

Random Knot Diagrams

Jason Cantarella (UGA)

Harrison Chapman (UGA), Matt Mastin (Mailchimp, Inc.)

Crucial Assist: Eric Rawdon (St. Thomas)

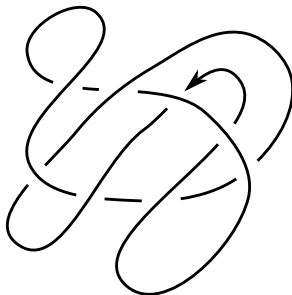
Rocky Mountain Algebraic Combinatorics Seminar

June 23, 2016

Natural questions about knot diagrams

Question

What fraction of 8-crossing diagrams are trefoils?

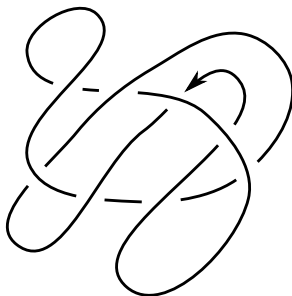


Natural questions about knot diagrams

Question

What fraction of 8-crossing diagrams are trefoils?

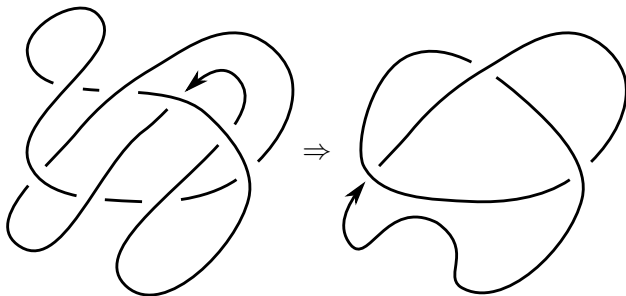
12.48%



Natural questions about knot diagrams

Question

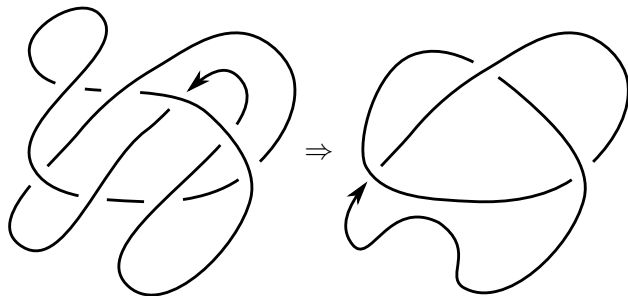
What is the average minimal crossing # of an 8-crossing diagram?



Natural questions about knot diagrams

Question

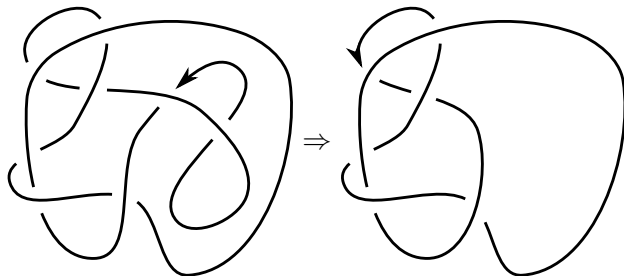
What is the average minimal crossing # of an 8-crossing diagram?
0.52



Natural questions about knot diagrams

Definition

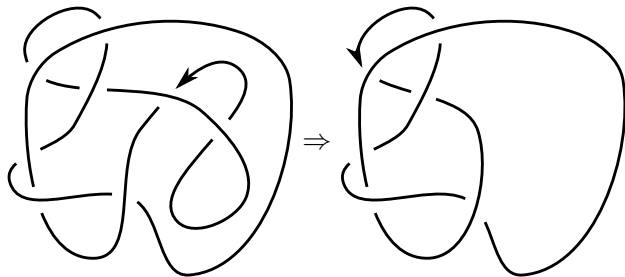
The **untwisting** operator deletes all 1-crossing connect summands of a diagram. (Equivalently, performs all “available” Reidemeister I moves.)



Natural questions about knot diagrams

Question

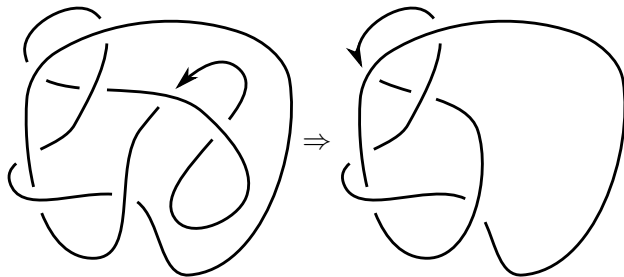
What is the average crossing # of a untwisted 8-crossing diagram?



Natural questions about knot diagrams

Question

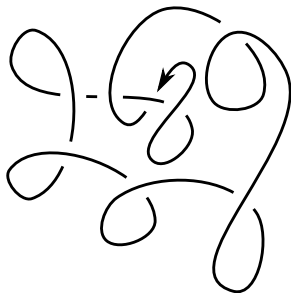
What is the average crossing # of a untwisted 8-crossing diagram?
2.20



Natural questions about knot diagrams

Question

How many 8-crossing diagrams can be untwisted to the unknot?

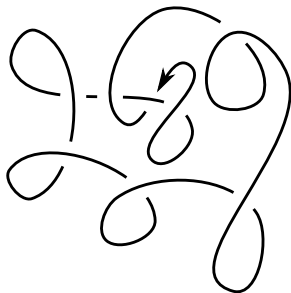


Natural questions about knot diagrams

Question

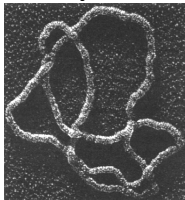
How many 8-crossing diagrams can be untwisted to the unknot?

42.05%

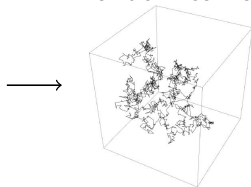


Ansatz

Polymers



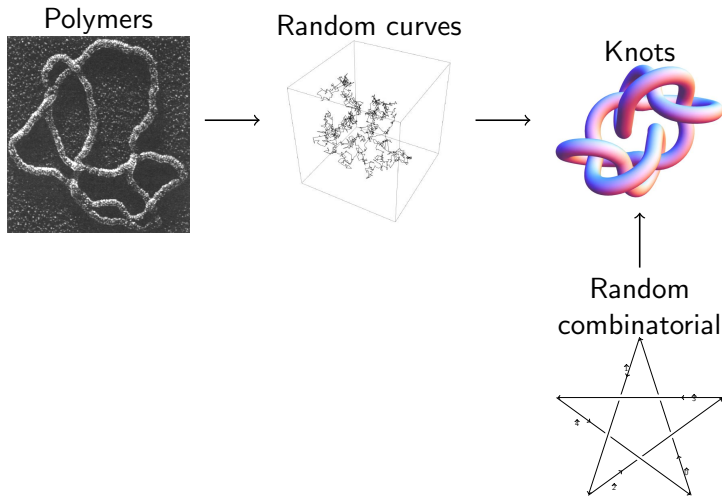
Random curves



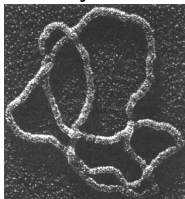
Knots



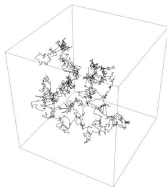
Combinatorial approaches



Polymers



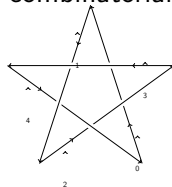
Random curves



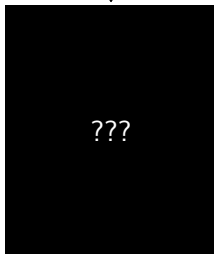
Knots



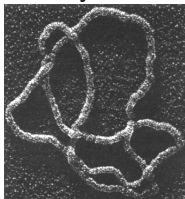
Random
combinatorial



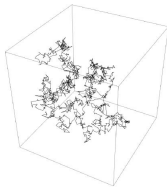
???



Polymers



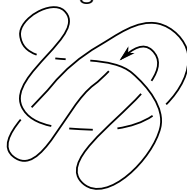
Random curves



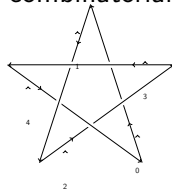
Knots



Random
diagrams



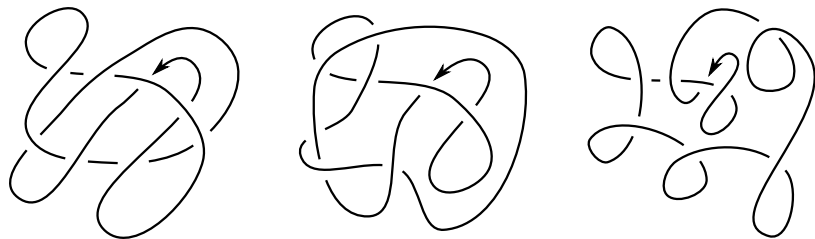
Random
combinatorial



Random diagrams

Definition

In the **random diagram model** of random knotting, a n -crossing diagram is drawn uniformly from the finite set of n -crossing knot diagrams.



How to enumerate knot diagrams (like a topologist)

Definition

A **knot shadow** is a equivalence class of generic immersions of the unoriented S^1 into the sphere S^2 up to diffeomorphism of S^2 .

Plan to Enumerate Diagrams

- 1 *Enumerate shadows (and discard isomorphic shadows)*
- 2 *Assign crossing and orientation information (and discard crossing patterns related by an automorphism of the shadow)*

How to enumerate knot diagrams (like a topologist)

Definition

A **knot shadow** is a equivalence class of generic immersions of the unoriented S^1 into the sphere S^2 up to diffeomorphism of S^2 .

Plan to Enumerate Diagrams

- 1 *Enumerate shadows (and discard isomorphic shadows)*
- 2 *Assign crossing and orientation information (and discard crossing patterns related by an automorphism of the shadow)*

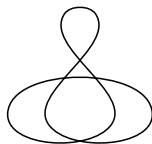
Observation (known to all combinatoricists, but new to me)

Symmetry stinks.

Tabulating knot shadows: plantri, two ways

Proposition

Knot shadows \leftrightarrow 1-component 4-valent embedded planar multigraphs up to embedded isomorphism

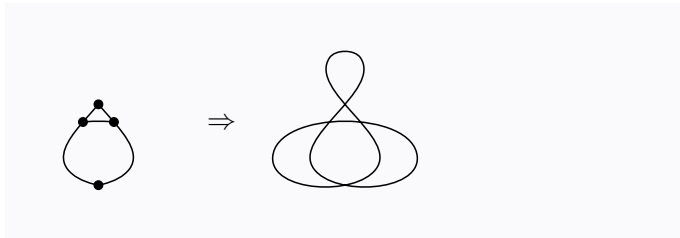


Tabulating knot shadows: plantri, two ways

Proposition

Knot shadows \leftrightarrow 1-component 4-valent embedded planar multigraphs up to embedded isomorphism

- 1 Add loops and edges to planar simple graphs (slow)

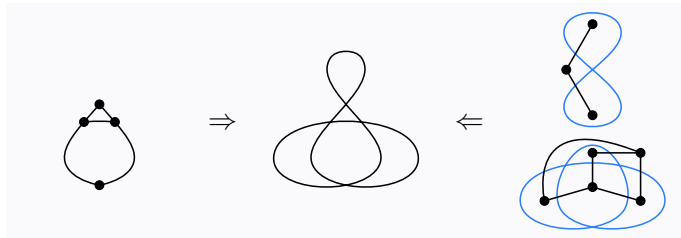


Tabulating knot shadows: plantri, two ways

Proposition

Knot shadows \leftrightarrow 1-component 4-valent embedded planar multigraphs up to embedded isomorphism

- 1 Add loops and edges to planar simple graphs (slow)
- 2 Generate multiquadrangulations of sphere by careful pattern of connect sums, take dual graphs (fast)

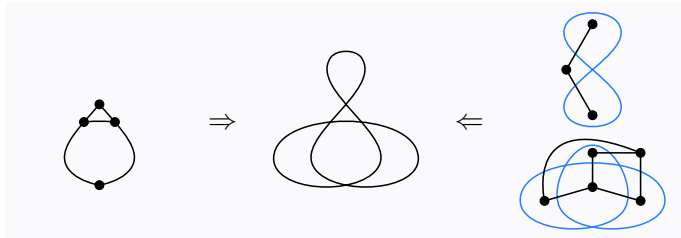


Tabulating knot shadows: plantri, two ways

Proposition

Knot shadows \leftrightarrow 1-component 4-valent embedded planar multigraphs up to embedded isomorphism

- 1 Add loops and edges to planar simple graphs (slow)
- 2 Generate multiquadrangulations of sphere by careful pattern of connect sums, take dual graphs (fast)



Actually generate all **link shadows**, then restrict to knot shadows

Cloud computing for mathematicians

Several thousand hours of CPU time were required to produce about 1.8 billion diagrams. How to organize this kind of computation?



Moral: If you know Linux and a little python, you can play in the cloud.

AWS Services (you only need three)

Amazon Web Services

Compute

EC2

Virtual Servers in the Cloud

EC2 Container Service

Run and Manage Docker Containers

Elastic Beanstalk

Run and Manage Web Apps

Lambda

Run Code in Response to Events

Storage & Content Delivery

S3

Scalable Storage in the Cloud

CloudFront

Global Content Delivery Network

Elastic File System

Fully Managed File System for EC2

Glacier

Archive Storage in the Cloud

Snowball

Large Scale Data Transport

Storage Gateway

Hybrid Storage Integration

Database

RDS

Managed Relational Database Service

DynamoDB

Managed NoSQL Database

ElastiCache

In-Memory Cache

Redshift

Fast, Simple, Cost-Effective Data Warehousing

DMS

Developer Tools

CodeCommit

Store Code in Private Git Repositories

CodeDeploy

Automate Code Deployments

CodePipeline

Release Software using Continuous Delivery

Management Tools

CloudWatch

Monitor Resources and Applications

CloudFormation

Create and Manage Resources with Templates

CloudTrail

Track User Activity and API Usage

Config

Track Resource Inventory and Changes

OpsWorks

Automate Operations with Chef

Service Catalog

Create and Use Standardized Products

Trusted Advisor

Optimize Performance and Security

Security & Identity

Identity & Access Management

Manage User Access and Encryption Keys

Directory Service

Host and Manage Active Directory

Inspector

Analyze Application Security

WAF

Filter Malicious Web Traffic

Internet of Things

AWS IoT

Connect Devices to the Cloud

Game Development

GameLift

Deploy and Scale Session-based Multiplayer Games

Mobile Services

Mobile Hub

Build, Test, and Monitor Mobile Apps

Cognito

User Identity and App Data Synchronization

Device Farm

Test Android, iOS, and Web Apps on Real Devices in the Cloud

Mobile Analytics

Collect, View and Export App Analytics

SNS

Push Notification Service

Application Services

API Gateway

Build, Deploy and Manage APIs

AppStream

Low Latency Application Streaming

CloudSearch

Managed Search Service

Elastic Transcoder

Easy-to-Use Scalable Media Transcoding

SES

Email Sending and Receiving Service

SQS

Message Queue Service

Resource Groups

Learn more

A resource group is a collection of resources that share one or more tags. Create a group for each project, application, or environment in your account.

Create a GroupTag Editor

Additional Resources

Getting Started

Read our [documentation](#) or view our [training](#) to learn more about AWS.

AWS Console Mobile App

View your resources on the go with our AWS Console mobile app, available from [Amazon Appstore](#), [Google Play](#), or [iTunes](#).

AWS Marketplace

Find and buy software, launch with 1-Click and pay by the hour.

AWS re:Invent Announcements

Explore the next generation of AWS cloud capabilities. [See what's new](#)

Service Health

✓ All services operating normally.

EC2: A better computer (or lots of them) by the hour.

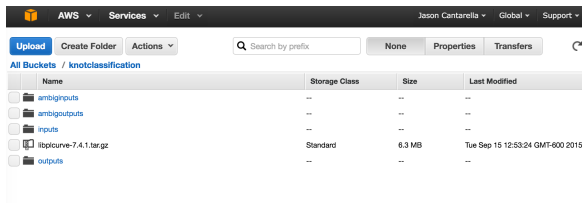
Cloud computing runs on virtual Linux machines. You are root on the machine, and can install software, edit configuration files, compile code, or attach a license for Mathematica or MAGMA.

Type	Processor	Cores	Memory	Price
c4.large	Haswell	2	3.75 gb	1.9 cents/hour
c4.8xlarge	Haswell	36	60 gb	59 cents/hour
r3.large	Ivy Bridge	2	15.75 gb	2.5 cents/hour
r3.8xlarge	Ivy Bridge	32	244 gb	58 cents/hour
x1.32xlarge	Haswell	128	2 tb	397 cents/hour

At peak, we had 100 c4.large instances running simultaneously. The machine may shut down at any time; you are only charged for full hours.

S3: Permanent Internet Accessible Storage, Cheap

If the machines are temporary, where does the data go once they are done computing? To your desktop? (But what if it crashes?) The solution is the "Simple Storage Service":



The cost is 3 cents/gb for instant access, but as low as 0.7 cents/gb for archival storage. Access is by web interface or command line tools (cp, ls, ...). The filesystem is distributed, backed up, and always accessible. There are no limits on the number of files in a "bucket".

SQS: How to get everyone working

SQS is a message queue— the messages describe work to do. You can write to and read from an sqs queue with the Python library boto.

- Worker reads message (should describe < 1 hour of work).
- SQS “locks” the message for one hour.
- If the worker completes the job, the worker should delete the message.
- If the worker dies, the message unlocks and is returned to the queue.
- Messages returned to queue more than n times are set aside for debugging.

Your first million messages (per month) are free. SQS is distributed, backed up, fault-tolerant and always working.

How can you play?



You get \$75 in credits when you sign up. I spent $< \$10$ on this computation.

Results! (Compared to counts in literature)

oriented	$n = 0$	1	2	3	4	5
S^2, S^1	1	1	3	9	37	182
S^2	1	1	2	6	21	99
S^1	1	1	2	6	21	97
—	1	1	2	6	19	76

Curves on S^2 . The number of types

V.I. Arnol'd. *Topological Invariants of Plane Curves*

A008989 Number of immersions of unoriented circle into unoriented sphere with n double points.

1, 1, 2, 6, 19, 76, 376, 2194 [list](#) [graph](#) [rcfs](#) [listen](#) [history](#) [text](#) [internal format](#)

OFFSET

0,3

REFERENCES

V. I. Arnold, Topological Invariants of Plane Curves..., American Math.

LINKS

[Table of \$n, a\(n\)\$ for \$n=0..7\$.](#)

CROSSREFS

Sequence in context: [A159119](#) [A181770](#) [A138800](#) * [A057240](#) [A079564](#) [A079453](#)

Adjacent sequences: [A008986](#) [A008987](#) [A008988](#) * [A008990](#) [A008991](#) [A008992](#)

KEYWORD

nonn

AUTHOR

[N. J. A. Sloane](#).

EXTENSIONS

Two more terms from Guy H. Valette (guy.valette(AT)skynet.be), Feb 09 20

STATUS

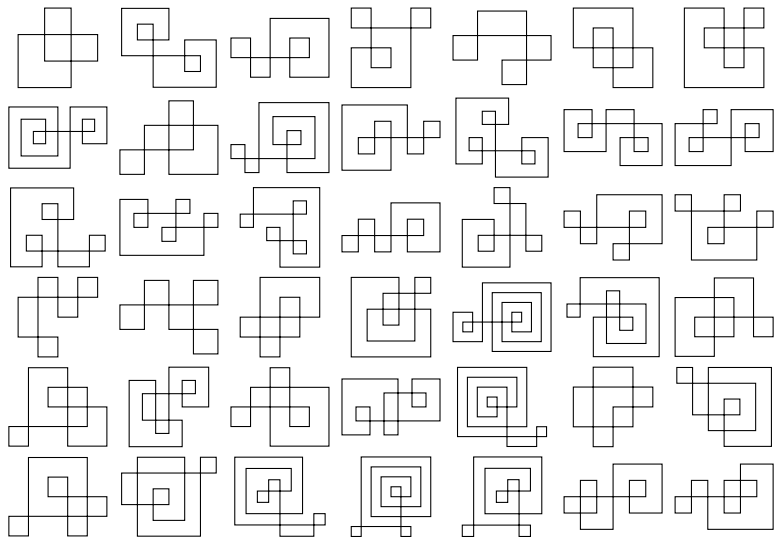
approved

OEIS A008989

n	# knot shadows
0	1
1	1
2	2
3	6
4	19
5	76
6	376
7	2194
8	14614
9	106421
10	823832

We have not found any existing counts of **diagrams**.

The space of shadows



Monogons are very common

Mean Number of Monogons in a Shadow

Cr	3	4	5	6	7	8	9	10
	$\frac{12}{6}$	$\frac{48}{19}$	$\frac{213}{76}$	$\frac{1196}{376}$	$\frac{7714}{2194}$	$\frac{56540}{14614}$	$\frac{448584}{106421}$	$\frac{3758456}{823832}$
	2.	2.53	2.8	3.18	3.52	3.87	4.22	4.56

Fraction of Shadows Containing a Monogon

Cr	3	4	5	6	7	8	9	10
	$\frac{5}{6}$	$\frac{18}{19}$	$\frac{74}{76}$	$\frac{371}{376}$	$\frac{2178}{2194}$	$\frac{14562}{14614}$	$\frac{106216}{106421}$	$\frac{822989}{823832}$
	0.833	0.947	0.974	0.987	0.993	0.996	0.998	0.999

Bigons are very common

Mean Number of Bigons in a Shadow

Cr	3	4	5	6	7	8	9	10
	$\frac{6}{6}$	$\frac{18}{19}$	$\frac{88}{76}$	$\frac{470}{376}$	$\frac{3037}{2194}$	$\frac{21925}{14614}$	$\frac{173342}{106421}$	$\frac{1450209}{823832}$
	1.	0.947	1.16	1.25	1.38	1.5	1.63	1.76

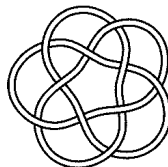
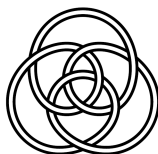
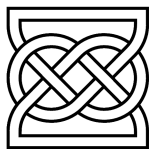
Fraction of Shadows Containing a Bigon

Cr	3	4	5	6	7	8	9	10
	$\frac{3}{6}$	$\frac{11}{19}$	$\frac{52}{76}$	$\frac{275}{376}$	$\frac{1714}{2194}$	$\frac{11892}{14614}$	$\frac{89627}{106421}$	$\frac{712961}{823832}$
	0.5	0.579	0.684	0.731	0.781	0.814	0.842	0.865

Basic polyhedra 8^* , 9^* , and 10^*

Proposition

Conway's basic polyhedra 8^ , 9^* , and 10^* are the only shadows in ≤ 10 crossings with no monogons or bigons.*



8_{18} (left), 9_{40} (middle), 10_{123} (right)

Assign crossings, orientation, identify

- 1 Orient each component. (2 choices)
- 2 Assign over-under information to each vertex. (2^n choices)

n	# knot shadows	2^{n+1} (# shadows)	# knot diagrams
3	6	96	36
4	19	608	276
5	76	4,864	2,936
6	376	48,128	35,872
7	2,194	561,664	484,088
8	14,614	7,482,368	6,967,942
9	106,421	108,975,104	105,555,336
10	823,832	1,687,207,936	1,664,142,836

Assign crossings, orientation, identify

- 1 Orient each component. (2 choices)
- 2 Assign over-under information to each vertex. (2^n choices)

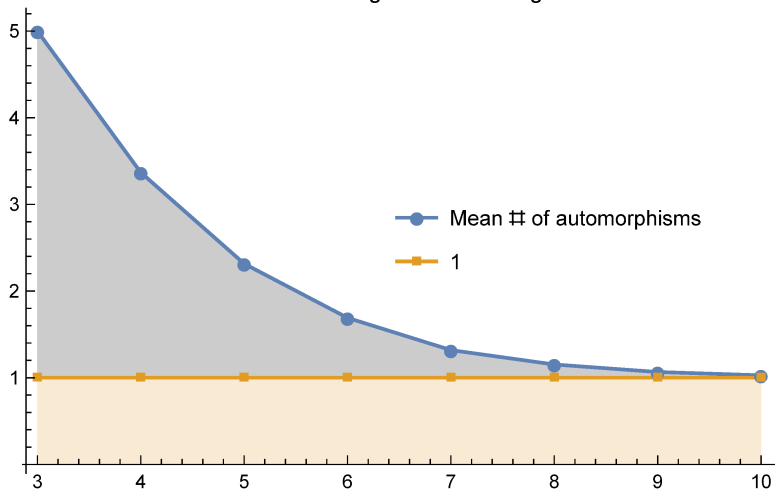
n	# knot shadows	2^{n+1} (# shadows)	# knot diagrams
3	6	96	36
4	19	608	276
5	76	4,864	2,936
6	376	48,128	35,872
7	2,194	561,664	484,088
8	14,614	7,482,368	6,967,942
9	106,421	108,975,104	105,555,336
10	823,832	1,687,207,936	1,664,142,836

Observation

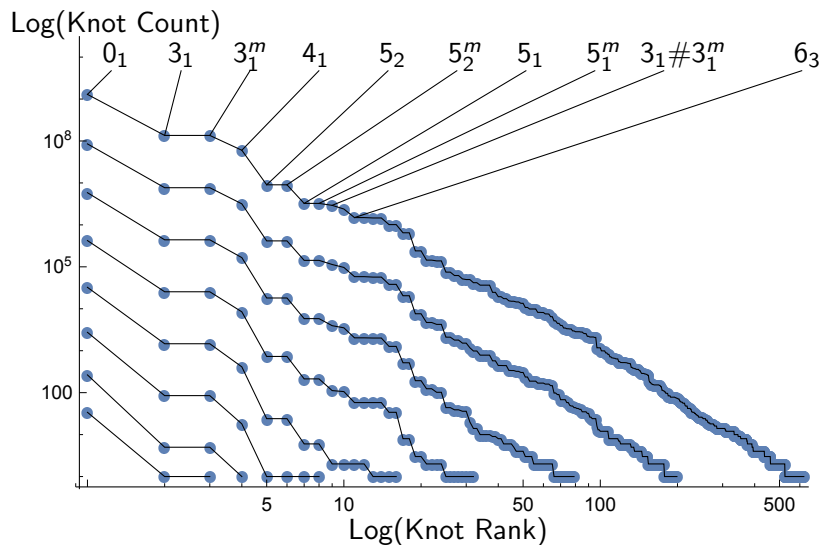
Symmetry becomes rare, quickly!

Size of the automorphism group of a random diagram

Mean number of automorphisms
versus crossing number of diagram



Knotting in diagrams

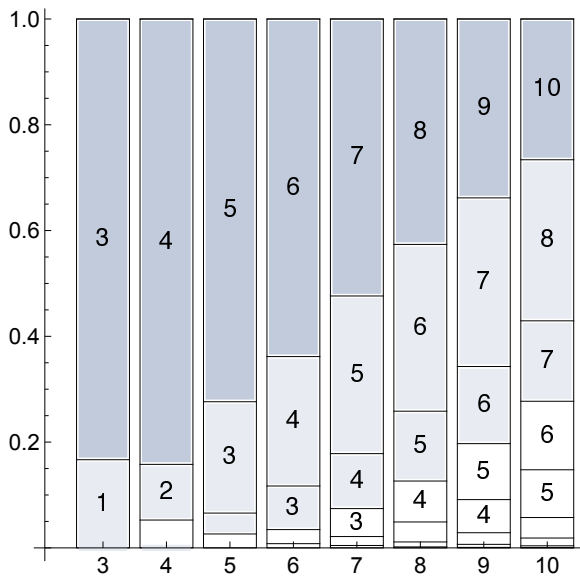


Unknot fraction

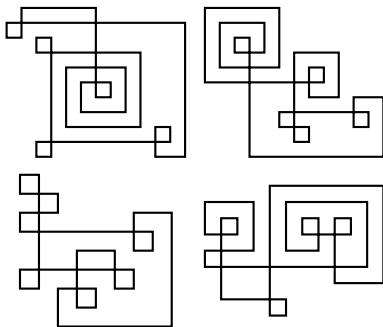
Cr	Unknots	(decimal)
3	$\frac{17}{18}$	0.94
4	$\frac{265}{276}$	0.96
5	$\frac{343}{367}$	0.93
6	$\frac{4057}{4484}$	0.90
7	$\frac{105583}{121022}$	0.87
8	$\frac{2926416}{3483971}$	0.84
9	$\frac{42626767}{52777668}$	0.81
10	$\frac{1291291155}{1664142836}$	0.78

Unknots are very common, even among 10 crossing diagrams.
Why?

Most diagrams are (very) composite



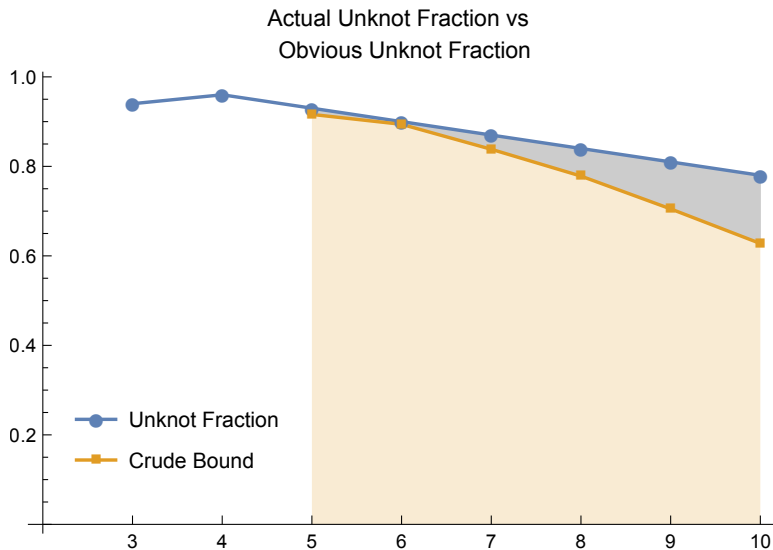
Maximally composite diagrams are “treelike”



Question

Treelike diagrams can't be knotted with any assignment of crossings. Does this (crude) bound explain the unknot fraction?

Pretty much.



A question on unknotting

Theorem ([Frisch-Wassermann-Delbrück Conjecture]
Sumners-Whittington 1988)

The ratio of unknots in random n -edge self-avoiding lattice polygons tends to zero exponentially with n .

Theorem (Chapman)

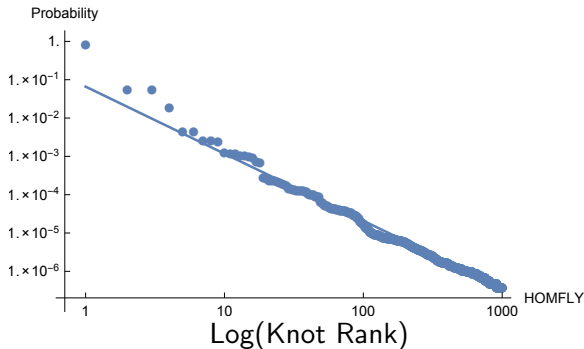
The ratio of unknots in diagrams tends to zero exponentially as n increases.

Future Direction: So what about those log-log plots?

Proposition (with Shonkwiler, 2015)

The symplectic structure on polygon space yields a fast direct sampling algorithm for closed equilateral polygons.

Log(Knot Freq)

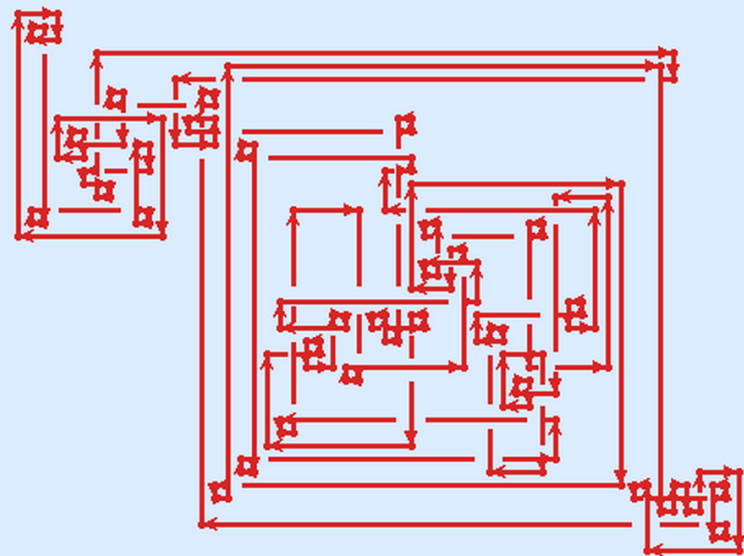


Future Direction: You can play, too!

- Knot Probabilities in Random Diagrams Cantarella, Chapman, Mastin. arXiv:1512.05749
- All data (and pictures for all the diagrams) available at www.jasoncantarella.com/wordpress/papers/
- A Fast Direct Sampling Algorithm for Random Equilateral Polygons Cantarella, Duplantier, Shonkwiler, Uehara. arXiv:1510.02466

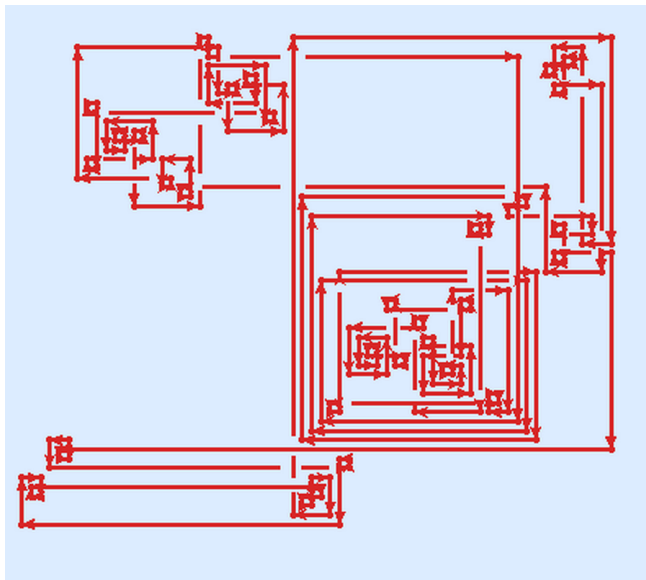
Future Direction: Uniform sampling of large diagrams

Harrison Chapman has results on sampling large diagrams:



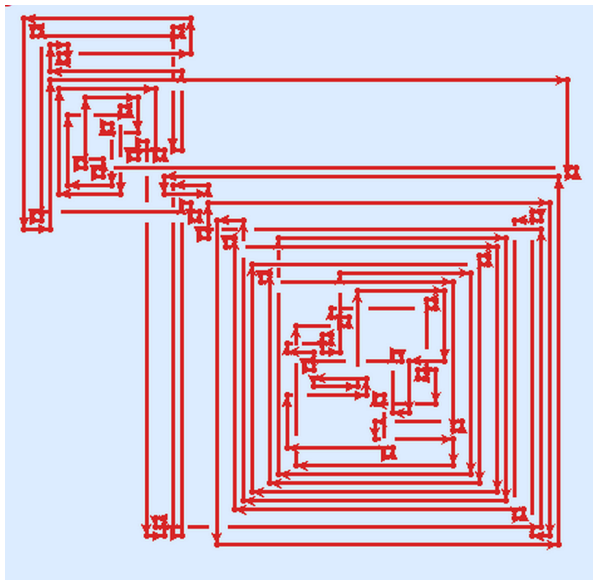
Future Direction: Uniform sampling of large diagrams

Harrison Chapman has results on sampling large diagrams:

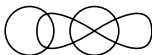
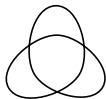


Future Direction: Uniform sampling of large diagrams

Harrison Chapman has results on sampling large diagrams:



Thank you!



This research was supported in part by NSF grant DMS-1344994 (RTG in Algebra, Algebraic Geometry, and Number Theory, at the University of Georgia).