

Random Knot Diagrams

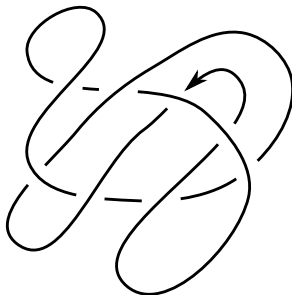
Harrison Chapman (UGA - Graduate student)
joint w/ Jason Cantarella (UGA), Matt Mastin (Wake Forest)

AMS Western Spring Sectionals 2015 (UNLV) – April 18, 2015

Natural questions about knot diagrams

Question

What fraction of 8-crossing diagrams are trefoils?

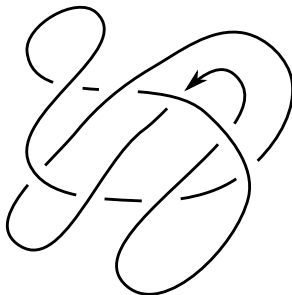


Natural questions about knot diagrams

Question

What fraction of 8-crossing diagrams are trefoils?

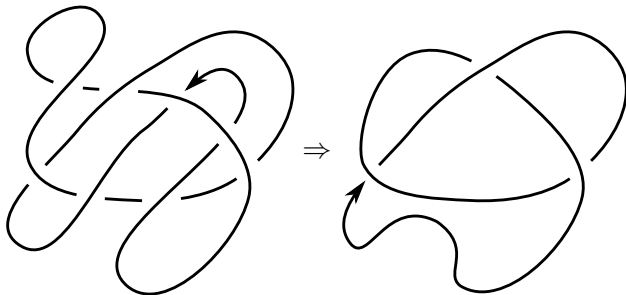
12.48%



Natural questions about knot diagrams

Question

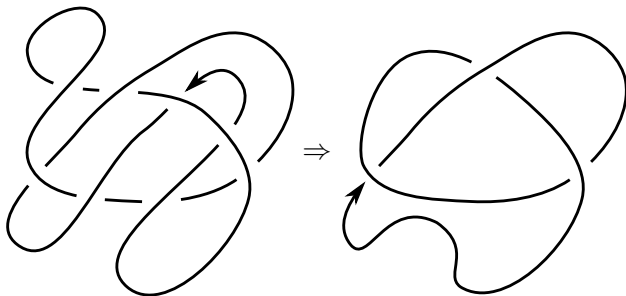
What is the average minimal crossing # of an 8-crossing diagram?



Natural questions about knot diagrams

Question

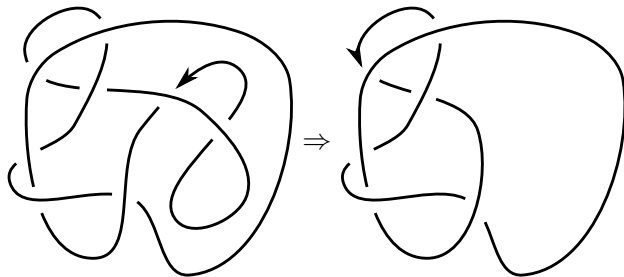
What is the average minimal crossing # of an 8-crossing diagram?
0.52



Natural questions about knot diagrams

Definition

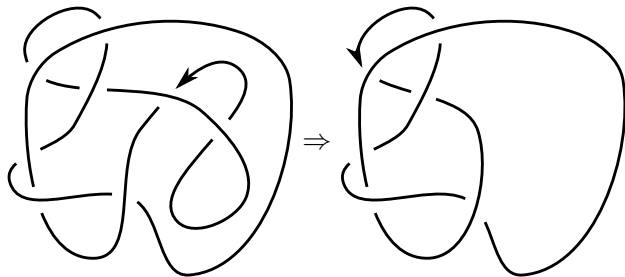
Define an operation on diagrams, **untwisting**: Recursively RI untwist loops in a diagram until there are no more.



Natural questions about knot diagrams

Question

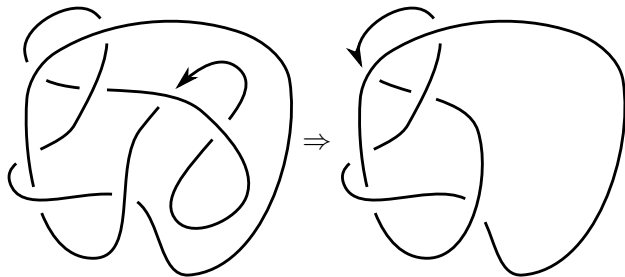
What is the average crossing # of a untwisted 8-crossing diagram?



Natural questions about knot diagrams

Question

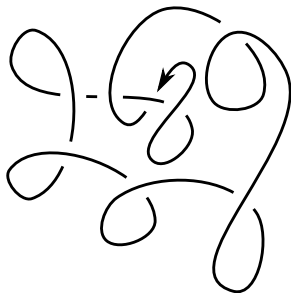
What is the average crossing # of a untwisted 8-crossing diagram?
2.20



Natural questions about knot diagrams

Question

How many 8-crossing diagrams can be untwisted to the unknot?

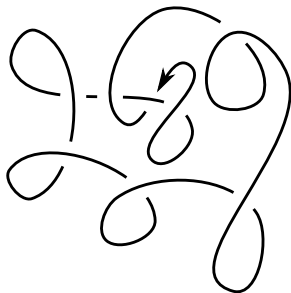


Natural questions about knot diagrams

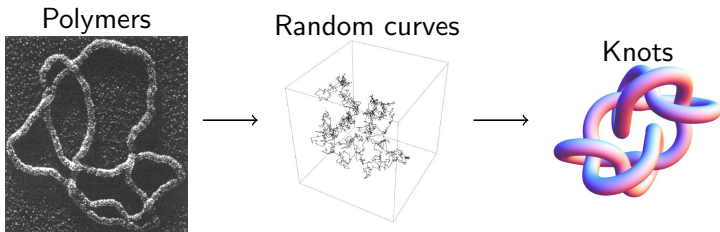
Question

How many 8-crossing diagrams can be untwisted to the unknot?

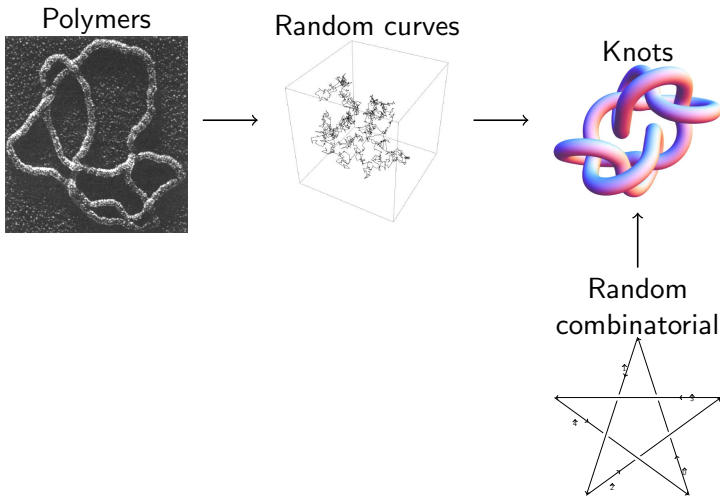
42.05%



Ansatz

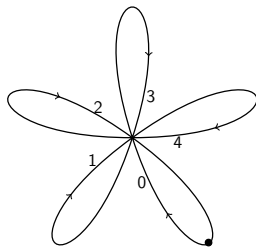


Combinatorial approaches

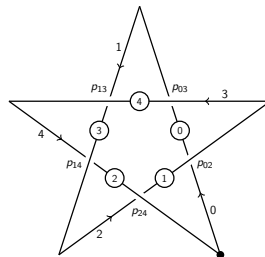


The Petaluma model

Satisfying theorems have been proven for the Petaluma model



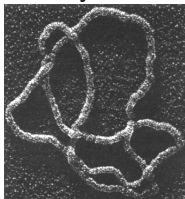
Petal diagram



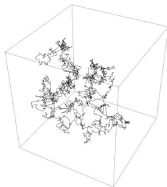
Star diagram

(Diagram from Evan-Zohar, Hass, et al.)

Polymers



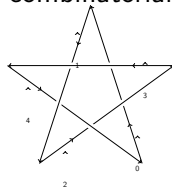
Random curves



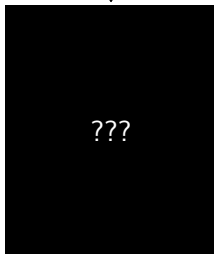
Knots



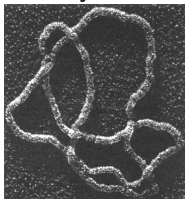
Random
combinatorial



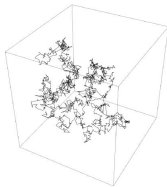
???



Polymers



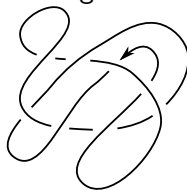
Random curves



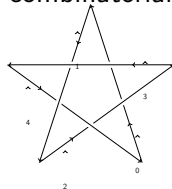
Knots



Random
diagrams



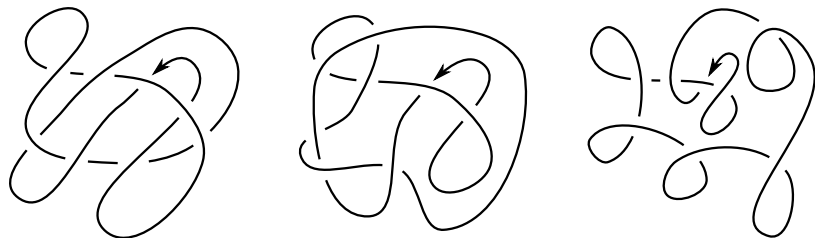
Random
combinatorial



Random diagrams

Definition

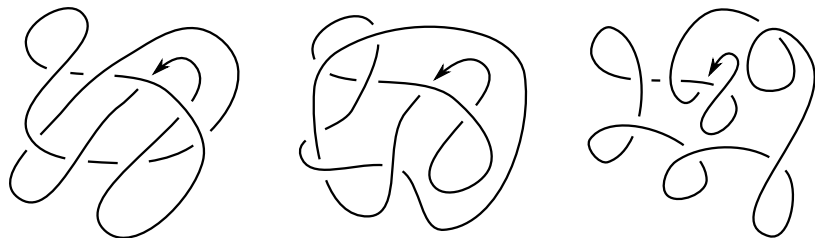
In the **random diagram model** of random knotting, a n -crossing diagram is drawn uniformly from the finite set of n -crossing knot diagrams.



Random diagrams

Definition

A **knot diagram** is a equivalence class of generic immersions of the oriented S^1 into the sphere S^2 together with over-under strand information at each double point up to diffeomorphism of S^2 .



How to enumerate knot diagrams

Definition

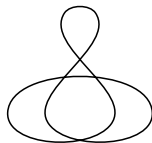
A **knot shadow** is a generic equivalence class of immersions of the unoriented S^1 into the sphere S^2 up to diffeomorphism of S^2 .

- 1 Enumerate shadows
- 2 Assign crossing and orientation information and identify equivalent diagrams

Tabulating knot shadows

Proposition

Knot shadows \leftrightarrow 1-component 4-valent embedded planar multigraphs up to embedded isomorphism

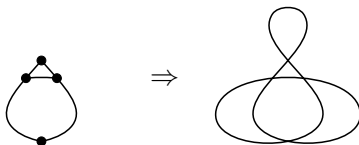


Tabulating knot shadows

Proposition

Knot shadows \leftrightarrow 1-component 4-valent embedded planar multigraphs up to embedded isomorphism

- 1 Add loops and edges to planar simple graphs

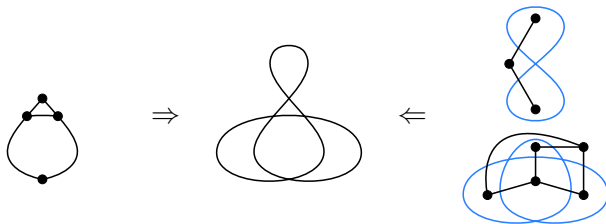


Tabulating knot shadows

Proposition

Knot shadows \leftrightarrow 1-component 4-valent embedded planar multigraphs up to embedded isomorphism

- 1 Add loops and edges to planar simple graphs
- 2 Generate multiquadrangulations by connect sum, take dual graphs

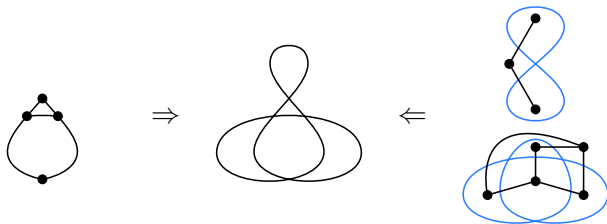


Tabulating knot shadows

Proposition

Knot shadows \leftrightarrow 1-component 4-valent embedded planar multigraphs up to embedded isomorphism

- 1 Add loops and edges to planar simple graphs
- 2 Generate multiquadrangulations by connect sum, take dual graphs



Actually generate all **link shadows**, then restrict to knot shadows

Assign crossings, orientation, identify

- 1 Orient each component. (2 choices)
- 2 Assign over-under information to each vertex. (2^n choices)

n	# knot shadows	2^{n+1} (# shadows)	# knot diagrams
3	6	96	36
4	19	608	276
5	76	4,864	2,936
6	376	48,128	35,872
7	2,194	561,664	484,088
8	14,614	7,482,368	6,967,942
9	106,421	108,975,104	105,555,336

Verifying against existing shadow counts

oriented	$n=0$	1	2	3	4	5
S^2, S^1	1	1	3	9	37	182
S^2	1	1	2	6	21	99
S^1	1	1	2	6	21	97
—	1	1	2	6	19	76

Curves on S^2 . The number of types

V.I. Arnol'd. *Topological Invariants of Plane Curves*

A008989 Number of immersions of unoriented circle into unoriented sphere with n double points.

1, 1, 2, 6, 19, 76, 376, 2194 [list](#) [graph](#) [rcf](#) [links](#) [history](#) [text](#) [internal format](#)

OFFSET

0,3

REFERENCES

V. I. Arnol'd, Topological Invariants of Plane Curves..., American Math.

LINKS

[Table of \$n, a\(n\)\$ for \$n=0..7\$.](#)

CROSSREFS

Sequence in context: [A159119](#) [A181770](#) [A138800](#) * [A057240](#) [A079564](#) [A079453](#)

Adjacent sequences: [A008986](#) [A008987](#) [A008988](#) * [A008990](#) [A008991](#) [A008992](#)

KEYWORD

nonn

AUTHOR

[N. J. A. Sloane](#).

EXTENSIONS

Two more terms from Guy H. Valette (guy.valette(AT)skynet.be), Feb 09 20

STATUS

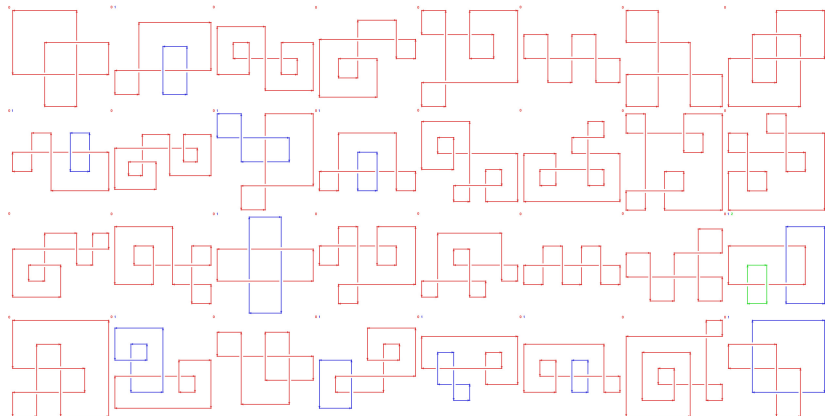
approved

n	# knot shadows
0	1
1	1
2	2
3	6
4	19
5	76
6	376
7	2194
8	14614
9	106421

OEIS A008989

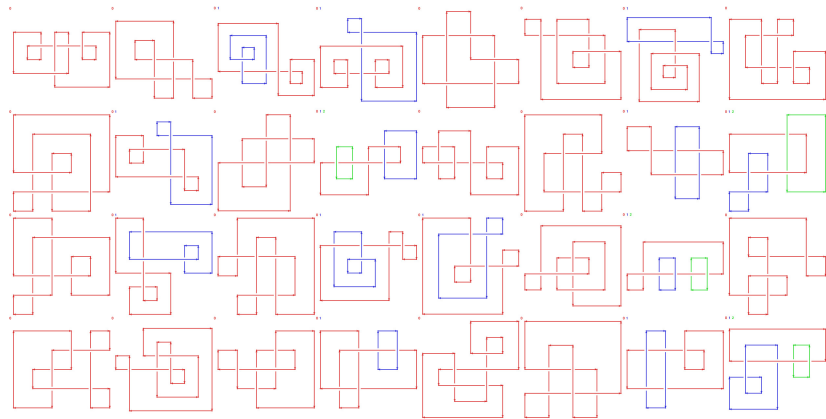
We have not found any existing counts of **diagrams**.

The space of shadows



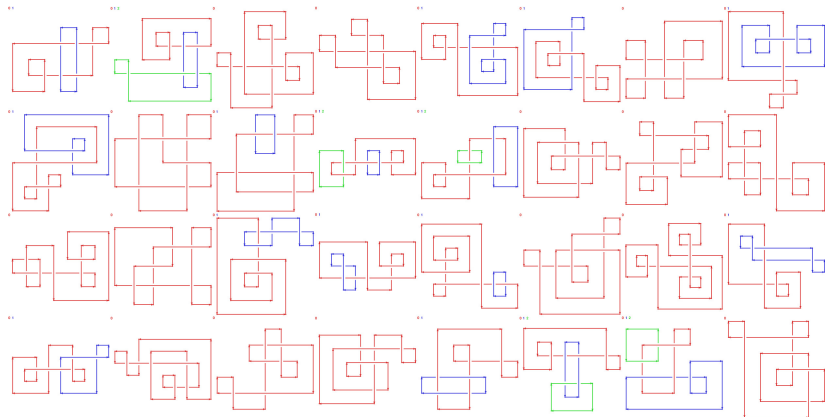
Link shadows. Pictures generated by Eric Lybrand (UGA undergrad).

The space of shadows



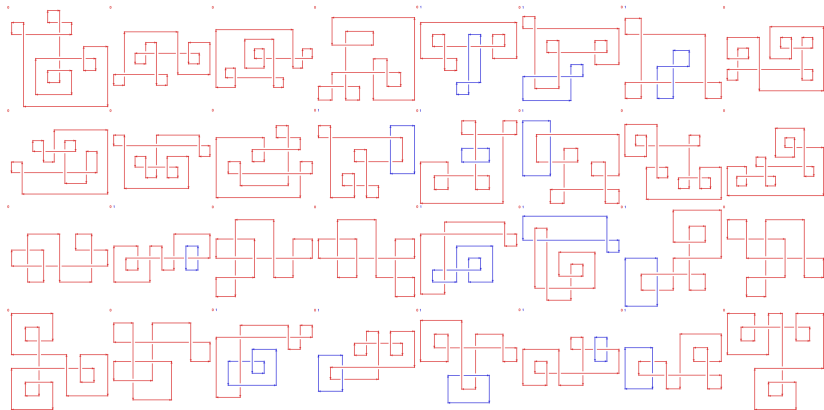
Link shadows. Pictures generated by Eric Lybrand (UGA undergrad).

The space of shadows



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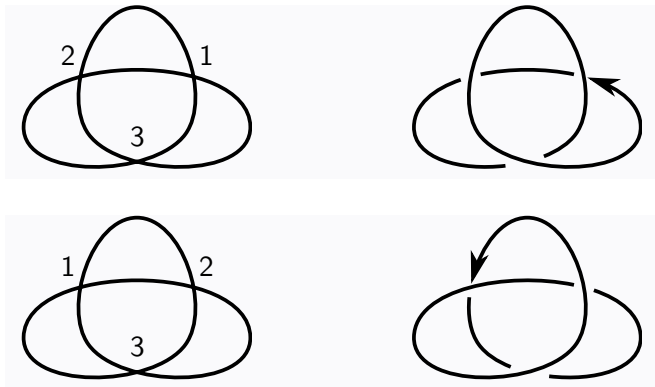
The space of shadows



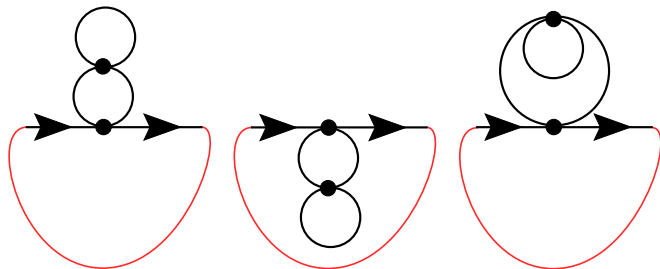
Link shadows. Pictures generated by Eric Lybrand (UGA undergrad).

Tabulation is difficult!

Accounting for symmetry is complicated.



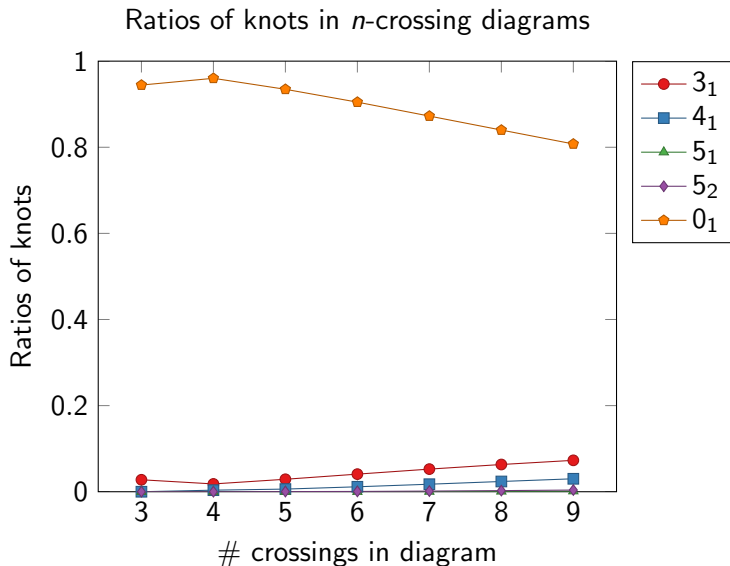
Breaking symmetries could make counting easier



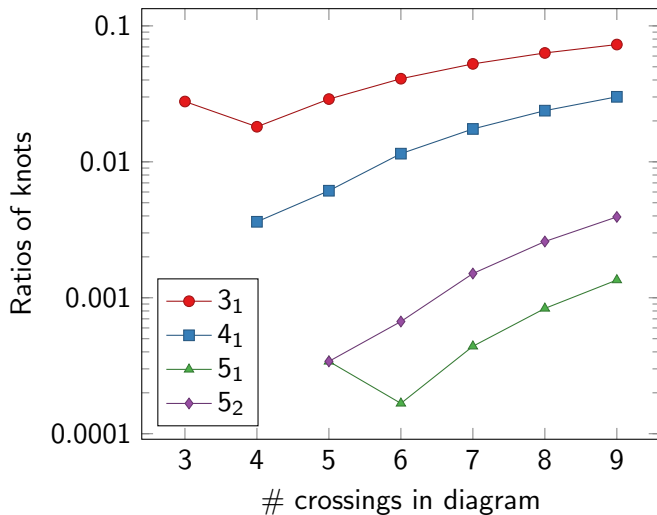
Two-leg diagrams counted by generating function (Bouttier, et al. 2003):

$$G_0 = \frac{24g - 1 + \sqrt{1 - 12g}}{9g(1 + \sqrt{1 - 12g})} = 1 + 2g + 9g^2 + 54g^3 + 378g^4 + \dots$$

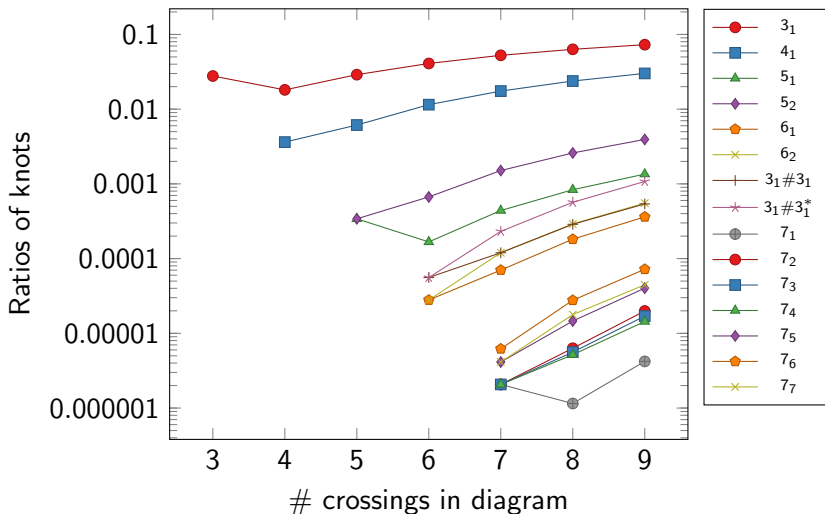
Knotting in diagrams



Ratios of knots in n -crossing diagrams (log scale)



Ratios of knots in n -crossing diagrams (log scale)



A question on unknotting

Theorem ((Frisch-Wassermann-Delbrück Conjecture)
Sumners-Whittington 1988)

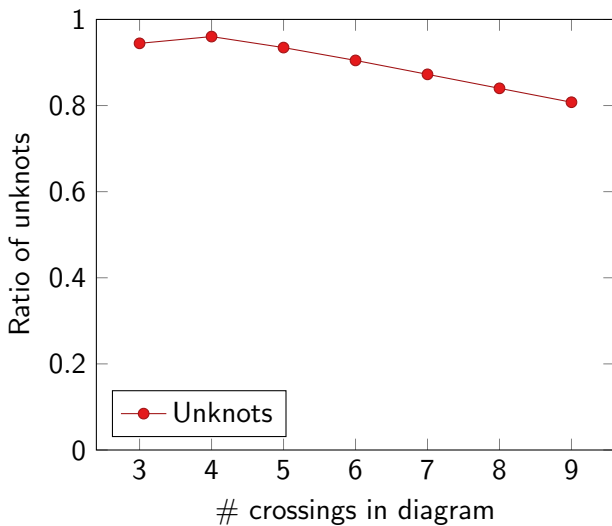
The ratio of unknots in random n -edge self-avoiding lattice polygons tends to zero exponentially with n .

Conjecture

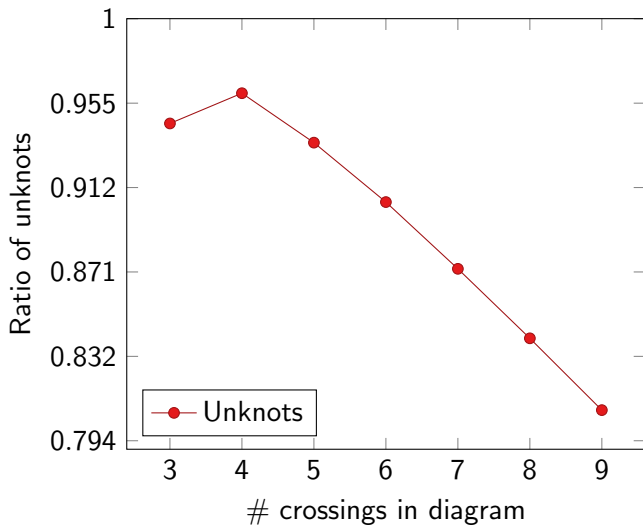
The ratio of unknots in diagrams tends to zero as n increases.
(Exponentially?)

Exist theorems on planar (simple) graphs (Dowden 2008) similar to the Kesten pattern theorem (Kesten 1963) for lattice walks

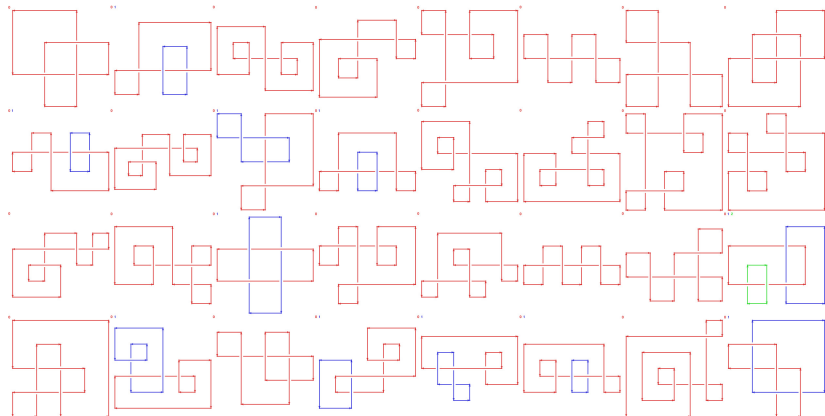
Ratio of unknots in n -crossing diagrams



Ratio of unknots in n -crossing diagrams (log scale)

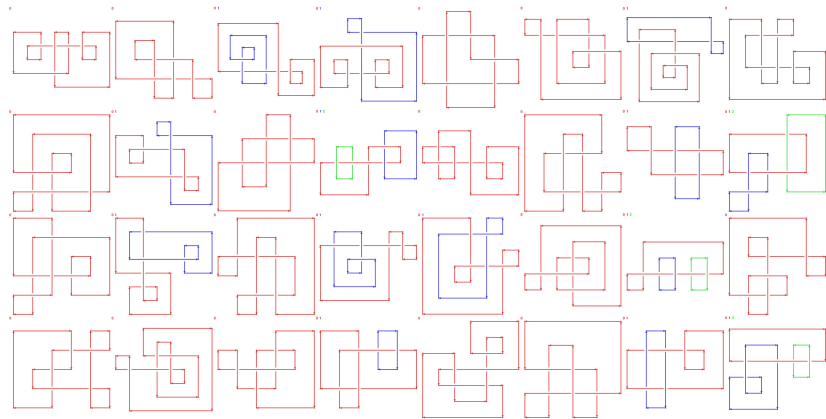


Why so many unknots?



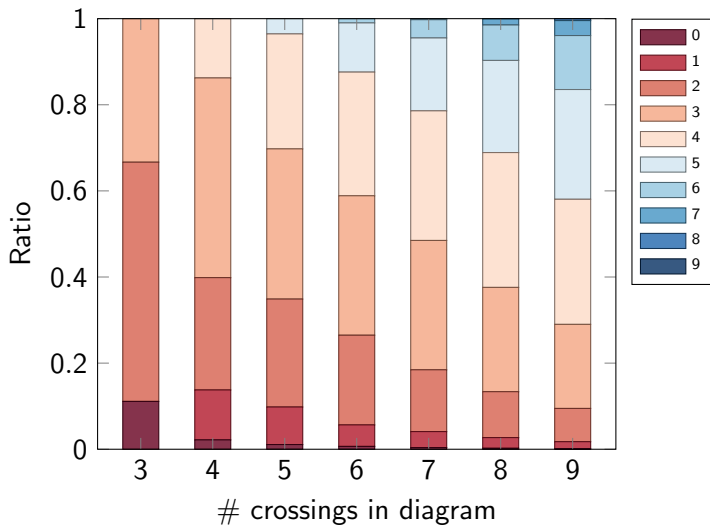
Link shadows. Pictures generated by Eric Lybrand (UGA undergrad).

Why so many unknots?

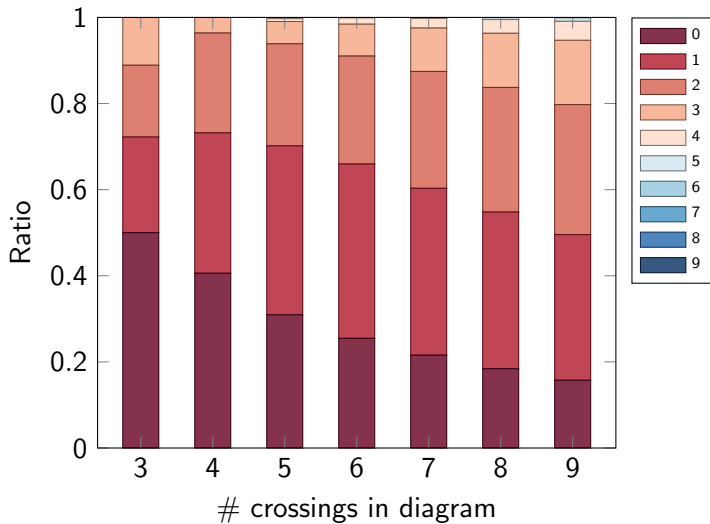


Link shadows. Pictures generated by Eric Lybrand (UGA undergrad).

Reidemeister-I loops (monogons) in diagrams



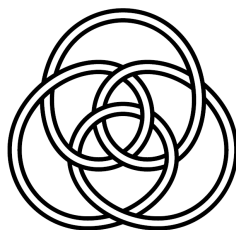
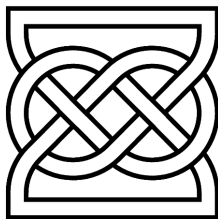
Bigons in diagrams



Basic polyhedra 8^* and 9^*

Proposition

Conway's basic polyhedra 8^ and 9^* are the only diagrams in ≤ 9 crossings with no monogons or bigons.*



8_{18} (left), 9_{40} (right).

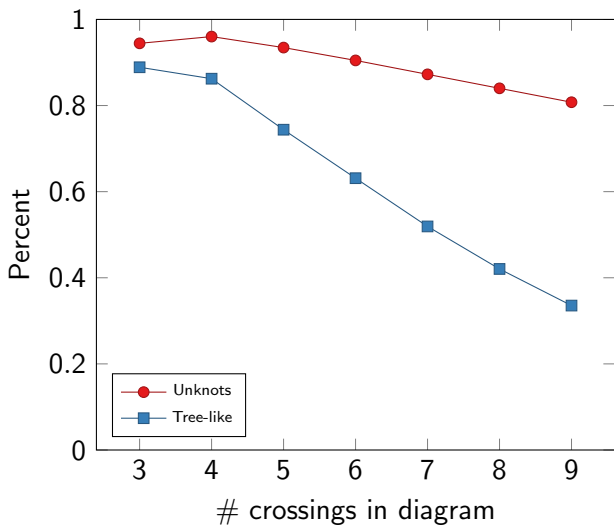
Some shadows are always unknots

A **tree-like curve** (Aicardi 1994) is a knot shadow which can be untwisted to the trivial shadow.

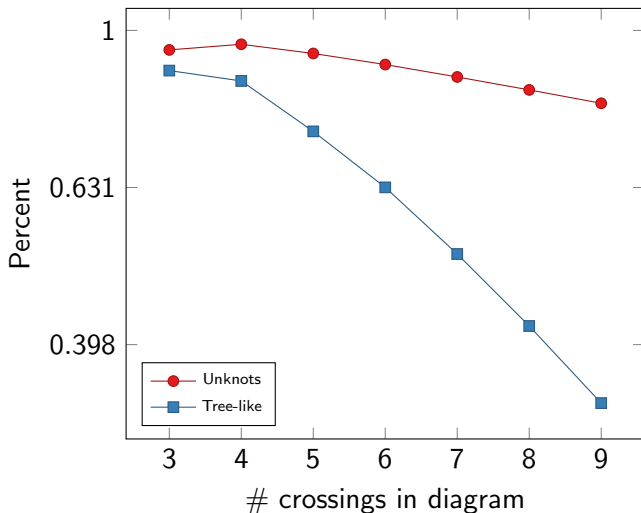


Tree-like curves \Rightarrow lower bound on unknottedness.

Ratio of unknots, tree-like curves in n -crossing diagrams

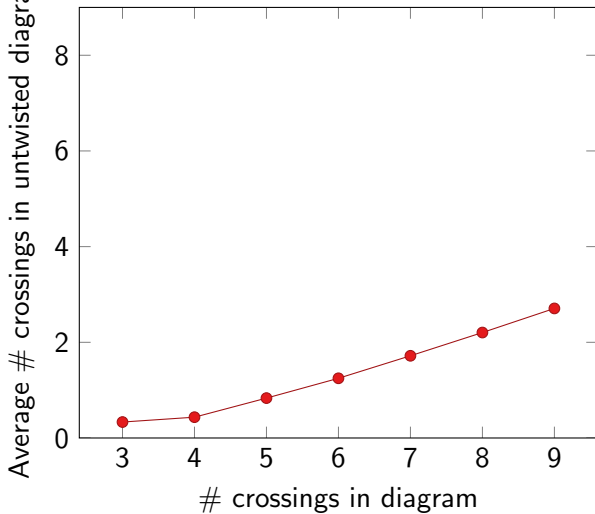


Ratio of unknots, tree-like curves in n -crossing diagrams (log scale)

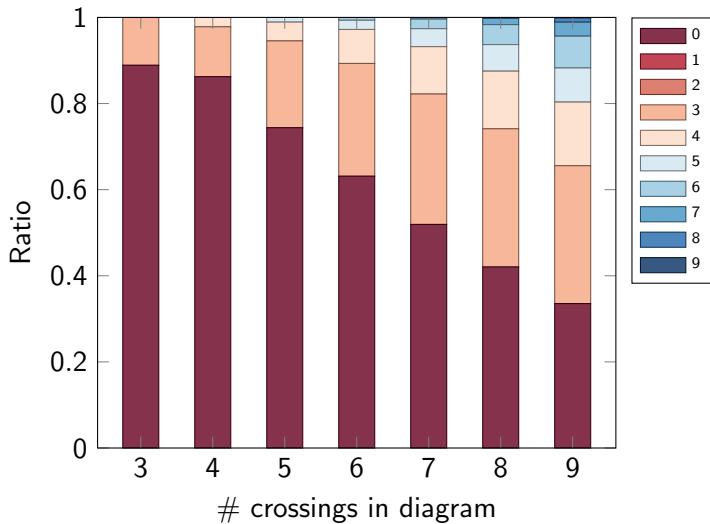


Tree-like curves alone explain only some of the unknot fraction

Crossing # vs. Average untwisted crossing #



Untwisted crossing

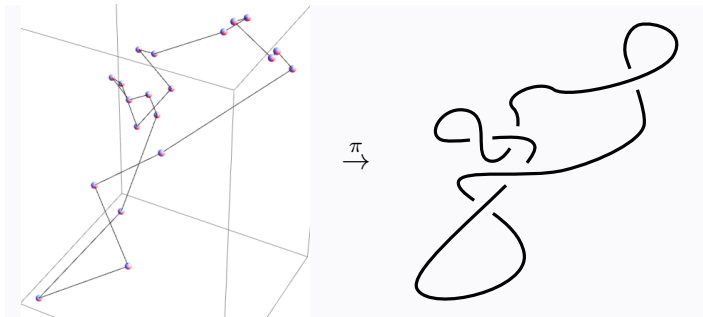


Questions to answer

Random curves project to diagrams.

Question

How does the pushforward measure differ from uniform diagram sampling? (c.f. Hua, Nguyen, Raghavan, Arsuaga, Vasquez 2005)



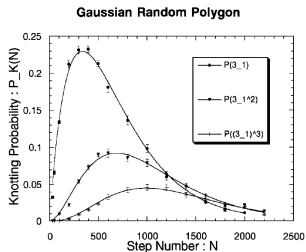
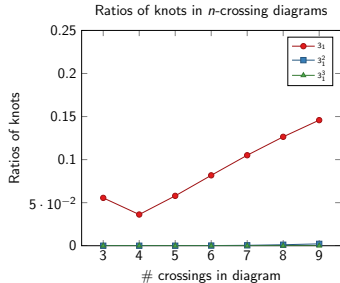
(from Shonkwiler)

Questions to answer

Question

Let $PD(n, K)$ be the probability of knot type K in a random diagram of n crossings and $PC(n, K)$ the probability of knot type K in a random polygon of n edges.

If n, m are such that $PD(n, 0_1) = PC(m, 0_1)$, is there a relationship between $PD(n, K) = PC(m, K)$ for other knot types K ?



(from Deguchi, et. al.)

Questions to answer

Fact

No one will enumerate the 100-crossing knot diagrams.

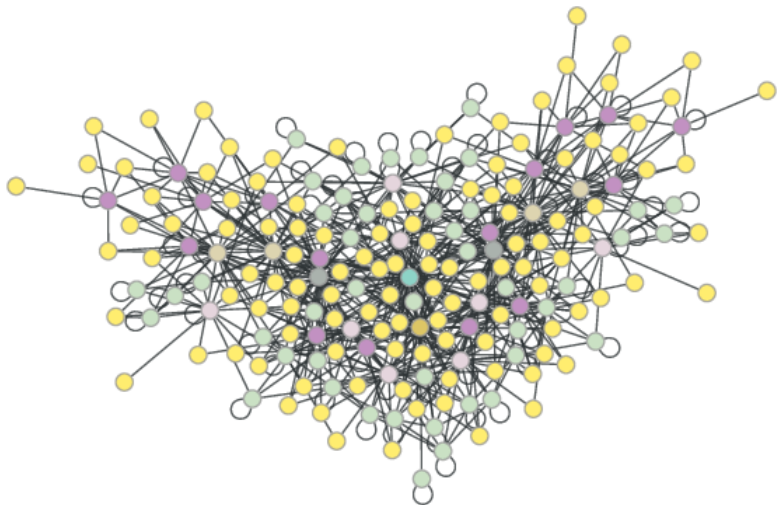
Question

Can we generate uniformly sampled random 100-crossing knot diagrams **another way**?

Future direction: Link diagrams

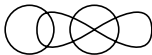
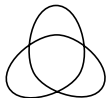
n	# link shadows	# knot shadows
0	1	1
1	1	1
2	3	2
3	7	6
4	30	19
5	124	76
6	733	376
7	4586	2194
8	33373	14614
9	259434	106421

Future direction: Knot distances



Thank you!

Coming soon: Cantarella, Chapman, Mastin. *Knot probabilities in random diagrams.*



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