# Isomorphisms of Multigraphs in Terms of Isomorphisms of Colored Simple Graphs

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Keywords:

**Definition 1.** A **multigraph** $G$ is a triple $G = (V, E, f)$ where $V$ is a finite set of *vertices*, $E$ is a finite set of *edges*, and $f$ is a map from $E$ to the power set of $V$, $\mathcal{P}(V)$, so that $f(e)$ is a subset of size either 1 or 2. The vertices in $f(e)$ are called the **endpoints** of $e$. Edges with $|f(e)| = 1$ are called **loops**.

An **isomorphism** from a graph $G_1$ to a graph $G_2$ is a pair of bijections $(\phi_V, \phi_E)$ with $\phi_V : V_1 \to V_2$ and $\phi_E : E_1 \to E_2$ such that $(\phi_V \circ f_1)(e) = (f_2 \circ \phi_E)(e)$ (as sets) for all $e \in E_1$.

A **coloring on the vertices** of a graph is a map, $C_V$, from $V$ to a set of colors $X$. An isomorphism $(\phi_V, \phi_E)$ from a vertex colored graph $G_1$ to a vertex colored graph $G_2$ **respects the vertex coloring** if, and only if, $C_{V_1}(v) = (C_{V_2} \circ \phi_V)(v)$ for all $v \in V_1$.

A **coloring on the edges** of a graph is a map, $C_E$, from $E$ to a set of colors $X$. An isomorphism $(\phi_V, \phi_E)$ from an edge colored graph $G_1$ to an edge colored graph $G_2$ **respects the edge coloring** if, and only if, $C_{E_1}(e) = (C_{E_2} \circ \phi_E)(e)$ for all $e \in E_1$.

**Definition 2.** Given a multigraph $G = (V, E, f)$ we define an associated graph $\bar{G} = (\bar{V}, \bar{E}, \bar{f})$ by the following construction.

- The vertices of $\bar{G}$ are the vertices of $G$, in other words, $\bar{V} = V$.
- The edges of $\bar{G}$ come from collapsing non-loop edges that have the same endpoints. We do this by defining the edges of $\bar{G}$ to be a set of preimages of $f$. In particular, set

$$\bar{E} = \{f^{-1}(f(v)) : |f(v)| = 2\}.$$

- $\bar{f}(\bar{e}) = f(e)$ where $e$ is any element of $\bar{e}$. Note that this is well-defined as all edges in the set $\bar{e}$ have the same endpoints by construction.

We also define a coloring on the vertices and edges of $\bar{G}$ as follows.

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- $C_{\bar{V}}(\bar{v}) =$ the number of edges $e \in E$ such that $f(e) = \{v\}$.
- $C_{\bar{E}}(\bar{e}) = |\bar{e}|$.

**Lemma 3.** *Given any graph $G$ the associated graph $\bar{G}$ is simple.*

*Proof.* An edge in $G$ is a loop precisely when $|f(v)| = 1$ which is excluded in the definition of $\bar{G}$. Similarly, duplicate edges are not allowed in $\bar{E}$ because it is defined to be the *set* of pre-images. Thus, $\bar{G}$ has no loops or multiedges and is therefore simple. $\square$

Since $\bar{G} = (\bar{V}, \bar{E}, \bar{f})$ is always simple it can be described by only its set of vertices $\bar{V}$ and a collection of two element subsets of $\bar{V}$ giving the edges. In this case, a bijection $\bar{\phi}_V : \bar{V}_1 \to \bar{V}_2$ induces a map on the edge sets by $\bar{\phi}_E(\bar{e}) = \{\bar{\phi}_V(\bar{v}), \bar{\phi}_V(\bar{w})\}$ where $\bar{v}$ and $\bar{w}$ are the endpoints of $\bar{e}$. Thus, an isomorphism between simple graphs can be described by only a bijection on the vertex sets.

The following theorem says that there is a map from the set of isomorphisms $\bar{G}_1 \to \bar{G}_2$ to the power set of the isomorphisms $G_1 \to G_2$ whose image is a partition.

**Theorem 4.** *Let $G_1$ and $G_2$ be multigraphs and $\bar{G}_1$ and $\bar{G}_2$ the associated colored graphs given in Definition 2. Then, given an isomorphism $\bar{\phi}$ from $\bar{G}_1$ to $\bar{G}_2$ that respects the vertex and edge coloring, we can construct a set of isomorphisms from $G_1$ to $G_2$, denoted $\Phi(\bar{\phi})$. In addition, $\Phi(\bar{\phi}_1)$ and $\Phi(\bar{\phi}_2)$ are disjoint if, and only if, $\bar{\phi}_1$ and $\bar{\phi}_2$ are distinct isomorphisms. Moreover, if $(\phi_V, \phi_E)$ is an isomorphism from $G_1$ to $G_2$, then there exists an isomorphism $\bar{\phi}$ from $\bar{G}_1$ to $\bar{G}_2$ such that $(\phi_V, \phi_E) \in \Phi(\bar{\phi})$.*

*Proof.* IDEA: $\phi_V^i = \bar{\phi}$ for all $i$ and the $\phi_E^i$ come from all choices of bijections between the *sets* labeling the edges of $\bar{G}_1$ and $\bar{G}_2$ and the different bijections on the loops at each vertex.

We will first describe the construction and then prove that it satisfies the properties in the theorem.

First note that $\bar{\phi}$ is a bijection from $\bar{V}_1 \to \bar{V}_2$, but by definition $\bar{V}_1 = V_1$ and $\bar{V}_2 = V_2$. So, $\bar{\phi}$ *is* a bijection from $V_1 \to V_2$.

Since $\bar{\phi}$ respects the edge coloring on $\bar{G}$ we know that $|\bar{e}| = |\bar{\phi}_{\bar{E}}(\bar{e})|$. Let $\Phi_{\bar{e}}$ be the set of bijections from $\bar{e} \to \bar{\phi}_{\bar{E}}(\bar{e})$ and let $\Phi_{\bar{E}} := \times_{\bar{e} \in \bar{E}} \Phi_{\bar{e}}$.

Let $\Phi_v$ be the set of bijections from the set of loops at $v$ to the set of loops at $\bar{\phi}(v)$ and let $\Phi_L := \times_{v \in V} \Phi_v$.

We can now define $\Phi(\bar{\phi})$ as follows.

$$\Phi(\bar{\phi}) := \{\bar{\phi}\} \times (\Phi_{\bar{E}} \times \Phi_L)$$

Consider an element $(\phi_V, (\phi_E, \phi_L))$ in $\Phi(\bar{\phi})$. We claim this gives an isomorphism from $G_1 \to G_2$. By construction $\phi_E$ is an (ordered) collection of maps from $E_1 \to E_2$ since $|\bar{e}|$ and $|\bar{\phi}_{\bar{E}}(\bar{e})|$ are subsets of edges in $E_1$ and $E_2$ respectively. Similarly, $\phi_L$ is an ordered collection of maps $E_1 \to E_2$. We must argue that together these maps give a bijection from $E_1 \to E_2$. The edge labels of $\bar{G}$ form a partition of the non-loop edges in $G$ and the collections of loops at each vertex of $G$ form a partition of the loops in $G$. Thus, $\phi_E$ gives a bijection from the non-loop edges in $E_1$ to the non-loop edges in $E_2$ and $\phi_L$ gives a bijection from the loops in $E_1$ to the loops in $E_2$. Thus $(\phi_E, \phi_L)$ gives a bijection from $E_1 \to E_2$.

It remains to show that $(\phi_V, (\phi_E, \phi_L))$ is an isomorphism from $G_1 \to G_2$. Let $e \in E_1$.

$\square$