On the asymptotics of uniformly random knot diagrams

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We study random knotting by considering knot and link diagrams as decorated, (rooted) combinatorial maps on spheres, and pulling them uniformly from among sets of a given number of vertices n. We prove some asymptotic results and examine how quickly this behavior occurs in practice. En route, we show how some asymptotic laws for unlabeled maps apply to decorated maps as well.

I. INTRODUCTION

There is a dearth of models for drawing random knots; self avoiding lattice walks [cite], random space polygons [cite], random braid words [cite], *Petaluma* [cite], et. al. In this paper we will discuss the *random diagram model* under which *knot diagrams* are drawn uniformly from the set of all diagrams with a given number of crossings. Alternating knot and link diagrams have been studied [cite] but little is published about the knottiness of arbitrary random diagrams of large size.

In this paper we begin by considering a slightly different object, *rooted diagrams*, which break symmetries (as opposed to in [1]). We are then able to prove that in the limit, knot diagrams behave similarly to rooted diagrams, so that these results carry over.

II. DEFINITIONS

II.1. Preliminaries

II.1.1. Knots, links, and tangles

A *knot* is an isotopy class of embedding of the circle into S^3. A *link* is an isotopy class of embedding of one or more circles into S^3. Both of the prior are considered up to *ambient isotopy* of the embedded circles. A *knot diagram* (resp. *link diagram*) is a generic immersion of a circle (resp. any number of circles) into the sphere S^2 (generic in that all intersection points are double points) together with over-under information at each double strand. The study of links and knots is well known to be equivalent to the study of link diagrams and knot diagrams (more formally defined below) up to the so-called *Reidemeister moves*, shown in figure 1.

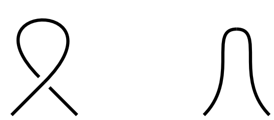

FIG. 1. The three Reidemeister moves.

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A $2k$-*leg curve* is a generic immersion of k intervals into the sphere so that all $2k$ interval ends lie in the same face of the sphere. A $2k$-*tangle* is a $2k$-leg curve together with over-under information at each double point.

FIG. 2. A 6-leg curve, and a 6-tangle with its underlying structure.

II.1.2. Combinatorial maps

We wish to examine the underlying *map* structure of knot and link diagrams. If Σ is a surface, then its Euler characteristic $\chi(\Sigma)$ is a topological invariant. The type g of a surface is defined by $\chi(\Sigma) = 2 - 2g$ (this definition agrees with the genus g of orientable surfaces).

Definition. *A* map with n vertices *M is a graph $\Gamma(M)$ embedded on a surface Σ of type g so that every connected component of $\Sigma \setminus M$ is a topological disk. The connected components of $\Sigma \setminus M$ are called the* faces *of M. The map M is* 4-regular *or* quartic *if every vertex in the underlying graph $\Gamma(M)$ has degree 4. A* rooted map *is a map together with a single edge marked with a direction.*

If Σ is the sphere, then the map M is called planar*.*

In this paper we will only consider Σ to be an orientable surface (of genus g). Considering $g \neq 0$ extends results to *virtual* diagrams, but we will be most concerned with results for classical knot theory, where $\Sigma = S^2$.

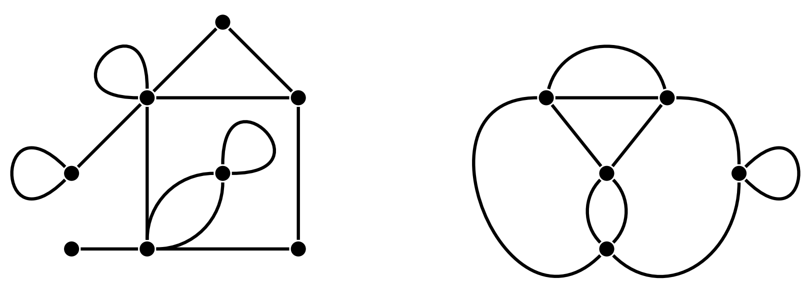

FIG. 3. Two planar maps. The map on the right is in the class of knot shadows.

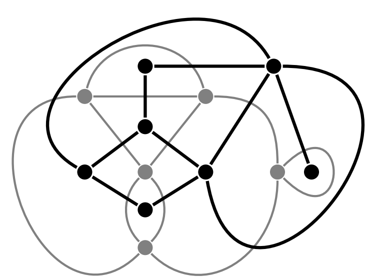

FIG. 4. Planar quadrangulation which is dual to a knot shadow.

There are alternate, equivalent definitions of rooting a map. Let M be a rooted map with root edge ρ which points from vertices v_1 to v_2. Then the unique face f which ρ goes counterclockwise around is the *root face*. The root face is the *exterior face*, and is typically drawn as such. The vertex v_2 at the tip of ρ is the *root vertex*.

Maps have a well defined notion of the *dual map*; a map $M = (V, E, F)$ has dual $M^* = (F, E^*, V)$, where there is an edge $(f_1, f_2) \in E^*$ if f_1 is adjacent to f_2 in M (faces are adjacent if they share an edge on their boundaries). The dual graph of a 4-regular map is a *quadrangulation*, i.e. a map for which every face has four bounding edges. A map is *simple* if it contains no parallel edges or self loops (its underlying graph is simple). Given a rooted map M, its dual is rooted as follows. Let ρ be the root edge of M pointing from v_1 to the root vertex v_2 be adjacent to the face f_1 and the root face f_2. Then (f_1, f_2) is the dual root edge and directed from f_1 to f_2, and f_2, v_2 are the dual root vertex and root face, respectively (and the dual of a dual rooted map is the original rooted map).

A map M is a CW-complex for the surface in which it embeds. Given a cycle γ of edges in M,

Definition. *A rooted map P is a* submap *of a larger (possibly unrooted) map M if there exists a cycle of k (possibly repeated) edges in M so that one of the two halves of M separated by the cycle is identical*

II.2. Diagrams and shadows

By breaking symmetries with a root, we may study certain classes of planar maps by way of the celebrated bijection with *blossom trees*[2]. We carry this idea to link and knot diagrams:

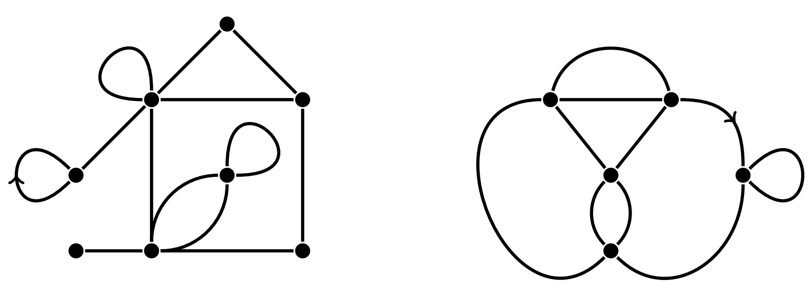

FIG. 5. Two rooted planar maps. The map on the right is in the class of rooted knot shadows.

Definition. *A* (rooted) link diagram with *n* crossings *is a 4-regular (rooted) planar map of n vertices together with a choice of over-under strand information at each vertex. The class of rooted link diagrams with n crossings is denoted $\mathcal{L}_n$.*

The class of maps which represent rooted link shadows *in n crossings, i.e. maps which can be found as the underlying map structure of a rooted link diagram are denoted $\mathscr{L}_n$.*

It is well understood that the class of rooted link shadows in n-vertices is identical to the class of 4-regular planar maps in n vertices is identical to $\mathscr{L}_n$; furthermore, the class of rooted planar quadrangulations is dual to $\mathscr{L}_n$.

Definition. *A* (rooted) knot diagram *is a (rooted) link diagram which consists of only one knot component. The class of rooted knot diagrams with n crossings is denoted by $\mathcal{K}_n$.*

The class of maps which represent rooted knot shadows *in n crossings are denoted $\mathscr{K}_n$.*

Knot shadows $\mathscr{K}_n$ represent a curious, small subclass of $\mathscr{L}_n$.

To prove asymptotics for shadows, diagrams, and arbitrary other flavors, we introduce the idea of a labeled map;

Definition. *Given a triple of label sets $S = (S_V, S_E, S_F)$, an S-*map *or S-*labeled map *is a map $M = (V, E, F)$ together with a triple of maps (s_V, s_E, s_F), $s_* : * \to S_*$ which label each vertex, edge, and face of M with an element of S.*

Unless mentioned otherwise, we will only consider vertex-labeled maps, and so abuse notation with $S := S(S, \{0\}, \{0\})$. Call a link (resp. knot) shadow labeled with a set S as link (knot) S-shadow. We will show in corollary 3 that link (resp. knot) diagrams are exactly link (knot) $\{+,-\}$-shadows. With this in mind we can define other useful classes of S-shadows, like pseudodiagrams:

Definition. *A link (resp. knot) pseudodiagram is a link (knot) shadow, together with over-under information at some subset of vertices in the shadow. Equivalently (c.f. corollary 3), a pseudodiagram is a $\{+, 0, -\}$-shadow.*

Definition. *A link* HOMFLY skein resolution diagram *is a link shadow where each vertex is either of crossing type (and has over-under information), or of smooth type, where it represents the valid skein tangency which keeps edge orientations consistent. Equivalently, it is a $\{+, -, s\}$-shadow.*

The number of link components in a HOMFLY skein resolution diagram may differ from the number of link components in the shadow. Hence the class of knot HOMFLY skein resolution diagrams is *different* from the class of knot pseudodiagrams, despite having the same labeling set.

TODO: I'd like to be able to say here (or later) that we can include labels which correspond to tangencies (rather than crossings!). I argue that from a consistent orientation on the edges of the shadow, labels which represent either skein a or skein b can be consistently parsed.

Rooted (knot or link) diagrams are equivalently viewed as *two-leg diagrams* or *2-tangles* as illustrated below.

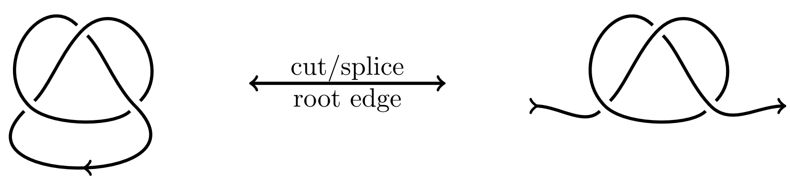

FIG. 6. Rooted diagrams of the trefoil and its mirror image

Additionally, rooted diagrams can be viewed as *four-leg diagrams*, or 4-*tangles* by deleting the root crossing. This identification is not injective for diagrams as it forgets the sign of the removed crossing.

A (rooted) shadow is *prime* if it cannot be disconnected by removing two edges (i.e. it is at least 3-connected); otherwise it is *composite.* A rooted shadow is *two-leg-prime* if it cannot be disconnected by removing two edges, one being the root edge. Diagrams are (two-leg-)prime if their underlying shadow structure is.

There is a bijection between blossom trees and rooted link shadows.

Proposition 1. *There is a consistent way to order the components of a rooted link shadow. There is a consistent way to index the vertices of a rooted link shadow. There is a consistent way to index and orient the edges of a rooted link shadow so that the directed edges meet head-to-tail across the vertices of the shadow.*

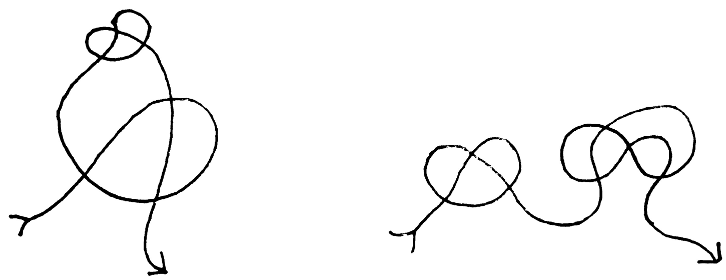

FIG. 7. A composite shadow which is two-leg-prime (left). A shadow which is not two-leg-prime (right).

Proof. Let L be a rooted link shadow. Begin by labelling the edges of L with the link component in which they lie. Index the root vertex and the root edge by 1. □

Corollary 2. *There is a consistent way to order and orient the components of a rooted link diagram. There is a consistent way to index the crossings of a rooted link diagram. There is a consistent way to index the edges of a rooted link diagram. These are the same as those for the underlying shadow.*

Proof. These are all induced on the diagram from its shadow. □

Corollary 3. *Rooted link diagrams are in bijection with rooted link shadows labelled with* $S = \{+, -\}$.

Proof. Given a rooted link diagram, there is a consistent orientation of its components. There is hence a labelling of the underlying rooted shadow with $S = \{+, -\}$ (they are the standard crossing signs). This process is reversible since the consistent component orientation for the diagram is identical to that of the shadow. □

III. ASYMPTOTIC STRUCTURE THEOREMS FOR DIAGRAMS

It is believed and numerically evident [3] that the number of link diagrams in a random link diagram grows exponentially, hence a random link diagram is almost certainly not a knot diagram.

III.1. A pattern theorem for classes of link diagrams

III.1.1. A pattern theorem

Theorem 2 in [4] provides a pattern theorem for knot and link shadows, provided a strategy of attaching a desired pattern. However, care is required in the case of knot or link *diagrams*, in which each vertex takes a value in the set $\{+, -\}$. In fact, we turn our attention to the dual case in which *faces* are labelled with an arbitrary set.

Theorem 4. *Let S be a set and $\mathscr{M}[S]$ be some class of S-maps on a surface of type g and let P be a planar S-map with boundary that can be found as a submap of maps in $\mathscr{M}[S]$. Let $M(x)$ be the generating function by number of edges for $\mathscr{M}$. Let $H(x)$ be the generating function by number of edges for those maps M in $\mathscr{M}[S]$ that contain less than $ce(M)$ pairwise disjoint copies of P. Suppose that we can embed P in a possibly larger rooted planar labeled map with boundary Q and attach copies of Q to each map K counted by $H(x)$ in such a way that*

1. *for some fixed positive integer k, at least $\lfloor e(K)/k \rfloor$ possible non-conflicting places of attachment exist,*

2. *only S-maps in $\mathscr{M}[S]$ are produced,*

3. *for any map produced as such we can identify the copies of Q that have been added and they are all pairwise disjoint, and*

4. *given the copies that have been added, the original map and associated places of attachment are uniquely determined.*

If $1 > c > 0$ is sufficiently small, then $r(M) < r(H)$. The maps may be rooted or not.

The method of attachment is vague, but flexible. We will provide some examples which we use in our results for knot diagrams. The proof extends the proof of the original theorem for maps, and makes use of a lemma:

Lemma ([4], lemma 3)**.** *If*

1. *$F(z) \neq 0$ is a polynomial with non-negative coefficients and $F(0) = 0$,*

2. *$H(w)$ has a power series expansion with non-negative coefficients and $0 < r(H) < \infty$,*

3. *for some positive integer k the linear operator $\mathscr{L}$ is given by $\mathscr{L}(w^n) = z^n (F(z)/z)^{\lfloor n/k \rfloor}$, and*

4. *$G(z) = \mathscr{L}(H(w))$,*

then $r(H)^k = r(G)^{k-1} F(r(G))$.

The proof of the theorem then remains almost unchanged from the original theorem, although care will be necessary in defining attachment.

Proof of theorem 4. Let $G(z)$ be the generating function which counts S-maps $\mathscr{G}[S]$ which are the result of attaching some number between 0 and $\lfloor n/k \rfloor$ copies of Q to S-face maps $\mathscr{H}[S]$ counted by $H(x)$. The method of attachment leads to the relation $G(z) = \mathscr{L}(H(w))$, where $F(z) = z + z^q$ and q is the number of edges added when a copy of Q is attached, as

$$G(z) = \sum_{X \in \mathscr{G}[S]} z^{e(X)} = \sum_{Y \in \mathscr{H}[S]} z^{e(Y)} \left(1 + z^{q-1}\right)^{\lfloor n/k \rfloor} = \mathscr{L}(H(w)).$$

Let g_n be the coefficients of $G(z)$.

Suppose $M \in \mathscr{M}[S]$ contains m copies of Q. By property (3) of our attachment, $m \leq n$. If M had been produced from some S-map K in $\mathscr{H}[S]$ by our attachment process, we can find all possible K by removing at least $m - cn$ copies of Q from M. It is possible to bound from above the number of ways to do this by

$$\sum_{j \geq m-cn} \binom{m}{j} = \sum_{k<cn} \binom{m}{k} < \sum_{k<cn} \binom{n}{k} \leq n \binom{n}{cn} \leq \frac{n(ne)^{cn}}{cn^{cn}} = n \left(\frac{e}{c}\right)^{cn} =: t_n.$$

If $M(x) = \sum m_n x^n$, then $m_n \geq g_n$ and $t_n > 1$ for sufficiently large n, so $m_n \geq g_n/t_n$. Hence,

$$1/r(M) \geq \limsup_{n\to\infty} (g_n/t_n)^{1/n} = \lim_{n\to\infty} (t_n)^{-1/n} \limsup_{n\to\infty} (g_n)^{1/n} \geq (c/e)^c / r(G).$$

By the prior lemma, $r(H)^k = r(G)^k(1 + r(G)^{q-1})$ so that

$$r(H)/r(M) \ \geq (1 + r(G)^{q-1})^{1/k}(c/e)^c.$$

As $\lim_{c\to 0^+} (c/e)^c = 1$ and $r(G)^k(1 + r(G)^{q-1}) = r(H)^k \geq 1/12^k$, it follows that $r(H)/r(M) > 1$ for sufficiently small c, completing the proof of the theorem. □

The conclusion is about radii of convergence of two power series, and may appear an esoteric result. However, application of the Cauchy-Hadamard theorem, together with one additional hypothesis, gives a more familiar tune:

Corollary 5 ([4]). *Suppose all of the hypotheses of theorem 4 and additionally that $\mathscr{M}[S]$ grows smoothly, i.e. that $\lim_{n\to\infty} m_n^{1/n}$ exists. Then there exists constants $c > 0$ and $d < 1$ and $N > 0$ so that for all $n \geq N$,*

$$\frac{h_n}{m_n} < d^n.$$

I.e., the pattern P is ubiquitous.

Because of Euler's formula, the number of vertices, edges, or faces in a link shadow or planar quadrangulation is entirely determined by choosing any one cardinality. Hence, we can size shadows by the number of vertices and still keep the above results.

The crux of applying this theorem to link diagrams then falls upon determining an "attachment" operation which satisfies the hypotheses, along with patterns valid for a given class of shadows. We can generally define attachment operations for different kinds of tangles in the dual. By abuse of notation, let $S = (\{0\}, \{0\}, S)$ be a set of labels for the faces of the dual (for now, we are concerned about diagrams, which only have labeled vertices).

1. **Connect sum.** Let L and Q' be rooted S-quadrangulations. Orient the remaining edges of L canonically by proposition 1. Define the *connect sum* of Q' into L at an edge $e \in L$, $L\#_e Q'$, by

    (a) Cut and split the edge e, creating a map L' and leaving a distinguished, oriented bigon f. Denote the two edges formed by splitting e by e_1, e_2, so that the loop $e_1(-e_2)$ is a counterclockwise cycle around f. If e was the root of L, make e_1 the new root of L'.

    (b) Cut and split the root edge ϵ of Q', creating a map Q and leaving a distinguished, oriented bigon g. Denote the two edges formed by splitting ϵ by ϵ_1, ϵ_2, so that the loop $\epsilon_1(-\epsilon_2)$ is a counterclockwise cycle around g. Make ϵ_2 the new root of Q.

    (c) Glue the map Q into the map L''s distinguished face f along the boundary of the distinguished bigon g so that e_1 and ϵ_2 are mapped to the same edge and so that the orientations of the boundaries align.

    (d) Forget about all edge orientations except for the root edge of L.

    Notice that none of the original faces in L and Q' are changed; hence the result is a new rooted S-quadrangulation. Any given S-quadrangulation in $2n$ edges has precisely $2n$ different non-conflicting sites for connect summation (i.e. $k = 1$ for this attachment operation). This process is reversible, given a 2-cycle which bounds an instance of Q' (collapse the disk to a single edge). If L^* and $(Q')^*$ each consist of only one link component, then $(L\#_e Q')^* =: L^*\#_e(Q')^*$ will as well. In fact, this attachment into the quadrangulation is precisely dual to the usual link connect sum from knot theory. If Q is 2-leg prime, then no two copies of Q can intersect (as in this case there are precisely two paths of length 1 in Q between its two boundary vertices).

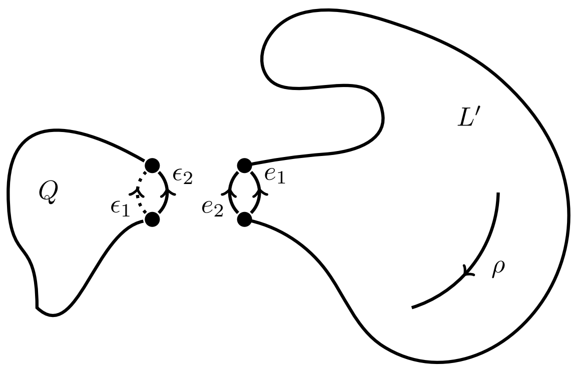

FIG. 8. The connect sum operation. Q' and L' are viewed as CW-complexes, and their boundaries are appropriately identified.

2. **4-tangle replacement.** Let L be a rooted S-quadrangulation, and Q a rooted S-quadrangulation with square boundary. Then given a face $f \in L$, we can define a *4-tangle replacement* of f with Q by identifying the boundary of Q with the boundary of f in some manner. Precisely how to equate the boundaries will depend on the case at hand; what matters is that there be at least one valid way to glue Q into any face f and produce a new S-quadrangulation in the chosen class. Indeed, the result will always be a new S-quadrangulation. Any S-quadrangulation in $2n$ edges has n faces, and hence at least n non-conflicting attachment locations (i.e., $k = 2$).

   To make the process reversible for all S (not just $S = \{0\}$), we must pick slightly different Q. Given a non-trivial S-map P, choose a planar rooted labeled map with boundary Q which cannot intersect with a copy of itself so that P is a unique copy of itself in Q, where all faces which are not in P are labeled with a special label $*$. Attachment of Q into the face f of an S-quadrangulation L then consists first of the actual attachment operation described above, and then relabeling all faces with label $*$ with the original label of f. The process is then reversed by

   (a) Given a 4-cycle bounding a (relabeled) copy Q' of Q in L, tentatively delete the copy and replace it with a bare face f.

   (b) Identify the unique copy of P inside of Q.

   (c) If every face of $Q \setminus P$ does not have the same label x, then this would not in fact have been a place of attachment (hence reversal needn't be possible). If they do, then label the new face f with the label x.

   Care must be taken in choice of Q in which to embed arbitrary P here; an important fact to remember is that submap insertion of any S-quadrangulation with boundary P will not introduce any shorter paths through Q, else there would be a 3-cycle in P (which is impossible as it is a quadrangulation).

   If one is dealing with 4-tangle replacement within a class of knot S-shadow duals, the following lemma is helpful;

   Lemma 6. *Given a link S-shadow L, a vertex v in L, and an 4-leg S-curve T with 2 link components, it is always possible to replace v by T in at least one way so that the result has the same number of components as L.*

Proof. TODO: This proof needs work

The vertex v in L and the tangle T are each either of crossing (abab) type or tangency (aabb) type. If they agree in type, replace v by T so that the strands agree. If they differ and $L \setminus v$ is of type abab, replace T in with type abba. If they differ and $L \setminus v$ is of type aabb, replace T with type abab. In all cases, the number of components is preserved. □

There are applications of this attachment in proving the weak pattern theorem for certain classes of maps:

i. Given arbitrary face labels S, a prime link dual S-shadow L and a prime 4-leg dual S-curve P, define Q as in figure 9. Observe that if there exist two copies Q' and Q'' of Q

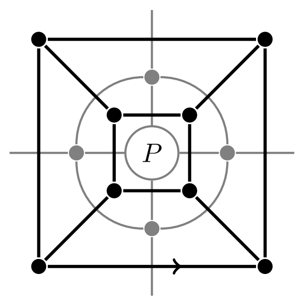

FIG. 9. The labeled quadrangulation Q (in black) is dual to encircling the 4-leg curve P with a link component. The four faces bounding P have the distinguished label $*$.

in a prime link dual shadow L, then they cannot intersect: All paths between boundary vertices of Q are either of length 3, or greater. If there is an intersection between Q' and Q'', then without loss of generality one of two things happens: (1) There is a path along the root face of Q'' of 3 edges lies in Q' (since there are no paths of length 1 or 2) and runs between two adjacent vertices a and b of Q'; but then the remaining boundary edge of Q'' must run from a to b; but this gives the existence of a 2-cycle in L which we said was prime and hence has no 2-cycles. (2) The entire boundary cycle of Q'' lies within Q' and necessarily runs between two opposing vertices a and c of Q'. But then these two opposing vertices must be the same vertex in L; but then there is the 2-cycle a to b to $c = a$ in L, which again was chosen to have no 2-cycles.

ii. Given face labels S, a prime *knot* dual S-shadow K, and a nontrivial prime 4-leg dual S-curve with 2 components P, define Q as in figure 10, embedding P in such a way that Q has precisely 2 link components. Observe that, as in the case above, the shortest paths which both start and end on the boundary of Q are of length 3 between adjacent vertices, or length 4 between opposite vertices. As we are taking K to be prime, the same reasoning shows that copies of Q in K can not overlap. On the other hand, we must now be careful that the 4-tangle replacement into a knot dual S-shadow does not introduce new link components. Fortunately, there is always at least one way to insert a 4-leg dual S-curve while keeping constant the number of link components.

III.1.2. Strategy for proving smooth growth

The theorem in the prior section by itself does not sufficiently prove *ubiquity* as required to prove asymmetry. One may worry about bad cases; e.g., one in which . Indeed, we require that

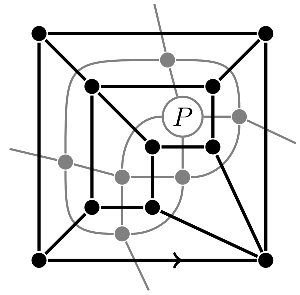

FIG. 10. The labeled quadrangulation Q (in black) in which to embed P. The remaining faces bounding P have the distinguished label $*$.

the class of maps *grow smoothly*, i.e. that (for $m_n = |\mathscr{M}_n|$) the limit

$$\lim_{n\to\infty} m_n^{1/n}$$

exists.

Bender, et al. [4] give a powerful proof strategy for proving smooth growth of a sequence. We adapt that to prove the following theorem.

Theorem 7. *Let $\mathscr{C}$ be a class of combinatorial objects with generating function $\sum_{n=0}^{\infty} c_n\, z^n$; let the radius of convergence of the OGF be r, and $\mathscr{D}$ some other class with generating function $\sum_{n=0}^{\infty} d_n\, z^n$. Suppose that $0 > r \leq 1$ and let $C_i > 0$ and $1 - r > \delta > 0$ be arbitrary.*

Suppose there is a composition operation $\times$ on elements $A, B \in \mathscr{C} \cup \mathscr{D}$ so that,

1. *$A \times B \in \mathscr{C}$,*
2. *there exists some fixed $k \in \mathbb{Z}_{\geq 0}$ so that $|A \times B| = |A| + |B| + k$, and*
3. *given any $C \in \mathscr{C}$, there is at most one maximal factorization $D_1 \times D_2 \times \cdots \times D_s = C$ with $D_i \in \mathscr{D}$.*

Suppose there exists $R \geq 0$ so that for $n \geq R$ there exists $\ell \in \mathbb{Z}_{\geq 0}$ and maps $\psi_0 : \mathscr{C}_n \hookrightarrow \mathscr{D}_{n+\ell}$ and $\psi_1 : \mathscr{C}_n \hookrightarrow \mathscr{D}_{n+\ell+1}$. Then the limit

$$\lim_{n\to\infty} c_n^{1/n}$$

exists.

It is known that there are at most 12^n planar maps, and so in our cases we will always have $r \geq 1/12$.

Proof. The proof breaks down into 3 steps;

1. *Show that there exists some $n \geq 0$ with $c_n > C_1(r+\delta)^{-n}$.* This step follows from the Cauchy-Hadamard theorem, which says that

$$\limsup_{n\to\infty} c_n^{1/n} = r^{-1}.$$

By the definition of lim sup, we have that if $a < r^{-1}$, then for any $M \geq R$ we have that there is some $n \geq M$ with $c_n^{1/n} > a$. For instance, we know that $(r+\delta/2)^{-1} < r^{-1}$, hence for any M we have some $n \geq M$ with $c_n > (r+\delta/2)^{-n}$. Notice now that as

$$\left(\frac{r+\delta}{r+\delta/2}\right) > 1,$$

there must be some $M \geq R$ so that for all $m \geq M$

$$\left(\frac{r+\delta}{r+\delta/2}\right) > C_1^{1/m}, \text{ implying that } (r+\delta/2)^{-m} > C_1(r+\delta)^{-m},$$

whence we then have (by lim sup) some $n \geq M$ with $c_n > (r+\delta/2)^{-n} > C_1(r+\delta)^{-n}$.

2. *Show that there exists some $m \geq 0$ with $d_m > C_2(r+\delta)^{-m}$ and $d_{m+1} > C_2(r+\delta)^{-(m+1)}$.* Notice that $(r+\delta) < 1$ and so for any $m \geq 0$, $(r+\delta)^{-m} < (r+\delta)^{-(m+1)}$. As there exist injections ψ_0, ψ_1 from $\mathscr{C}_n$ into $\mathscr{D}_{n+\ell}$ and $\mathscr{D}_{n+\ell+1}$, setting $m = n+\ell$ and $C_1 = C_2(r+\delta)^{-(m+n-1)}$ we have that

$$d_m \geq |\operatorname{im}\psi_0| = c_n > C_1(r+\delta)^{-n} = C_2(r+\delta)^{-(m+1)} > C_2(r+\delta)^{-m}$$

and

$$d_{m+1} \geq |\operatorname{im}\psi_1| = c_n > C_1(r+\delta)^{-n} = C_2(r+\delta)^{-(m+1)}.$$

3. *Show that there exists some N so that for any $n \geq N$, $c_n > (r+\delta)^{-n}$.* Consider k from the hypothesis. Let $C_2 = (r+\delta)^{-k}$. Let $N = (m+k)(m+k+1)$. Then if $n \geq N$, we can write n as a linear combination $a(m+k) + b(m+k+1) = am + b(m+1) + (a+b)k$, with $a, b \geq 0$. Observe that $c_n > d_m^a d_{m+1}^b$ as there exists a subset of objects $S \subset \mathscr{C}_n$ which can be expressed uniquely as a product of a elements of $\mathscr{D}_m$ and b elements of $\mathscr{D}_{m+1}$ (and $|S| > d_m^a d_{m+1}^b$). Then

$$c_n > d_m^a d_{m+1}^b > C_2^a(r+\delta)^{-am} C_2^b (r+\delta)^{-b(m+1)} = (r+\delta)^{-(am+b(m+1)+(a+b)k)} = (r+\delta)^{-n}.$$

To finish the proof we realize that this last step implies that the lim inf is r^{-1} and hence the limit result follows. Observe that $\liminf_{n\to\infty} c_n^{1/n} = r^{-1}$ if for any $\epsilon > 0$, there exists N so that for all $n \geq N$, $c_n^{1/n} > r^{-1} - \epsilon = \frac{1-r\epsilon}{r}$. We may assume that $\epsilon < 1$ since otherwise the inequality is clear since $c_n \geq 0$ always. So we are done if we can choose δ so that

$$\frac{r^2\epsilon}{1-r\epsilon} > \delta,$$

as then we have from our prior result that $c_n^{1/n} > (r+\delta)^{-1} > r^{-1} - \epsilon$. Indeed, we have $r^2\epsilon > 0$ and $1 > 1 - r\epsilon > 0$ so that the left hand side of the inequality is positive; but we may choose $\delta > 0$ as small as we desire. Hence for $N = (m+k)(m+k+1)$, the result that $\lim_{n\to\infty} c_n^{1/n} = r^{-1}$ follows. $\square$

III.1.3. Smooth growth for knot and link diagrams

The class $\mathscr{L}$ of rooted link shadows has been counted exactly. Rooted link shadows are in bijection with rooted 4-regular planar maps and the coefficients of the generating function are known [5]. If $l_n = |\mathscr{L}_n|$, then:

$$l_n = \frac{2(3^n)}{(n+2)(n+1)}\binom{2n}{n}.$$

On the other hand, asymptotics of knot shadows are as of yet unknown. We are still however able to prove that they grow smoothly, as to prove our result of asymptotic asymmetry.

Theorem 8. *The class $\mathscr{K}$ of rooted knot shadows grows smoothly. I.e., the limit $\lim_{n\to\infty} k_n^{1/n}$ exists (and is equal to $1/r(K)$).*

Proof. As mentioned above, a very loose bound on $r = r(K)$ is $1/12 \le r \le 1$ as the number of planar maps in general is bounded by 12^n. Let $C_i > 0$ and $1 - r > \delta > 0$ be arbitrary. We need to define a composition $\times$ and subclass $\mathscr{D}$ of shadows which are prime under $\times$.

Define the concatenation $K = K_1 \times K_2$ on shadows in $\mathscr{K}$ by gluing the front leg of K_1 to the hind leg of K_2. Hence, we will take $\mathscr{D}$ to be the class of knot shadows which remain at least

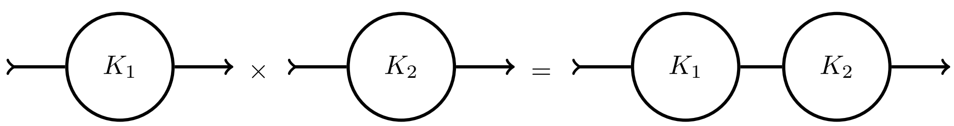

FIG. 11. The two-leg sum operation $\times$. If both A and B are two-leg-prime, then $A \times B$ has exactly one separating edge.

2-connected after removing the root edge.

Certainly $A \times B \in \mathscr{K}$ as we obtain a new 2-leg shadow. As $\times$ introduces no crossings, we have $k = 0$ and $|A \times B| = |A| + |B|$. Finally, a 2-leg shadow K either lies in $\mathscr{D}$ or has $\ell - 1$ disconnecting edges. Cutting these edges produces the disjoint union of ℓ well-ordered 2-leg shadows (well ordered from their position in the long curve K) which is the unique ordered +-decomposition of K into elements of $\mathscr{D}$.

Let φ be the map which twists the root edge, making the loop the new root (using the appropriate induced orientation). Then $\varphi : \mathscr{K}_* \hookrightarrow \mathscr{D}_{*+1}$, since deleting the root and smoothing the pointed edge produces a knot shadow, which must be at least 2-connected. Then we take $\psi_0 = \varphi$

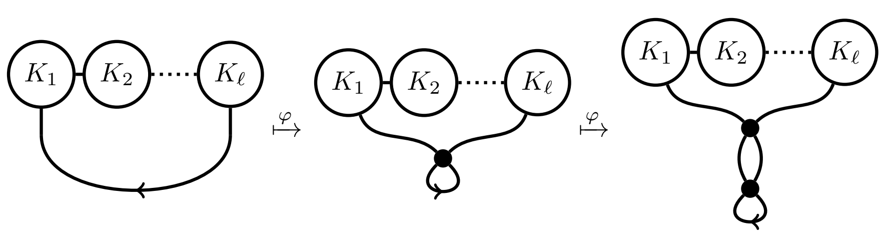

FIG. 12. The map φ adds a vertex and ensures that the new map is 2-leg-prime.

and $\psi_1 = \varphi^2$. This setup satisfies the hypotheses and hence proves the theorem. $\square$

Corollary 9. *There exists $N \geq 0$ and a constant $d < 1$ so that for $n \geq N$,*

$$\mathbb{P}(\text{a knot diagram } K \text{ is an unknot}) < d^n.$$

For any prime 2-tangle P, there exists $N \geq 0$ and constants $d < 1$, $c > 0$ so that for $n \geq N$,

$$\mathbb{P}(\text{a knot diagram } K \text{ contains} \leq cn \text{ copies of } P \text{ as connect summands}) < d^n.$$

Proof. The first statement will follow immediately from the second, given a prime 2-tangle corresponding to a prime knot diagram which is not an unknot. The second is a corollary of theorems 5 and 8: Let P be a prime 2-tangle which can be found as a connect summand of a knot diagram (i.e., it has one link component). If m_n is the number of knot diagrams, then there exists $c > 0$, $d > 1$, and $N > 0$ so that for all $n \geq N$, $\frac{h_n}{m_n} < d^n$, where h_n is the number of knot diagrams which contain at most cn copies of P as connect summands. This ratio is precisely the probability in the second statement. □

III.1.4. Smooth growth for prime knot and link diagrams

If, however, we are considering a class $\mathscr{P}$ of prime or reduced rooted diagrams, the method of proof for smoothness does not immediately carry over; it is possible that φ introduces numerous isthmi, in which case our diagrams created in the final step would not even be reduced. In the case where $\mathscr{P}$ is the class of prime rooted link shadows, exact counts are known from their bijection with simple quadrangulations [6];

$$s_n = \frac{4(3n)!}{n!(2n+2)!}.$$

In other cases again smoothness is more complicated to prove, although we can use a similar argument to that in the case of all knot shadows.

Proposition 10. *Prime knots*

Proof. Step i is again immediate, so we begin with step ii. Let ψ, ψ' respectively be maps which take the root vertex to the two 4-tangle shadows:

Observe that neither ψ nor ψ' remove primeness or reducedness. Their images provide an injection into the spaces with 2 and 3 additional crossings, respectively. So take m appropriately.

Define the operation $+$ now by the detour-glom. Notice that primeness is preserved and the process is splittable; given the root edge we can identify the bendy edges and rebuild the old two shadows. Notice that $|A+B| = |A|+|B|+4$. Now let $C_3 \geq 1$ and $C_2 = C_3(r+\delta)^{-4}$. Then if there exist nonnegative integers a, b such that $n = am + b(m+1) + (a+b)4 = a(m+4) + b(m+5)$, i.e. if $n \geq (m+4)(m+5)$, then

$$p_n > p_{m+4}^a p_{m+5}^b > C_2^{a+b}(r+\delta)^{-(am+b(m+1))} > C_3^{a+b}(r+\delta)^{-(a(m+4)+b(m+5))} > C_3(r+\delta)^{-n}.$$

□

III.2. Asymmetry of diagrams and consequences

The following theorem of Richmond and Wormald [7] provides a sufficient set of criteria for almost all elements of $\mathscr{K}$ to have trivial automorphism group.

Theorem 11 (Richmond-Wormald 1996)**.** *Let $\mathscr{C}$ be a class of rooted maps on a surface. Suppose that there is an outer-cyclic rooted planar map M_1 such that in all maps in $\mathscr{C}$, all copies of M_1 are pairwise disjoint, and such that*

1. *M_1 has no reflective symmetry in the plane preserving the unbounded face,*
2. *there exist constants $c > 0$ and $d < 1$ such that the proportion of n-vertex maps in $\mathscr{C}$ that do not contain at least cn pairwise disjoint copies of M_1 is at most d^n for n sufficiently large (M is "ubiquitous"), and*
3. *for any map M in $\mathscr{C}$ containing a copy of M_1, all maps obtainable by removing M_1 and gluing it back in to the same face are in $\mathscr{C}$ (M is "free").*

Then the proportion of n-vertex maps in $\mathscr{C}$ with nontrivial automorphisms is exponentially small.

It has been suggested without proof in [8, 9] that classes of knot shadows are almost surely asymmetric. We will prove this for $\mathscr{K}$ by proving it for its dual $\mathscr{K}^*$, a class of quadrangulations of the sphere. Specifically, we will take M_1 to be the dual of the underlying planar map of the following 2-tangle: Clearly M_1 has no reflective symmetry by inspection, and certainly any of the

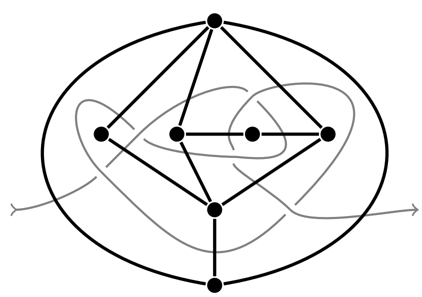

FIG. 13. The dual 2-leg curve M_1 (black), and one of its representations as a 2-tangle (gray).

ways of replacing M_1 keep the object in the class of quadrangulations dual to knot maps. Finally, the ubiquity condition is exactly the pattern theorem for 2-tangles proved in the prior section! The same pattern proves asymmetry for certain other classes of knots or links; for example, reduced diagrams.

Additionally, we can use our 4-tangle replacement scheme to create a M_1 which shows asymmetry of prime knot (link) diagrams. Take M_1 as in figure 14. Then M_1 consists of exactly two link components and is of abab type; any way of replacing a vertex in a knot shadow with a 4-leg curve of abab type keeps the number of link components constant. Furthermore, M_1 is ubiquitous in prime (knot) diagrams as it is an application of corollary 9 to P, the square with an additional 2-path joining two of its opposite vertices.

Application of the above theorems provides us with the following corollary which enables us to transfer any asymptotic results on rooted diagrams to unrooted diagrams.

Corollary 12. *Let L be a uniform random variable taking values in the space $\mathcal{K}_n$ or $\mathcal{L}_n$. Then there exist constants $C, \alpha > 0$ so that $\mathbb{P}(\operatorname{aut} L \neq 1) < Ce^{-\alpha n}$. Hence, rooted diagrams behave like unrooted diagrams.*

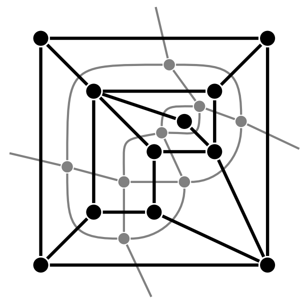

FIG. 14. Choice of M_1 for proving that prime knot shadows are asymmetric

Indeed, link diagrams with n vertices are dual to quadrangulations with $n+2$ faces; there are $n+2$ ways of choosing the "exterior" root face and then 4 ways of rooting the edges around this chosen face. Hence if $\tilde{\ell}_n$, $\tilde{k}_n$ are the counts of unrooted link or knot diagrams we have that in the limit,

$$\tilde{\ell}_n \underset{n\to\infty}{\sim} \frac{\ell_n}{4(n+2)} \text{ and } \tilde{k}_n \underset{n\to\infty}{\sim} \frac{k_n}{4(n+2)}.$$

Corollary 13. *A random knot or link diagram has the pattern theorem. Namely, a random knot diagram is almost surely composite and almost surely knotted, and a random link diagram is almost surely not a knot diagram.*

IV. NUMERICAL RESULTS ON KNOTTING

One may be concerned that the "asymptotic" behavior proved in the prior section only applies to knot diagrams with an absurd number of crossings (in the sense that no physical knot should be expected to be so complicated). However, exact and numerical results show that this behavior is attained very quickly. For example, almost all 10-crossing knot diagrams have no nontrivial automorphisms!

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