

Isomorphisms of Multigraphs in Terms of Isomorphisms of Colored Simple Graphs

Jason Cantarella, Harrison Chapman, Eric Lybrand, Hollis
Neel and Malik Henry,^{*} Matt Mastin,[†] and Eric Rawdon(?)[†]

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Definition 1. A **multigraph** G is a triple $G = (V, E, f)$ where V is a finite set of *vertices*, E is a finite set of *edges*, and f is a map from E to the power set of V , $\mathcal{P}(V)$, so that $f(e)$ is a subset of size either 1 or 2. The vertices in $f(e)$ are called the **endpoints** of e . Edges with $|f(e)| = 1$ are called **loops**.

An **isomorphism** from a graph G_1 to a graph G_2 is a pair of bijections (ϕ_V, ϕ_E) with $\phi_V : V_1 \rightarrow V_2$ and $\phi_E : E_1 \rightarrow E_2$ such that $(\phi_V \circ f_1)(e) = (f_2 \circ \phi_E)(e)$ (as sets) for all $e \in E_1$.

A **coloring on the vertices** of a graph is a map, C_V , from V to a set of colors X . An isomorphism (ϕ_V, ϕ_E) from a vertex colored graph G_1 to a vertex colored graph G_2 **respects the vertex coloring** if, and only if, $C_{V_1}(v) = (C_{V_2} \circ \phi_V)(v)$ for all $v \in V_1$.

A **coloring on the edges** of a graph is a map, C_E , from E to a set of colors X . An isomorphism (ϕ_V, ϕ_E) from an edge colored graph G_1 to an edge colored graph G_2 **respects the edge coloring** if, and only if, $C_{E_1}(e) = (C_{E_2} \circ \phi_E)(e)$ for all $e \in E_1$.

Definition 2. Given a multigraph $G = (V, E, f)$ we define an associated graph $\bar{G} = (\bar{V}, \bar{E}, \bar{f})$ by the following construction.

- The vertices of \bar{G} are the vertices of G , in other words, $\bar{V} = V$.
- The edges of \bar{G} come collapsing non-loop edges that have the same endpoints. We do this by defining the edges of \bar{G} to be a set of preimages of f . In particular, set

$$\bar{E} = \{f^{-1}(f(v)) : |f(v)| = 2\}.$$

- $\bar{f}(\bar{e}) = f(e)$ where e is any element of \bar{e} . Note that this is well-defined as all edges in the set $\bar{f}(\bar{e})$ have the same endpoints by construction.

We also define a coloring on the vertices and edges of \bar{G} as follows.

^{*}University of Georgia, Mathematics Department, Athens GA

[†]Wake Forest University, Mathematics Department, Athens GA

- $C_{\bar{V}}(\bar{v}) =$ the number of edges $e \in E$ such that $f(e) = \{v\}$.
- $C_{\bar{E}}(\bar{e}) = f(e)$ where e is any element of \bar{e} .

Lemma 3. *Given any graph G the associated graph \bar{G} is simple.*

Proof. An edge in G is a loop precisely when $|f(v)| = 1$ which is excluded in the definition of \bar{G} . Similarly, duplicate edges are not allowed in \bar{E} because it is defined to be the *set* of pre-images. Thus, \bar{G} has no loops or multiedges and is therefore simple. \square

Since $\bar{G} = (\bar{V}, \bar{E}, \bar{f})$ is always simple it can be described by only its set of vertices \bar{V} and a collection of two element subsets of \bar{V} giving the edges. In this case, a bijection $\bar{\phi}_V : \bar{V}_1 \rightarrow \bar{V}_2$ induces a map on the edge sets by $\bar{\phi}_E(\bar{e}) = \{\bar{\phi}_V(\bar{v}), \bar{\phi}_V(\bar{w})\}$. Thus, an isomorphism between simple graphs can be described by only a bijection on the vertex sets.

Theorem 4. *Let G_1 and G_2 be multigraphs and \bar{G}_1 and \bar{G}_2 the associated colored graphs given in Definition 2. Then, given an isomorphism $\bar{\phi}$ from \bar{G}_1 to \bar{G}_2 that respects the vertex and edge coloring, we can construct a set of isomorphisms $\Phi(\bar{\phi}) = \{(\phi_V^1, \phi_E^1), \dots, (\phi_V^k, \phi_E^k)\}$ from G_1 to G_2 . In addition, $\Phi(\bar{\phi}_1)$ and $\Phi(\bar{\phi}_2)$ are disjoint if, and only if, $\bar{\phi}_1$ and $\bar{\phi}_2$ are distinct isomorphisms. Moreover, if (ϕ_V, ϕ_E) is an isomorphism from G_1 to G_2 , then there exists an isomorphism $\bar{\phi}$ from \bar{G}_1 to \bar{G}_2 such that (ϕ_V, ϕ_E) will be produced by this construction.*

Proof. We will first describe the construction and then prove that it satisfies the properties in the theorem.

IDEA: $\phi_V^i = \bar{\phi}$ for all i and each ϕ_E^i comes from all choices of permutations of the elements of each *set* labeling each edge of \bar{G} . \square