Random Knot Diagrams

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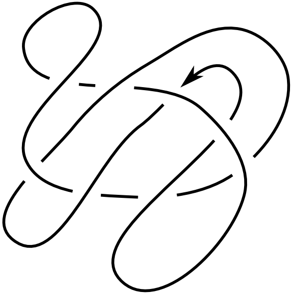
AMS Western Spring Sectionals 2015 (UNLV) – April 18, 2015

Natural questions about knot diagrams

Question

What fraction of 8-crossing diagrams are trefoils?

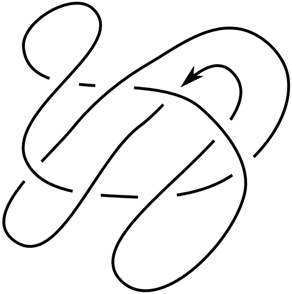


Natural questions about knot diagrams

Question

What fraction of 8-crossing diagrams are trefoils?

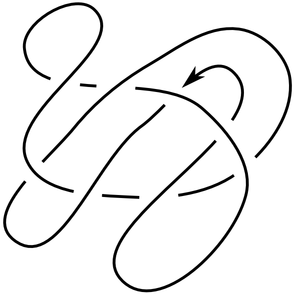
12.48%



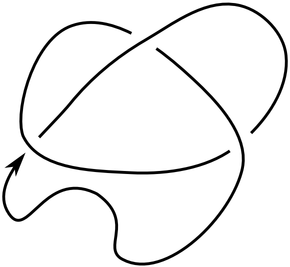
Natural questions about knot diagrams

Question

What is the average minimal crossing $\#$ of an 8-crossing diagram?



$\Rightarrow$

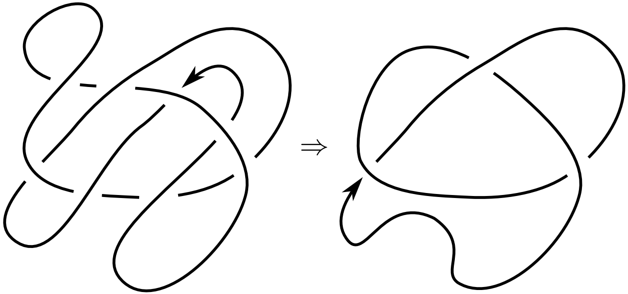


Natural questions about knot diagrams

Question

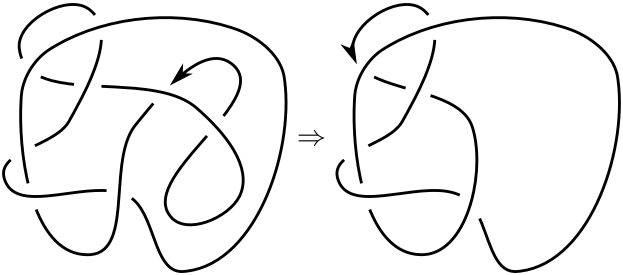
What is the average minimal crossing # of an 8-crossing diagram?

0.52



Natural questions about knot diagrams

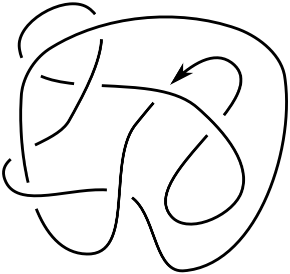
Define an operation on diagrams, **delooping**: Recursively RI untwist monogon loops in a diagram until there are no more.



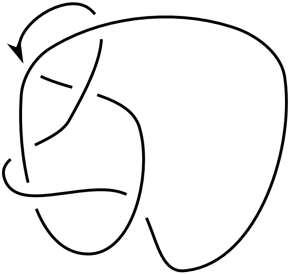
Natural questions about knot diagrams

Question

What is the average crossing $\#$ of a delooped 8-crossing diagram?



$\Rightarrow$

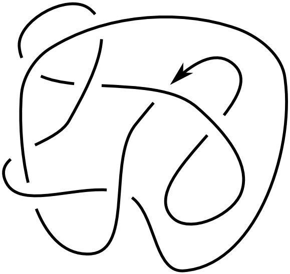


Natural questions about knot diagrams

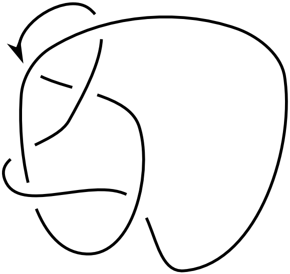
Question

What is the average crossing $\#$ of a delooped 8-crossing diagram?

2.20



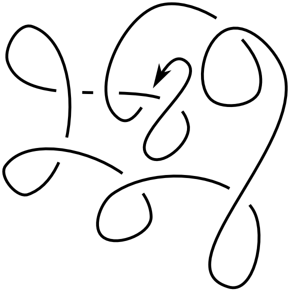
$\Rightarrow$



Natural questions about knot diagrams

Question

How many 8-crossing diagrams can be delooped to the unknot?

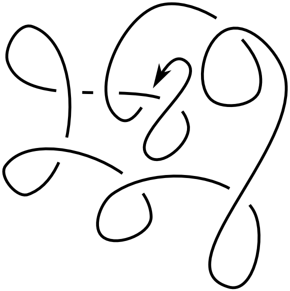


Natural questions about knot diagrams

Question

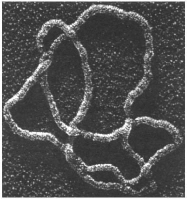
How many 8-crossing diagrams can be delooped to the unknot?

42.05%



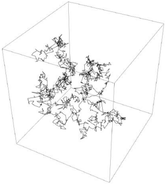
Ansatz

Polymers



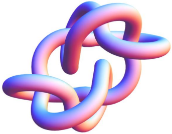
$\longrightarrow$

Random curves



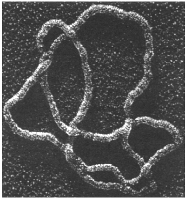
$\longrightarrow$

Knots

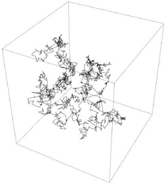


Combinatorial approaches

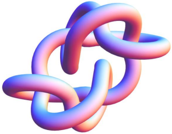
Polymers



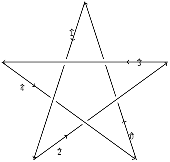
Random curves



Knots

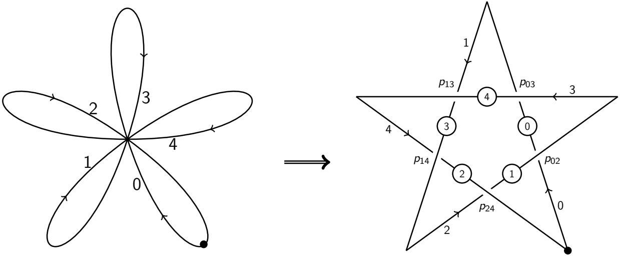


Random combinatorial



The Petaluma model

Satisfying theorems have been proven for the Petaluma model

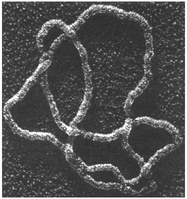


Petal diagram

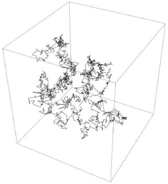
Star diagram

Figure: Diagram from Evan-Zohar, Hass, et al.

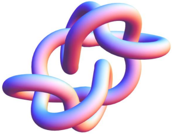
Polymers



Random curves

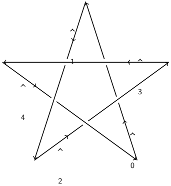


Knots

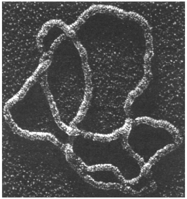


???

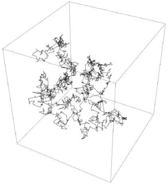
Random combinatorial



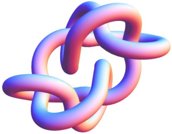
Polymers



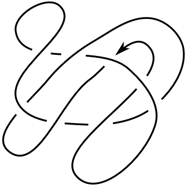
Random curves



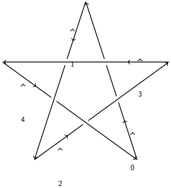
Knots



Random diagrams



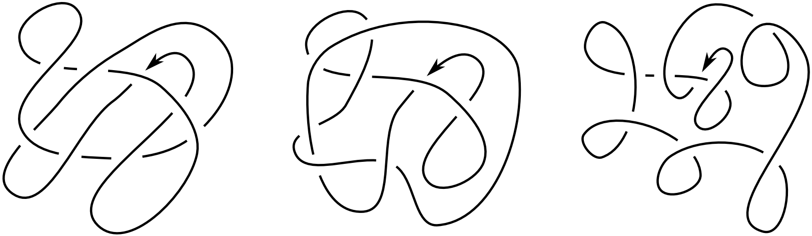
Random combinatorial



Random diagrams

Definition

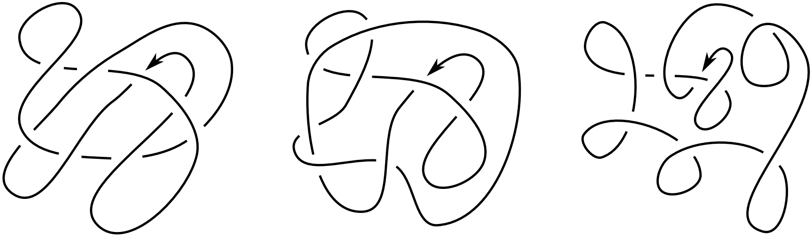
In the **random diagram model** of random knotting, a n-crossing diagram is drawn uniformly from the finite set of n-crossing knot diagrams.



Random diagrams

Definition

A **knot diagram** is a generic embedding of the oriented S^1 into the sphere S^2 together with over-under strand information at each double point.



Diagrams from shadows

Sample diagrams uniformly through tabulation:

1. Enumerate shadows (unoriented graph structure behind diagrams).
2. Expand shadows into diagrams.

How many shadows?

oriented	$n=0$	1	2	3	4	5
S^2, S^1	1	1	3	9	37	182
S^2	1	1	2	6	21	99
S^1	1	1	2	6	21	97
–	1	1	2	6	19	76

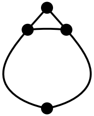
Curves on S^2. The number of types

A008989	Number of immersions of unoriented circle into unoriented sphere with n double points.
	1, 1, 2, 6, 19, 76, 376, 2194 (list; graph; refs; listen; history; text; internal format)
OFFSET	0,3
REFERENCES	V. I. Arnold, Topological Invariants of Plane Curves..., American Math.
LINKS	Table of n, a(n) for n=0..7.
CROSSREFS	Sequence in context: A150119 A181770 A138800 * A057240 A079564 A079453 Adjacent sequences: A008986 A008987 A008988 * A008990 A008991 A008992
KEYWORD	nonn
AUTHOR	N. J. A. Sloane.
EXTENSIONS	Two more terms from Guy H. Valette (guy.valette(AT)skynet.be), Feb 09 20
STATUS	approved

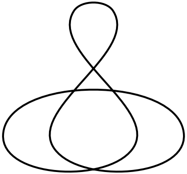
n	# knot shadows
0	1
1	1
2	2
3	6
4	19
5	76
6	376*
7	2194*
8	14614**
9	106421**

Tabulating knot shadows

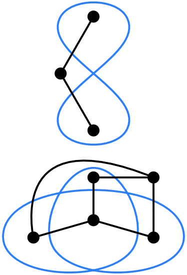
Generated table of knot shadows two different ways as a check. Both methods use features from McKay and Brinkmann's `plantri`.



$\Rightarrow$



$\Leftarrow$



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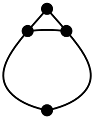
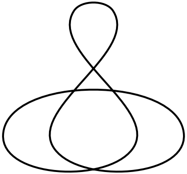
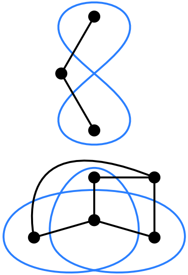


$\Rightarrow$

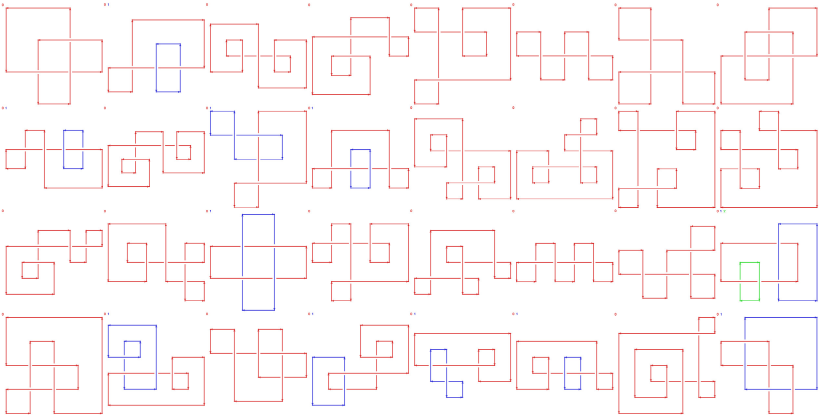


$\Leftarrow$



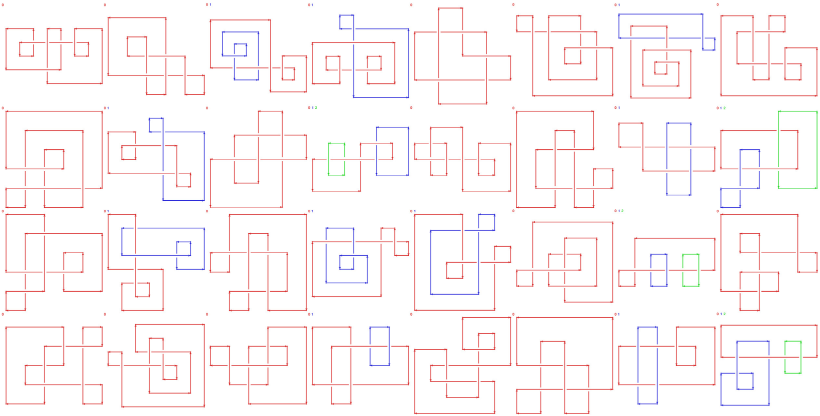
Actually generate all **link shadows**, then restrict to knot shadows

The space of shadows



Link shadows. Pictures generated by Eric Lybrand (UGA undergrad).

The space of shadows



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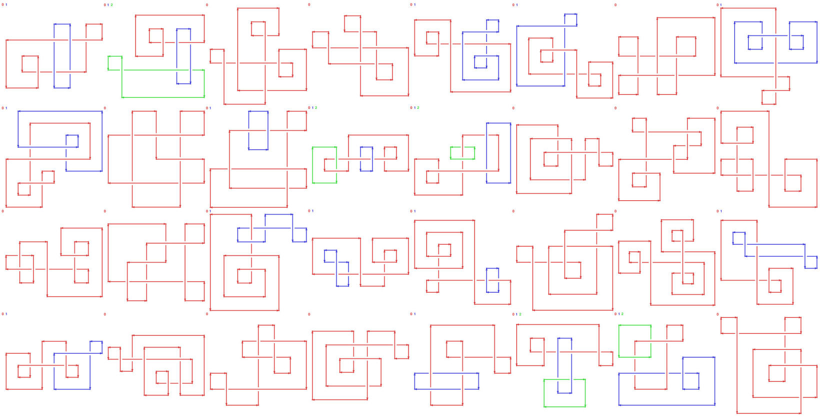
The space of shadows



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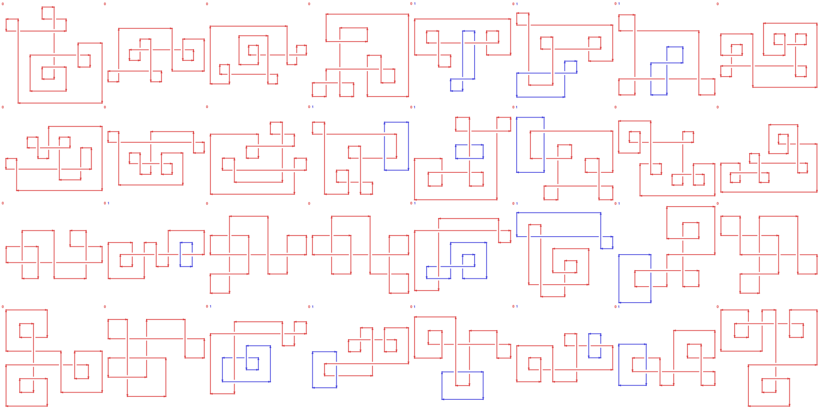
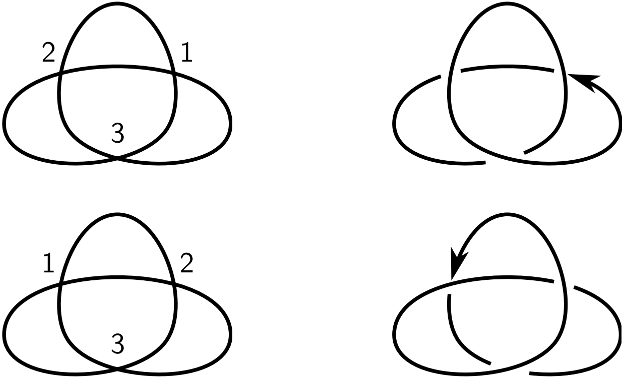
The space of shadows



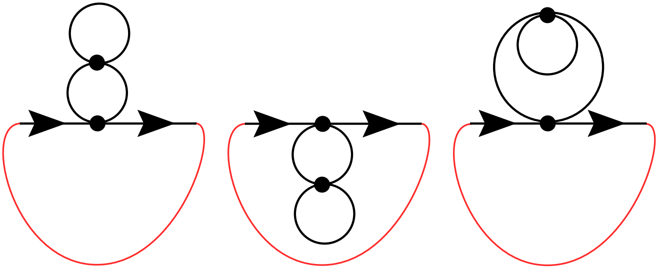
Link shadows. Pictures generated by Eric Lybrand (UGA undergrad).

Tabulation is difficult!

Accounting for symmetry is complicated.



Breaking symmetries could make counting easier



Two-leg diagrams counted by generating function (Bouttier, et. al):

$$G_0 = \frac{24g - 1 + \sqrt{1-12g}}{9g(1+\sqrt{1-12g})} = 1 + 2g + 9g^2 + 54g^3 + 378g^4 + \cdots$$

From shadows to diagrams

Expansion of n-crossing shadows to diagrams procedure:

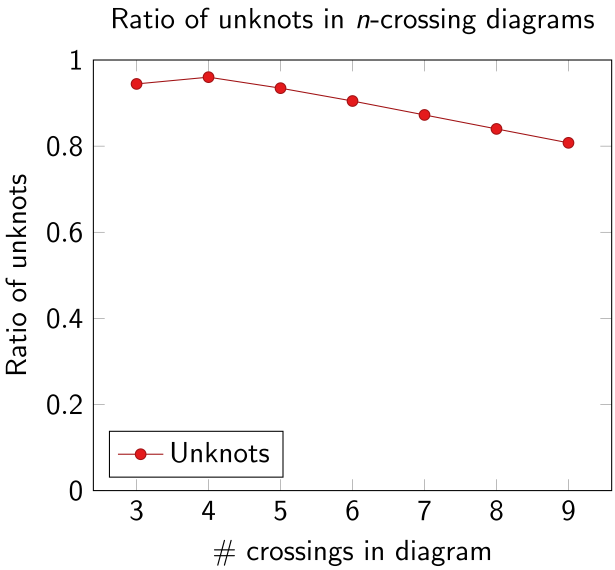
1. Orient each component. (2 choices)
2. Assign over-under information to each vertex. (2^n choices)
3. Group diagrams by isomorphism.

How many knot diagrams?

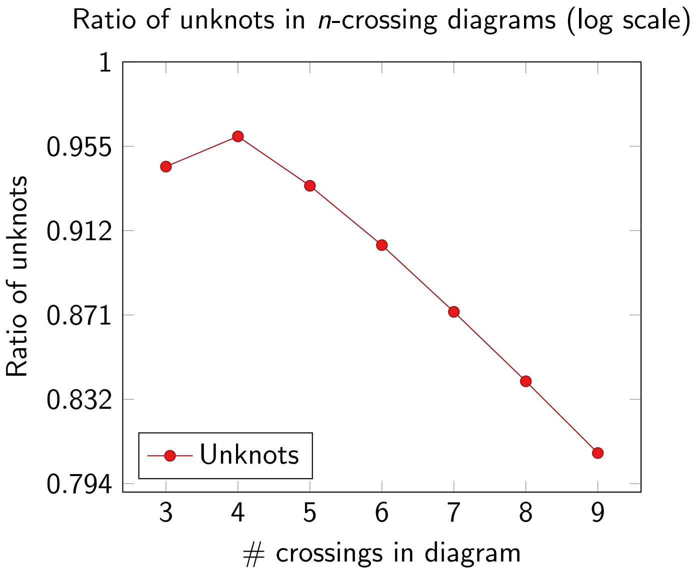
n	# knot shadows	2^{n+1} (# shadows)	# knot diagrams
3	6	96	36
4	19	608	276
5	76	4,864	2,936
6	376	48,128	35,872
7	2,194	561,664	484,088
8	14,614	7,482,368	6,967,942
9	106,421	108,975,104	105,555,336

Knotting probabilities

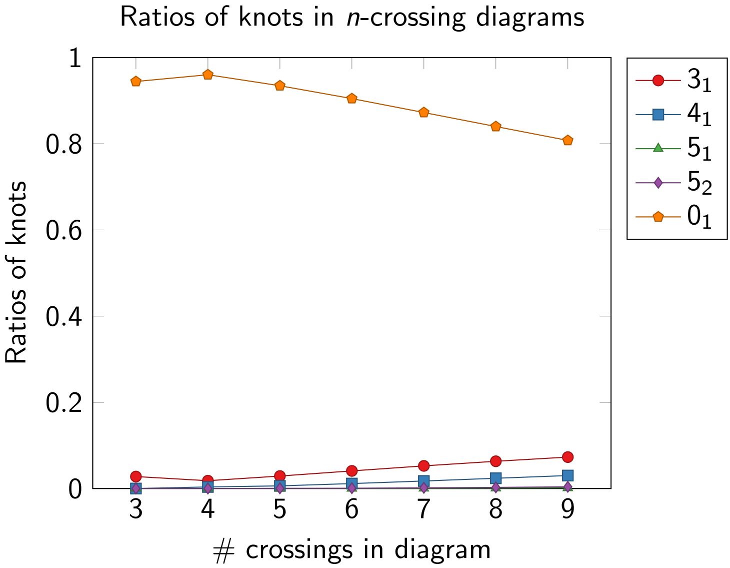
- Able to run searches across entire space computationally.
- Can check knot type of each diagram (HOMFLY is typically enough for our low crossing number)
- Possible to run many different types of searches



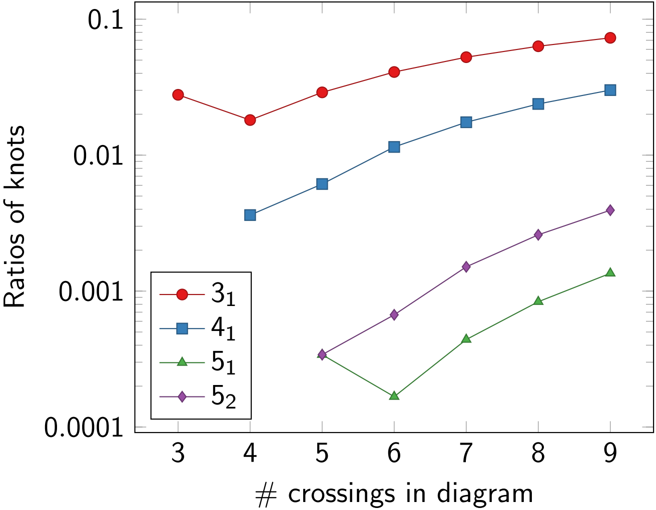
Ratio of unknots in n-crossing diagrams
Ratio of unknots
crossings in diagram
0
0.2
0.4
0.6
0.8
1
3
4
5
6
7
8
9
Unknots



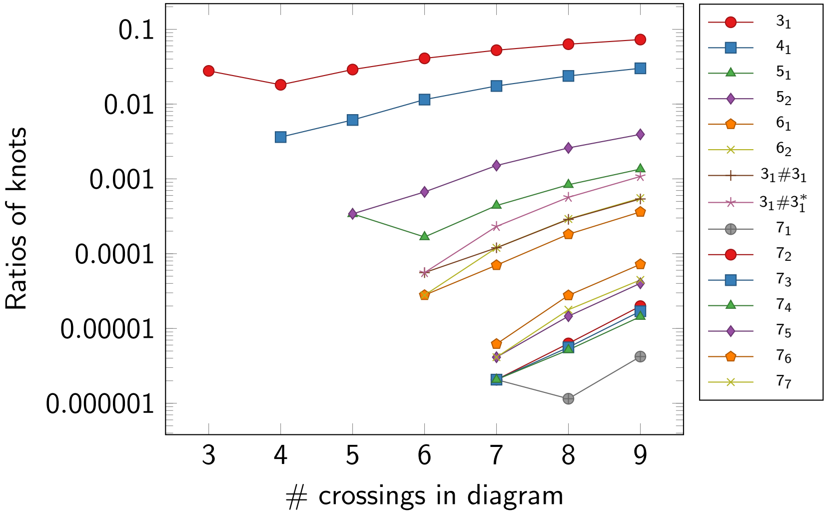
Ratio of unknots in n-crossing diagrams (log scale)
Ratio of unknots
1
0.955
0.912
0.871
0.832
0.794
3
4
5
6
7
8
9
crossings in diagram
Unknots

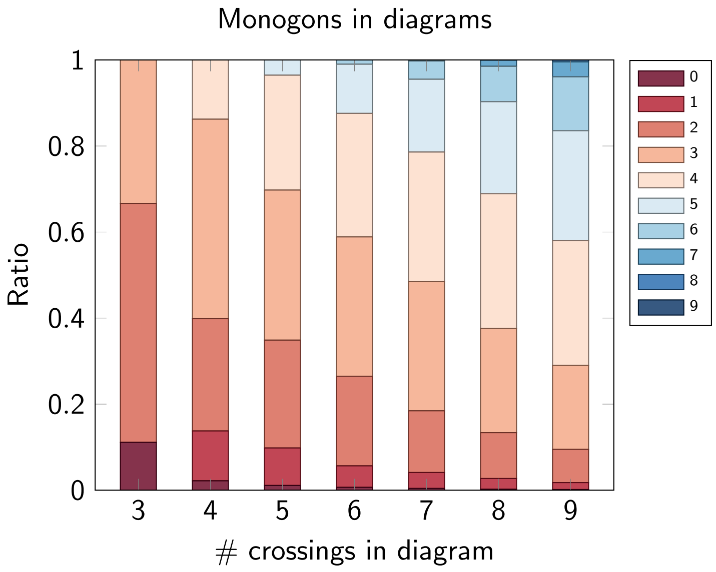

Ratios of knots in n-crossing diagrams
Ratios of knots
0
0.2
0.4
0.6
0.8
1
3
4
5
6
7
8
9
crossings in diagram
3_1
4_1
5_1
5_2
0_1

Ratios of knots in *n*-crossing diagrams (log scale)

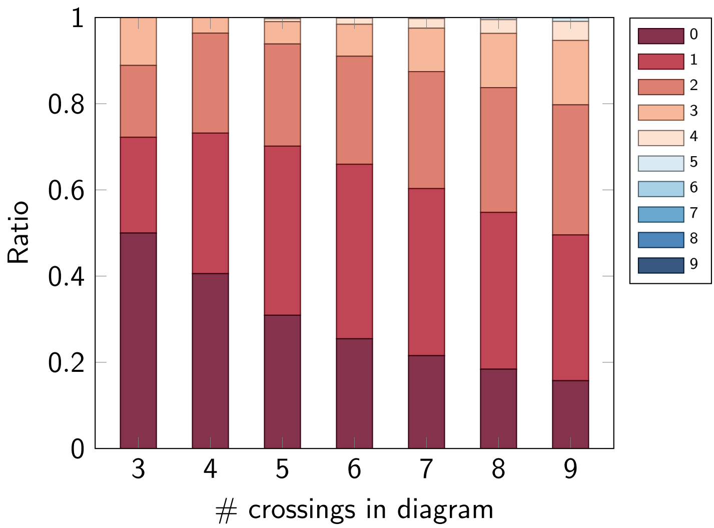


Ratios of knots in *n*-crossing diagrams (log scale)

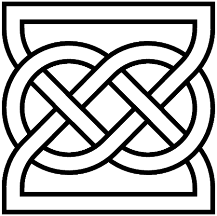



Monogons in diagrams
Ratio
1
0.8
0.6
0.4
0.2
0
3
4
5
6
7
8
9
crossings in diagram
0
1
2
3
4
5
6
7
8
9

Bigons in diagrams



Basic polyhedra 8^* and 9^*



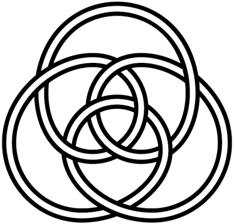
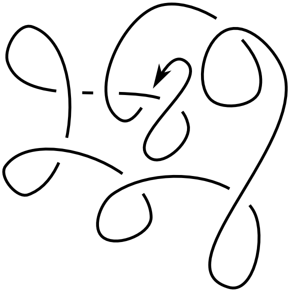


Figure: 8_{18} (left), 9_{40} (right).

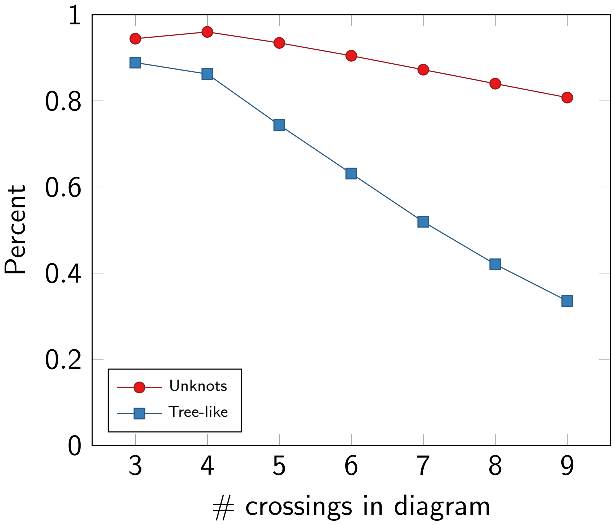
Tree-like curves

A **tree-like curve** is a knot shadow which can be delooped to the trivial shadow.

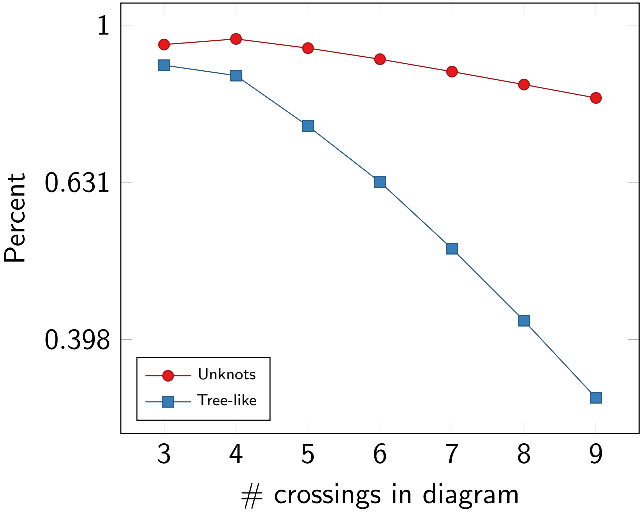


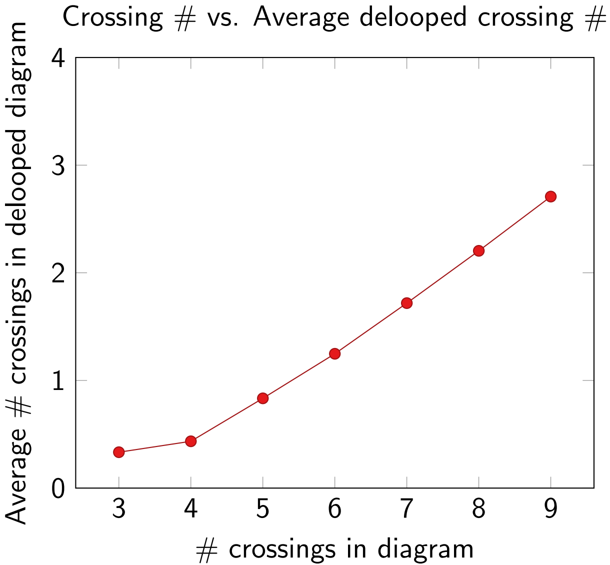
Tree-like curves $\Rightarrow$ lower bound on unknottedness.

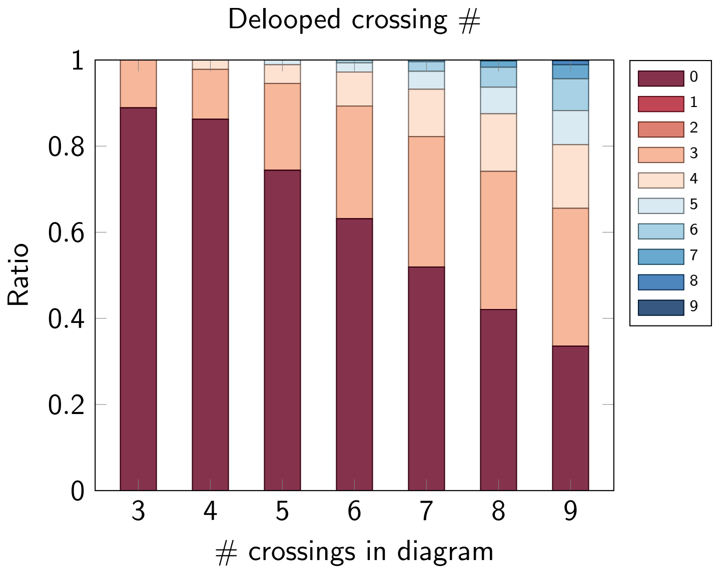
Ratio of unknots, tree-like curves in n-crossing diagrams



Ratio of unknots, tree-like curves in n-crossing diagrams (log scale)




Crossing # vs. Average delooped crossing #
Average # crossings in delooped diagram
crossings in diagram
0
1
2
3
4
3
4
5
6
7
8
9


Delooped crossing #
Ratio
1
0.8
0.6
0.4
0.2
0
3
4
5
6
7
8
9
crossings in diagram
0
1
2
3
4
5
6
7
8
9

Questions to answer

Theorem (Sumners-Wittington)

The ratio of unknots in random n-edge self-avoiding lattice polygons tends to zero exponentially with n.

Conjecture

The ratio of unknots in diagrams tends to zero as n increases. (Exponentially?)

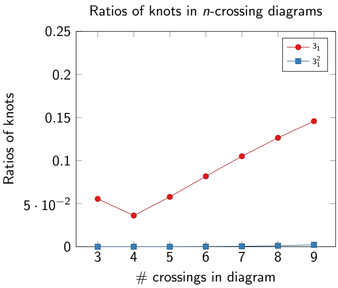
Questions to answer

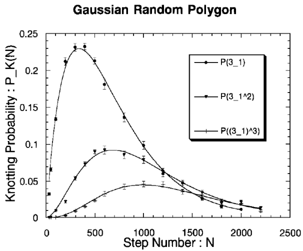
Random curves project to diagrams.

Question

How does the pushforward measure differ from uniform diagram sampling?

Questions to answer





(from Deguchi, et. al.)

Questions to answer

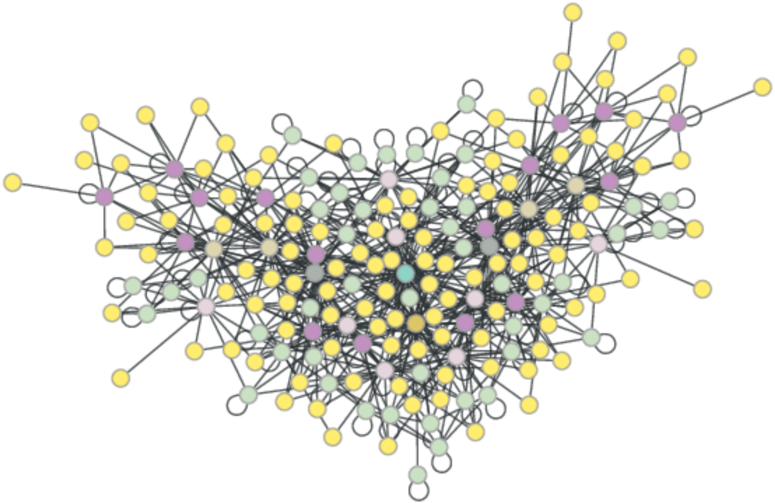
Question

Can we sample diagrams uniformly without enumeration?

Link diagrams

n	# link shadows	# knot shadows
0	1	1
1	1	1
2	3	2
3	7	6
4	30	19
5	124	76
6	733	376
7	4586	2194
8	33373	14614
9	259434	106421

Knot distances



Thank you!

Coming soon: *Knot probabilities in random diagrams*.

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