Random Knot Diagrams

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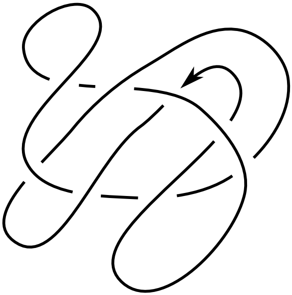
Crucial Assist: Eric Rawdon (St. Thomas)

AMS Fall Western Section Meeting, 10/24/15

Natural questions about knot diagrams

Question

What fraction of 8-crossing diagrams are trefoils?



Natural questions about knot diagrams

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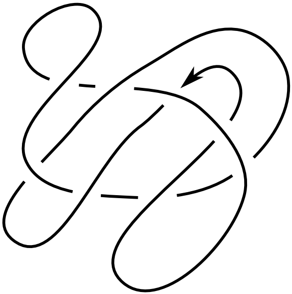
What fraction of 8-crossing diagrams are trefoils?

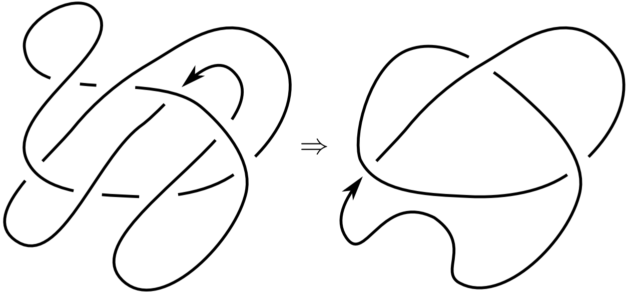
12.48%



Natural questions about knot diagrams

Question

What is the average minimal crossing # of an 8-crossing diagram?

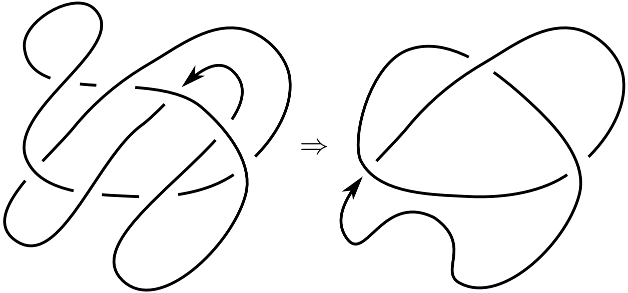


Natural questions about knot diagrams

Question

What is the average minimal crossing # of an 8-crossing diagram?

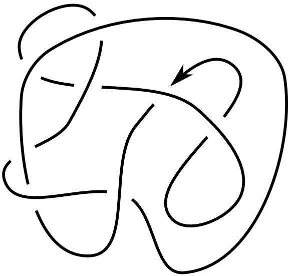
0.52



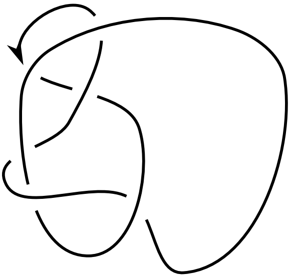
Natural questions about knot diagrams

Definition

The **untwisting** operator deletes all 1-crossing connect summands of a diagram. (Equivalently, performs all “available” Reidemeister I moves.)



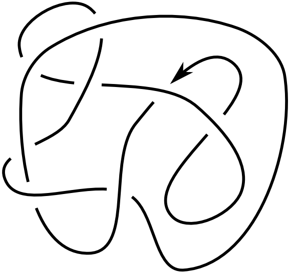
$\Rightarrow$



Natural questions about knot diagrams

Question

What is the average crossing # of a untwisted 8-crossing diagram?



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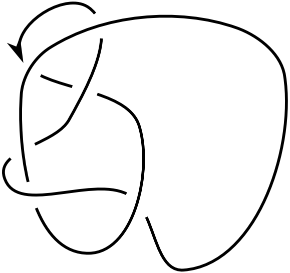
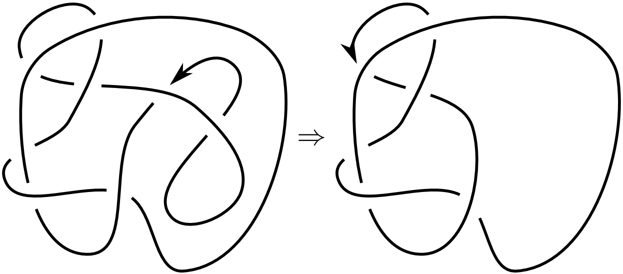


Natural questions about knot diagrams

Question

What is the average crossing # of a untwisted 8-crossing diagram?

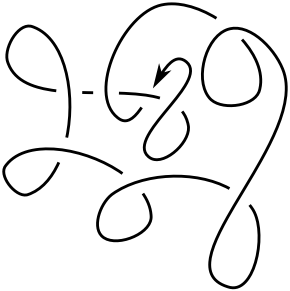
2.20



Natural questions about knot diagrams

Question

How many 8-crossing diagrams can be untwisted to the unknot?

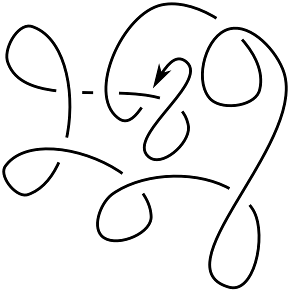


Natural questions about knot diagrams

Question

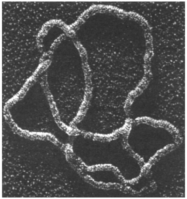
How many 8-crossing diagrams can be untwisted to the unknot?

42.05%



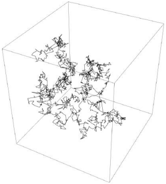
Ansatz

Polymers



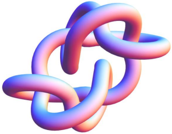
$\longrightarrow$

Random curves



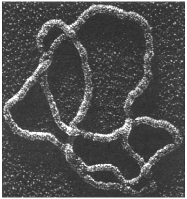
$\longrightarrow$

Knots

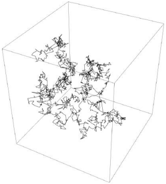


Combinatorial approaches

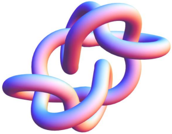
Polymers



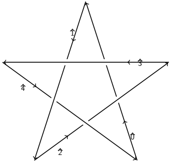
Random curves



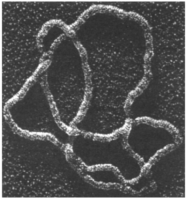
Knots



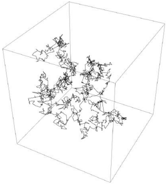
Random combinatorial



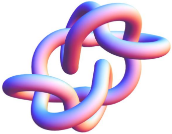
Polymers



Random curves

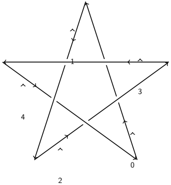


Knots

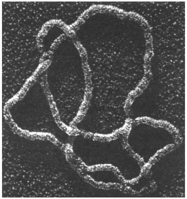


???

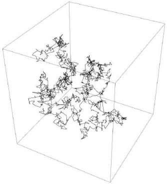
Random combinatorial



Polymers



Random curves



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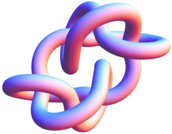
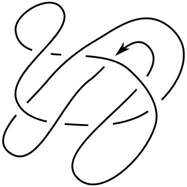


Random diagrams



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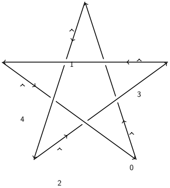
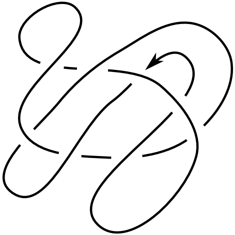
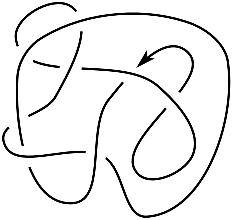
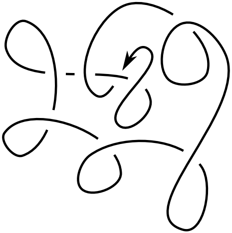


Random diagrams

Definition

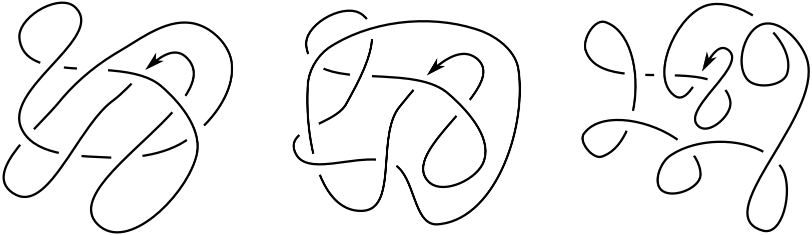
In the **random diagram model** of random knotting, a *n*-crossing diagram is drawn uniformly from the finite set of *n*-crossing knot diagrams (cf. Dunfield).

Random diagrams

Definition

A **knot diagram** is a equivalence class of generic immersions of the oriented S^1 into the sphere S^2 together with over-under strand information at each double point up to diffeomorphism of S^2.



How to enumerate knot diagrams (like a topologist)

Definition

A **knot shadow** is a equivalence class of generic immersions of the unoriented S^1 into the sphere S^2 up to diffeomorphism of S^2.

Plan to Enumerate Diagrams

1. *Enumerate shadows (and discard isomorphic shadows)*
2. *Assign crossing and orientation information (and discard crossing patterns related by an automorphism of the shadow)*

How to enumerate knot diagrams (like a topologist)

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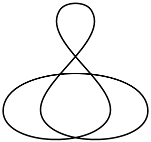
Observation

Symmetry stinks.

Tabulating knot shadows: `plantri`, two ways

Proposition

Knot shadows $\leftrightarrow$ *1-component 4-valent embedded planar multigraphs up to embedded isomorphism*



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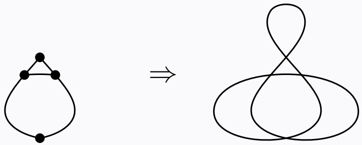
1. Add loops and edges to planar simple graphs (slow)



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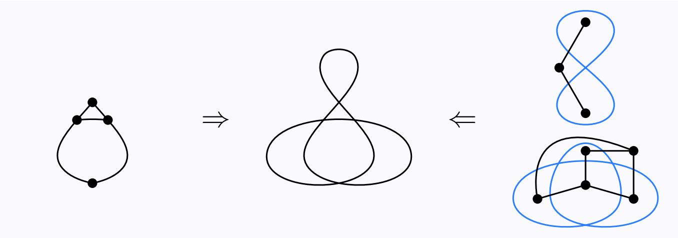
2. Generate multiquadrangulations of sphere by careful pattern of connect sums, take dual graphs (fast)



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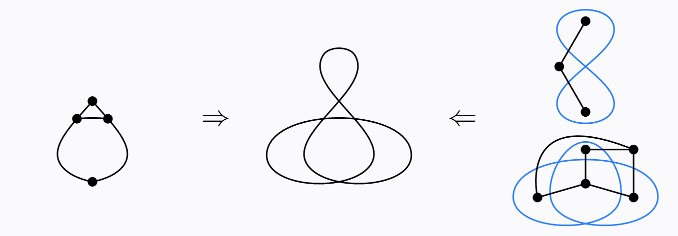

Actually generate all **link shadows**, then restrict to knot shadows

Verifying against existing shadow counts

oriented	$n=0$	1	2	3	4	5
S^2, S^1	1	1	3	9	37	182
S^2	1	1	2	6	21	99
S^1	1	1	2	6	21	97
–	1	1	2	6	19	76

Curves on S^2. The number of types

V.I. Arnol'd. *Topological Invariants of Plane Curves*

A008989 Number of immersions of unoriented circle into unoriented sphere with n double points.

1, 1, 2, 6, 19, 76, 376, 2194 (list; graph; refs; listen; history; text; internal format)

OFFSET	0,3
REFERENCES	V. I. Arnold, Topological Invariants of Plane Curves..., American Math.
LINKS	Table of n, a(n) for n=0..7.
CROSSREFS	Sequence in context: A150119 A181770 A138800 * A057240 A079564 A079453 Adjacent sequences: A008986 A008987 A008988 * A008990 A008991 A008992
KEYWORD	nonn
AUTHOR	N. J. A. Sloane.
EXTENSIONS	Two more terms from Guy H. Valette (guy.valette(AT)skynet.be), Feb 09 20
STATUS	approved

OEIS A008989

n	# knot shadows
0	1
1	1
2	2
3	6
4	19
5	76
6	376
7	2194
8	14614
9	106421
10	823832

OU, $g=5$	0	0	0	0	0	0	0	0	22 524 176	
UU, total	1	3	12	86	894	14 715	313 364	8 139 398	245 237 925	8 382 002 270
UU, $g=0$	1	2	6	19	76	376	2194	14 614	106 421	823 832
UU, $g=1$	0	1	5	45	335	3101	29 415	295 859	3 031 458	
UU, $g=2$	0	0	1	22	427	7557	124 919	1 961 246	29 479 410	

Coquereaux, Zuber, arxiv:1507.03163

Assign crossings, orientation, identify

1. Orient each component. (2 choices)
2. Assign over-under information to each vertex. (2^n choices)

n	# knot shadows	2^{n+1} (# shadows)	# knot diagrams
3	6	96	36
4	19	608	276
5	76	4,864	2,936
6	376	48,128	35,872
7	2,194	561,664	484,088
8	14,614	7,482,368	6,967,942
9	106,421	108,975,104	105,555,336
10	823,832	1,687,207,936	1,664,142,836

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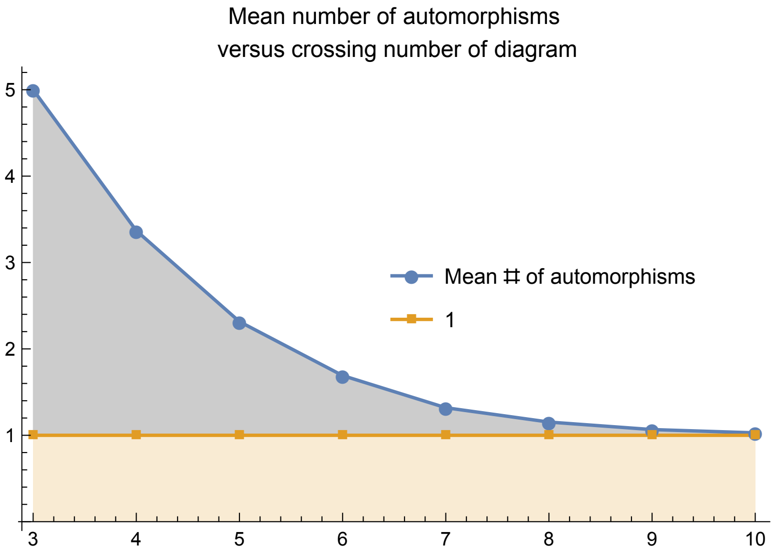
10	823,832	1,687,207,936	1,664,142,836

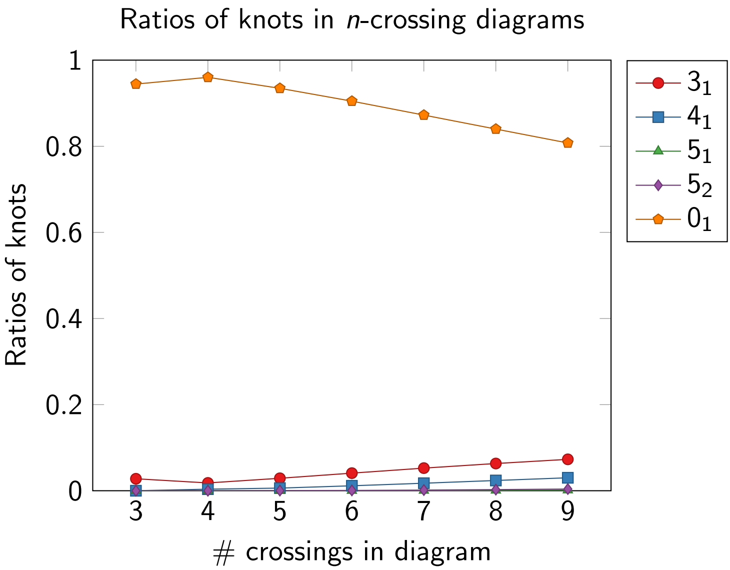
Observation

Symmetry becomes rare, quickly!

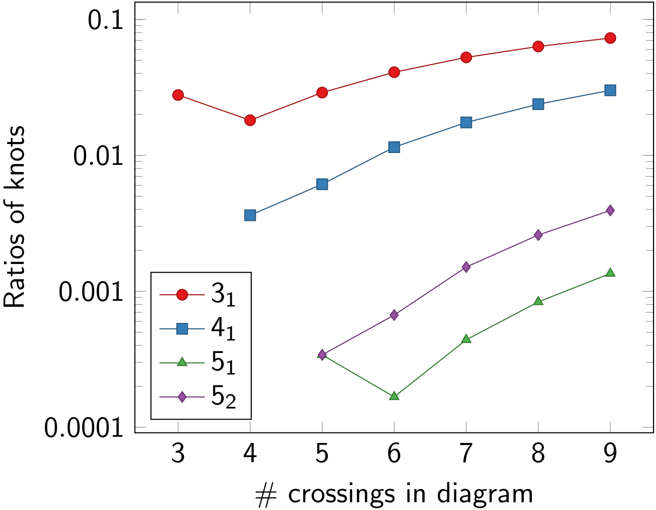
Size of the automorphism group of a random diagram



Knotting in diagrams



Ratios of knots in n-crossing diagrams (log scale)



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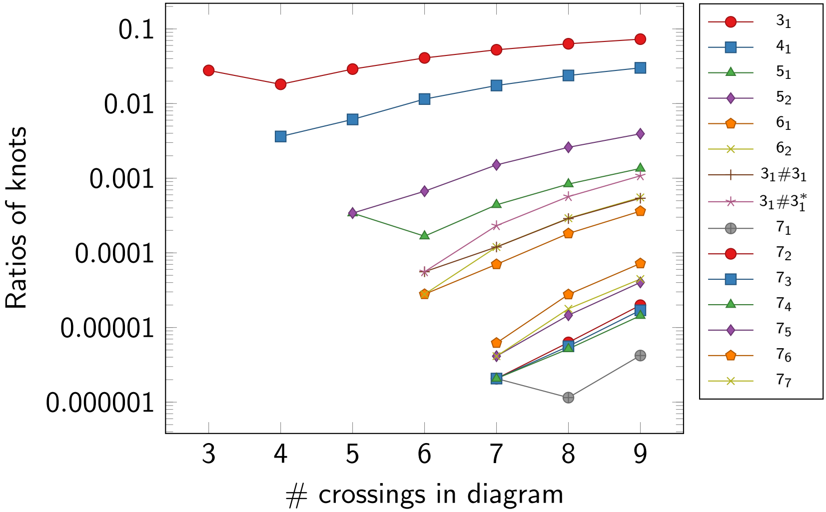
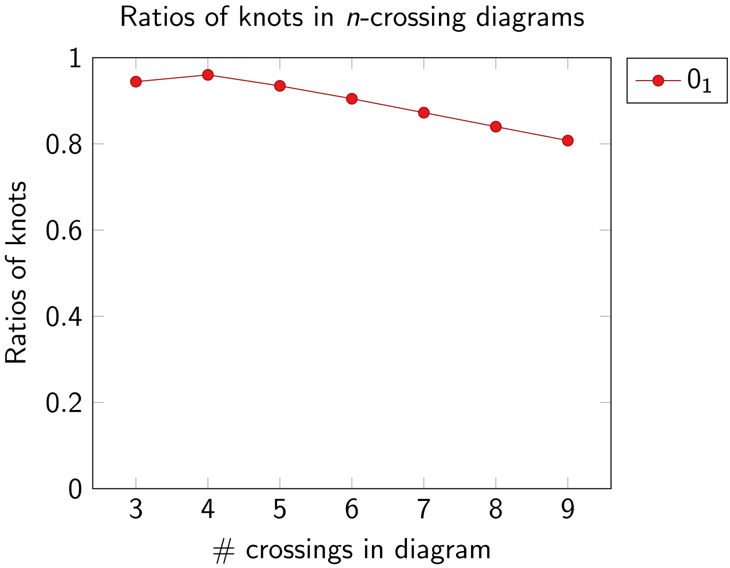
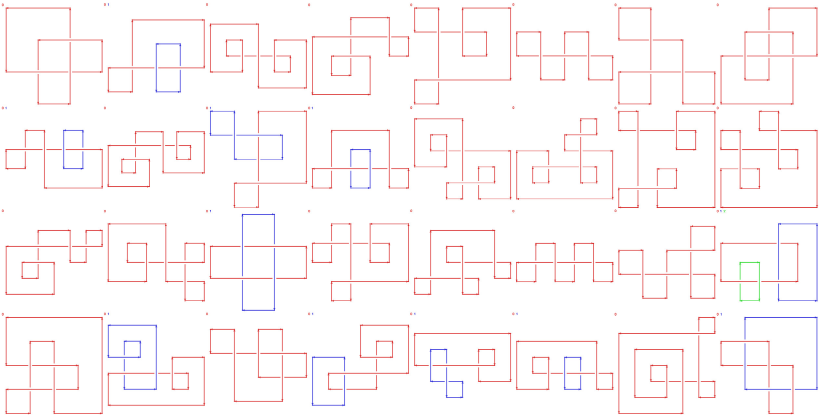


Why so many unknots?



The space of shadows



Link shadows. Pictures generated by Eric Lybrand (UGA undergrad).

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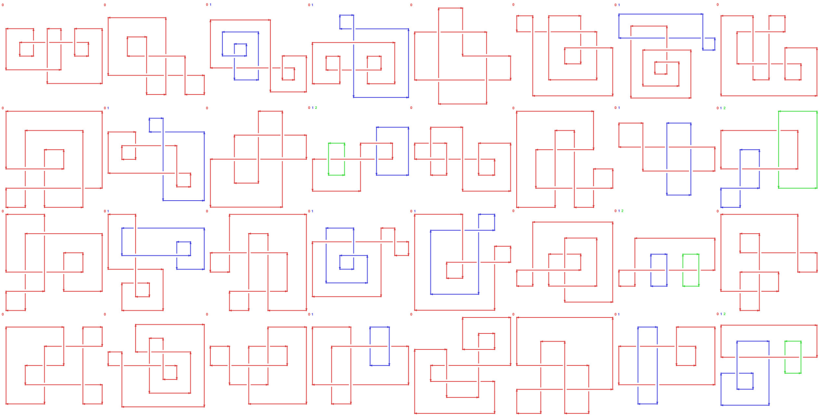

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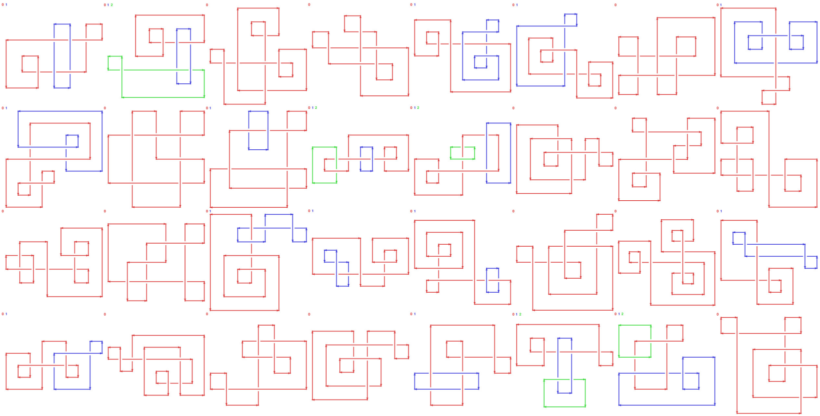

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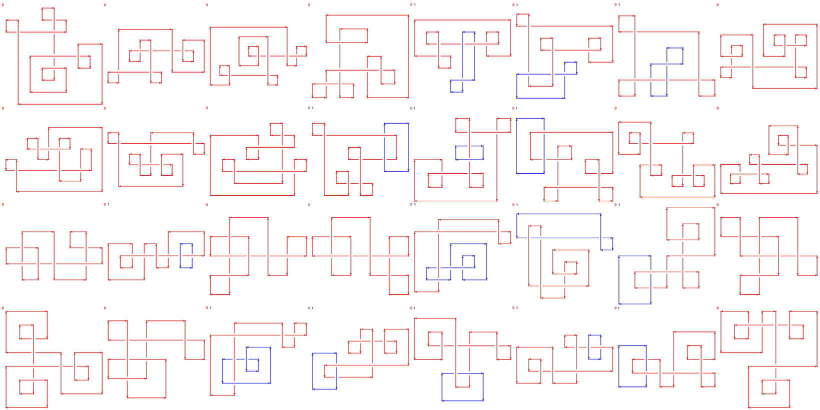

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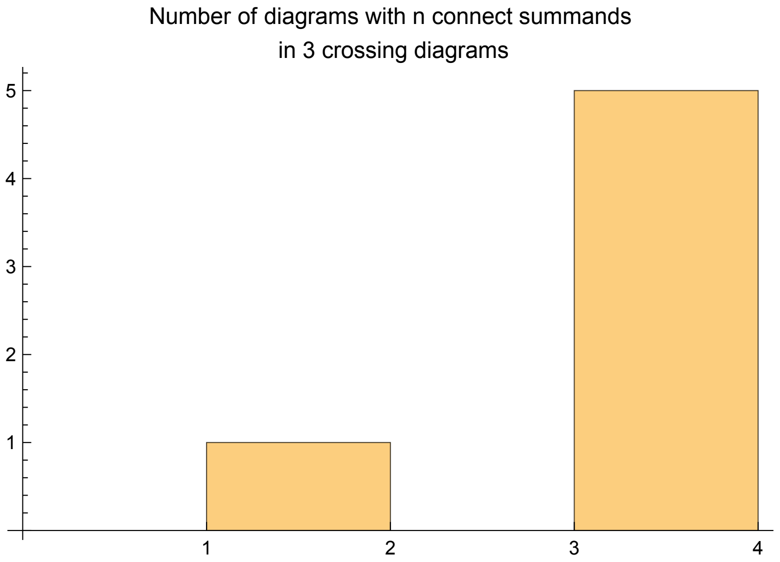
Most diagrams are (very) composite



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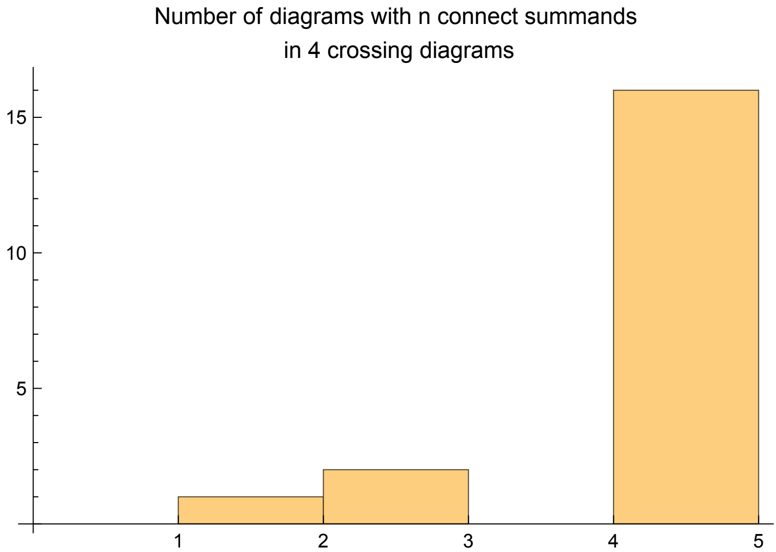

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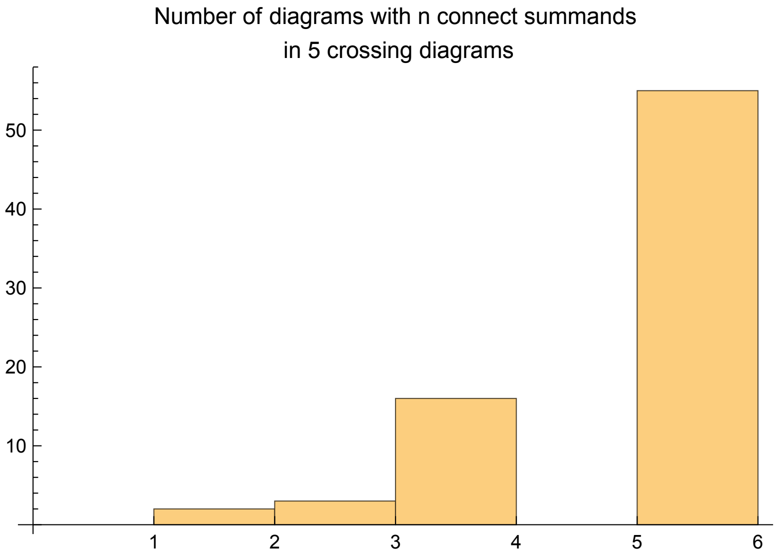

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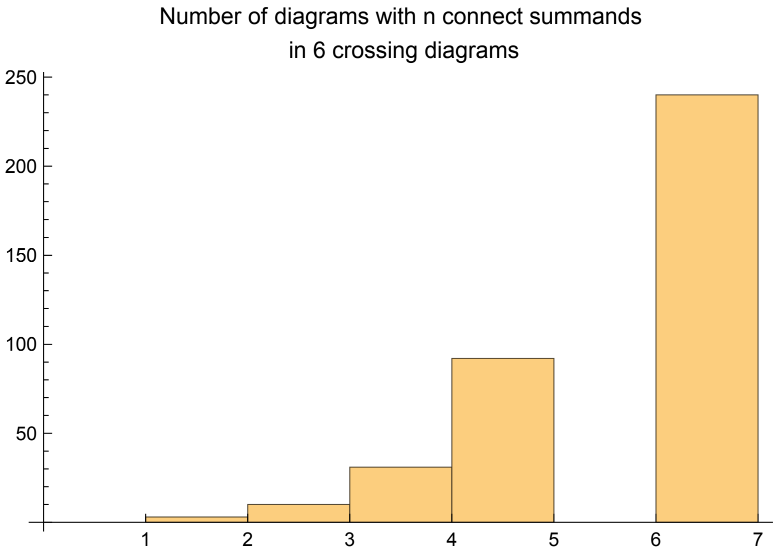

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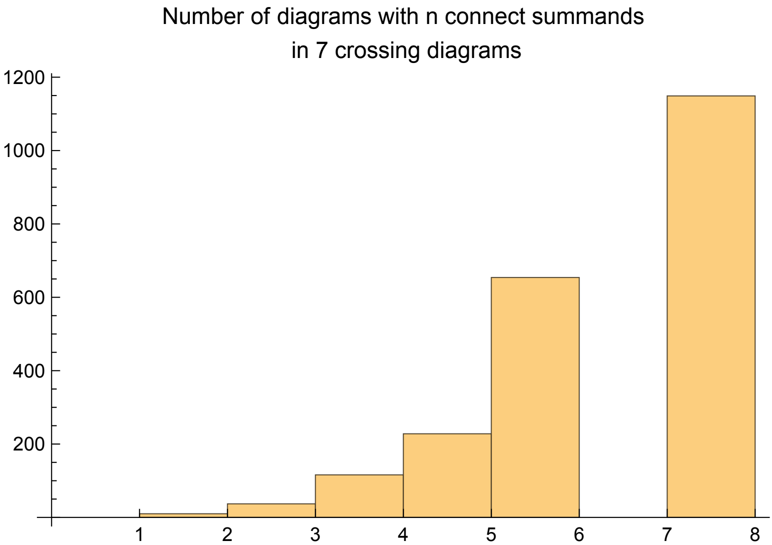

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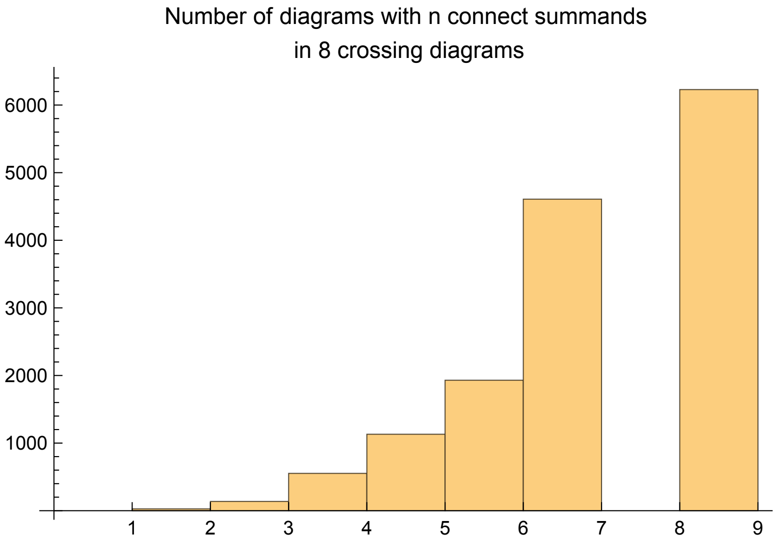

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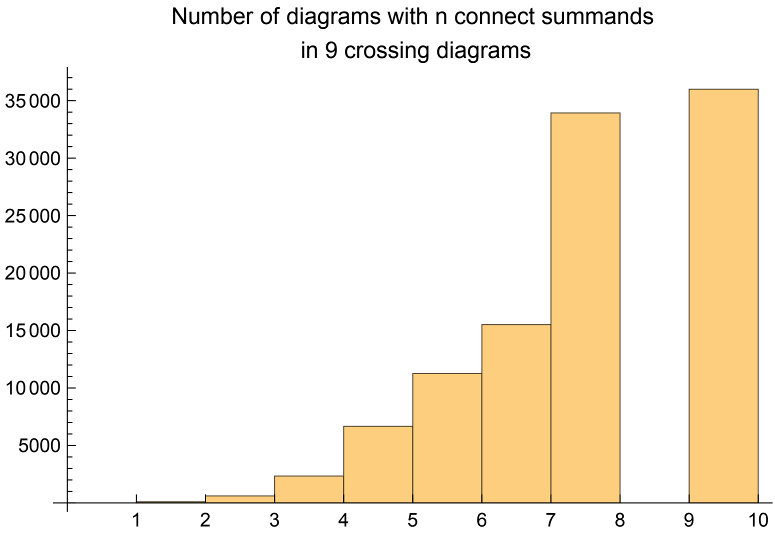

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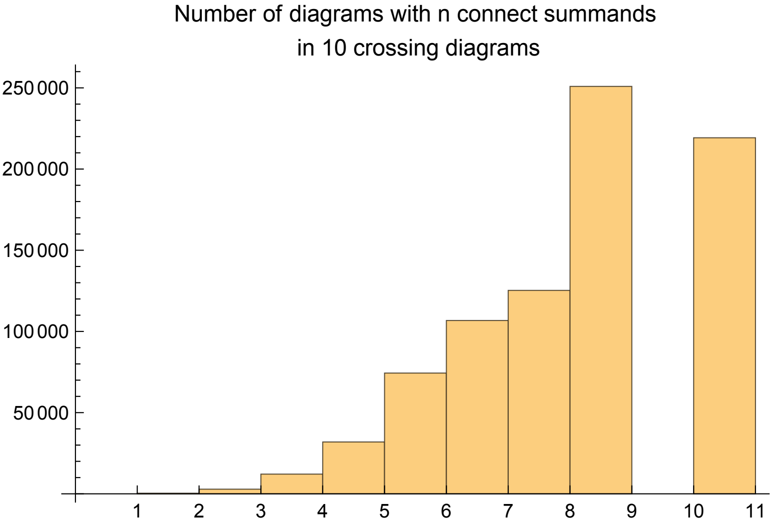
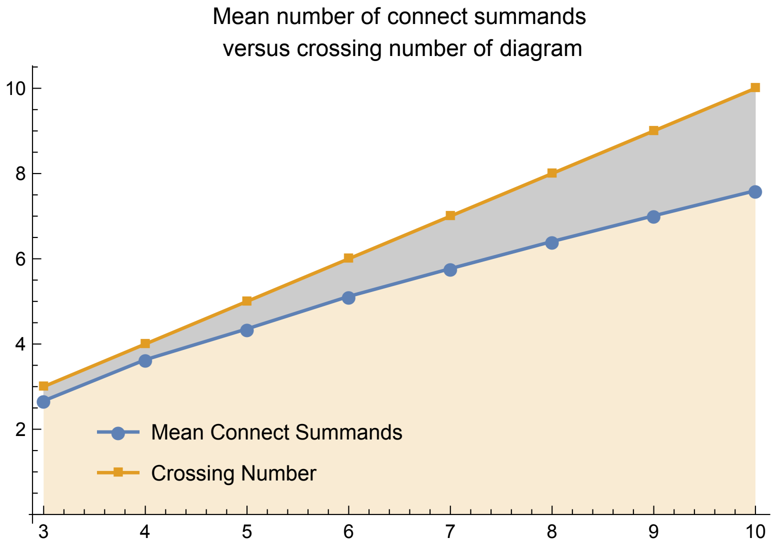


Mean number of connect summands $\simeq$ crossing number



Obvious Unknots

Proposition

If $d(n,k)$ is the number of diagrams with n crossings and k connect summands, and $d(n)$ is the number of all n crossing diagrams, then the unknot fraction among all n crossing diagrams is at least

$$\frac{1}{d(n)}\left(d(n,n)+\frac{3}{4}d(n,n-2)+\frac{7}{8}d(n,n-3)\right)$$

Proof.

Any diagram in $d(n,n)$ is a connect sum of all 1-crossing diagrams, and so can be simplified to the unknot via RI moves.

$\square$

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$$\frac{1}{d(n)}\left(d(n,n)+\frac{3}{4}d(n,n-2)+\frac{7}{8}d(n,n-3)\right)$$

Proof.

Any diagram in $d(n,n-2)$ is a connect sum of 1-crossing diagrams, and a prime 3-crossing diagram (turns out there's only one– the trefoil diagram). This diagram is knotted iff those three crossings have the same sign, which occurs $1/4$ of the time.

$\square$

Obvious Unknots

Proposition

If $d(n,k)$ is the number of diagrams with n crossings and k connect summands, and $d(n)$ is the number of all n crossing diagrams, then the unknot fraction among all n crossing diagrams is at least

$$\frac{1}{d(n)}\left(d(n,n)+\frac{3}{4}d(n,n-2)+\frac{7}{8}d(n,n-3)\right)$$

Proof.

You can make a similar argument for $d(n,n-3)$. $\square$

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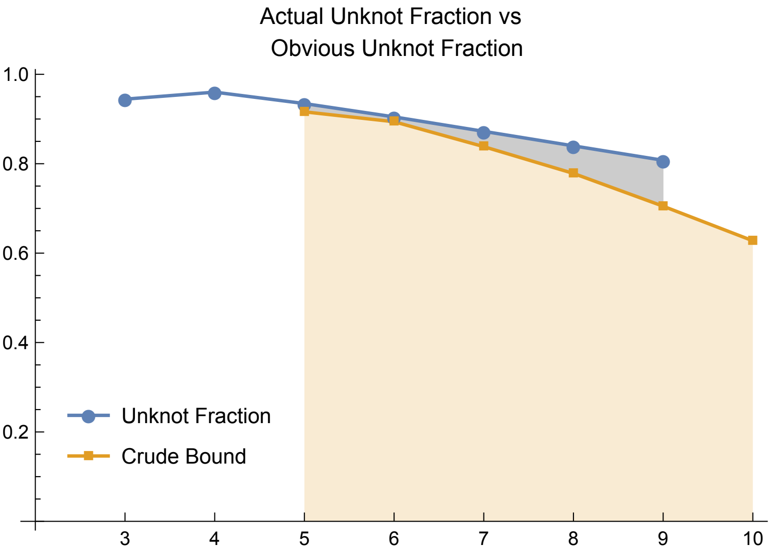
Question

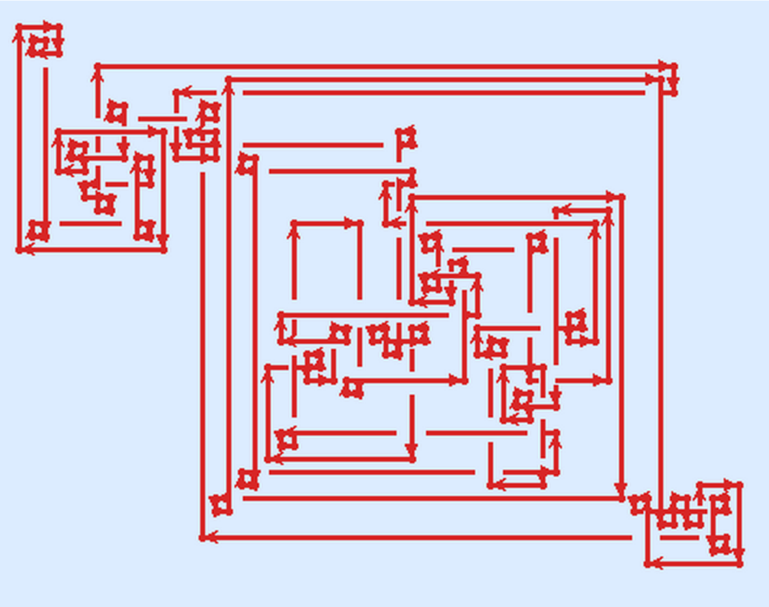
Does this (crude) bound explain the unknot fraction?

Yes.

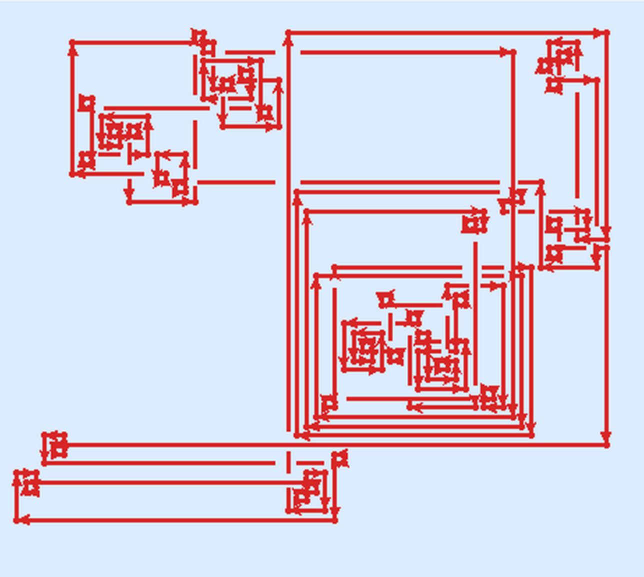


Actual Unknot Fraction vs
Obvious Unknot Fraction
1.0
0.8
0.6
0.4
0.2
3
4
5
6
7
8
9
10
Unknot Fraction
Crude Bound

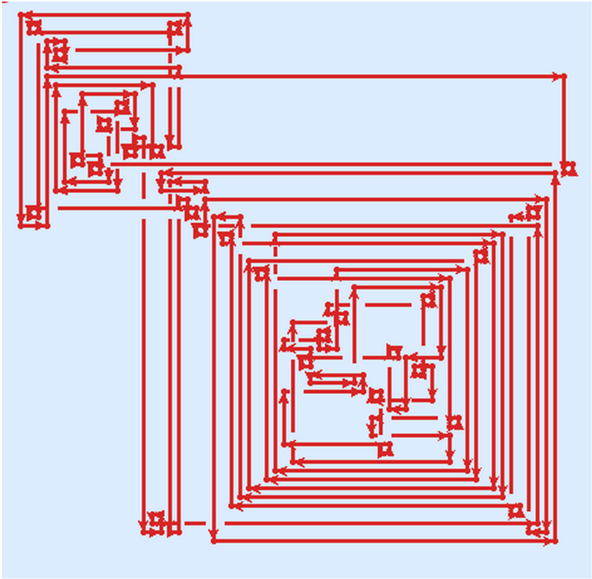
Future Direction: Uniform sampling of large diagrams



Future Direction: Uniform sampling of large diagrams



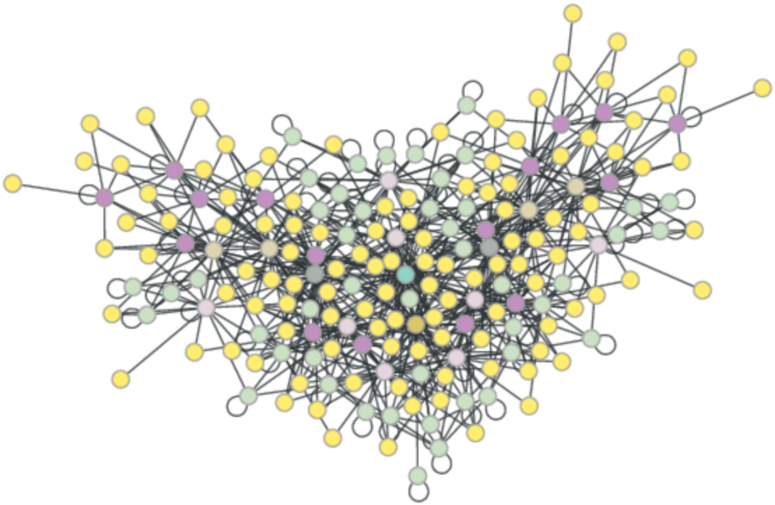
Future Direction: Uniform sampling of large diagrams



Future direction: Link diagrams

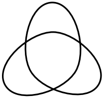
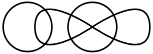
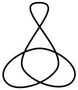
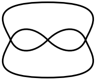
n	# link shadows	# knot shadows
0	1	1
1	1	1
2	3	2
3	7	6
4	30	19
5	124	76
6	733	376
7	4586	2194
8	33373	14614
9	259434	106421
10	2152298	823832

Future direction: Knot distances



Thank you!

Coming soon: Cantarella, Chapman, Mastin. *Knot probabilities in random diagrams*.

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