# Random Planar Diagrams

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AMS Western Spring Sectionals 2015 (UNLV) – April 18, 2015

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# Knotting and Polymers

## Random curve distributions

The classical workflow for understanding knotting in random polymers is via random distributions on different spaces of curves.

- Random space polygons. (Fixed edge length, equilateral, confined, etc.)
- Random closed self-avoiding lattice walks.

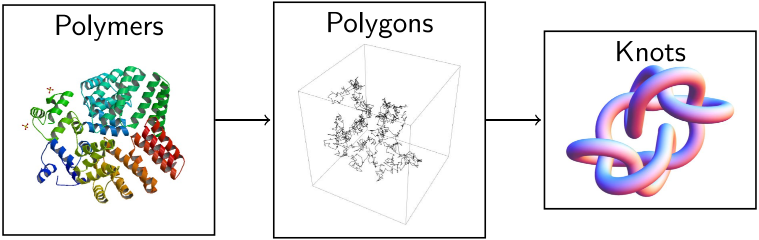



Figure: Typical random curve workflow

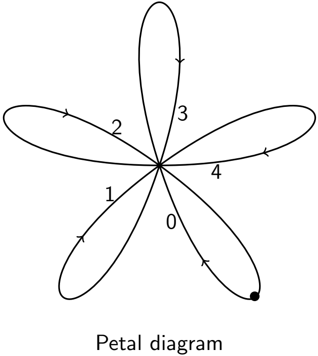
# Combinatorial knot distributions

There are are also combinatorial distributions in use to study random knotting. Examples include

- Petaluma model (Evan-Zohar, Hass, et al.)
- Random braid words

These combinatorial models are recent.

# The Petaluma model



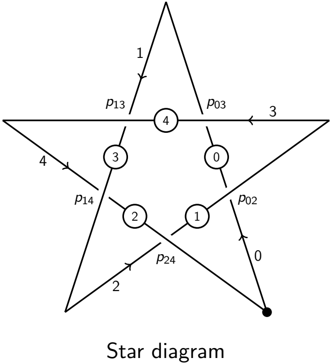




Figure: Petal diagram and corresponding star diagram for the trefoil. (Diagram from Evan-Zohar, et al.)

# The void

There is no clear connection between the two models. For example, nearly no random polygons are expected to produce a star diagram from the Petaluma model.

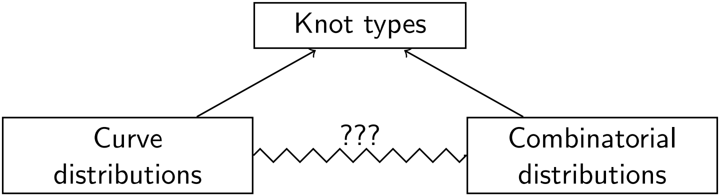



Figure: There is no convenient middle between the two methods

# The random diagram model

Every space curve can project to a diagram, **and** diagrams are combinatorial objects.

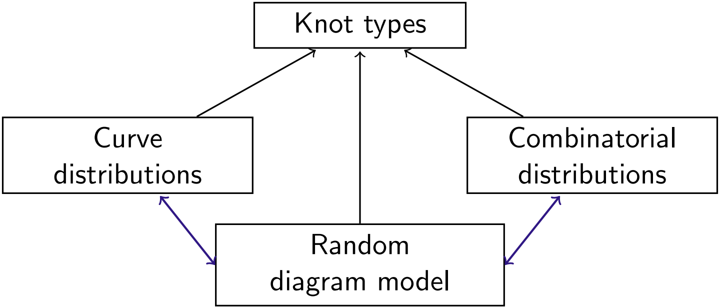



Figure: We're trying to fill the void with the random diagram model

# Random diagrams

**Definition**

In the **random diagram model** of random knotting, a $n$-crossing diagram is drawn uniformly from the finite set of $n$-crossing knot diagrams.

## Diagrams from shadows

We sample diagrams uniformly through tabulation:

1. Enumerate shadows, the underlying graph structure behind diagrams.
2. Expand shadows into diagrams.

# Knot shadows and circle immersions

Knot shadows in $n$ crossings correspond to unoriented, generic immersions of the circle into the sphere with $n$ double points, up to unoriented diffeomorphism.

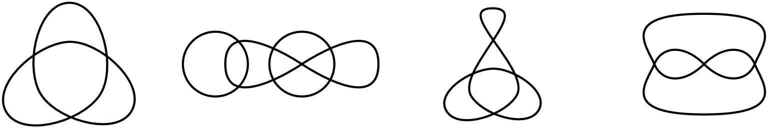


Figure: Knot shadows. The shadow on the left is equivalent to the shadow on the right.

# How many shadows?

| n | # knot shadows |
|---|---|
| 0 | 1 |
| 1 | 1 |
| 2 | 2 |
| 3 | 6 |
| 4 | 19 |
| 5 | 76 |
| 6 | 376* |
| 7 | 2194* |
| 8 | 14614** |
| 9 | 106421** |

Table: Counts on knot shadows. Numbers are large, but finite.

# How many knot shadows?

Counts of knot shadows with $n$ crossings match Arnol'd's counts of immersions of the unoriented circle into the unoriented sphere with $n$ double points (OEIS A008989).

*Caveat:* Arnol'd's list is for $n = 0$ to $n = 5$; the terms for $n = 6$ and $n = 7$ are attributed to Guy H. Valette with no clear source.

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Table: $*$: Attributed to Guy H. Valette. $**$: New; values not in OEIS.

# Tabulating knot shadows

We have generated our table of knot shadows two different ways as a computational check.

Both methods use features from McKay and Brinkmann's `plantri` software.

# Duals to quadrangulations

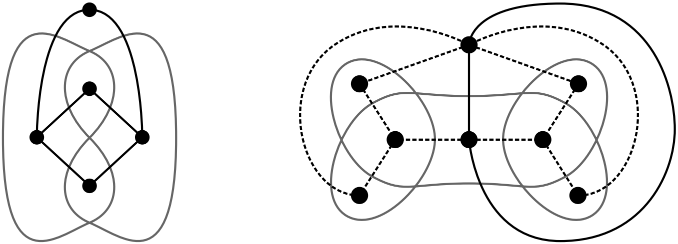

Figure: We find all quadrangulations of the sphere in $n$ faces. Shadows are dual to quadrangulations.

# Planar graph expansions

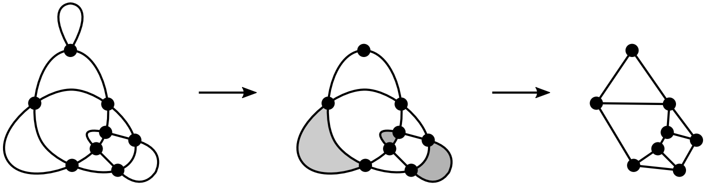

Figure: Reduction to a planar graph of degree $\leq 4$ and connectivity $\geq 1$. Expansion is the inverse.

# The space of shadows

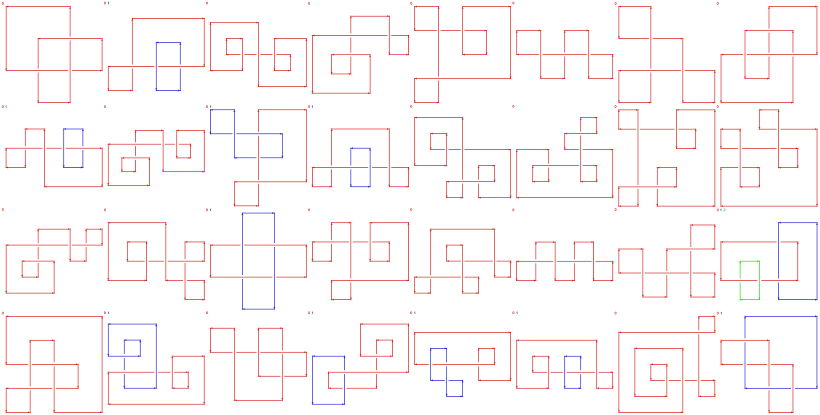

Figure: Link shadows. Pictures generated by Eric Lybrand (UGA) with with SnapPy. A map of all shadows with between 3 and 6 crossings is here.

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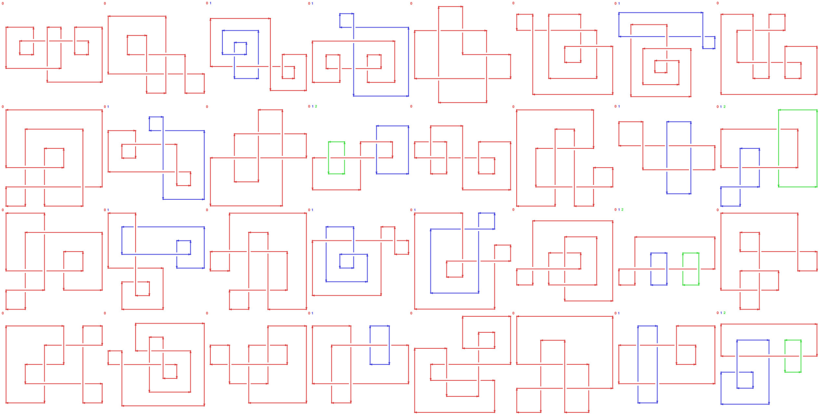

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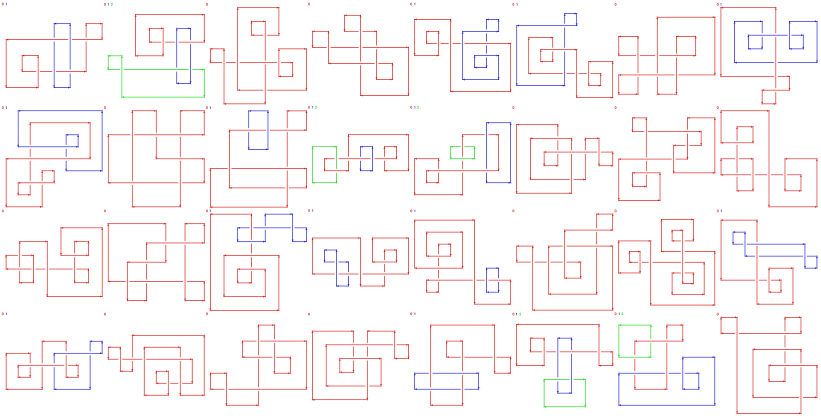

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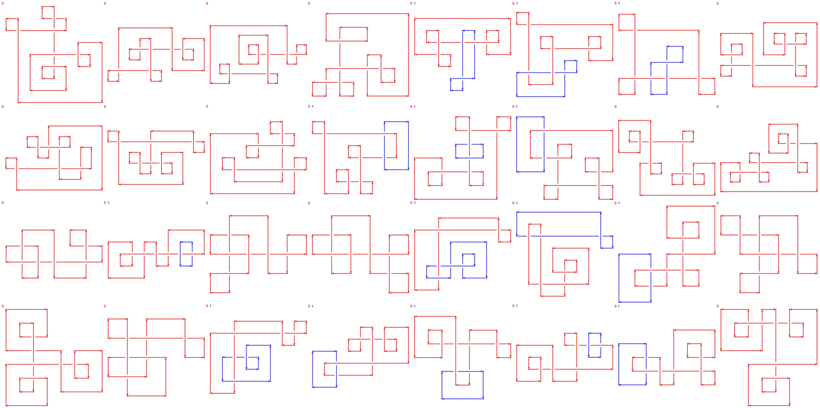

Figure: Link shadows. Pictures generated by Eric Lybrand (UGA) with with SnapPy. A map of all shadows with between 3 and 6 crossings is here.

# Tabulation is difficult!

As of yet this problem is a complicated computational problem; it takes a day on a desktop computer to classify all 9-crossing diagrams. It seems that this is a very difficult problem, because we group by isomorphism! It is difficult at both the shadow stage and the diagram stage

Trefoil shadow has 6-fold symmetry [Figure]
Trefoil [++-] has 3-fold symmetry [Figure]

Figure: The trefoil shadow has 6-fold symmetry (Left). This diagram with trefoil shadow has 3-fold symmetry (Right).

# Breaking symmetries could make counting easier

There exist counts for shadows/diagrams with broken symmetries (rooted diagrams). For example, there is a correspondence:

Figure: Two-leg diagrams (left) correspond to rooted shadows (right).

The diagrams on the left are counted by a generating function (Bouttier, et. al).

# From shadows to diagrams

Expansion of shadows to diagrams consists of three steps:

1. Orient each component. ($2^{\#\text{components}}$ choices)
2. Assign over-under information to each vertex. ($2^{\#\text{crossings}}$ choices)
3. Group diagrams into diagram isomorphism classes.

# How many knot diagrams?

| n | # knot shadows | # knot diagrams | # knot iso. classes |
|---|---|---|---|
| 3 | 6 | 96 | 36 |
| 4 | 19 | 608 | 276 |
| 5 | 76 | 4,864 | 2,936 |
| 6 | 376 | 48,128 | 35,872 |
| 7 | 2,194 | 561,664 | 484,088 |
| 8 | 14,614 | 7,482,368 | 6,967,942 |
| 9 | 106,421 | 108,975,104 | in process |

Table: Counts of knot shadows and diagrams

# Questions we can answer

Q: What fraction of 8-crossing diagrams are trefoils?

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Q: What is the average crossing number of a 9-crossing diagram after all immediate R1 moves?

2.702.

# Knotting probabilities

Advantage of combinatorial model; given enough computing power, we can definitively answer questions about diagrams

- Advantage of a combinatorial model: Able to run searches across entire space computationally.
- We can check knot type of each diagram (HOMFLY is typically enough for our low crossing number)

# Unknottedness in diagrams

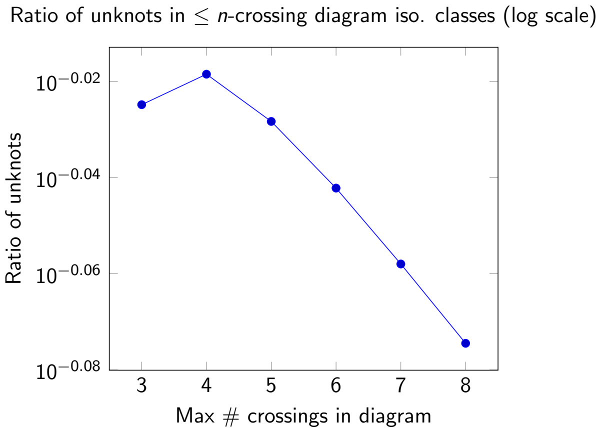


Figure: Unknot ratio may decrease exponentially.

# Knotting in diagrams

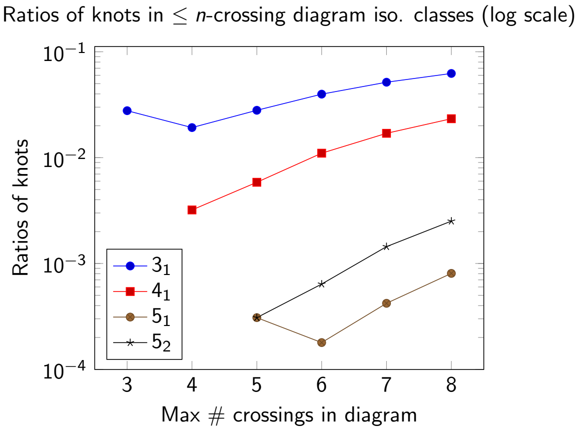


Figure: All ratios of knot types are still increasing. Knot $5_2$ is more common than knot $5_1$.

# Knotting in diagrams

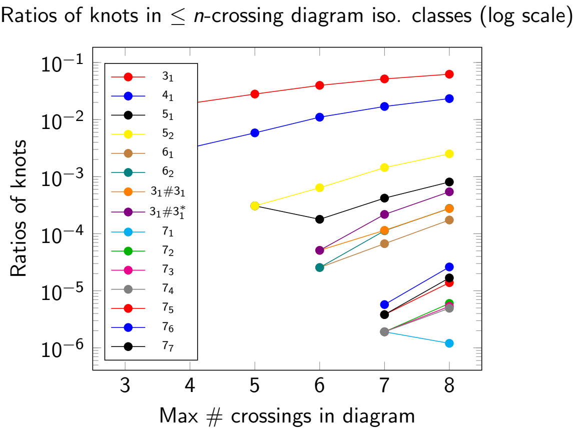


Figure: Ratios of knot types are still increasing.

# Counting monogons and bigons in knot shadows

| $n$ | shadows | 1-gon | 2-gon | neither |
|---|---|---|---|---|
| 3 | 6 | 5 (83.33%) | 3 (50%) | 0 |
| 4 | 19 | 18 (94.74%) | 11 (57.89%) | 0 |
| 5 | 76 | 74 (97.37%) | 52 (68.42%) | 0 |
| 6 | 376 | 371 (98.67%) | 275 (73.14%) | 0 |
| 7 | 2,194 | 2,178 (99.27%) | 1,714 (78.12%) | 0 |
| 8 | 14,614 | 14,562 (99.64%) | 11,892 (81.37%) | 1 |
| 9 | 106,421 | 106,216 (99.81%) | 89,627 (84.22%) | 1 |

Table: Counts of knot shadows with monogons, bigons, or neither. The 8- and 9- crossing shadows are Conway's $8^*$ and $9^*$, respectively.

# Basic polyhedra $8^*$ and $9^*$

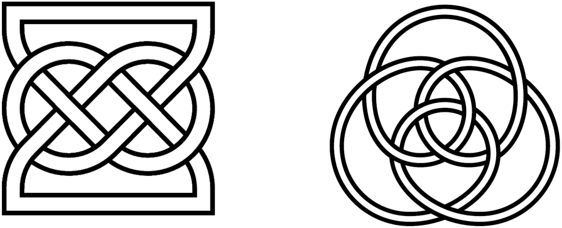

Figure: Knots with the two knot shadows in $\leq 9$ crossings which are planar simple graphs: $8_{18}$ (left), $9_{40}$ (right).

# Tree-like curves

A **tree-like curve** is a knot shadow which is a connect sum of one-crossing diagrams (figure-eight shadows).

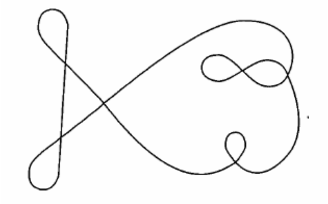

Figure: A tree-like curve.

Tree-like curves provide a lower bound on unknottedness: Diagrams with tree-like shadow are always unknotted.

# Tree-like curves

| n | # knot shadows | # tree-like | % tree-like |
|---|---|---|---|
| 1 | 1 | 1 | 100.00% |
| 2 | 2 | 2 | 100.00% |
| 3 | 6 | 5 | 83.33% |
| 4 | 19 | 16 | 84.21% |
| 5 | 76 | 55 | 72.37% |
| 6 | 376 | 240 | 63.83% |
| 7 | 2,194 | 1149 | 52.37% |
| 8 | 14,614 | 6,229 | 42.62% |
| 9 | 106,421 | 35,995 | 33.82% |

Table: Counts of knot shadows and tree-like curves

# Unknottedness and tree-like shadows

Ratio of unknots, tree-like curves in $\leq n$-crossing diagram iso. classes (log scale)

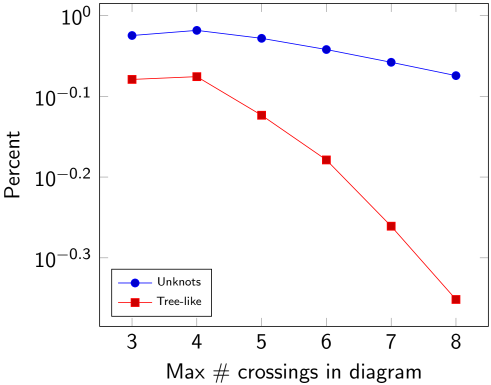


Figure: The number of unknots is bounded by the number of tree-like diagrams.

# Delooped crossing number

| $n$ | Average delooped crossing # |
|---|---|
| 3 | 0.50 |
| 4 | 0.53 |
| 5 | 0.92 |
| 6 | 1.25 |
| 7 | 1.72 |
| 8 | 2.19 |
| 9 | 2.70 |

Table: Average delooped crossing number over shadows with $n$ crossings.

# Future directions

- Most analysis here is on knot diagrams; what can we say about link diagrams?
- How does the random diagram model compare to other models?
  - *Petaluma* model (Evan-Zohar, Hass, Linial, and Nowik)
  - Random space polygons, random equilateral space polygons, random confined space polygons
  - Random closed self-avoiding lattice walks
- Uniform sampling of diagrams of higher crossing number: Can we avoid outright enumeration?

# Link diagrams

Counts for link diagrams

# Knot distances

Can study pure knot theoretic things, not just probabilistic things–transitions between knots
bat graph [figure]

# Thank you!