Asymptotics of Knot Diagrams

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University of Georgia

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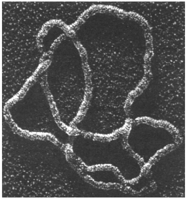
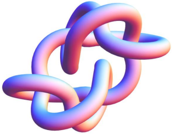
Polymers



$\longrightarrow$

Knots

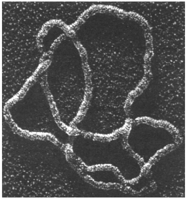


Knot theory

Definition

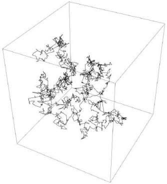
A **knot** is an embedding of the circle S^1 into S^3, up to ambient isotopy. "String can move but not pass through itself."

Polymers



$\longrightarrow$

Random curves



$\longrightarrow$

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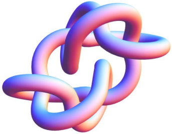

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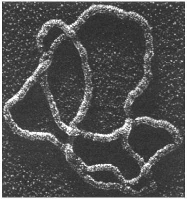

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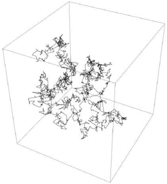

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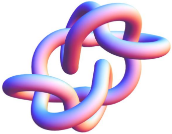
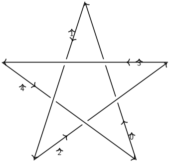
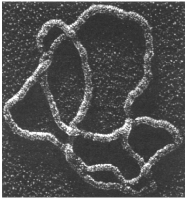


$\uparrow$

Random combinatorial



Polymers



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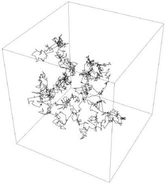

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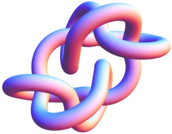

???

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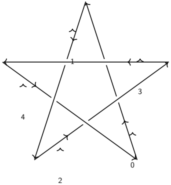

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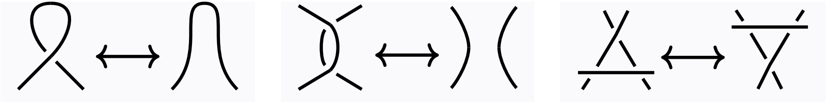
Theorem (Reidemeister)

A **knot** *is an equivalence class of* **knot diagrams** *up to changes by the three* **Reidemeister moves**.

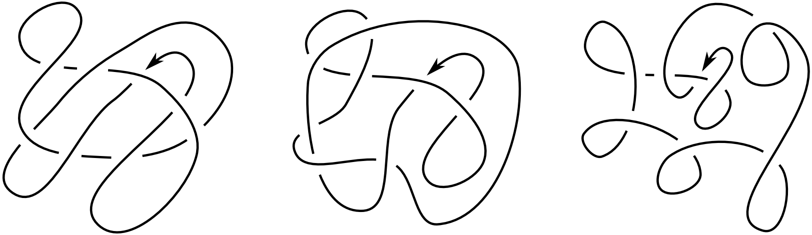


The three Reidemeister moves.

Random diagrams

Definition

In the **random diagram model** of random knotting, a n-crossing diagram is drawn uniformly from the finite set of n-crossing knot diagrams.



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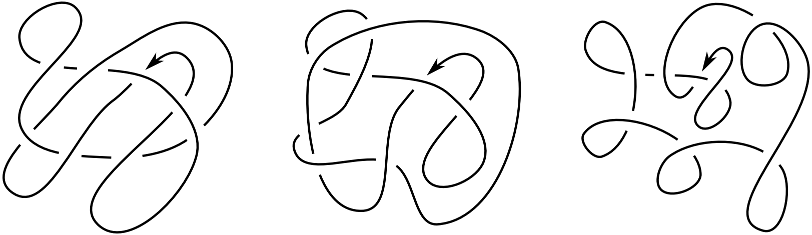
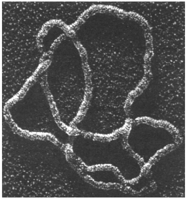


Model is similar to ones considered by Diao-Ernst-Ziegler (2004) and Dunfield (2014; in progress)

Polymers



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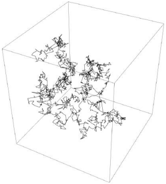

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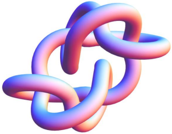

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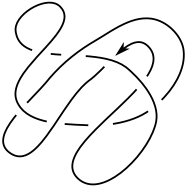

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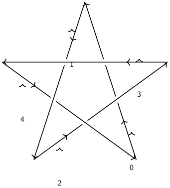
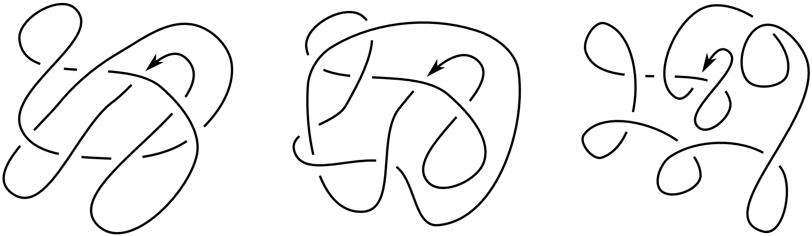


Knot diagrams

Definition

A **knot diagram** is a spherically embedded 4-regular graph together with extra "over-under" information at each vertex.



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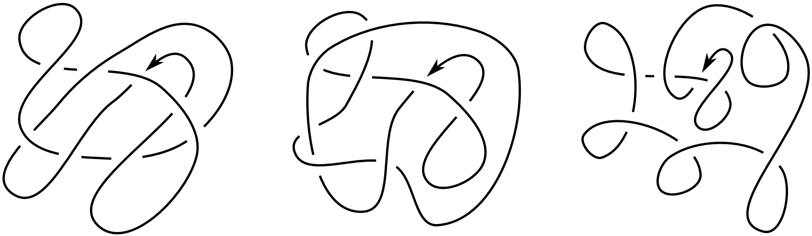

Notation

A graph embedded on a sphere is called a **planar map**.

The Frisch-Wasserman-Delbrück Conjecture

Definition

The equivalence class of knots containing the closed trivial loop is the **unknot**. A representative of this class is called **unknotted**. Otherwise, it is **knotted**.

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Conjecture (Frisch-Wasserman 1961, Delbrück 1962)

The probability that a randomly embedded circle in $\mathbb{R}^3$ *is knotted tends to one as* n *tends to infinity.*

Theorem (Sumners-Whittington 1988)

The FWD conjecture holds for n*-step self avoiding polygons in* $\mathbb{R}^3$*.*

Can prove the conjecture for other space curve models of random knotting, too

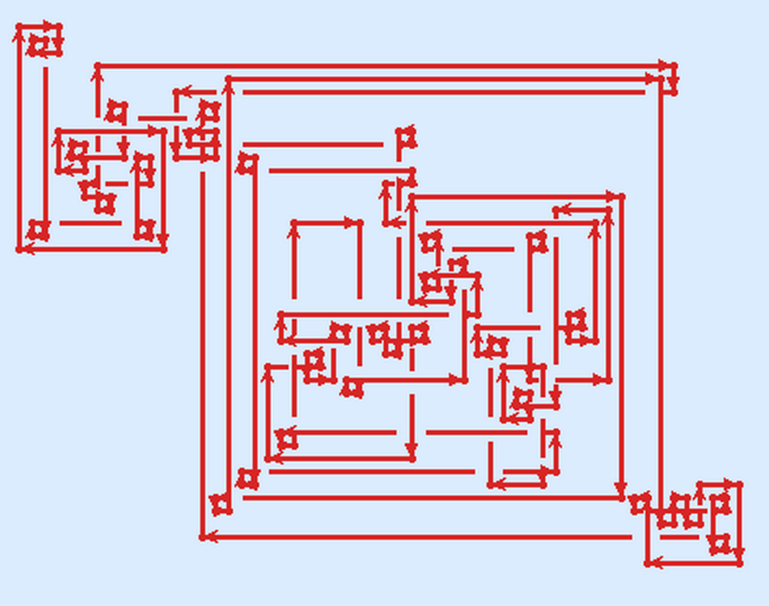
The Frisch-Wasserman-Delbrück Conjecture

Let's reinterpret FWD for our model:

Conjecture (Frisch-Wasserman-Delbrück)

The probability that a knot diagram with n crossings is knotted tends to one as n tends to infinity.

Is this knotted?



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How to prove this? Same idea as Sumners-Whittington's proof!

Idea

Substructure ("patterns") appear linearly often as the size of objects grows.

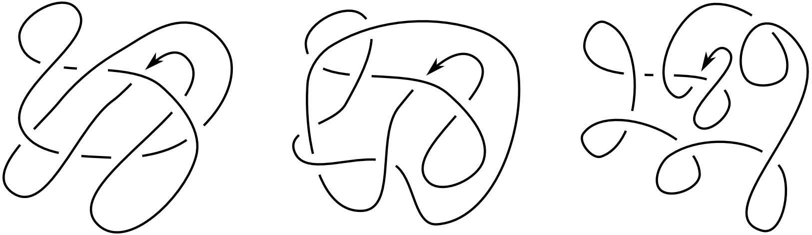
Then pick patterns that assure knottiness.

Symmetries are tough

Symmetries make working with diagrams **difficult**! So kill them...

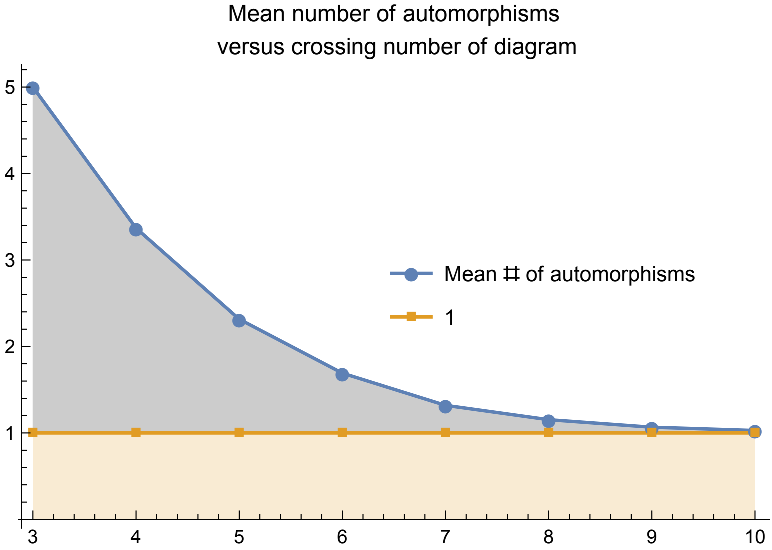
Definition

A **rooted knot diagram** is a knot diagram together with a choice of edge and a choice of direction.



No more nontrivial automorphisms since root must map to itself.

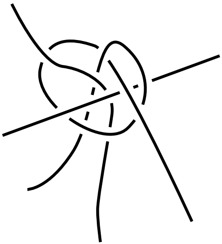
...Is that really okay?

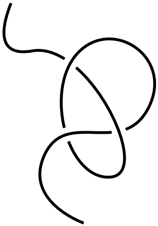


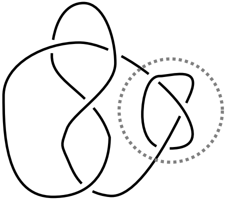
Patterns du jour

Definition

- A $2k$-**tangle** is a diagram-like object having $2k$ half-edges which lie in the exterior face.
- A tangle is contained in a diagram D if there exists some disk which, when intersected with D, produces the tangle.







A pattern theorem for knot diagrams

Indeed (adapting a proof of Bender-Gao-Richmond 1992),

Theorem (C.)

Let $\mathcal{K}$ be the class of rooted knot diagrams and $\mathcal{K}_n$ be the set of rooted knot diagrams with n crossings. Let P be a tangle which is appropriately "admissible." Then there exist constants $c > 0$ and $d < 1$ so that

$$\mathbb{P}(D \text{ in } \mathcal{K}_n \text{ contains} \leq cn \text{ copies of } P \text{ as a subtangle}) < d^n.$$

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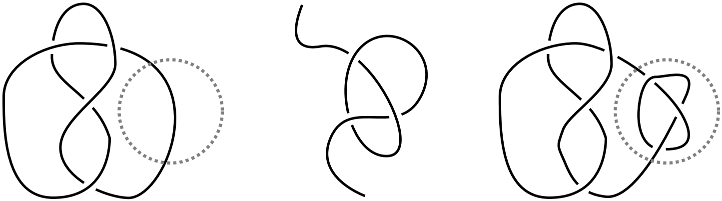
Key requirement for proof.

There is an "attachment" operation on diagrams which produces a new diagram containing P so that for some k depending on the attachment, a diagram in n crossings has n/k valid attachment sites. $\square$

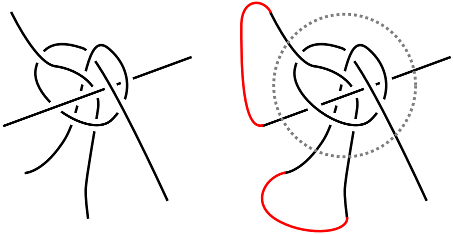
Admissible tangles

Knot diagrams and any 2-tangle of one component; expansion operation: connect summation



Admissible tangles

Knot diagrams and any $2k$-tangle of k components; expansion operation: connect summation (after placing into a 2-tangle)



A technical lemma

Caveat

It's actually required in the proof of the pattern theorem that $\mathcal{K}$ grows **smoothly**; *that*

$$\lim_{n\to\infty} |\mathcal{K}_n|^{1/n} = \limsup_{n\to\infty} |\mathcal{K}_n|^{1/n}.$$

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Asymptotics of knot diagrams are wholly unknown! There are only conjectures...

Conjecture (Schaeffer-P. Zinn-Justin 2004)

The number of rooted knot diagrams grows like

$$|\mathcal{K}_n| \underset{n\to\infty}{\sim} c\tau^n n^{\gamma-2}, \quad \textit{where } \gamma = -\frac{1+\sqrt{13}}{6} \approx -0.76759\ldots$$

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Fortunately (using methods of BGR 1992),

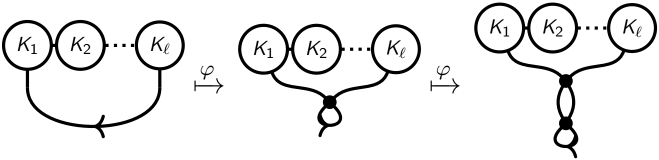
Lemma (C.)

The class of rooted knot diagrams grow smoothly.

Smooth growth for knot diagrams

(Very!) Rough idea of proof.

- It's possible to make $(n+1)$-crossing diagrams from n-crossing diagrams



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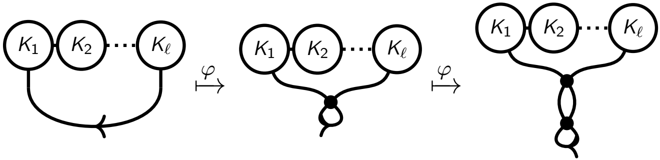
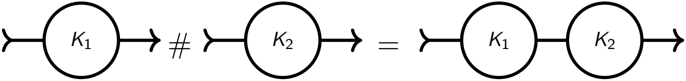
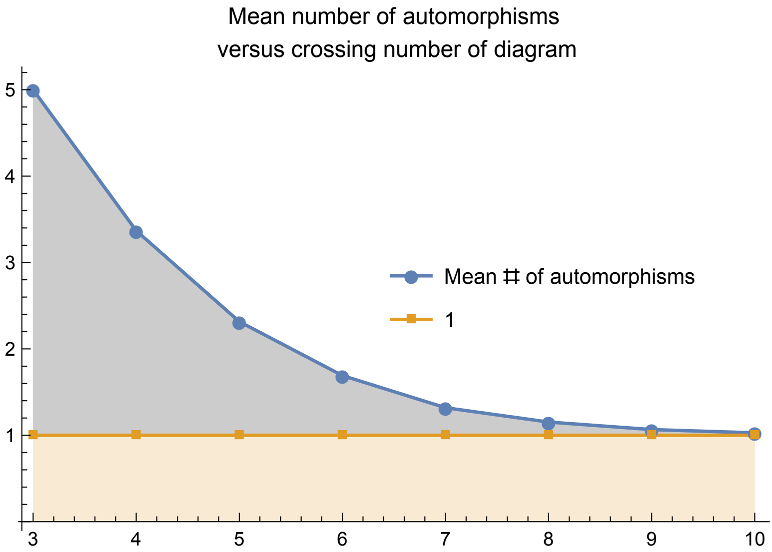


- And it's possible to create $(n+m)$-crossing diagrams from n-crossing diagrams and m-crossing diagrams.



Remember this?



Asymmetry of knot diagrams

The pattern theorem comes with a handy bonus (together with a theorem of Richmond-Wormald 1995):

Theorem (C.)

Almost all unrooted knot diagrams have only trivial automorphism group.

So for large n, rooted knot diagrams map $4n$-to-one to unrooted knot diagrams.

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Unrooted knot diagrams have the pattern theorem

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Unrooted knot diagrams are almost certainly knotted

Recap from work with Cantarella and Mastin

Idea (Cantarella-C.-Mastin)

Sample from the random (unrooted) knot diagram model via complete enumeration. (No other obvious methods)

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Idea

We can closely approximate the random unrooted diagram model by the random rooted diagram model. So just sample from the rooted diagram model.

The game plan

Fact

- *Can sample rooted 4-regular planar maps in* $O(n)$ *(Schaeffer 2003) [Great!]*

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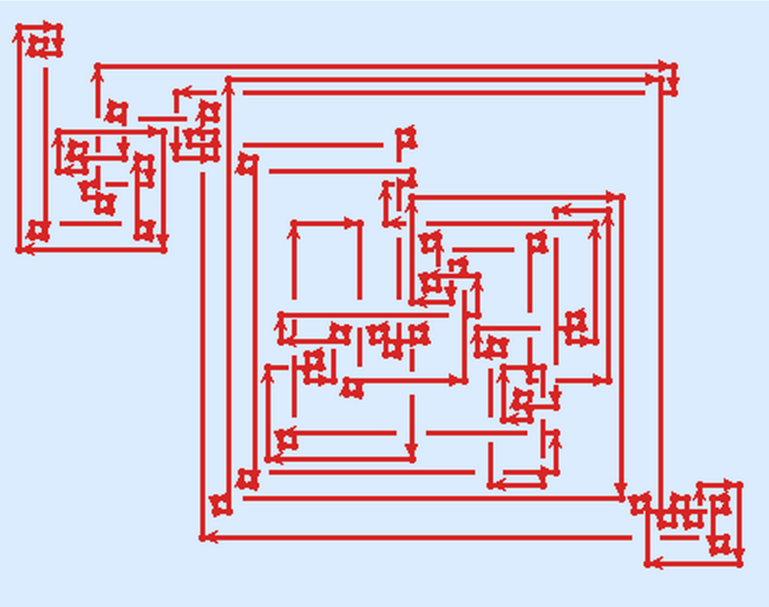
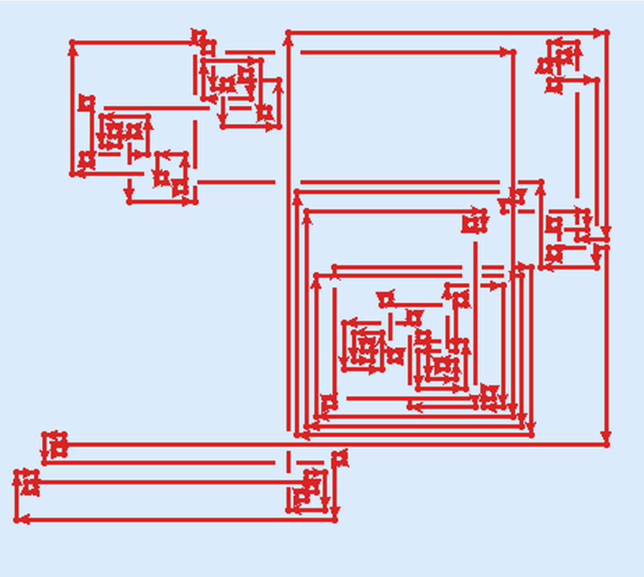
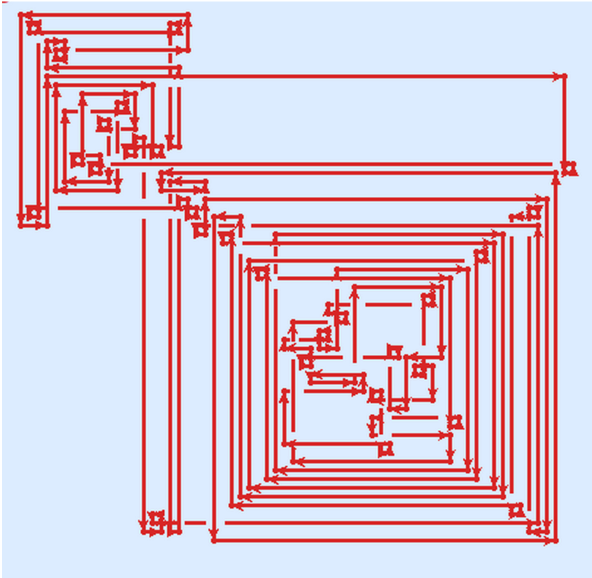
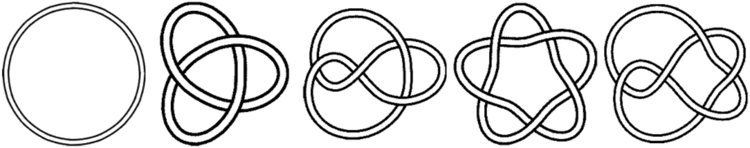
- *Can sample rooted 4-regular planar maps in $O(n)$ (Schaeffer 2003) [Great!]*
- *Can sample n crossing signs uniformly, quickly [Excellent!]*
- *Knot diagrams become asymptotically rare! [Rats!]*
- **However**, *we can still improve on CCM about ten-fold! [Whew...]*



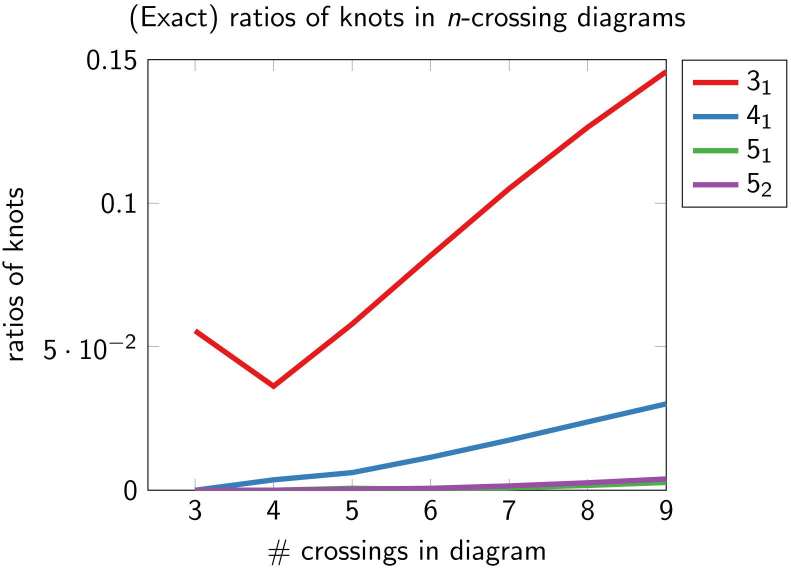




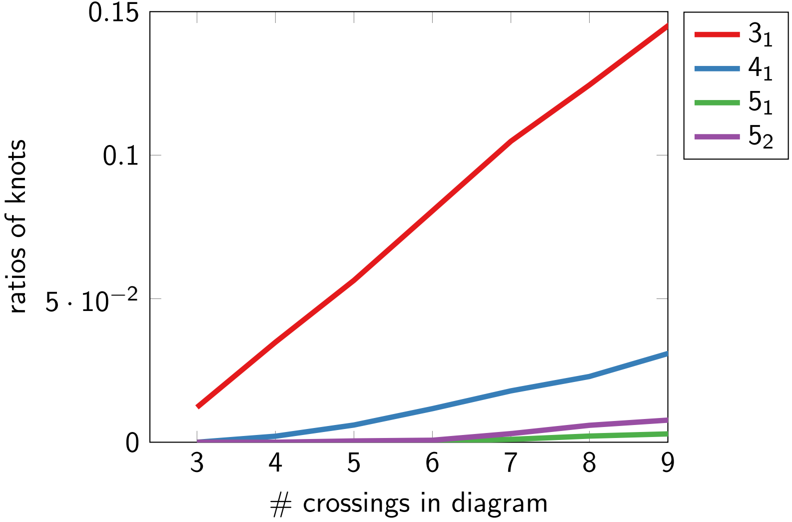
The first few knot types



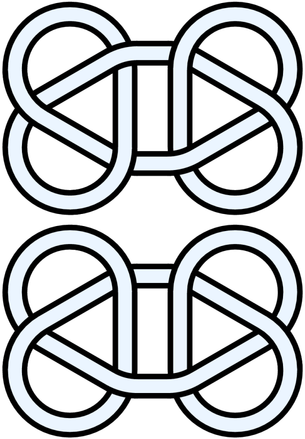
0_1 3_1 4_1 5_1 5_2


(Exact) ratios of knots in n-crossing diagrams
0.15
0.1
$5 \cdot 10^{-2}$
0
ratios of knots
3
4
5
6
7
8
9
crossings in diagram
3_1
4_1
5_1
5_2

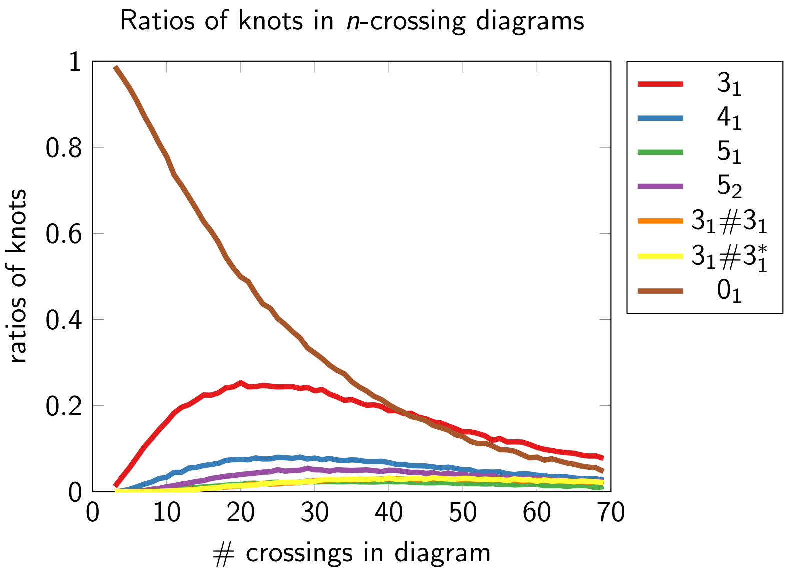
(Experimental) ratios of knots in n-crossing (rooted) diagrams



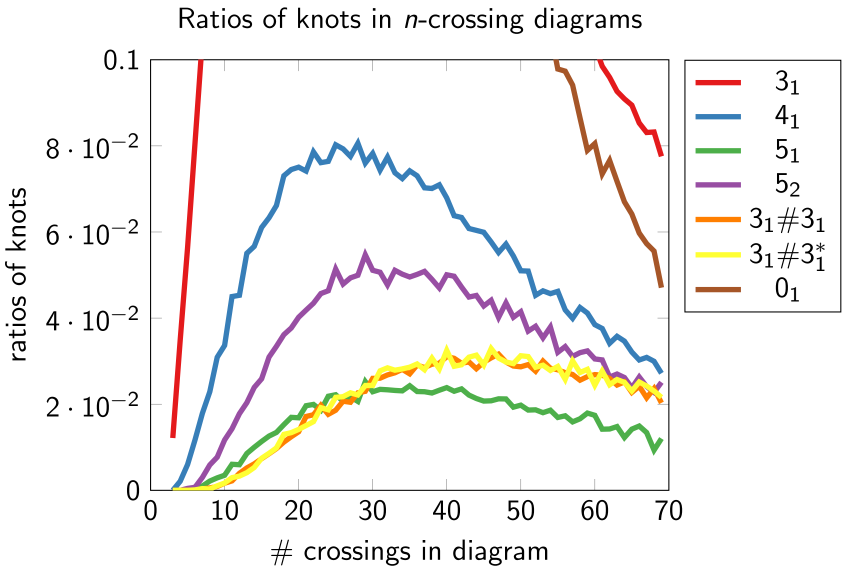
Let's throw in some composite knots

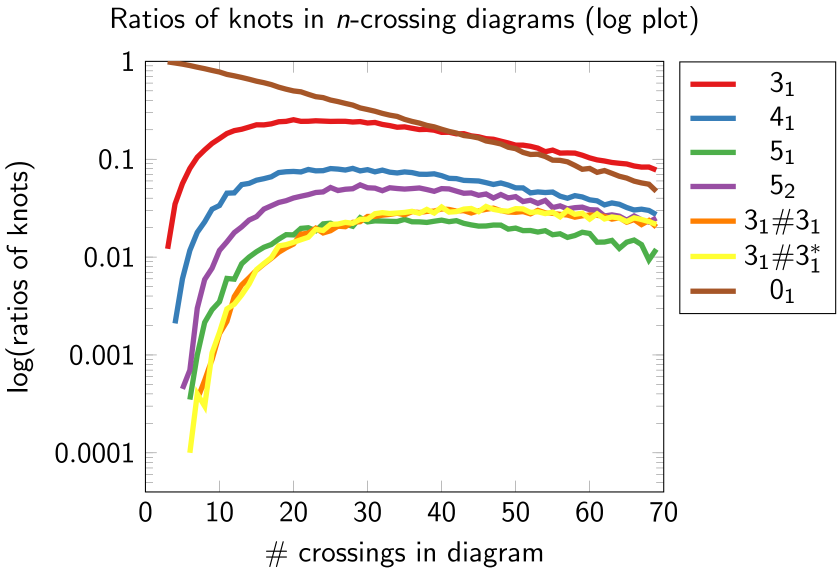


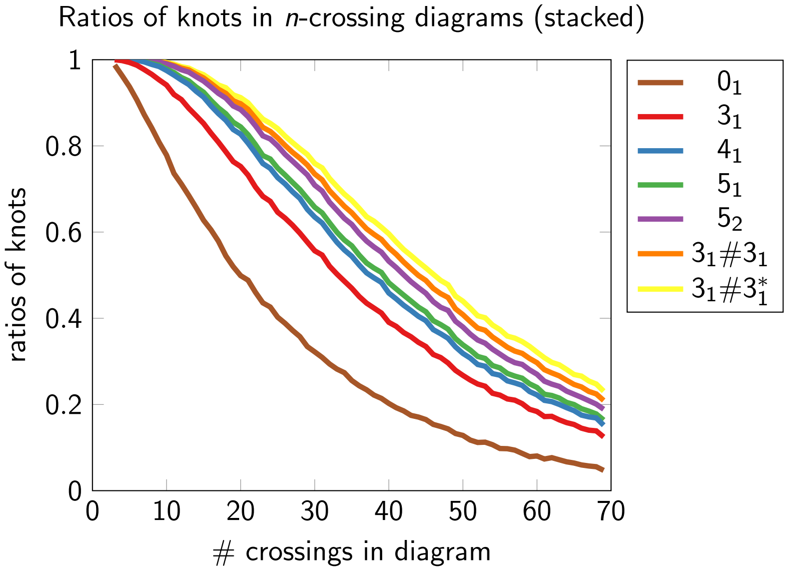
Granny knot $3_1 \# 3_1$ (top) vs square knot $3_1 \# 3_1^*$ (bottom)



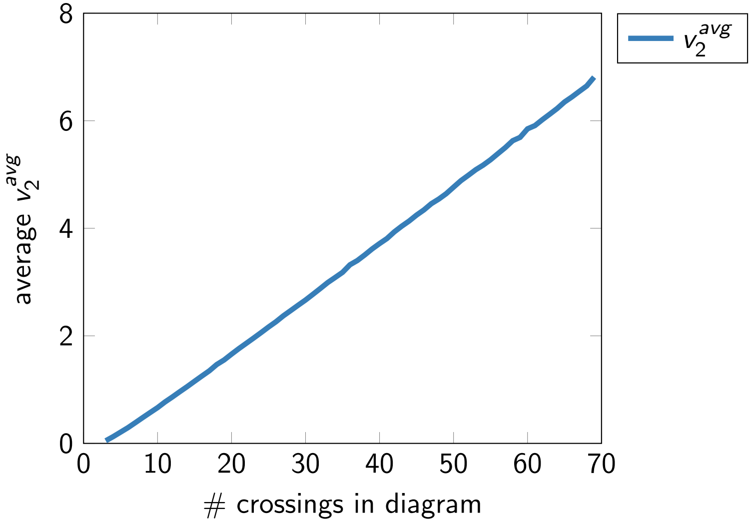
Ratios of knots in n-crossing diagrams
ratios of knots
crossings in diagram
0
0.2
0.4
0.6
0.8
1
0
10
20
30
40
50
60
70
3_1
4_1
5_1
5_2
$3_1\#3_1$
$3_1\#3_1^*$
0_1


Ratios of knots in n-crossing diagrams
ratios of knots
crossings in diagram
0.1
8 · 10^−2
6 · 10^−2
4 · 10^−2
2 · 10^−2
0
0
10
20
30
40
50
60
70
3_1
4_1
5_1
5_2
3_1#3_1
3_1#3_1^*
0_1


Ratios of knots in n-crossing diagrams (log plot)
1
0.1
0.01
0.001
0.0001
log(ratios of knots)
0
10
20
30
40
50
60
70
crossings in diagram
3_1
4_1
5_1
5_2
$3_1\#3_1$
$3_1\#3_1^*$
0_1


Ratios of knots in n-crossing diagrams (stacked)
ratios of knots
crossings in diagram
0
0.2
0.4
0.6
0.8
1
0
10
20
30
40
50
60
70
0_1
3_1
4_1
5_1
5_2
$3_1\#3_1$
$3_1\#3_1^*$

Average Vassiliev-2 invariant for n crossing immersions $S^1 \hookrightarrow S^3$



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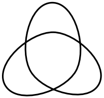
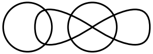
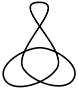
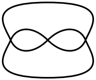
- Diagrams with different underlying structure (Knot diagrams are the circle; also theta curves, tadpoles, etc...)

Thank you!

Coming soon:

Cantarella, C-, Mastin. *Knot probabilities in random diagrams*.

C-. *Asymptotic laws for knot diagrams*.

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