Asymptotics of Random Knot Diagrams

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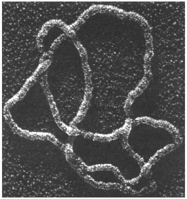
(some joint w/ Jason Cantarella and Matt Mastin)

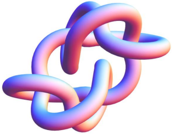
Special Session on Algebraic and Combinatorial Structures in Knot Theory
AMS Western Fall 2015 Sectional
CSU Fullerton, October 25, 2015

Polymers



$\longrightarrow$

Knots



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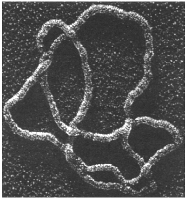
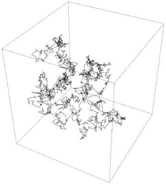


$\longrightarrow$

Random curves



$\longrightarrow$

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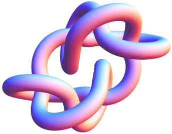

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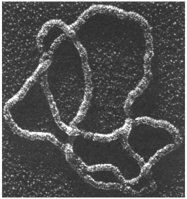

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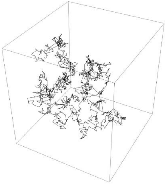

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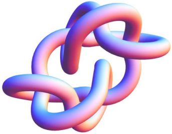
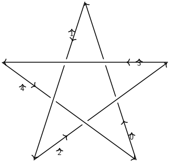
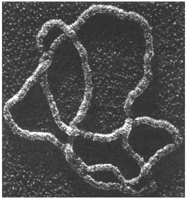


$\uparrow$

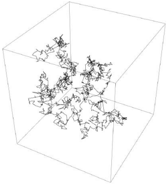
Random
combinatorial



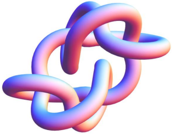
Polymers



Random curves

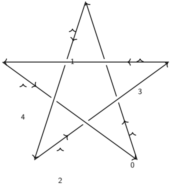


Knots



???

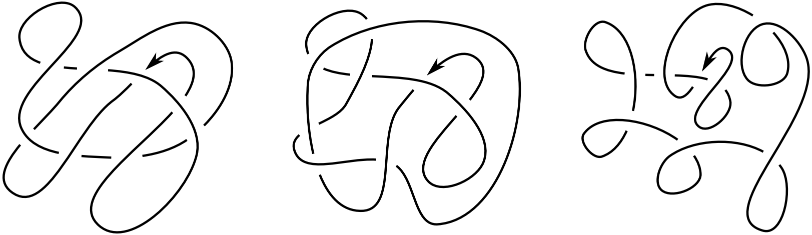
Random combinatorial



Random diagrams

Definition

In the **random diagram model** of random knotting, a n-crossing diagram is drawn uniformly from the finite set of n-crossing knot diagrams.



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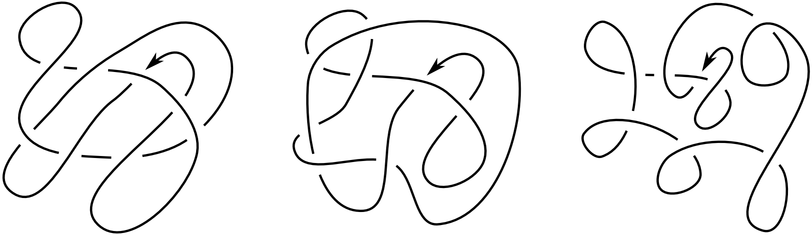
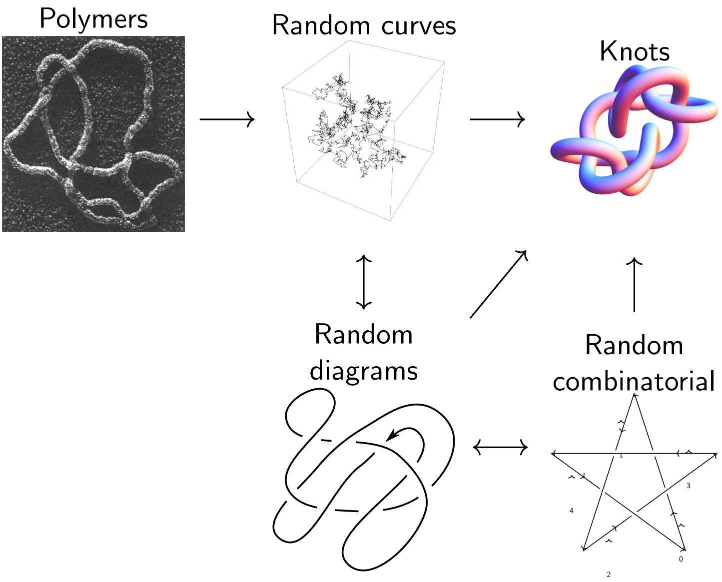


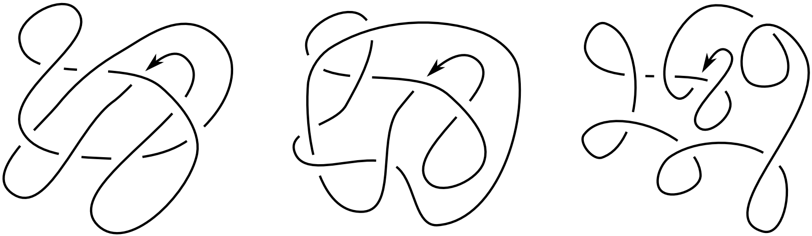
Model is similar to ones considered by Diao-Ernst-Ziegler (2004) and Dunfield (2014; in progress)


Polymers
Random curves
Knots
Random
diagrams
Random
combinatorial
1
3
4
0
2

Knot diagrams

Definition

A **knot diagram** is a spherically embedded 4-regular graph together with extra "over-under" information at each vertex.



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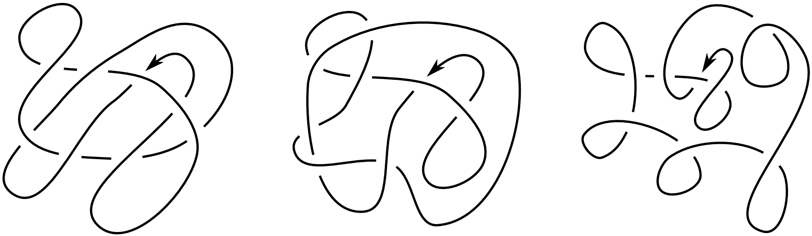

Notation

A graph embedded on a sphere is called a **planar map***.*

The Frisch-Wasserman-Delbrück Conjecture

Definition

The equivalence class of knots containing the closed trivial loop is the **unknot**. A representative of this class is called **unknotted**. Otherwise, it is **knotted**.

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Conjecture (Frisch-Wasserman 1961, Delbrück 1962)

The probability that a randomly embedded circle in $\mathbb{R}^3$ is knotted tends to one as n tends to infinity.

Theorem (Sumners-Whittington 1988)

The FWD conjecture holds for n-step self avoiding polygons in $\mathbb{R}^3$.

Can prove the conjecture for other space curve models of random knotting, too

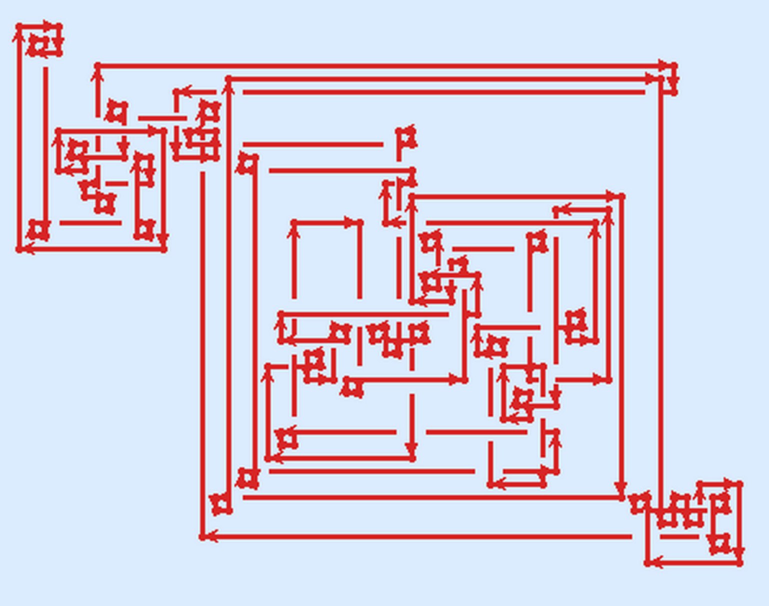
The Frisch-Wasserman-Delbrück Conjecture

Let's reinterpret FWD for our model:

Conjecture (Frisch-Wasserman-Delbrück)

The probability that a knot diagram with n crossings is knotted tends to one as n tends to infinity.

Is this knotted?



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How to prove this? Same idea as Sumners-Whittington's proof!

Idea

Substructure ("patterns") appear linearly often as the size of objects grows.

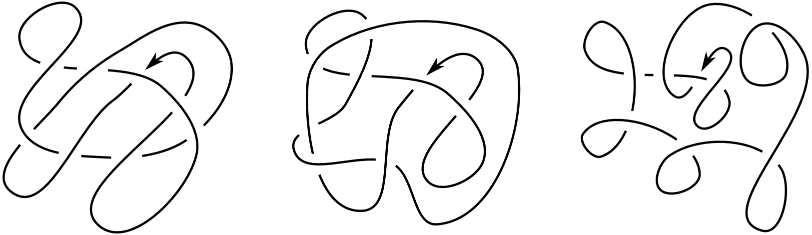
Then pick patterns that assure knottiness.

Symmetries are tough

Symmetries make working with diagrams **difficult**! So kill them...

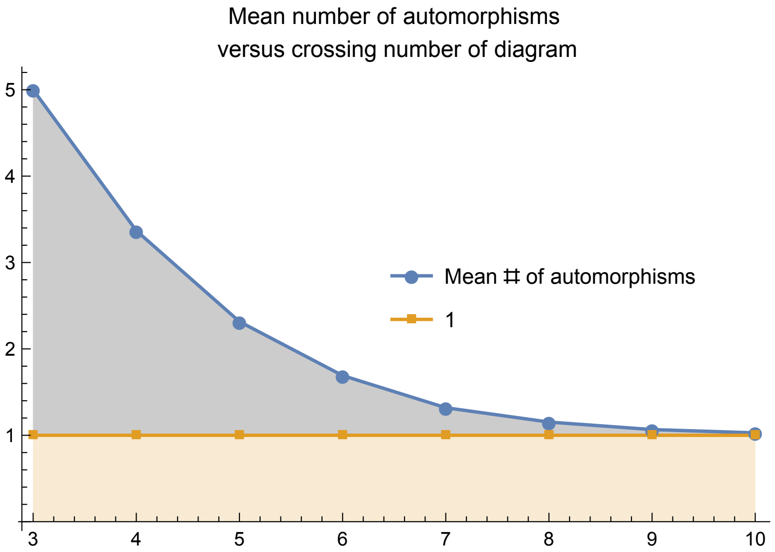
Definition

A **rooted knot diagram** is a knot diagram together with a choice of edge and a choice of direction.



No more nontrivial automorphisms since root must map to itself.

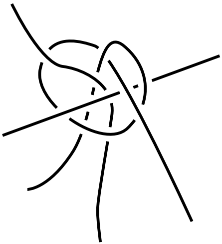
...Is that really okay?

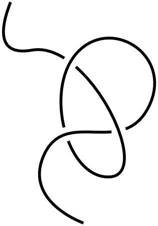


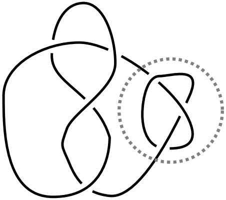
Patterns du jour

Definition

- A $2k$-**tangle** is a diagram-like object having $2k$ half-edges which lie in the exterior face.
- A tangle is contained in a diagram D if there exists some disk which, when intersected with D, produces the tangle.







A pattern theorem for knot diagrams

Indeed (adapting a proof of Bender-Gao-Richmond 1992),

Theorem (C.)

Let $\mathcal{K}_n$ be the set of rooted knot diagrams with n crossings.
Let P be a tangle which is appropriately "admissible."
Then there exist constants $c > 0$ and $d < 1$ so that

$$\mathbb{P}(D \text{ in } \mathcal{K}_n \text{ contains} \leq cn \text{ copies of } P \text{ as a subtangle}) < d^n.$$

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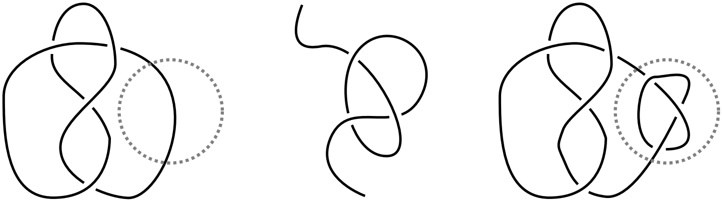
Key requirement for proof.

There is an "attachment" operation on diagrams which produces a new diagram containing P for which any diagram has n/k valid sites (k arbitrary) □

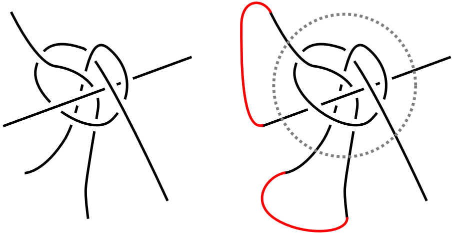
Admissible tangles

Knot diagrams and any 2-tangle of one component; expansion operation: connect summation



Admissible tangles

Knot diagrams and any $2k$-tangle of k components; expansion operation: connect summation (after placing into a 2-tangle)



A technical lemma

Caveat

It's actually required in the proof of the pattern theorem that $\mathcal{K}$ *grows* **smoothly**; *that*

$$\lim_{n\to\infty} |\mathcal{K}_n|^{1/n} = \limsup_{n\to\infty} |\mathcal{K}_n|^{1/n}.$$

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Asymptotics of knot diagrams are wholly unknown! There are only conjectures...

Conjecture (Schaeffer-P. Zinn-Justin 2004)

The number of rooted knot diagrams grows like

$$|\mathcal{K}_n| \underset{n\to\infty}{\sim} c\tau^n n^{\gamma-2}, \quad \textit{where } \gamma = -\frac{1+\sqrt{13}}{6} \approx -0.76759...$$

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Fortunately (using methods of BGR 1992),

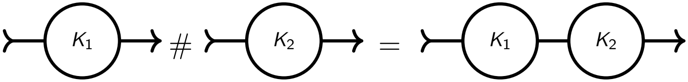
Lemma (C.)

The class of rooted knot diagrams grow smoothly.

Smooth growth for knot diagrams

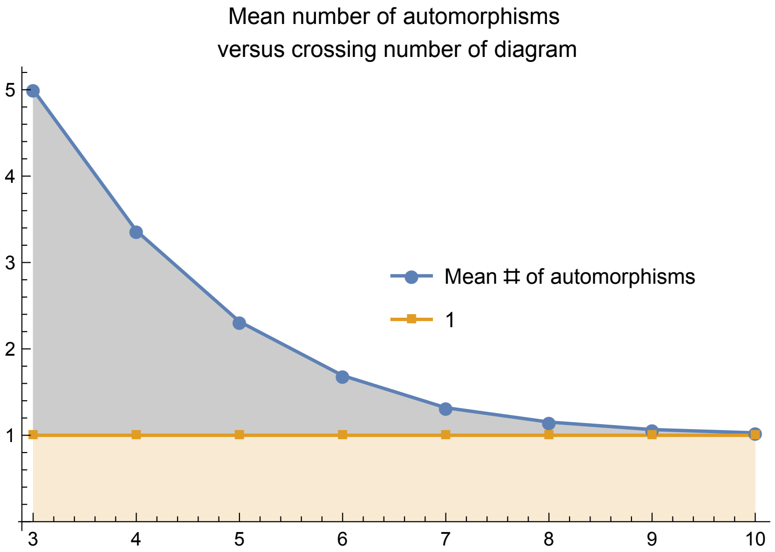
(Very!) Rough idea of proof.

Connect sum produces diagrams of size $n + m$ from diagrams of size n and m:



$\square$

Remember this?



Asymmetry of knot diagrams

The pattern theorem comes with a handy bonus (together with a theorem of Richmond-Wormald 1995):

Theorem (C.)

Almost all unrooted knot diagrams have only trivial automorphism group.

So for large n, rooted knot diagrams map $4n$-to-one to unrooted knot diagrams.

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Unrooted knot diagrams are almost certainly knotted

Recap from work with Cantarella and Mastin

Idea (Cantarella-C.-Mastin)

Sample from the random (unrooted) knot diagram model via complete enumeration. (No other obvious methods)

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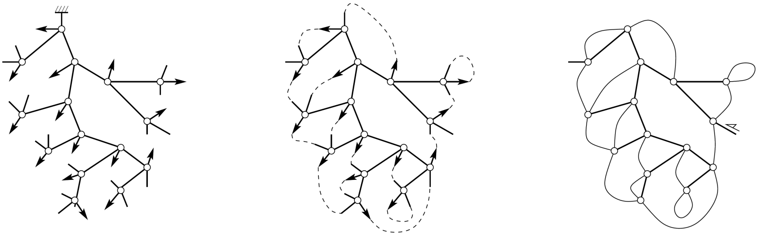
Idea

We can closely approximate the random unrooted diagram model by the random rooted diagram model. So just sample from the rooted diagram model.

A beautiful bijection

Rooted 4-valent maps are in bijection with blossom trees (easy to sample)



(Figure from Schaeffer, Zinn-Justin 2004)

Plan for random uniform sampling

Fact

- *Can sample rooted 4-regular planar maps in $O(n)$ (Schaeffer 2003) [Great!]*

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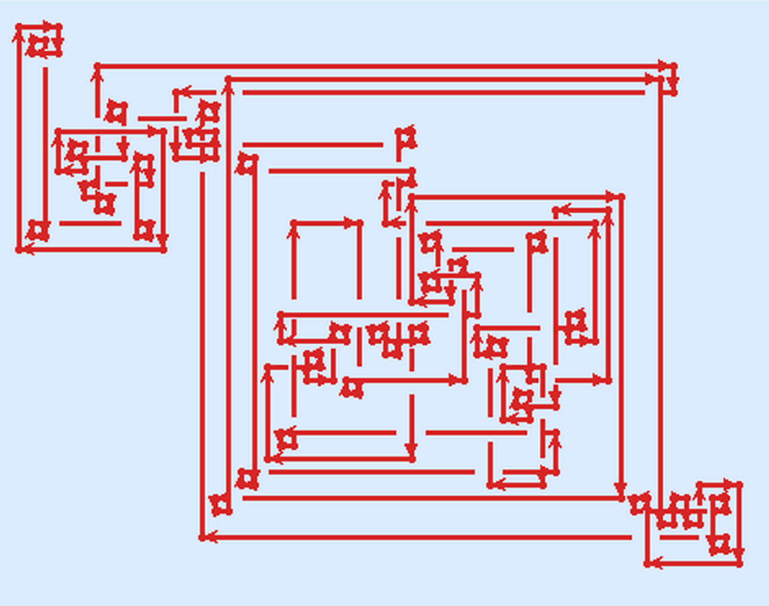
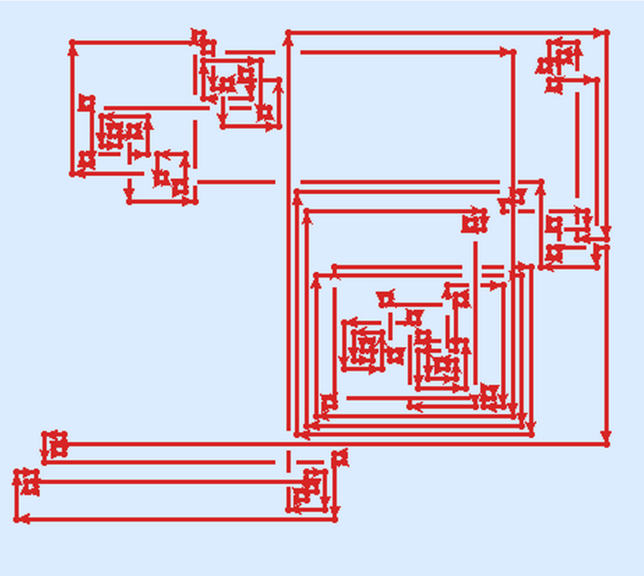
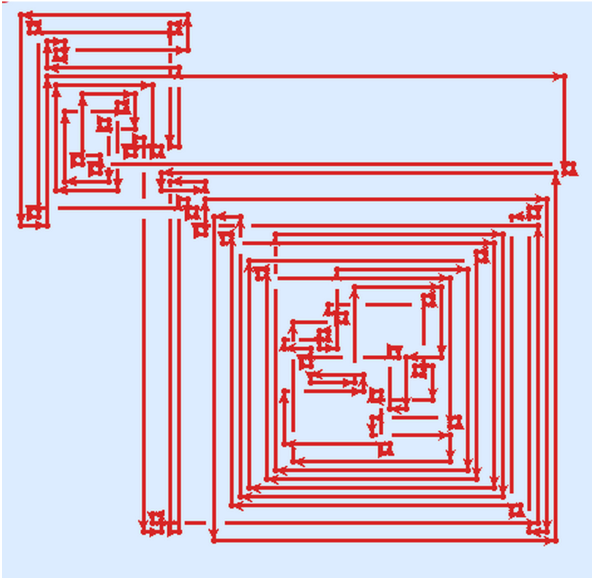
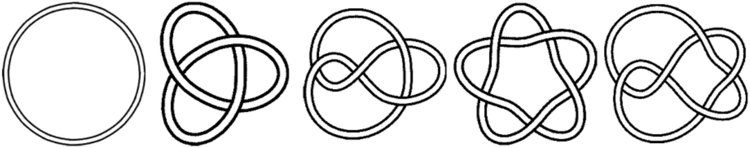
- **However**, *we can still improve on CCM about ten-fold! [Whew...]*





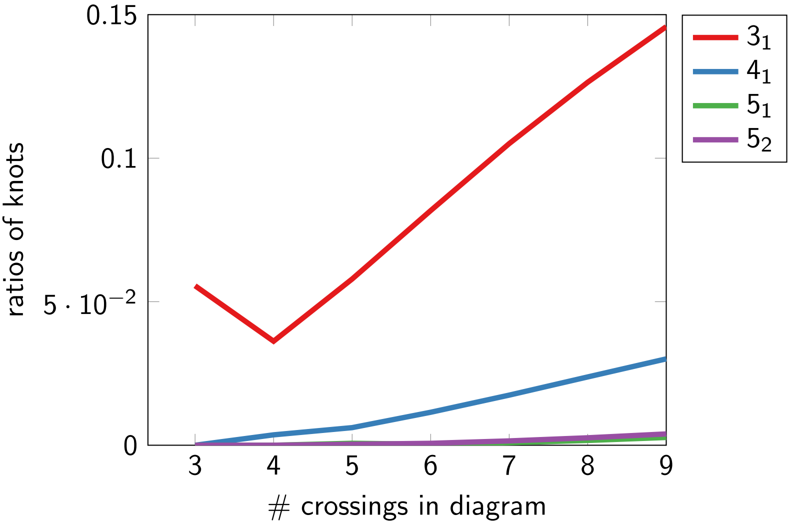


The first few knot types

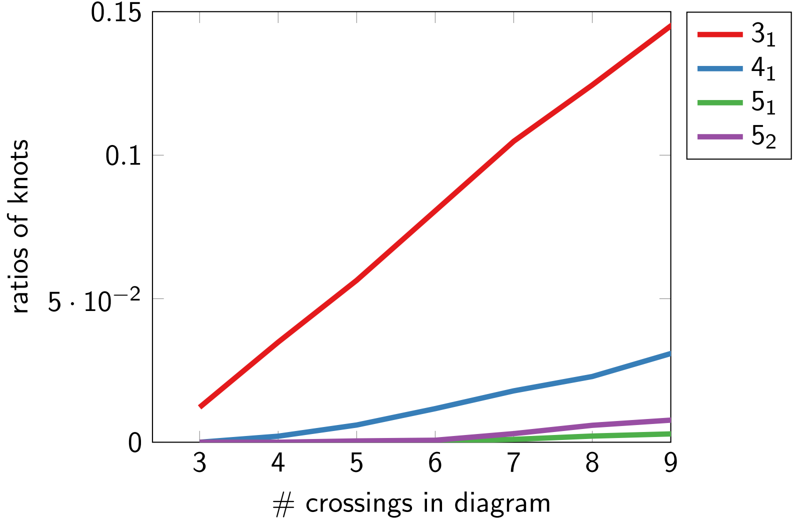


0_1 3_1 4_1 5_1 5_2

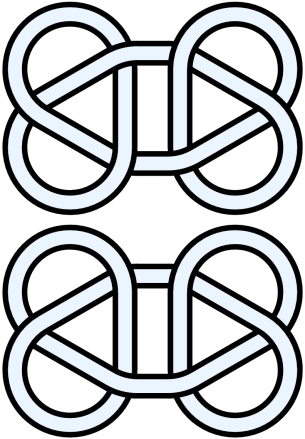
(Exact) ratios of knots in n-crossing diagrams



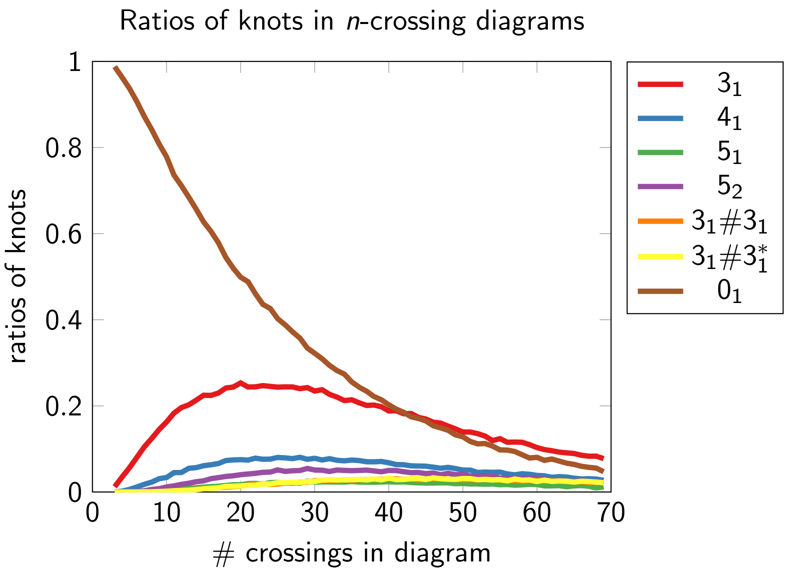
(Experimental) ratios of knots in n-crossing (rooted) diagrams



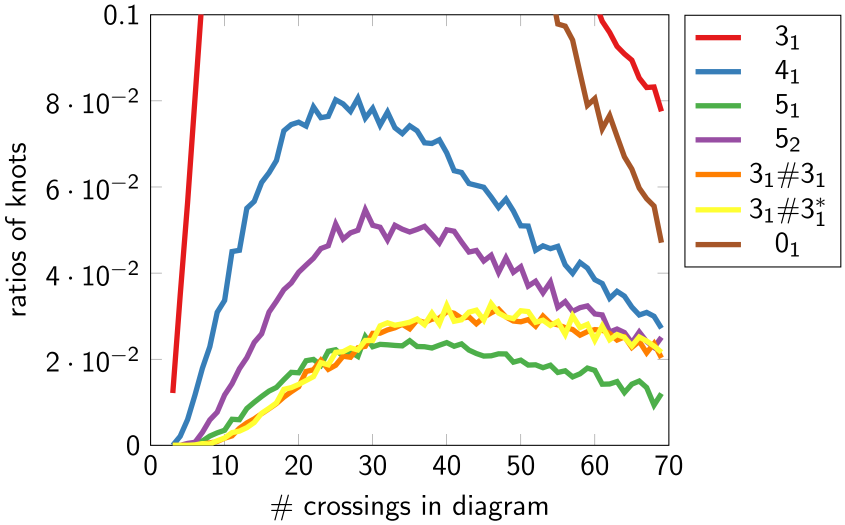
Let's throw in some composite knots

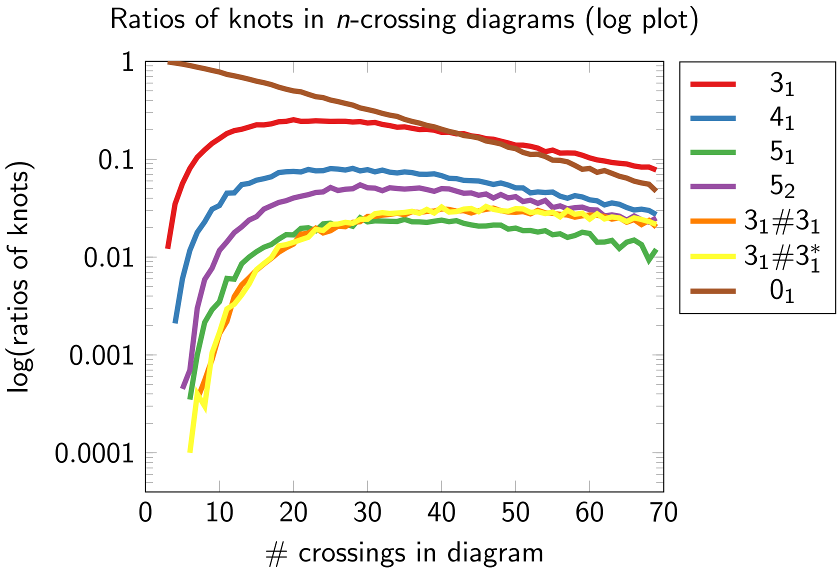


Granny knot $3_1\#3_1$ (top) vs square knot $3_1\#3_1^*$ (bottom)

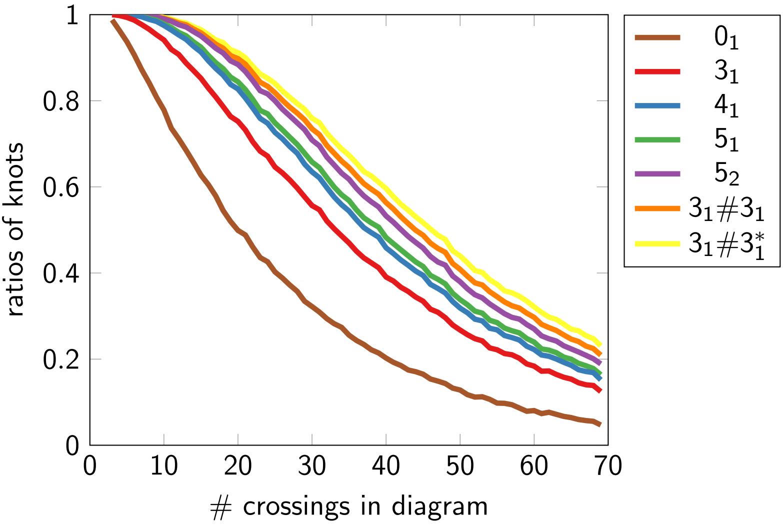

Ratios of knots in n-crossing diagrams
ratios of knots
crossings in diagram
0
0.2
0.4
0.6
0.8
1
0
10
20
30
40
50
60
70
3_1
4_1
5_1
5_2
$3_1\#3_1$
$3_1\#3_1^*$
0_1

Ratios of knots in n-crossing diagrams




Ratios of knots in n-crossing diagrams (log plot)
log(ratios of knots)
1
0.1
0.01
0.001
0.0001
0
10
20
30
40
50
60
70
crossings in diagram
3_1
4_1
5_1
5_2
$3_1\#3_1$
$3_1\#3_1^*$
0_1

Ratios of knots in n-crossing diagrams (stacked)



Future headings

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- What else that is true for other models (i.e. SAP models) can we prove for random knot diagrams?
- Diagrams with different underlying structure (Knot diagrams are the circle; also theta curves, tadpoles, etc...)

SAPs vs. Random Diagrams

Can hope to prove **other** theorems which are true for SAPs.

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Namely, the FWD theorem for diagrams shows that

$$\lim_{n\to\infty} (k_n^*)^{1/n} > \lim_{n\to\infty} |\{n\text{-crossing } 0_1 \text{ diagrams}\}|^{1/n},$$

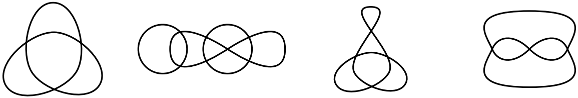
so there is some n and some non-trivial knot type which is more common than the unknot in $\mathcal{K}_n$.

Thank you!

Coming soon:

Cantarella, C-, Mastin. *Knot probabilities in random diagrams*.

C-. *Asymptotic laws for knot diagrams*.



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