Random Knot Diagrams

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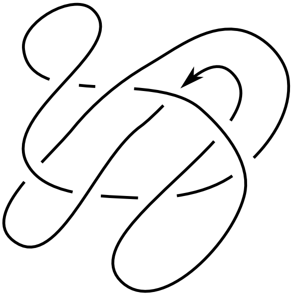
Harrison Chapman (UGA), Matt Mastin (Mailchimp, Inc.)
Crucial Assist: Eric Rawdon (St. Thomas)

Rocky Mountain Algebraic Combinatorics Seminar
June 23, 2016

Natural questions about knot diagrams

Question

What fraction of 8-crossing diagrams are trefoils?



Natural questions about knot diagrams

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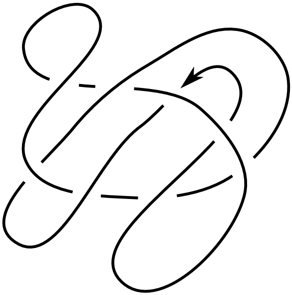
12.48%



Natural questions about knot diagrams

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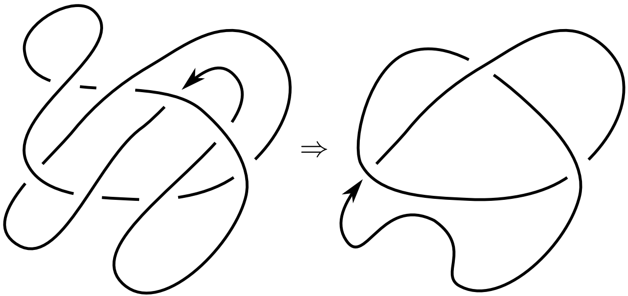
What is the average minimal crossing $\#$ of an 8-crossing diagram?



Natural questions about knot diagrams

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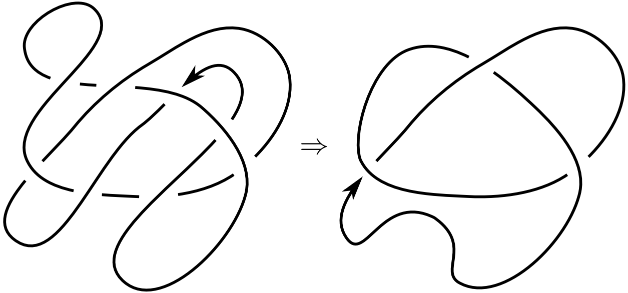
What is the average minimal crossing # of an 8-crossing diagram?

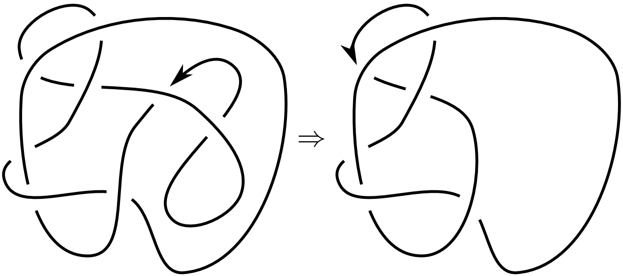
0.52



Natural questions about knot diagrams

Definition

The **untwisting** operator deletes all 1-crossing connect summands of a diagram. (Equivalently, performs all "available" Reidemeister I moves.)



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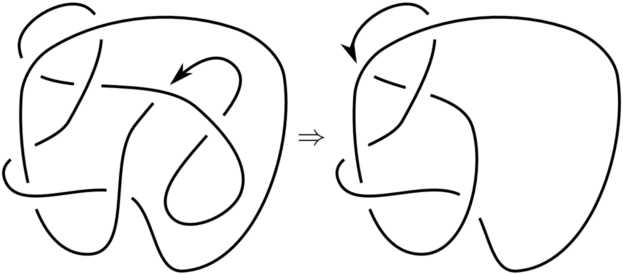

Natural questions about knot diagrams

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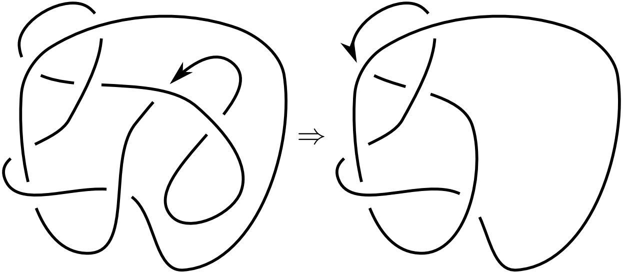
2.20



Natural questions about knot diagrams

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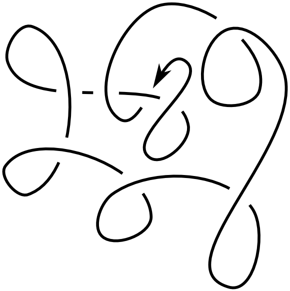
How many 8-crossing diagrams can be untwisted to the unknot?



Natural questions about knot diagrams

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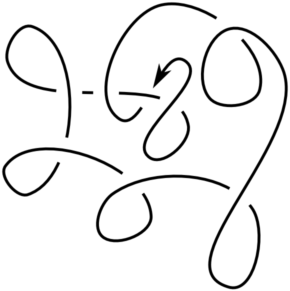
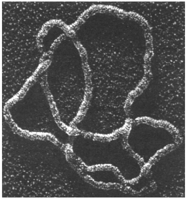
42.05%



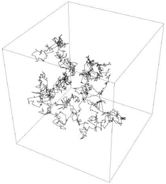
Ansatz

Polymers



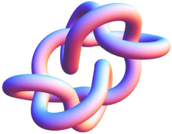
$\longrightarrow$

Random curves



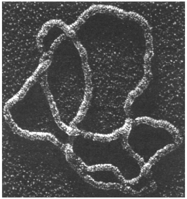
$\longrightarrow$

Knots



Combinatorial approaches

Polymers



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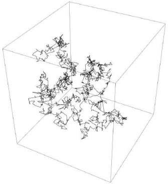
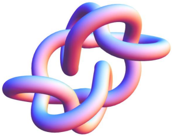
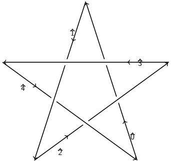


Knots



Random combinatorial



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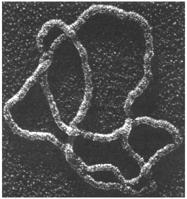
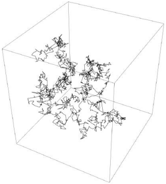
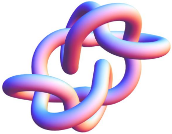


Random curves

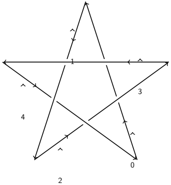


Knots



???

Random combinatorial



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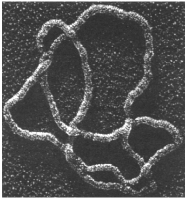

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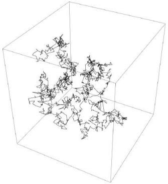

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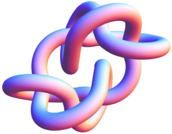
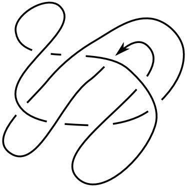
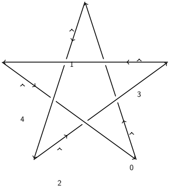


Random diagrams



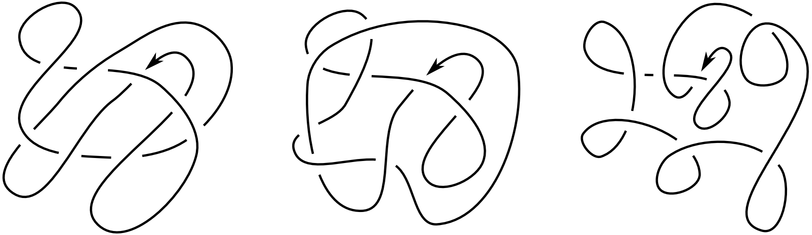
Random combinatorial



Random diagrams

Definition

In the **random diagram model** of random knotting, a n-crossing diagram is drawn uniformly from the finite set of n-crossing knot diagrams.



How to enumerate knot diagrams (like a topologist)

Definition

A **knot shadow** is a equivalence class of generic immersions of the unoriented S^1 into the sphere S^2 up to diffeomorphism of S^2.

Plan to Enumerate Diagrams

1. *Enumerate shadows (and discard isomorphic shadows)*
2. *Assign crossing and orientation information (and discard crossing patterns related by an automorphism of the shadow)*

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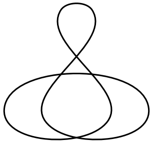
Observation (known to all combinatoricists, but new to me)

Symmetry stinks.

Tabulating knot shadows: `plantri`, two ways

Proposition

Knot shadows $\leftrightarrow$ *1-component 4-valent embedded planar multigraphs up to embedded isomorphism*



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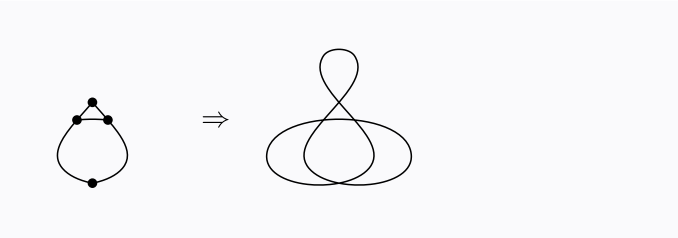

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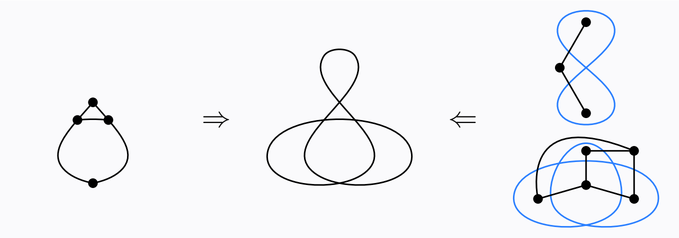
2. Generate multiquadrangulations of sphere by careful pattern of connect sums, take dual graphs (fast)



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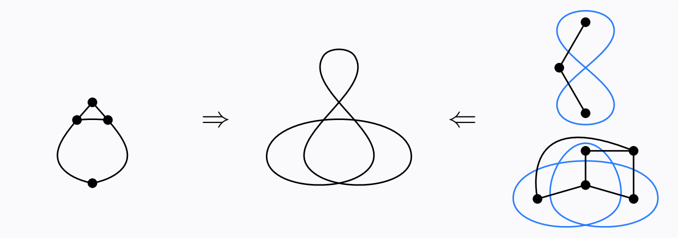


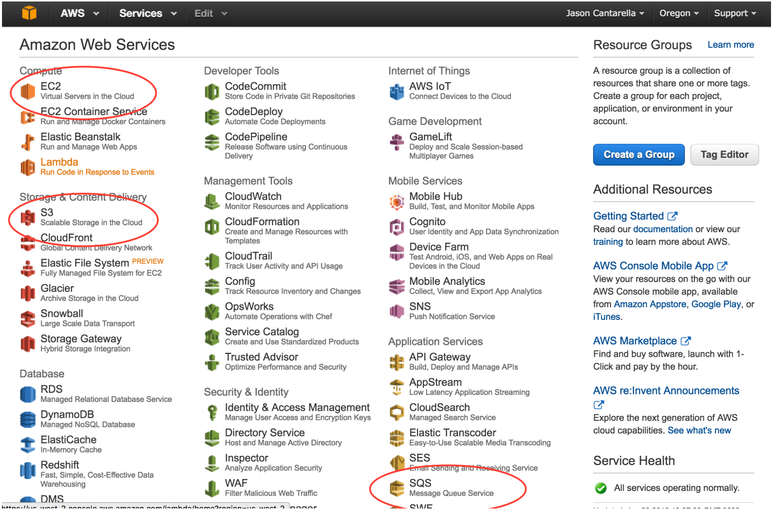
Actually generate all **link shadows**, then restrict to knot shadows

Cloud computing for mathematicians

Several thousand hours of CPU time were required to produce about 1.8 billion diagrams. How to organize this kind of computation?

Moral: If you know Linux and a little python, you can play in the cloud.

AWS Services (you only need three)



EC2: A better computer (or lots of them) by the hour.

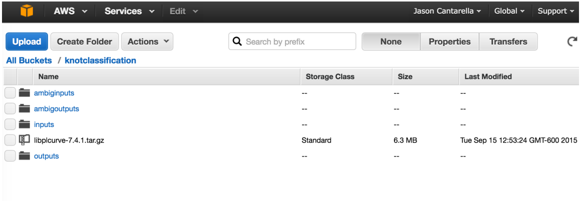
Cloud computing runs on virtual Linux machines. You are root on the machine, and can install software, edit configuration files, compile code, or attach a license for Mathematica or MAGMA.

Type	Processor	Cores	Memory	Price
c4.large	Haswell	2	3.75 gb	1.9 cents/hour
c4.8xlarge	Haswell	36	60 gb	59 cents/hour
r3.large	Ivy Bridge	2	15.75 gb	2.5 cents/hour
r3.8xlarge	Ivy Bridge	32	244 gb	58 cents/hour
x1.32xlarge	Haswell	128	2 tb	397 cents/hour

At peak, we had 100 c4.large instances running simultaneously. The machine may shut down at any time; you are only charged for full hours.

S3: Permanent Internet Accessible Storage, Cheap

If the machines are temporary, where does the data go once they are done computing? To your desktop? (But what if it crashes?) The solution is the "Simple Storage Service":



The cost is 3 cents/gb for instant access, but as low as 0.7 cents/gb for archival storage. Access is by web interface or command line tools (cp, ls, ...). The filesystem is distributed, backed up, and always accessible. There are no limits on the number of files in a "bucket".

SQS: How to get everyone working

SQS is a message queue– the messages describe work to do. You can write to and read from an sqs queue with the Python library boto.

- Worker reads message (should describe < 1 hour of work).
- SQS “locks” the message for one hour.
- If the worker completes the job, the worker should delete the message.
- If the worker dies, the message unlocks and is returned to the queue.
- Messages returned to queue more than n times are set aside for debugging.

Your first million messages (per month) are free. SQS is distributed, backed up, fault-tolerant and always working.

How can you play?

You get \$75 in credits when you sign up. I spent $<$ \$10 on this computation.

Results! (Compared to counts in literature)

oriented	$n=0$	1	2	3	4	5
S^2, S^1	1	1	3	9	37	182
S^2	1	1	2	6	21	99
S^1	1	1	2	6	21	97
–	1	1	2	6	19	76

Curves on S^2. The number of types

V.I. Arnol'd. *Topological Invariants of Plane Curves*

A008989 Number of immersions of unoriented circle into unoriented sphere with n double points.

1, 1, 2, 6, 19, 76, 376, 2194 (list; graph; refs; listen; history; text; internal format)

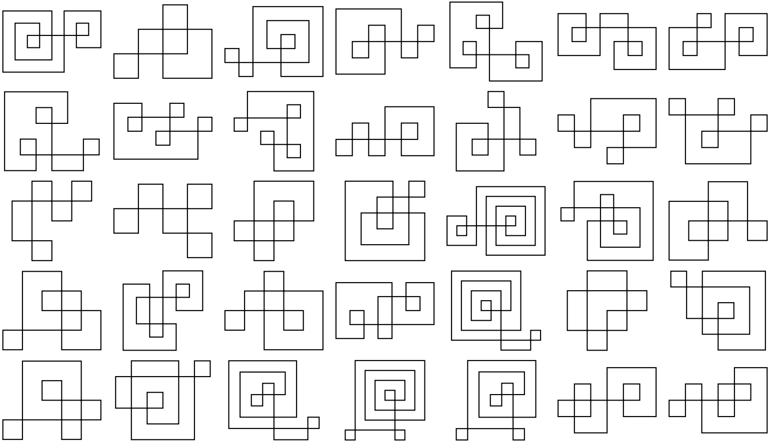
OFFSET	0,3
REFERENCES	V. I. Arnold, Topological Invariants of Plane Curves..., American Math.
LINKS	Table of n, a(n) for n=0..7.
CROSSREFS	Sequence in context: A150119 A181770 A138800 * A057240 A079564 A079453 Adjacent sequences: A008986 A008987 A008988 * A008990 A008991 A008992
KEYWORD	nonn
AUTHOR	N. J. A. Sloane.
EXTENSIONS	Two more terms from Guy H. Valette (guy.valette(AT)skynet.be), Feb 09 20
STATUS	approved

OEIS A008989

n	# knot shadows
0	1
1	1
2	2
3	6
4	19
5	76
6	376
7	2194
8	14614
9	106421
10	823832

We have not found any existing counts of **diagrams**.

The space of shadows



Monogons are very common

Mean Number of Monogons in a Shadow

Cr	3	4	5	6	7	8	9	10
	$\frac{12}{6}$	$\frac{48}{19}$	$\frac{213}{76}$	$\frac{1196}{376}$	$\frac{7714}{2194}$	$\frac{56540}{14614}$	$\frac{448584}{106421}$	$\frac{3758456}{823832}$
	2.	2.53	2.8	3.18	3.52	3.87	4.22	4.56

Fraction of Shadows Containing a Monogon

Cr	3	4	5	6	7	8	9	10
	$\frac{5}{6}$	$\frac{18}{19}$	$\frac{74}{76}$	$\frac{371}{376}$	$\frac{2178}{2194}$	$\frac{14562}{14614}$	$\frac{106216}{106421}$	$\frac{822989}{823832}$
	0.833	0.947	0.974	0.987	0.993	0.996	0.998	0.999

Bigons are very common

Mean Number of Bigons in a Shadow

Cr	3	4	5	6	7	8	9	10
	$\frac{6}{6}$	$\frac{18}{19}$	$\frac{88}{76}$	$\frac{470}{376}$	$\frac{3037}{2194}$	$\frac{21925}{14614}$	$\frac{173342}{106421}$	$\frac{1450209}{823832}$
	1.	0.947	1.16	1.25	1.38	1.5	1.63	1.76

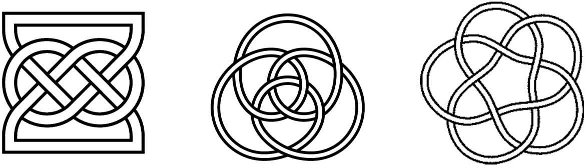
Fraction of Shadows Containing a Bigon

Cr	3	4	5	6	7	8	9	10
	$\frac{3}{6}$	$\frac{11}{19}$	$\frac{52}{76}$	$\frac{275}{376}$	$\frac{1714}{2194}$	$\frac{11892}{14614}$	$\frac{89627}{106421}$	$\frac{712961}{823832}$
	0.5	0.579	0.684	0.731	0.781	0.814	0.842	0.865

Basic polyhedra 8^*, 9^*, and 10^*

Proposition

Conway's basic polyhedra 8^*, 9^*, *and* 10^* *are the only shadows in* ≤ 10 *crossings with no monogons or bigons.*



8_{18} (left), 9_{40} (middle), 10_{123} (right)

Assign crossings, orientation, identify

1. Orient each component. (2 choices)
2. Assign over-under information to each vertex. (2^n choices)

n	# knot shadows	2^{n+1} (# shadows)	# knot diagrams
3	6	96	36
4	19	608	276
5	76	4,864	2,936
6	376	48,128	35,872
7	2,194	561,664	484,088
8	14,614	7,482,368	6,967,942
9	106,421	108,975,104	105,555,336
10	823,832	1,687,207,936	1,664,142,836

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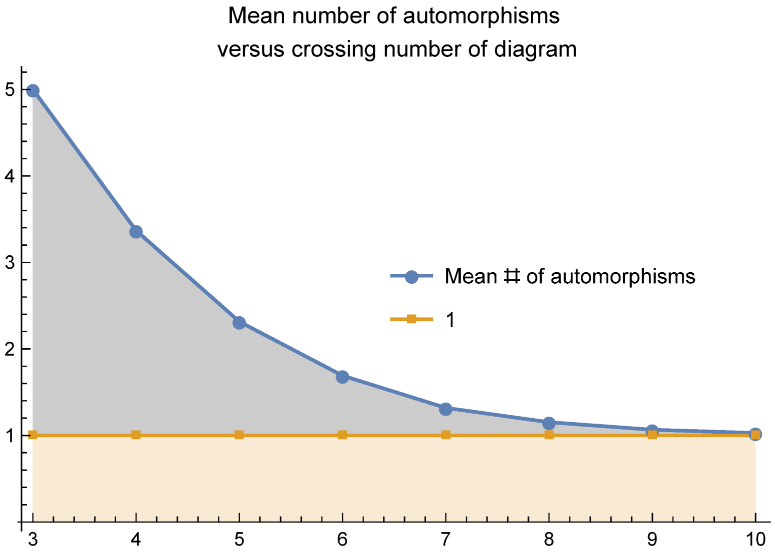
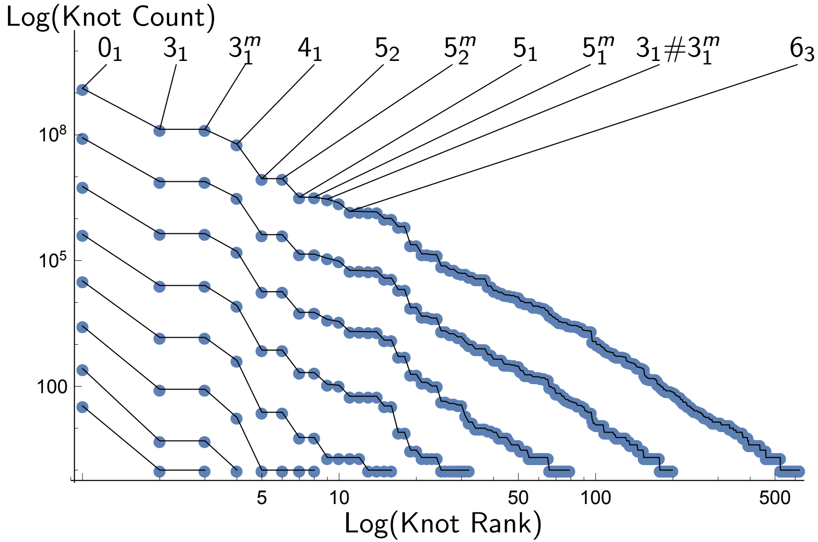
Observation

Symmetry becomes rare, quickly!

Size of the automorphism group of a random diagram



Knotting in diagrams

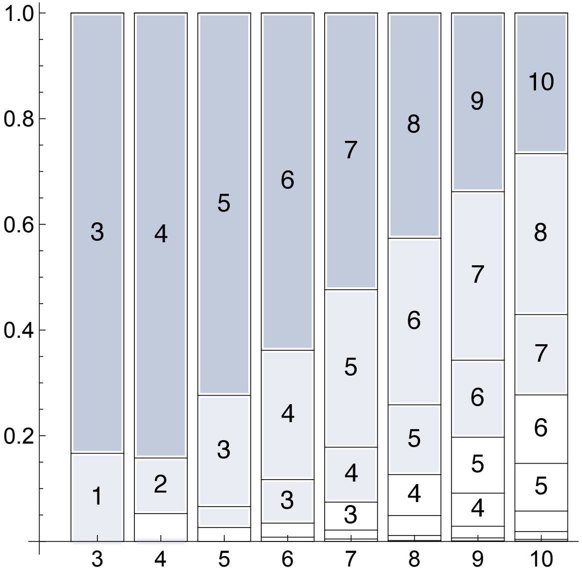


Unknot fraction

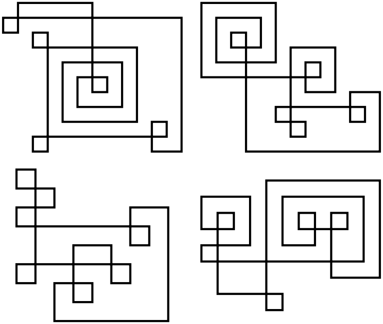
Cr	Unknots	(decimal)
3	$\frac{17}{18}$	0.94
4	$\frac{265}{276}$	0.96
5	$\frac{343}{367}$	0.93
6	$\frac{4057}{4484}$	0.90
7	$\frac{105583}{121022}$	0.87
8	$\frac{2926416}{3483971}$	0.84
9	$\frac{42626767}{52777668}$	0.81
10	$\frac{1291291155}{1664142836}$	0.78

Unknots are very common, even among 10 crossing diagrams. Why?

Most diagrams are (very) composite



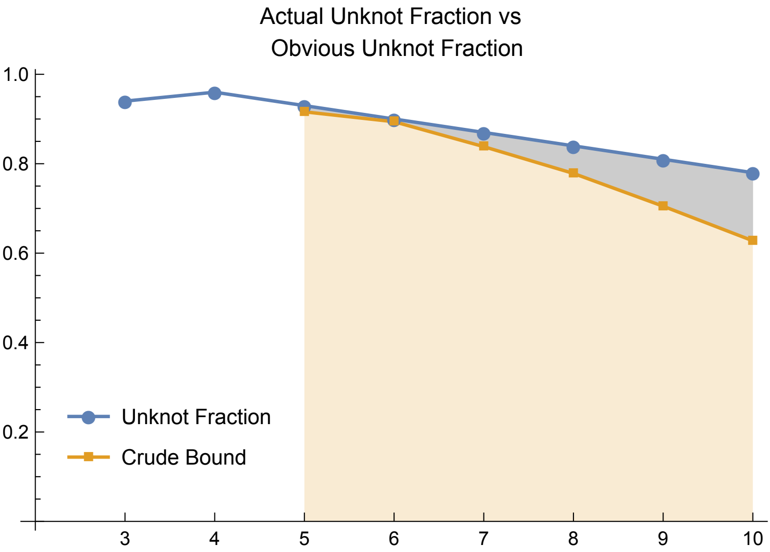
Maximally composite diagrams are "treelike"



Question

Treelike diagrams can't be knotted with any assignment of crossings. Does this (crude) bound explain the unknot fraction?

Pretty much.



Actual Unknot Fraction vs
Obvious Unknot Fraction
1.0
0.8
0.6
0.4
0.2
3
4
5
6
7
8
9
10
Unknot Fraction
Crude Bound

A question on unknotting

Theorem ([Frisch-Wassermann-Delbrück Conjecture] Sumners-Whittington 1988)

The ratio of unknots in random n-edge self-avoiding lattice polygons tends to zero exponentially with n.

Theorem (Chapman)

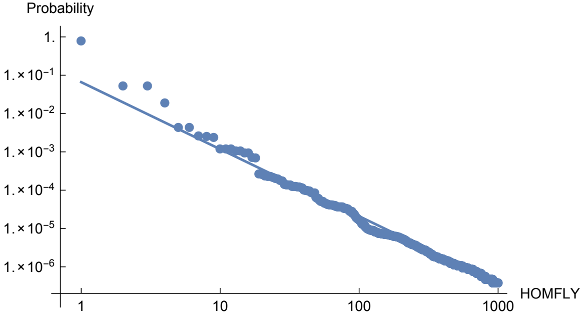
The ratio of unknots in diagrams tends to zero exponentially as n increases.

Future Direction: So what about those log-log plots?

Proposition (with Shonkwiler, 2015)

The symplectic structure on polygon space yields a fast direct sampling algorithm for closed equilateral polygons.

Log(Knot Freq)



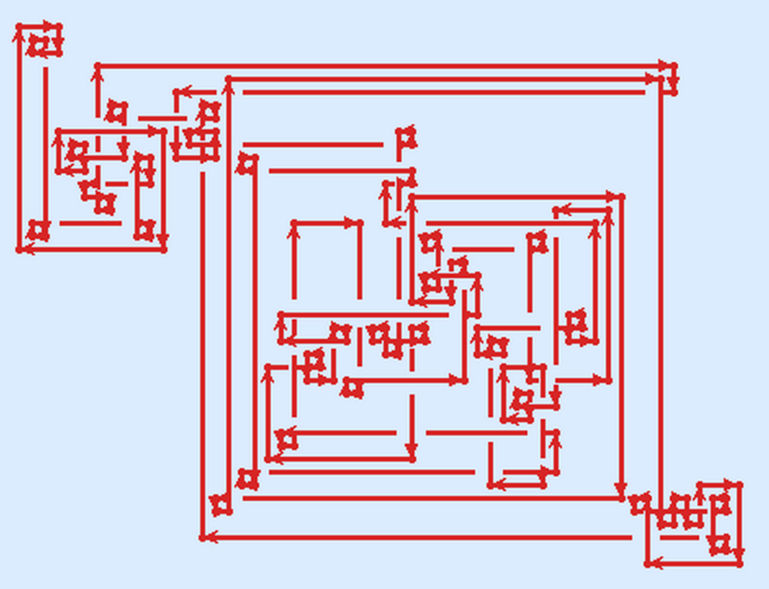
Log(Knot Rank)

Future Direction: You can play, too!

- Knot Probabilities in Random Diagrams Cantarella, Chapman, Mastin. arXiv:1512.05749
- All data (and pictures for all the diagrams) available at www.jasoncantarella.com/wordpress/papers/
- A Fast Direct Sampling Algorithm for Random Equilateral Polygons Cantarella, Duplantier, Shonkwiler, Uehara. arXiv:1510.02466

Future Direction: Uniform sampling of large diagrams

Harrison Chapman has results on sampling large diagrams:



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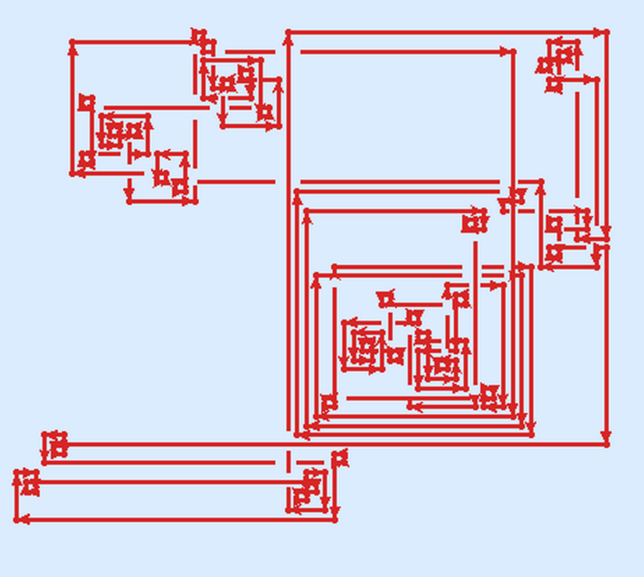
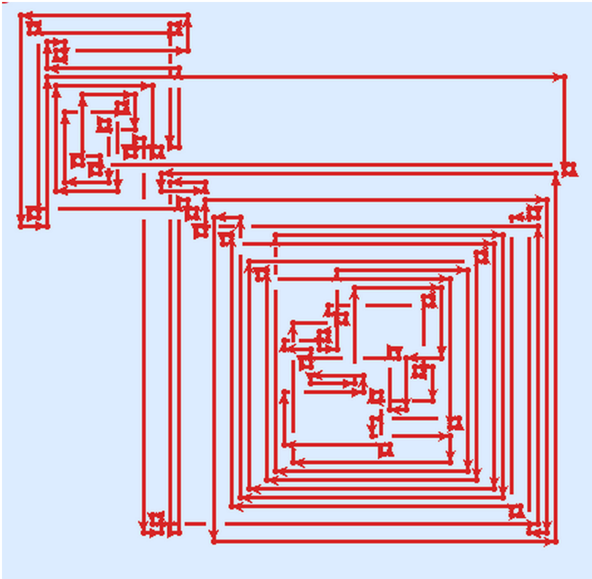
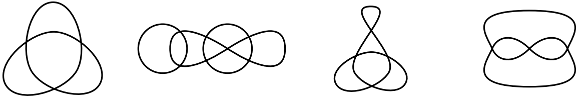


Future Direction: Uniform sampling of large diagrams

Harrison Chapman has results on sampling large diagrams:



Thank you!



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