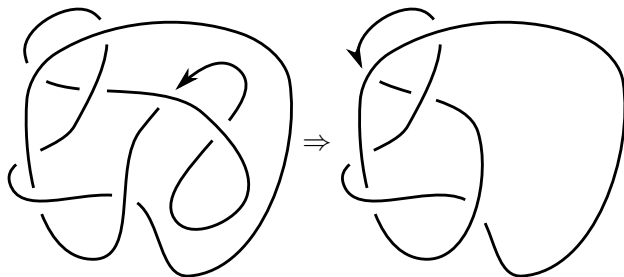
Define an operation on diagrams, **untwisting**: Recursively RI untwist loops in a diagram until there are no more.



# Natural questions about knot diagrams

## Question

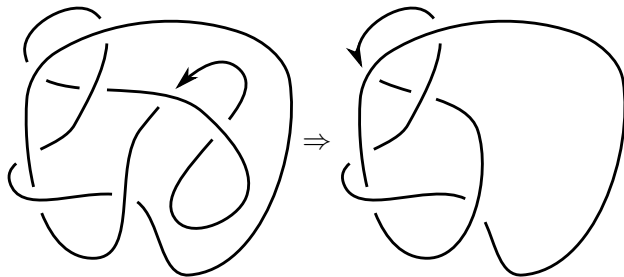
What is the average crossing # of a untwisted 8-crossing diagram?



# Natural questions about knot diagrams

## Question

What is the average crossing # of a untwisted 8-crossing diagram?  
2.20

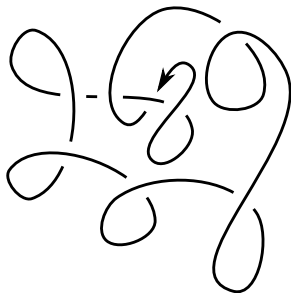




# Natural questions about knot diagrams

## Question

How many 8-crossing diagrams can be untwisted to the unknot?

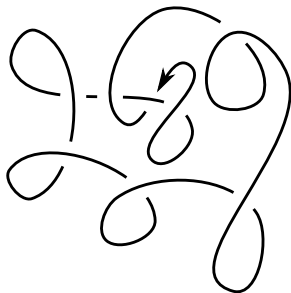


# Natural questions about knot diagrams

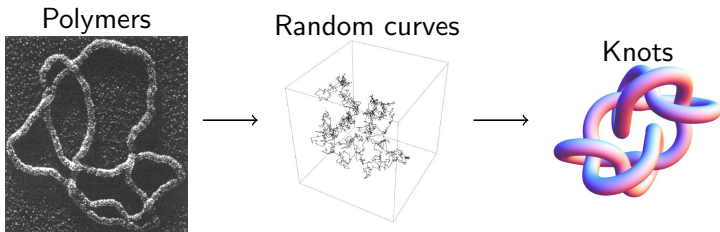
## Question

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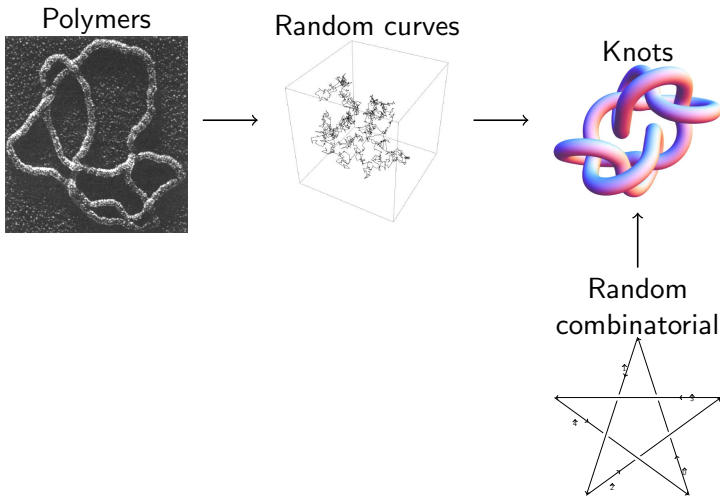
42.05%



# Ansatz

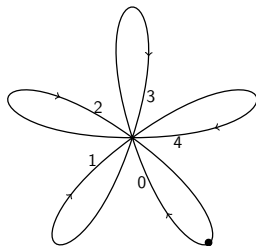


# Combinatorial approaches

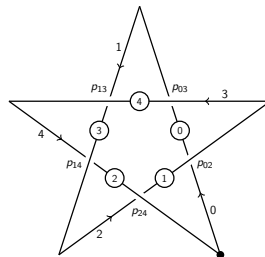


# The Petaluma model

Satisfying theorems have been proven for the Petaluma model



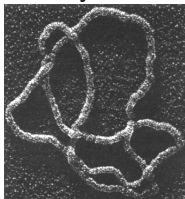
Petal diagram



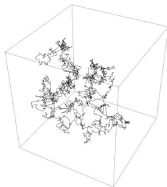
Star diagram

(Diagram from Evan-Zohar, Hass, et al.)

Polymers



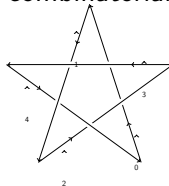
Random curves



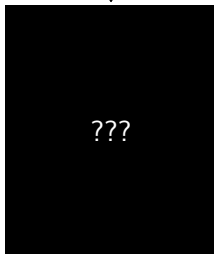
Knots



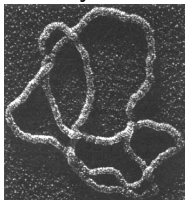
Random  
combinatorial



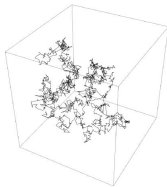
???



Polymers



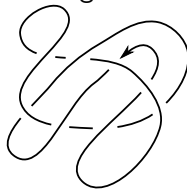
Random curves



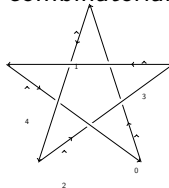
Knots



Random  
diagrams



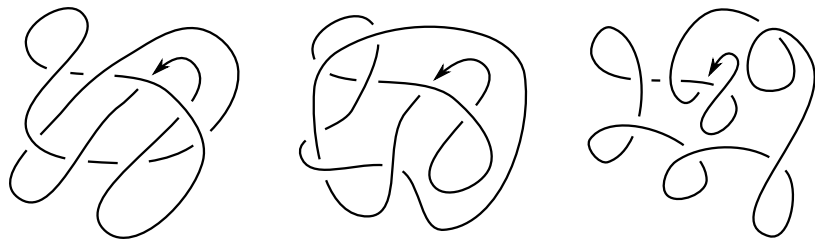
Random  
combinatorial



# Random diagrams

## Definition

In the **random diagram model** of random knotting, a  $n$ -crossing diagram is drawn uniformly from the finite set of  $n$ -crossing knot diagrams.

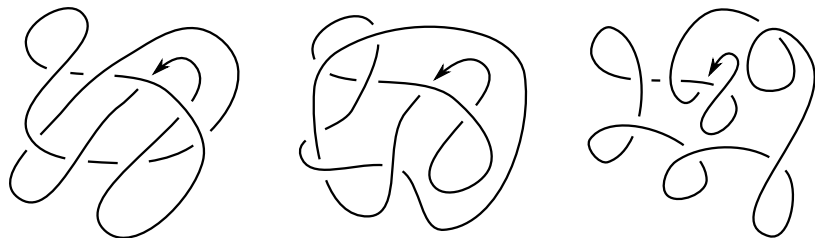




# Random diagrams

## Definition

A **knot diagram** is a equivalence class of generic immersions of the oriented  $S^1$  into the sphere  $S^2$  together with over-under strand information at each double point up to diffeomorphism of  $S^2$ .



# How to enumerate knot diagrams

## Definition

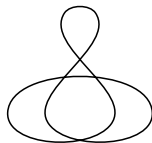
A **knot shadow** is a generic equivalence class of immersions of the unoriented  $S^1$  into the sphere  $S^2$  up to diffeomorphism of  $S^2$ .

- 1 Enumerate shadows
- 2 Assign crossing and orientation information and identify equivalent diagrams

# Tabulating knot shadows

## Proposition

*Knot shadows  $\leftrightarrow$  1-component 4-valent embedded planar multigraphs up to embedded isomorphism*

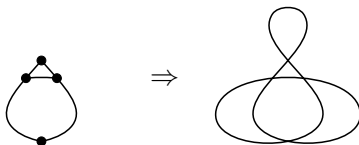


# Tabulating knot shadows

## Proposition

*Knot shadows  $\leftrightarrow$  1-component 4-valent embedded planar multigraphs up to embedded isomorphism*

- 1 Add loops and edges to planar simple graphs

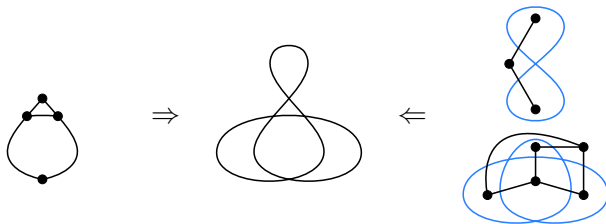


# Tabulating knot shadows

## Proposition

*Knot shadows  $\leftrightarrow$  1-component 4-valent embedded planar multigraphs up to embedded isomorphism*

- 1 Add loops and edges to planar simple graphs
- 2 Generate multiquadrangulations by connect sum, take dual graphs

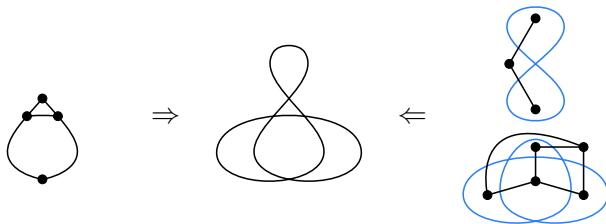


# Tabulating knot shadows

## Proposition

*Knot shadows  $\leftrightarrow$  1-component 4-valent embedded planar multigraphs up to embedded isomorphism*

- 1 Add loops and edges to planar simple graphs
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Actually generate all **link shadows**, then restrict to knot shadows

# Assign crossings, orientation, identify

- 1 Orient each component. (2 choices)
- 2 Assign over-under information to each vertex. ( $2^n$  choices)

n	# knot shadows	$2^{n+1}$ (# shadows)	# knot diagrams
3	6	96	36
4	19	608	276
5	76	4,864	2,936
6	376	48,128	35,872
7	2,194	561,664	484,088
8	14,614	7,482,368	6,967,942
9	106,421	108,975,104	105,555,336

# Verifying against existing shadow counts

oriented	$n = 0$	1	2	3	4	5
$S^2, S^1$	1	1	3	9	37	182
$S^2$	1	1	2	6	21	99
$S^1$	1	1	2	6	21	97
—	1	1	2	6	19	76

Curves on  $S^2$ . The number of types

V.I. Arnol'd. *Topological Invariants of Plane Curves*

A008989 Number of immersions of unoriented circle into unoriented sphere with  $n$  double points.

1, 1, 2, 6, 19, 76, 376, 2194 [list](#) [graph](#) [rcf](#) [links](#) [history](#) [text](#) [internal format](#)

OFFSET

0,3

REFERENCES

V. I. Arnol'd, Topological Invariants of Plane Curves..., American Math.

LINKS

[Table of  \$n, a\(n\)\$  for  \$n=0..7\$ .](#)

CROSSREFS

Sequence in context: [A159119](#) [A181770](#) [A138800](#) \* [A057240](#) [A079564](#) [A079453](#)

Adjacent sequences: [A008986](#) [A008987](#) [A008988](#) \* [A008990](#) [A008991](#) [A008992](#)

KEYWORD

nonn

AUTHOR

[N. J. A. Sloane](#).

EXTENSIONS

Two more terms from Guy H. Valette (guy.valette(AT)skynet.be), Feb 09 20

STATUS

approved

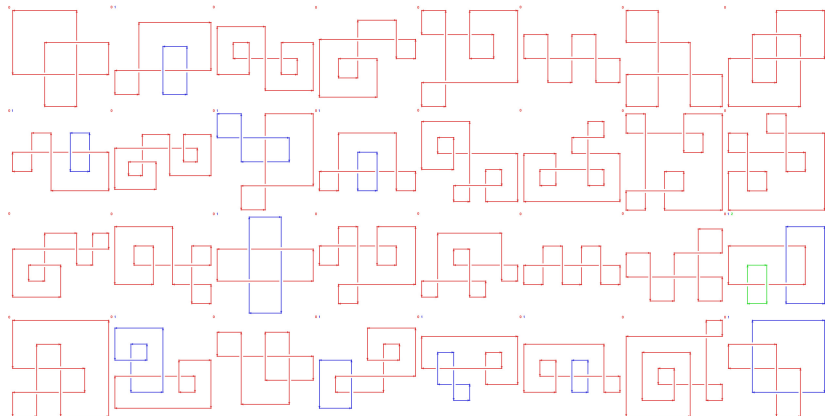
$n$	# knot shadows
0	1
1	1
2	2
3	6
4	19
5	76
6	376
7	2194
8	14614
9	106421

OEIS A008989

We have not found any existing counts of **diagrams**.

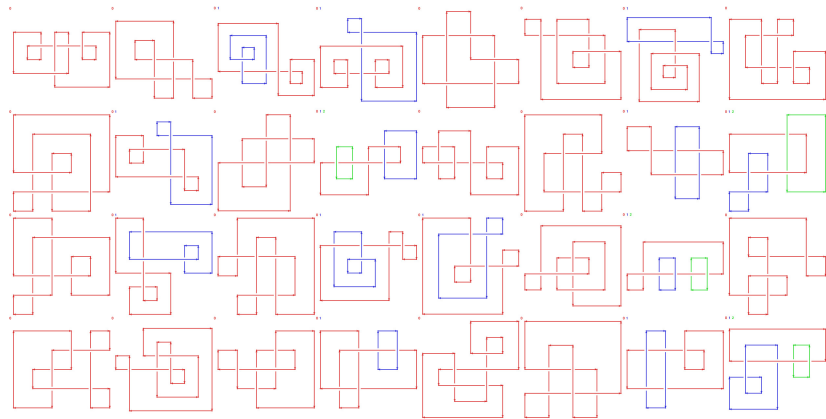


# The space of shadows



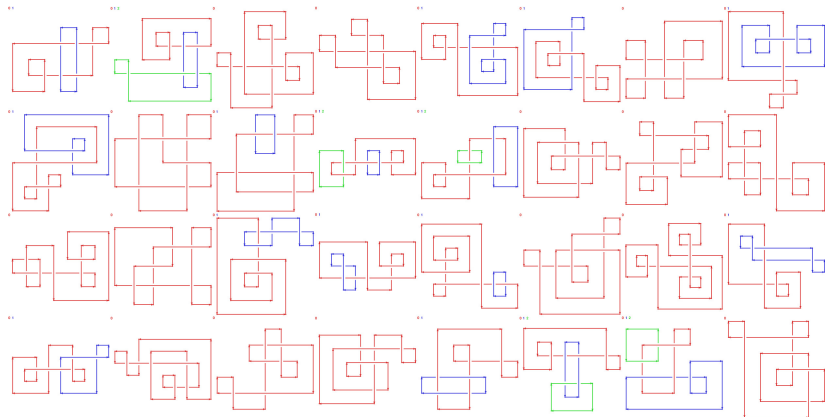
Link shadows. Pictures generated by Eric Lybrand (UGA undergrad).

# The space of shadows



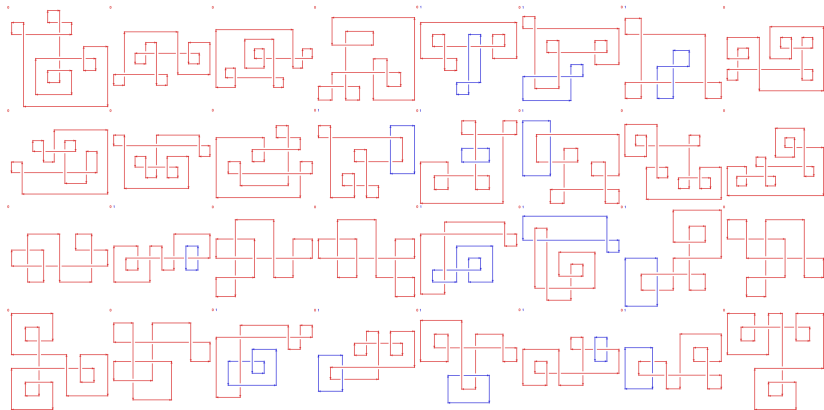
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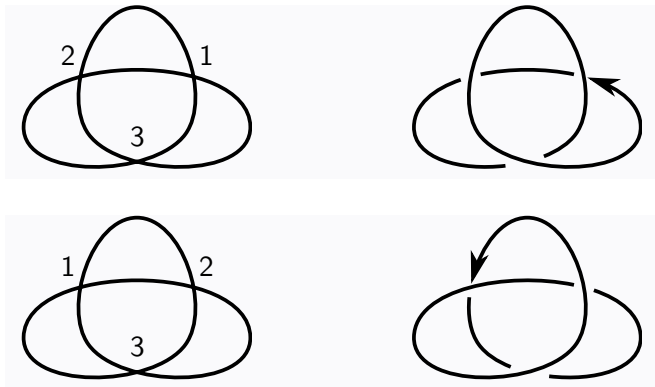
# The space of shadows



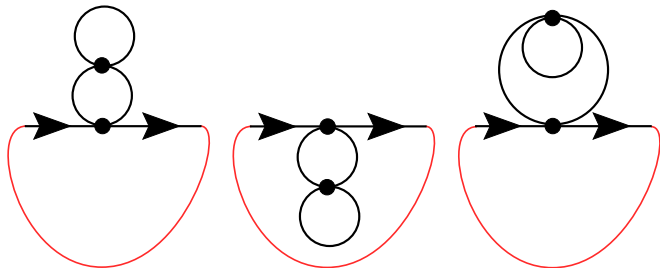
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# Tabulation is difficult!

Accounting for symmetry is complicated.



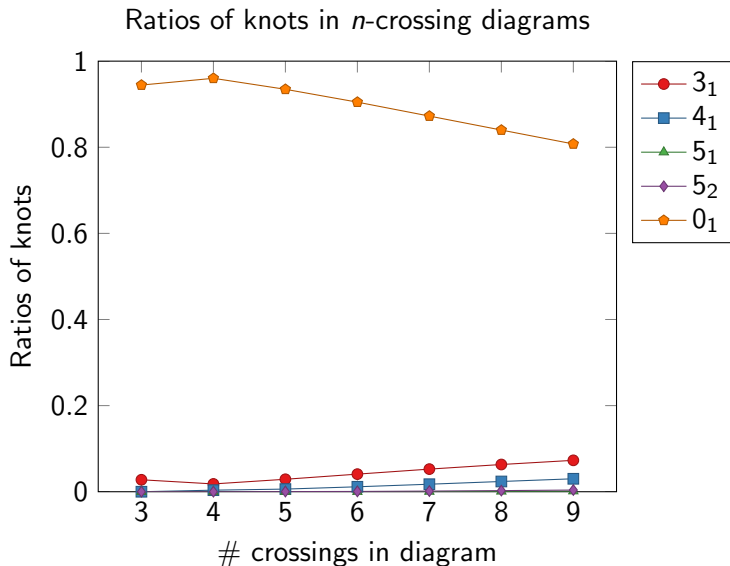
## Breaking symmetries could make counting easier



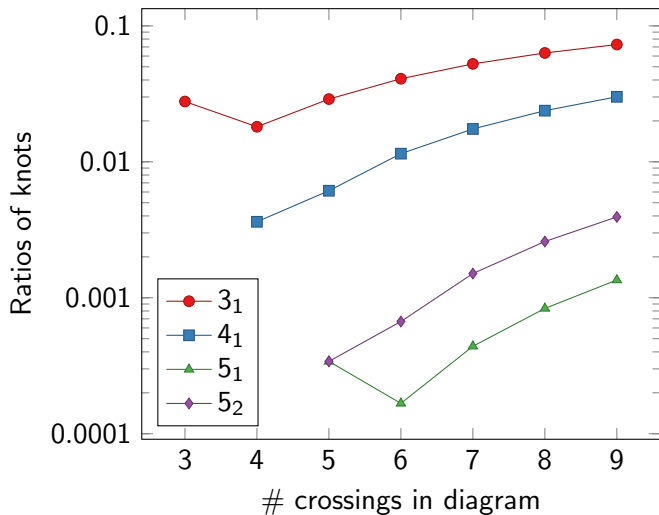
Two-leg diagrams counted by generating function (Bouttier, et al. 2003):

$$G_0 = \frac{24g - 1 + \sqrt{1 - 12g}}{9g(1 + \sqrt{1 - 12g})} = 1 + 2g + 9g^2 + 54g^3 + 378g^4 + \dots$$

# Knotting in diagrams

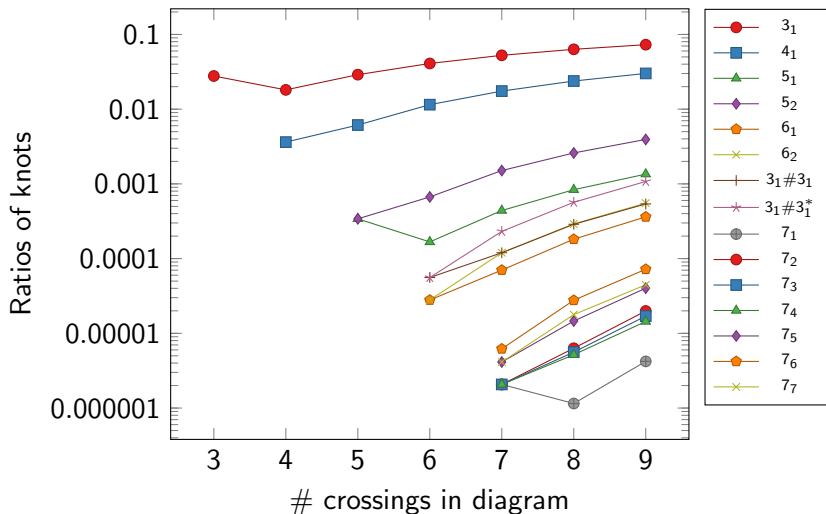


Ratios of knots in  $n$ -crossing diagrams (log scale)





Ratios of knots in  $n$ -crossing diagrams (log scale)



# A question on unknotting

Theorem ((Frisch-Wassermann-Delbrück Conjecture)  
Sumners-Whittington 1988)

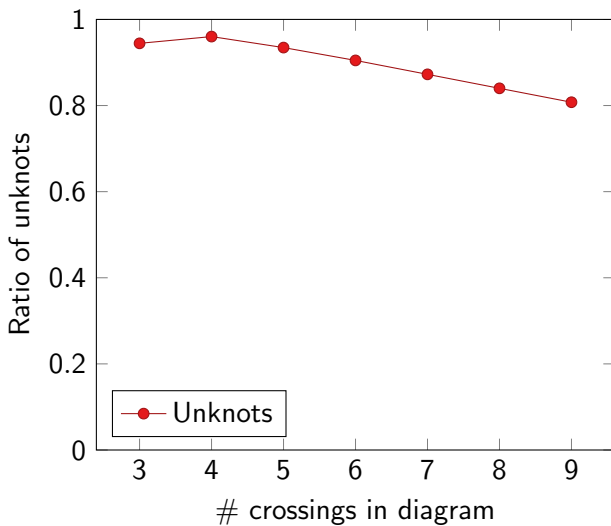
*The ratio of unknots in random  $n$ -edge self-avoiding lattice polygons tends to zero exponentially with  $n$ .*

## Conjecture

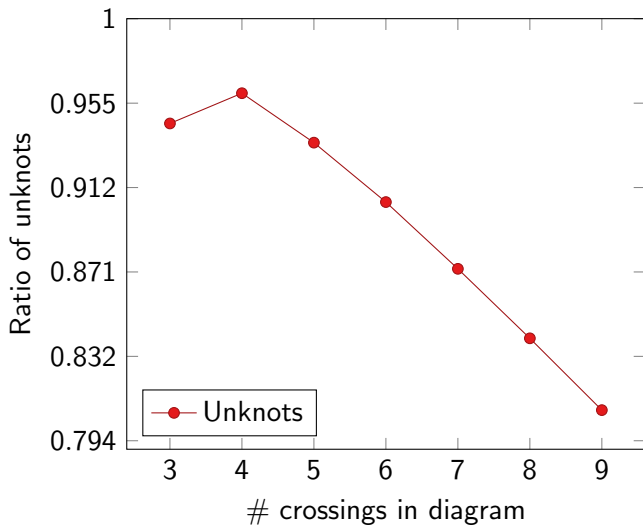
The ratio of unknots in diagrams tends to zero as  $n$  increases.  
(Exponentially?)

Exist theorems on planar (simple) graphs (Dowden 2008) similar to the Kesten pattern theorem (Kesten 1963) for lattice walks

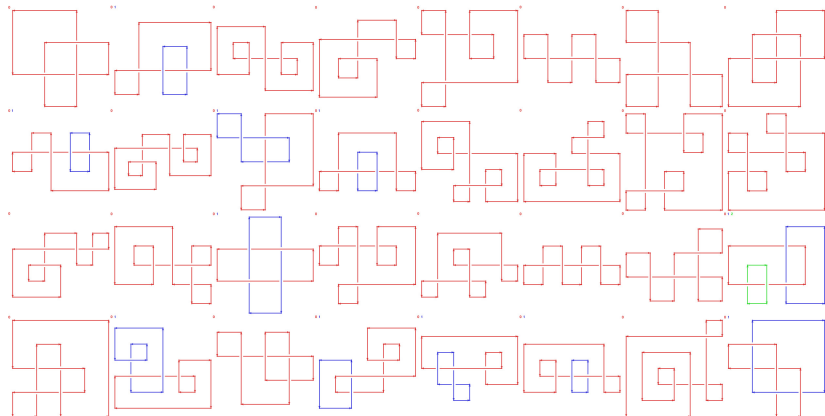
Ratio of unknots in  $n$ -crossing diagrams



Ratio of unknots in  $n$ -crossing diagrams (log scale)

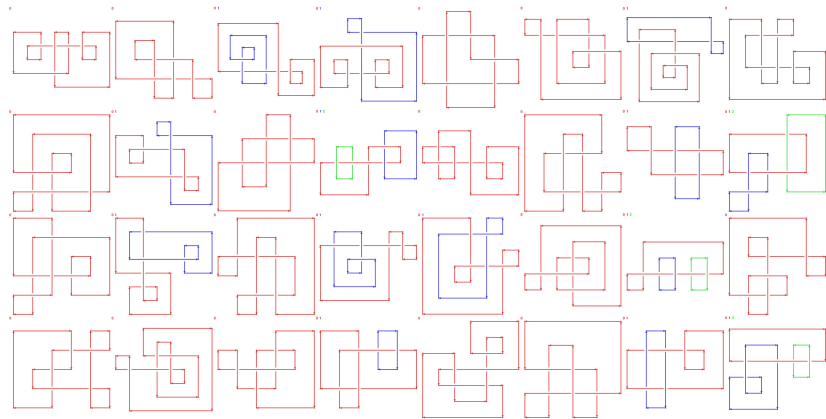


# Why so many unknots?



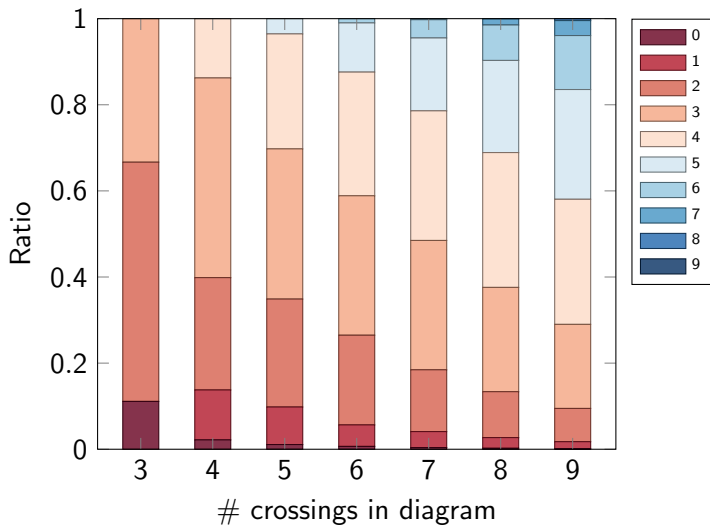
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# Why so many unknots?

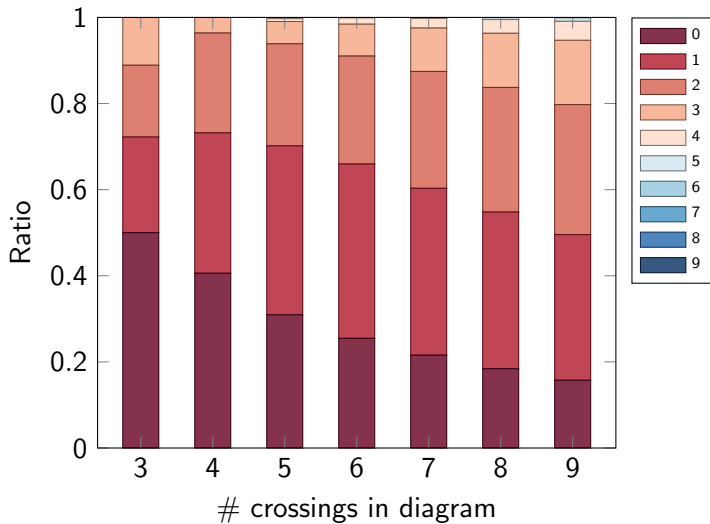


Link shadows. Pictures generated by Eric Lybrand (UGA undergrad).

## Reidemeister-I loops (monogons) in diagrams



## Bigons in diagrams

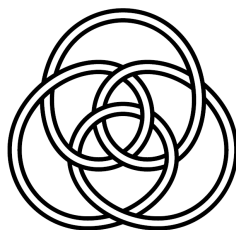
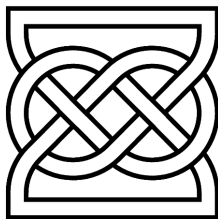




# Basic polyhedra $8^*$ and $9^*$

## Proposition

*Conway's basic polyhedra  $8^*$  and  $9^*$  are the only diagrams in  $\leq 9$  crossings with no monogons or bigons.*



$8_{18}$  (left),  $9_{40}$  (right).

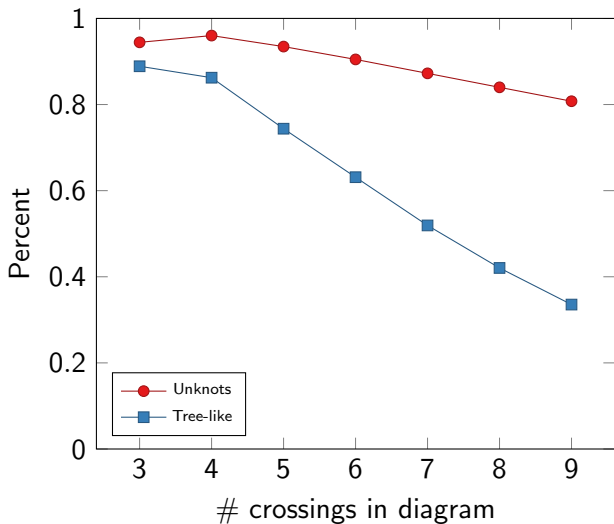
## Some shadows are always unknots

A **tree-like curve** (Aicardi 1994) is a knot shadow which can be untwisted to the trivial shadow.

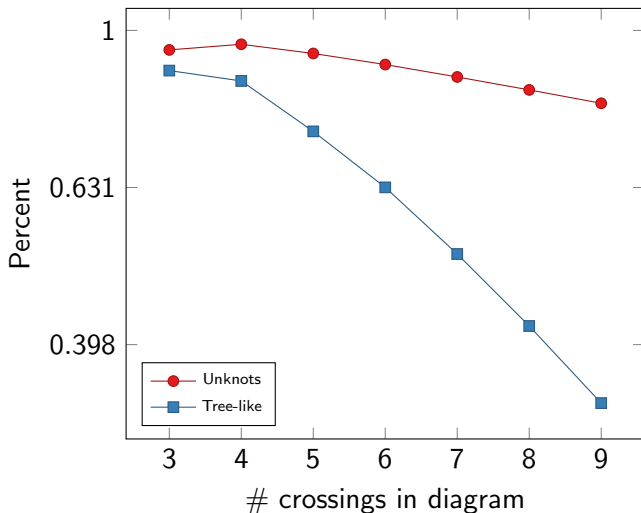


Tree-like curves  $\Rightarrow$  lower bound on unknottedness.

## Ratio of unknots, tree-like curves in $n$ -crossing diagrams

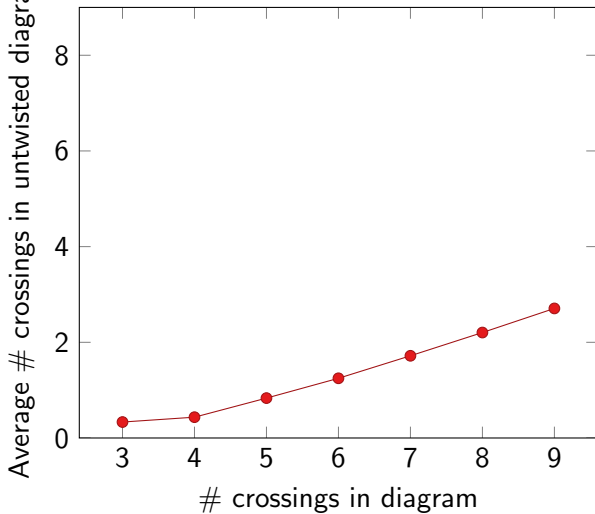


Ratio of unknots, tree-like curves in  $n$ -crossing diagrams (log scale)

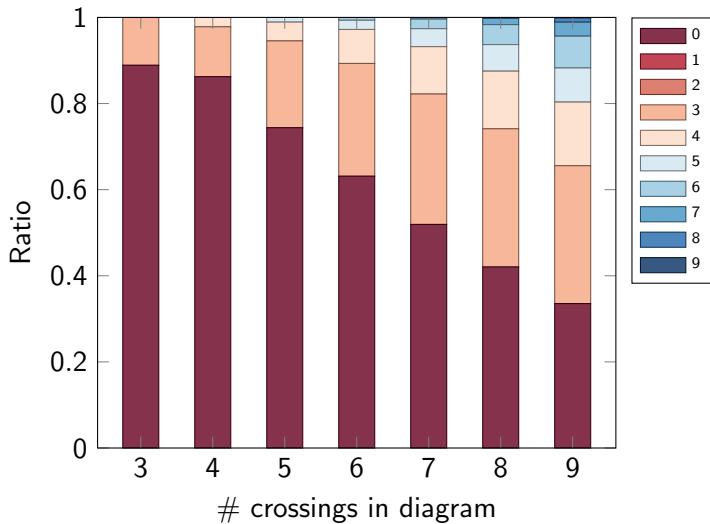


Tree-like curves alone explain only some of the unknot fraction

Crossing # vs. Average untwisted crossing #



## Untwisted crossing #

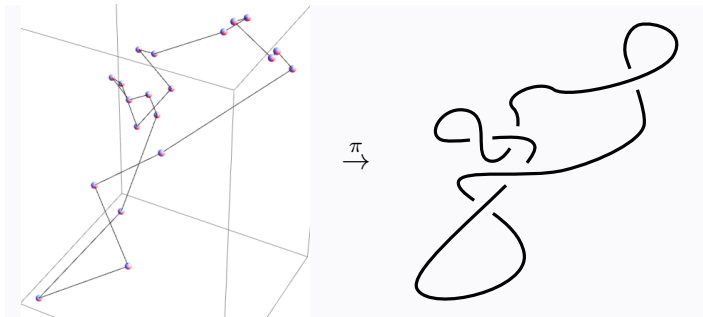


# Questions to answer

Random curves project to diagrams.

## Question

How does the pushforward measure differ from uniform diagram sampling? (c.f. Hua, Nguyen, Raghavan, Arsuaga, Vasquez 2005)



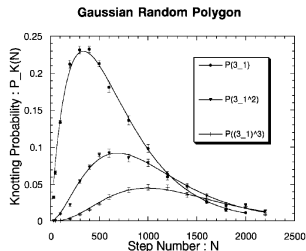
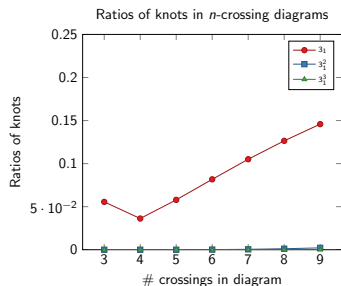
(from Shonkwiler)

# Questions to answer

## Question

Let  $PD(n, K)$  be the probability of knot type  $K$  in a random diagram of  $n$  crossings and  $PC(n, K)$  the probability of knot type  $K$  in a random polygon of  $n$  edges.

If  $n, m$  are such that  $PD(n, 0_1) = PC(m, 0_1)$ , is there a relationship between  $PD(n, K) = PC(m, K)$  for other knot types  $K$ ?



(from Deguchi, et. al.)



# Questions to answer

## Fact

No one will enumerate the 100-crossing knot diagrams.

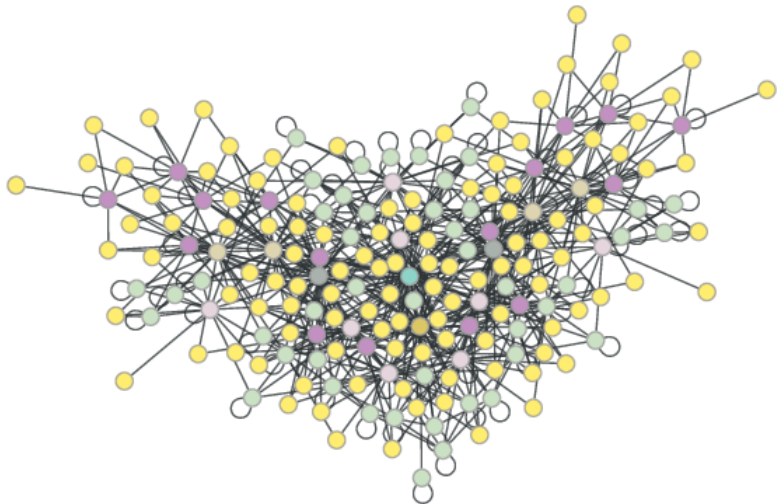
## Question

Can we generate uniformly sampled random 100-crossing knot diagrams **another way**?

## Future direction: Link diagrams

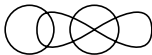
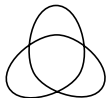
n	# link shadows	# knot shadows
0	1	1
1	1	1
2	3	2
3	7	6
4	30	19
5	124	76
6	733	376
7	4586	2194
8	33373	14614
9	259434	106421

## Future direction: Knot distances



# Thank you!

Coming soon: Cantarella, Chapman, Mastin. *Knot probabilities in random diagrams.*



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