Random Knot Diagrams

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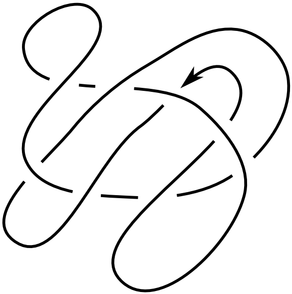
joint w/ Jason Cantarella (UGA), Matt Mastin (Wake Forest)

AMS Western Spring Sectionals 2015 (UNLV) – April 18, 2015

Natural questions about knot diagrams

Question

What fraction of 8-crossing diagrams are trefoils?

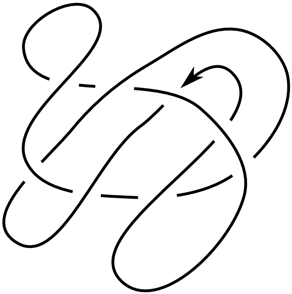


Natural questions about knot diagrams

Question

What fraction of 8-crossing diagrams are trefoils?

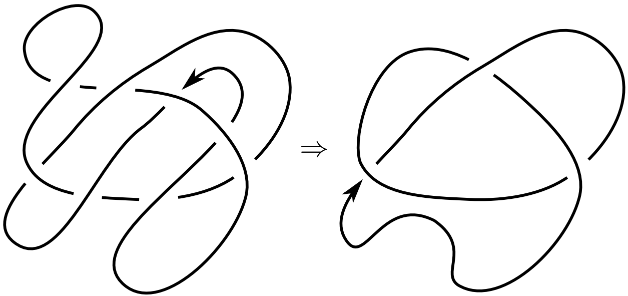
12.48%



Natural questions about knot diagrams

Question

What is the average minimal crossing $\#$ of an 8-crossing diagram?

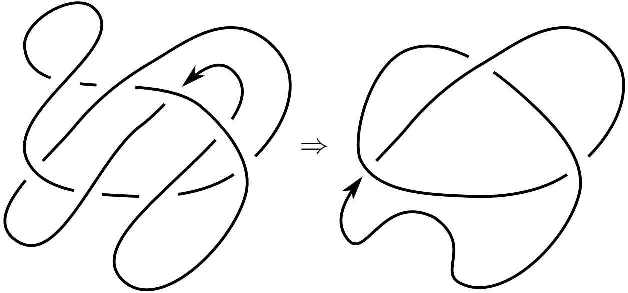


Natural questions about knot diagrams

Question

What is the average minimal crossing # of an 8-crossing diagram?

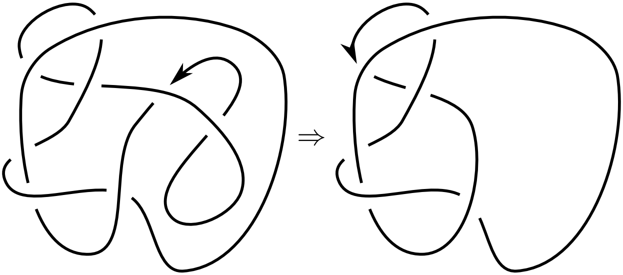
0.52



Natural questions about knot diagrams

Definition

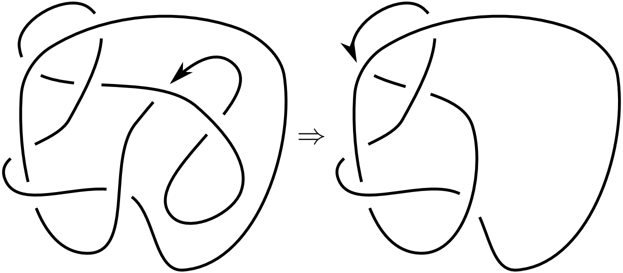
Define an operation on diagrams, **untwisting**: Recursively RI untwist loops in a diagram until there are no more.



Natural questions about knot diagrams

Question

What is the average crossing # of a untwisted 8-crossing diagram?

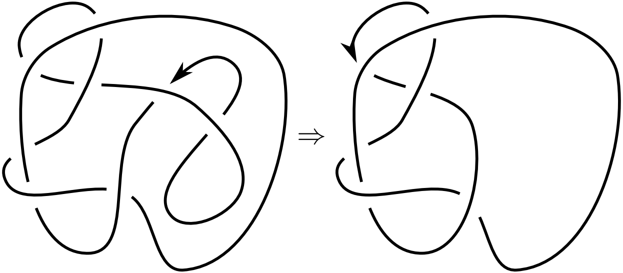


Natural questions about knot diagrams

Question

What is the average crossing # of a untwisted 8-crossing diagram?

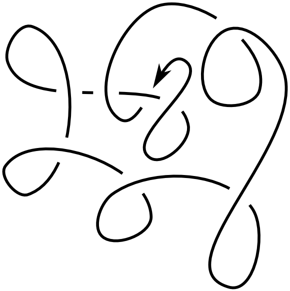
2.20



Natural questions about knot diagrams

Question

How many 8-crossing diagrams can be untwisted to the unknot?

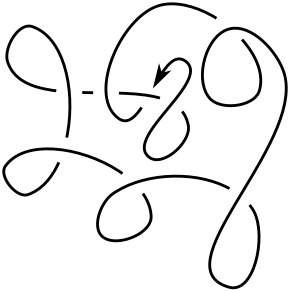


Natural questions about knot diagrams

Question

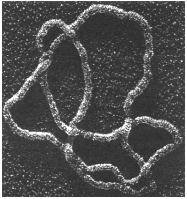
How many 8-crossing diagrams can be untwisted to the unknot?

42.05%



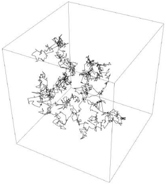
Ansatz

Polymers



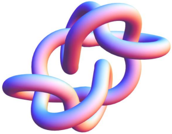
$\longrightarrow$

Random curves



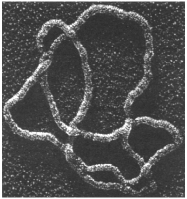
$\longrightarrow$

Knots

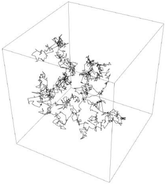


Combinatorial approaches

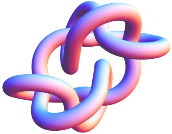
Polymers



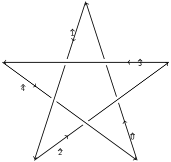
Random curves



Knots

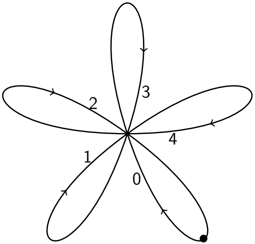


Random combinatorial

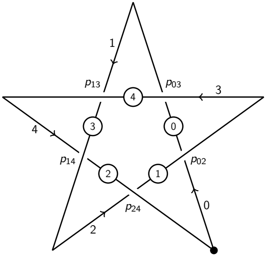


The Petaluma model

Satisfying theorems have been proven for the Petaluma model



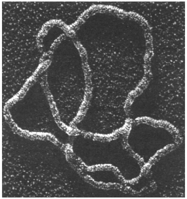
Petal diagram



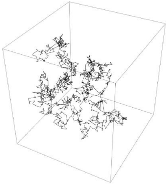
Star diagram

(Diagram from Evan-Zohar, Hass, et al.)

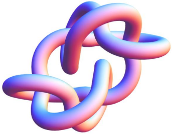
Polymers



Random curves

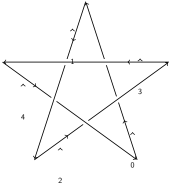


Knots

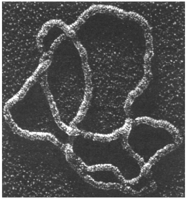


???

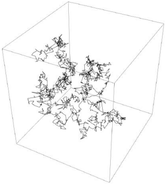
Random combinatorial



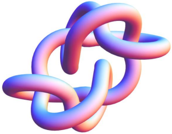
Polymers



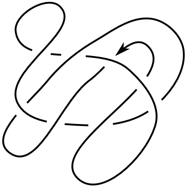
Random curves



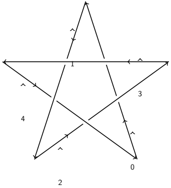
Knots



Random diagrams



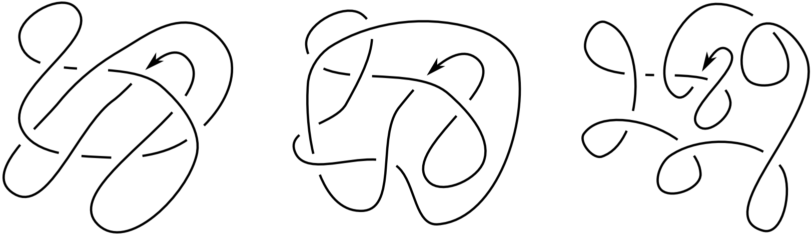
Random combinatorial



Random diagrams

Definition

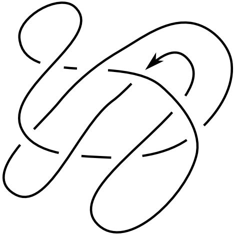
In the **random diagram model** of random knotting, a n-crossing diagram is drawn uniformly from the finite set of n-crossing knot diagrams.

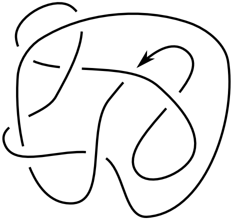


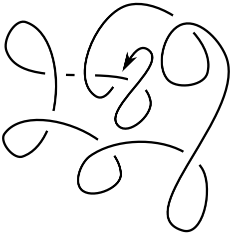
Random diagrams

Definition

A **knot diagram** is a generic embedding of the oriented S^1 into the sphere S^2 together with over-under strand information at each double point up to diffeomorphism of S^2.







How to enumerate knot diagrams

Definition

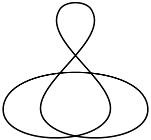
A **knot shadow** is a generic embedding of the unoriented S^1 into the sphere S^2 up to diffeomorphism of S^2.

1. Enumerate shadows
2. Assign crossing and orientation information and identify equivalent diagrams

Tabulating knot shadows

Proposition

Knot shadows $\leftrightarrow$ *1-component 4-valent embedded planar multigraphs up to embedded isomorphism*

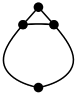


Tabulating knot shadows

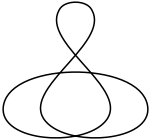
Proposition

Knot shadows $\leftrightarrow$ 1-component 4-valent embedded planar multigraphs up to embedded isomorphism

1. Add loops and edges to planar simple graphs



$\Rightarrow$

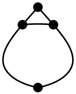


Tabulating knot shadows

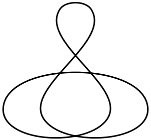
Proposition

Knot shadows $\leftrightarrow$ 1-component 4-valent embedded planar multigraphs up to embedded isomorphism

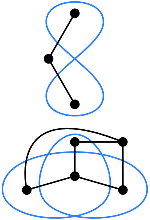
1. Add loops and edges to planar simple graphs
2. Generate multiquadrangulations by connect sum, take dual graphs



$\Rightarrow$



$\Leftarrow$



Tabulating knot shadows

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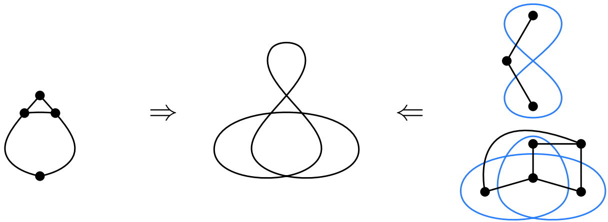


Actually generate all **link shadows**, then restrict to knot shadows

Assign crossings, orientation, identify

1. Orient each component. (2 choices)
2. Assign over-under information to each vertex. (2^n choices)

n	# knot shadows	2^{n+1} (# shadows)	# knot diagrams
3	6	96	36
4	19	608	276
5	76	4,864	2,936
6	376	48,128	35,872
7	2,194	561,664	484,088
8	14,614	7,482,368	6,967,942
9	106,421	108,975,104	105,555,336

Verifying against existing shadow counts

oriented	$n=0$	1	2	3	4	5
S^2, S^1	1	1	3	9	37	182
S^2	1	1	2	6	21	99
S^1	1	1	2	6	21	97
–	1	1	2	6	19	76

Curves on S^2. The number of types

V.I. Arnol'd. *Topological Invariants of Plane Curves*

A008989 Number of immersions of unoriented circle into unoriented sphere with n double points.

1, 1, 2, 6, 19, 76, 376, 2194 (list; graph; refs; listen; history; text; internal format)

OFFSET 0,3

REFERENCES V. I. Arnold, Topological Invariants of Plane Curves..., American Math.

LINKS Table of n, a(n) for n=0..7.

CROSSREFS Sequence in context: A150119 A181770 A138800 * A057240 A079564 A079453
Adjacent sequences: A008986 A008987 A008988 * A008990 A008991 A008992

KEYWORD nonn

AUTHOR N. J. A. Sloane.

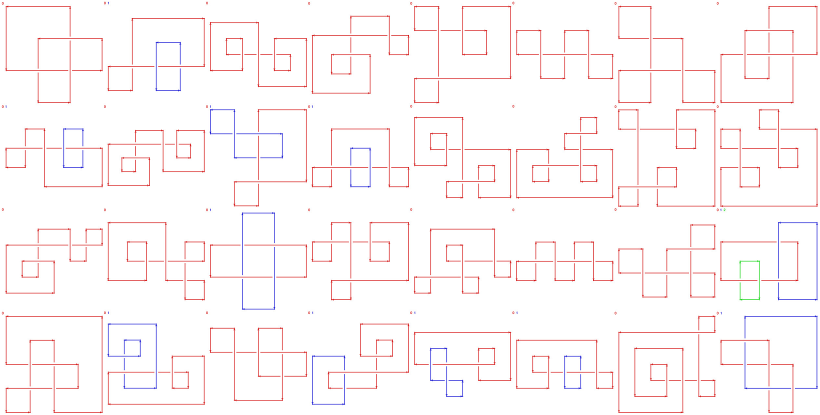
EXTENSIONS Two more terms from Guy H. Valette (guy.valette(AT)skynet.be), Feb 09 20

STATUS approved

OEIS A008989

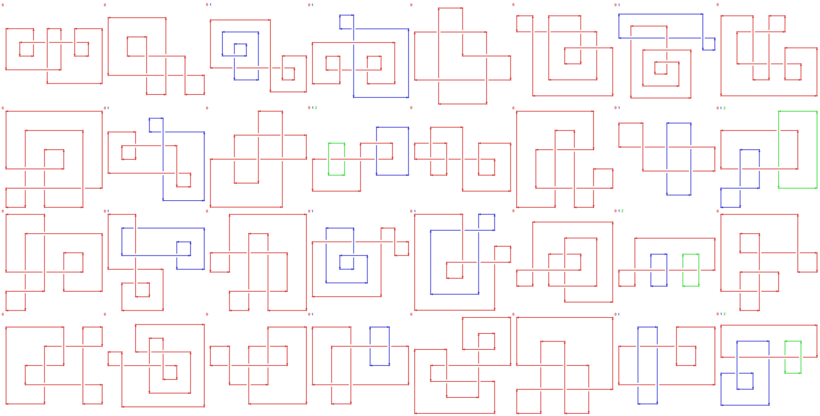
n	# knot shadows
0	1
1	1
2	2
3	6
4	19
5	76
6	376
7	2194
8	14614
9	106421

The space of shadows



Link shadows. Pictures generated by Eric Lybrand (UGA undergrad).

The space of shadows



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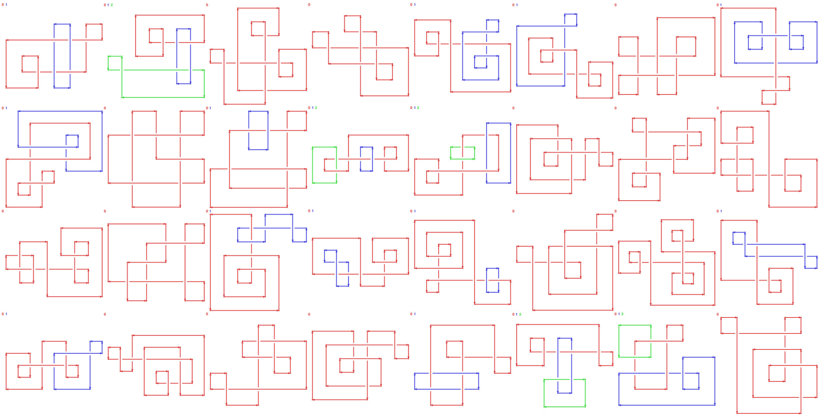
The space of shadows



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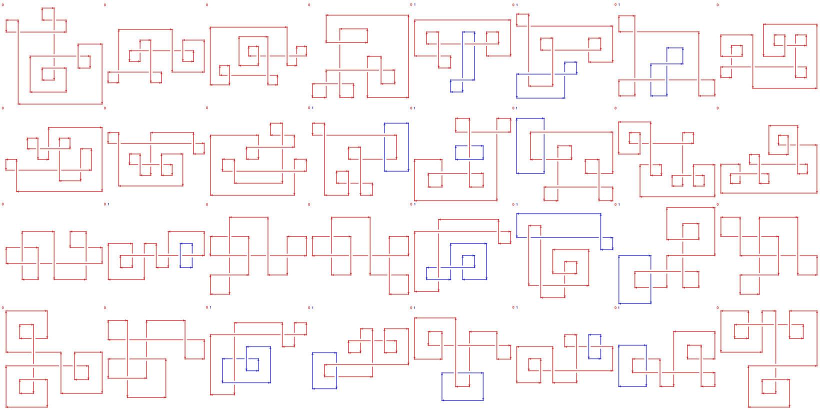
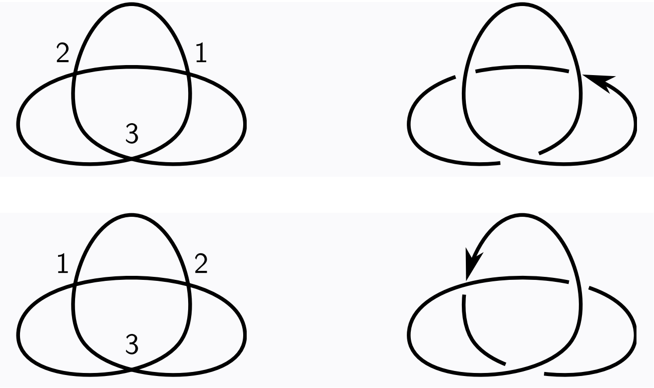
The space of shadows



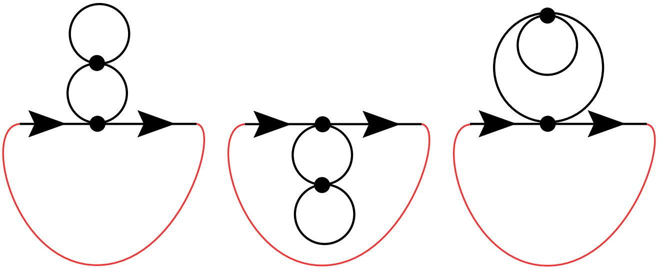
Link shadows. Pictures generated by Eric Lybrand (UGA undergrad).

Tabulation is difficult!

Accounting for symmetry is complicated.

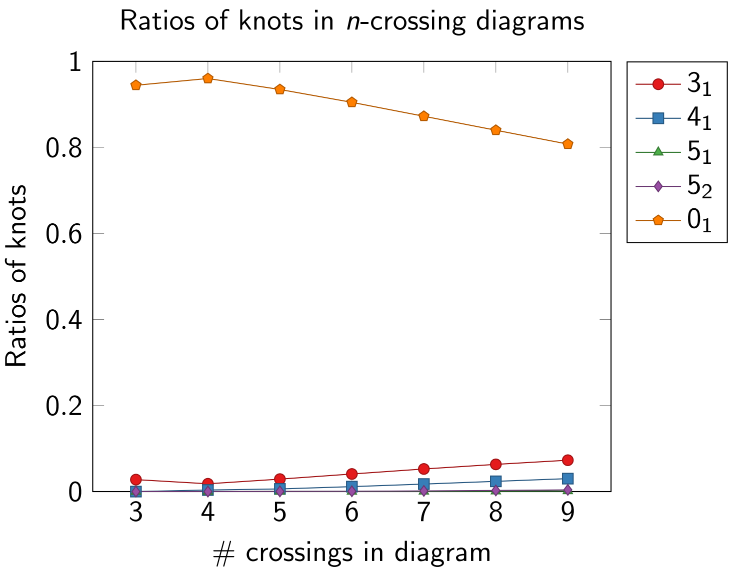


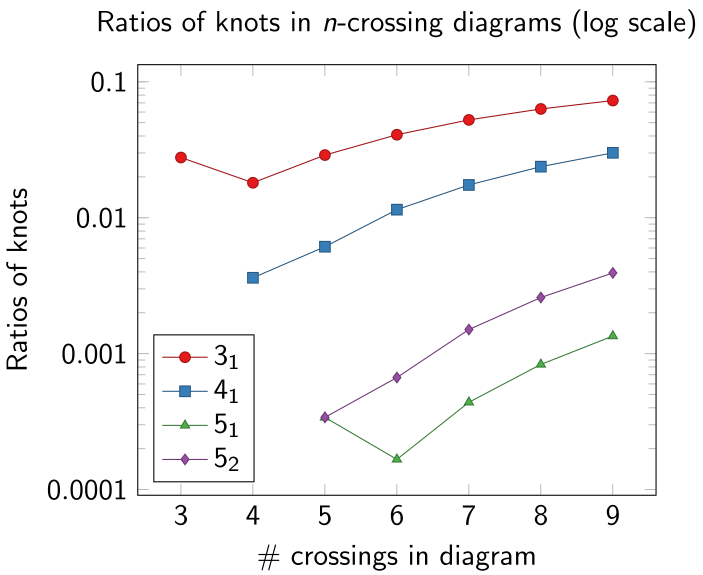
Breaking symmetries could make counting easier

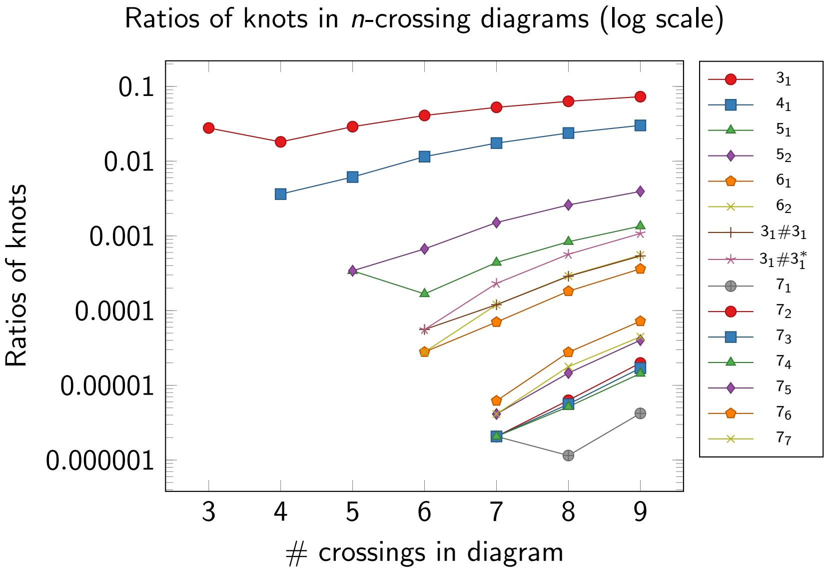


Two-leg diagrams counted by generating function (Bouttier, et al. 2003):

$$G_0 = \frac{24g - 1 + \sqrt{1-12g}}{9g(1+\sqrt{1-12g})} = 1 + 2g + 9g^2 + 54g^3 + 378g^4 + \cdots$$


Ratios of knots in n-crossing diagrams
Ratios of knots
crossings in diagram
0
0.2
0.4
0.6
0.8
1
3
4
5
6
7
8
9
3_1
4_1
5_1
5_2
0_1


Ratios of knots in n-crossing diagrams (log scale)
Ratios of knots
0.1
0.01
0.001
0.0001
3
4
5
6
7
8
9
crossings in diagram
3_1
4_1
5_1
5_2



Ratios of knots in n-crossing diagrams (log scale)
Ratios of knots
0.1
0.01
0.001
0.0001
0.00001
0.000001
3
4
5
6
7
8
9
crossings in diagram
3_1
4_1
5_1
5_2
6_1
6_2
$3_1\#3_1$
$3_1\#3_1^*$
7_1
7_2
7_3
7_4
7_5
7_6
7_7

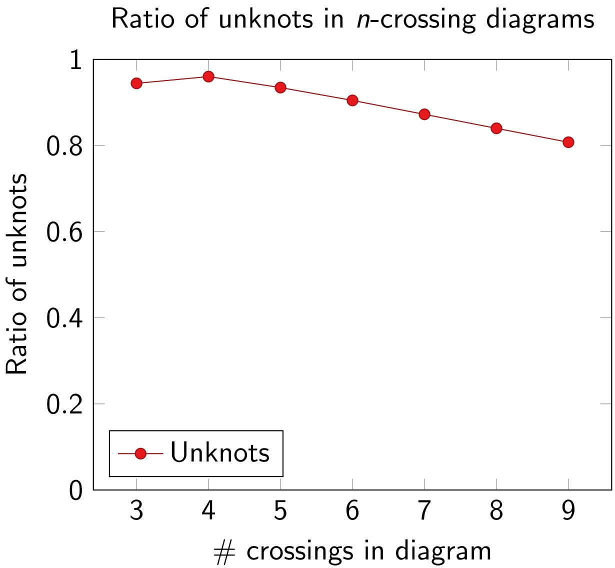
A question on unknotting

Theorem ((Frisch-Wassermann-Delbrück Conjecture) Sumners-Whittington 1988)

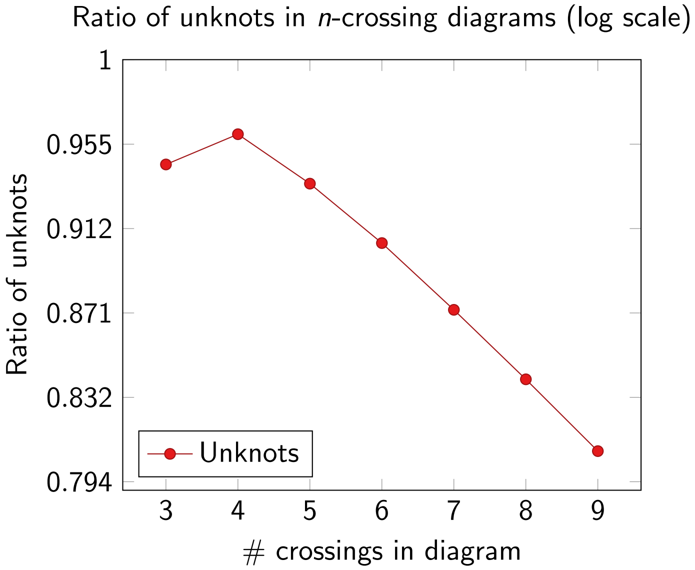
The ratio of unknots in random n-edge self-avoiding lattice polygons tends to zero exponentially with n.

Conjecture

The ratio of unknots in diagrams tends to zero as n increases. (Exponentially?)

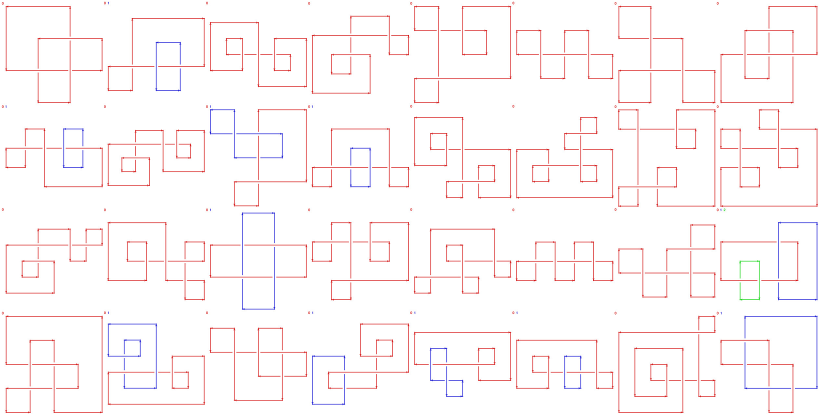


Ratio of unknots in n-crossing diagrams
Ratio of unknots
0
0.2
0.4
0.6
0.8
1
3
4
5
6
7
8
9
crossings in diagram
Unknots



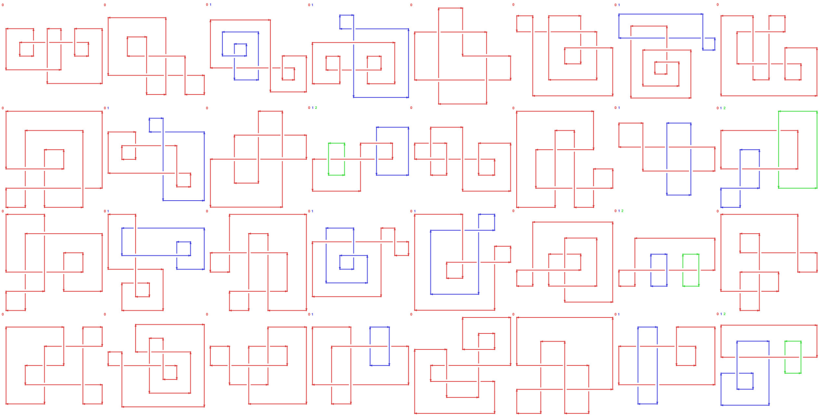
Ratio of unknots in n-crossing diagrams (log scale)
1
0.955
0.912
0.871
0.832
0.794
Ratio of unknots
Unknots
3
4
5
6
7
8
9
crossings in diagram

Why so many unknots?



Link shadows. Pictures generated by Eric Lybrand (UGA undergrad).

Why so many unknots?



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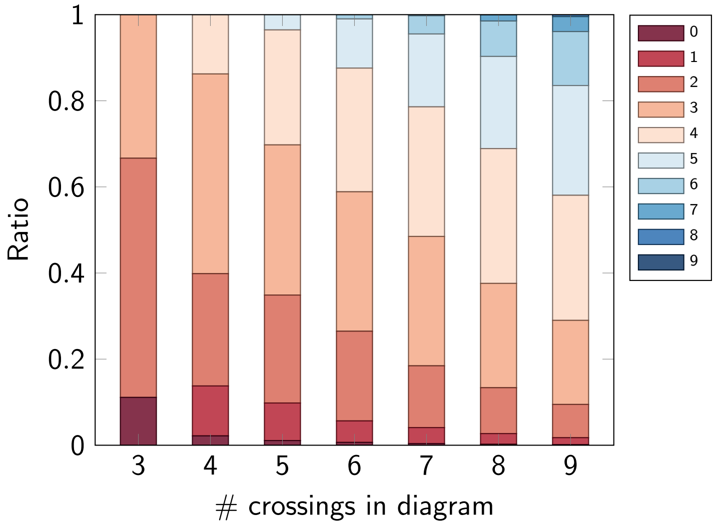
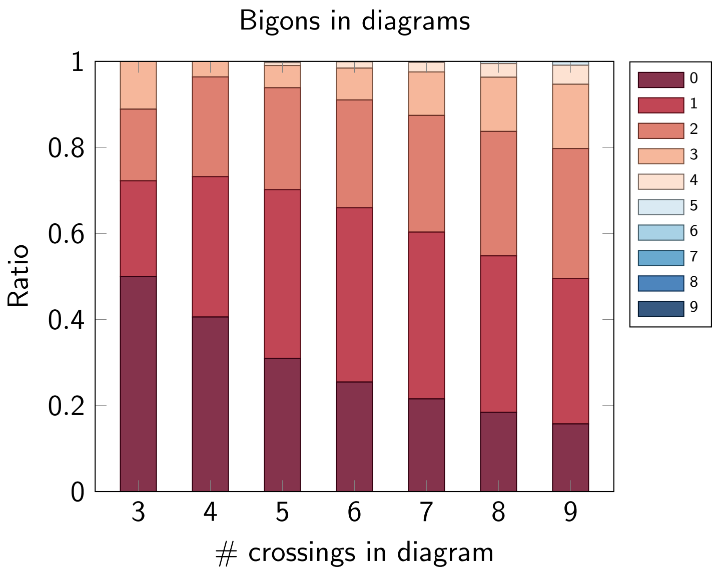
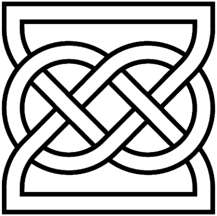
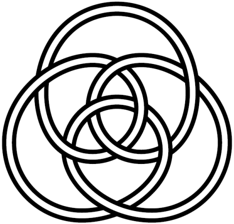
Reidemeister-I loops (monogons) in diagrams




Bigons in diagrams
Ratio
0
0.2
0.4
0.6
0.8
1
3
4
5
6
7
8
9
crossings in diagram
0
1
2
3
4
5
6
7
8
9

Basic polyhedra 8^* and 9^*

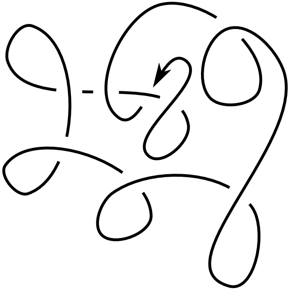




8_{18} (left), 9_{40} (right).

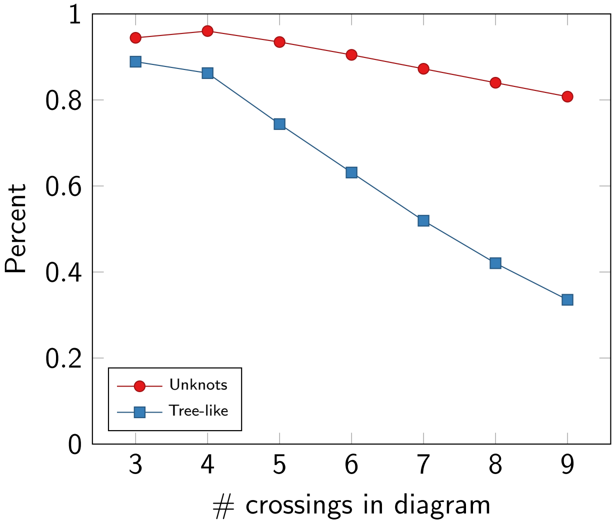
Some shadows are always unknots

A **tree-like curve** is a knot shadow which can be untwisted to the trivial shadow.

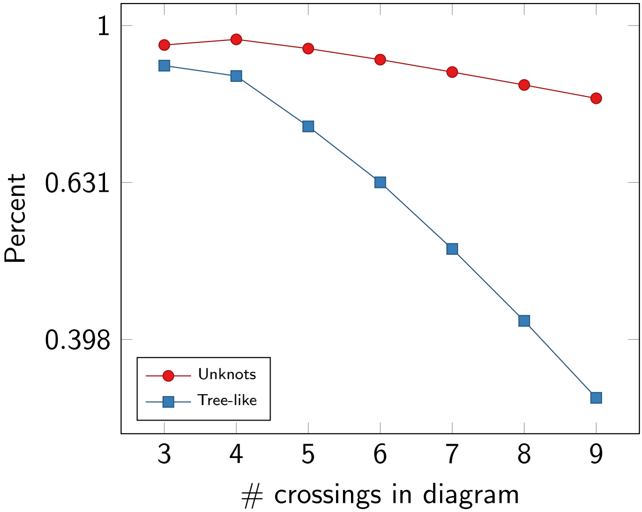


Tree-like curves $\Rightarrow$ lower bound on unknottedness.

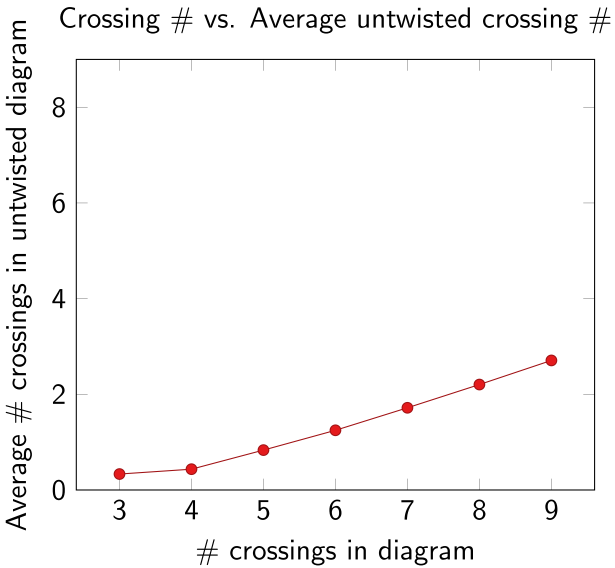
Ratio of unknots, tree-like curves in n-crossing diagrams



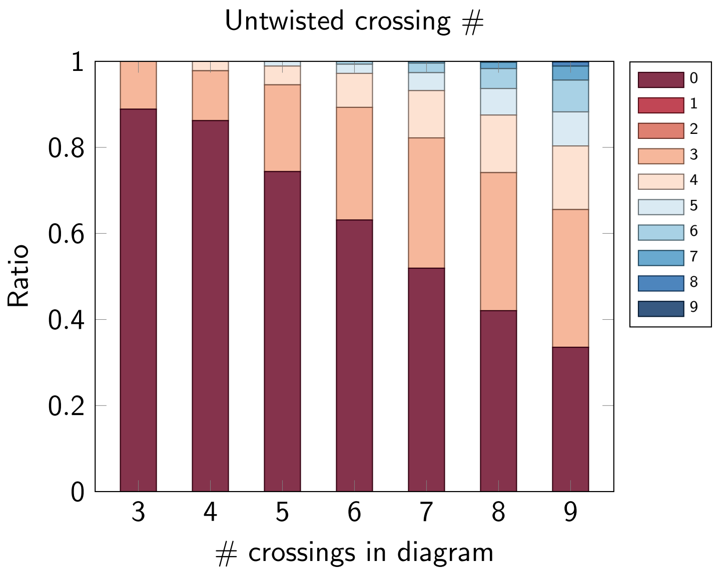
Ratio of unknots, tree-like curves in n-crossing diagrams (log scale)



Tree-like curves alone explain only some of the unknot fraction



Crossing # vs. Average untwisted crossing #
Average # crossings in untwisted diagram
crossings in diagram
0
2
4
6
8
3
4
5
6
7
8
9

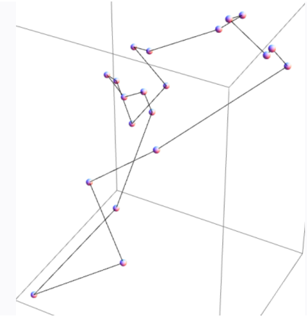

Untwisted crossing #
Ratio
0
0.2
0.4
0.6
0.8
1
3
4
5
6
7
8
9
crossings in diagram
0
1
2
3
4
5
6
7
8
9

Questions to answer

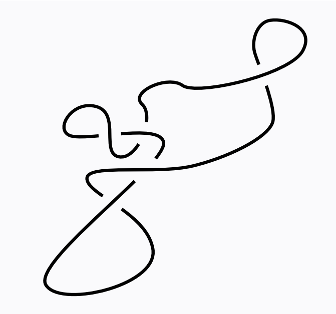
Random curves project to diagrams.

Question

How does the pushforward measure differ from uniform diagram sampling? (c.f. Hua, Nguyen, Raghavan, Arsuaga, Vasquez 2005)



$\xrightarrow{\pi}$



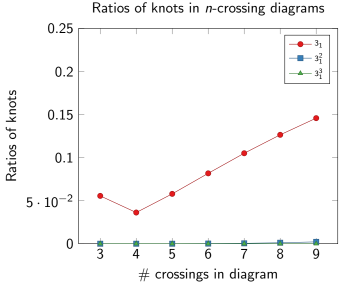
(from Shonkwiler)

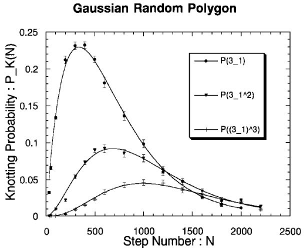
Questions to answer

Question

Let $PD(n, K)$ be the probability of knot type K in a random diagram of n crossings and $PC(n, K)$ the probability of knot type K in a random polygon of n edges.

If n, m are such that $PD(n, 0_1) = PC(m, 0_1)$, is there a relationship between $PD(n, K) = PC(m, K)$ for other knot types K?





(from Deguchi, et. al.)

Questions to answer

Fact

No one will realistically enumerate the 100-crossing knot diagrams.

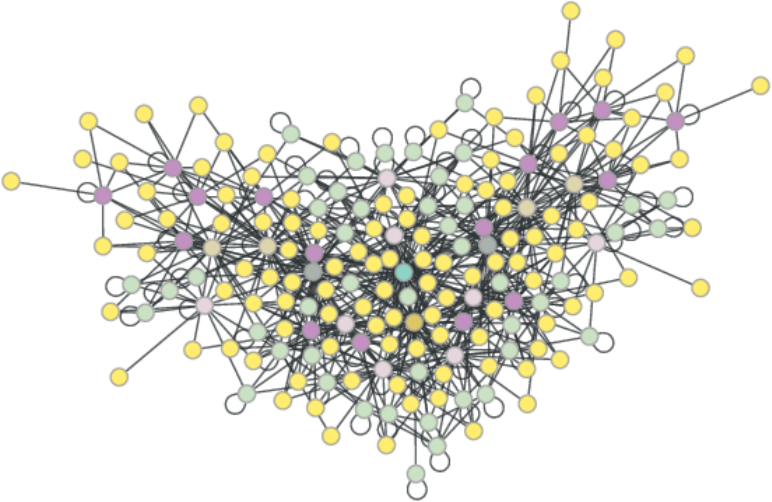
Question

Can we generate uniformly sampled random 100-crossing knot diagrams **another way**?

Future direction: Link diagrams

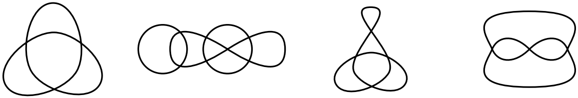
n	# link shadows	# knot shadows
0	1	1
1	1	1
2	3	2
3	7	6
4	30	19
5	124	76
6	733	376
7	4586	2194
8	33373	14614
9	259434	106421

Future direction: Knot distances



Thank you!

Coming soon: Cantarella, Chapman, Mastin. *Knot probabilities in random diagrams*.



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