

Random Planar Diagrams

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joint w/ Jason Cantarella (UGA), Matt Mastin (Wake Forest)

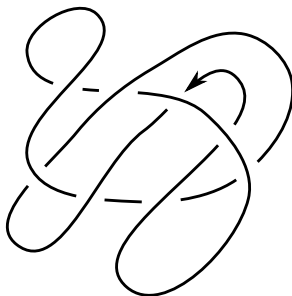
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Natural questions about knot diagrams

Question

What fraction of 8-crossing diagrams are trefoils?

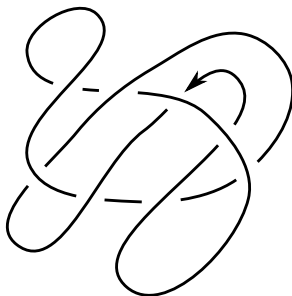


Natural questions about knot diagrams

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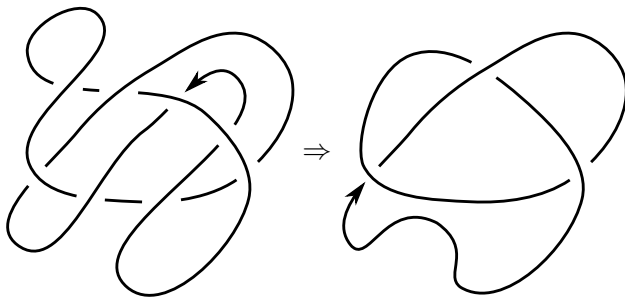
12.48%



Natural questions about knot diagrams

Question

What is the average minimal crossing # of an 8-crossing diagram?

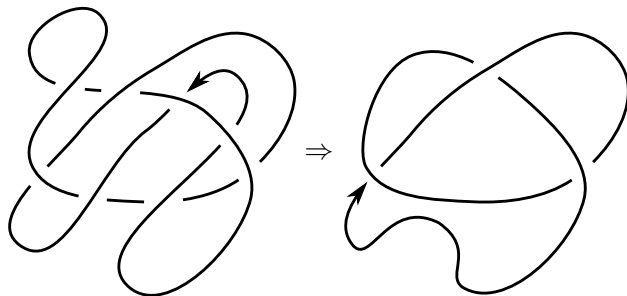


Natural questions about knot diagrams

Question

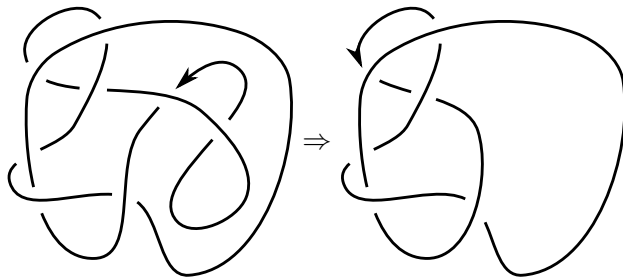
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12.48%



Natural questions about knot diagrams

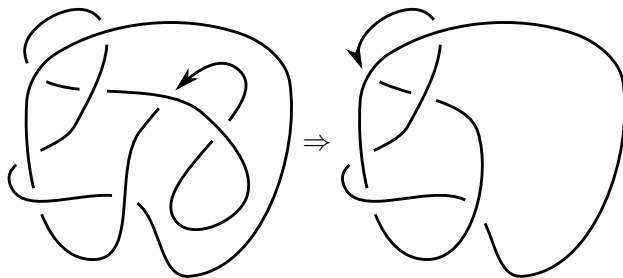
Define an operation on diagrams, **delooping**: Recursively RI untwist monogon loops in a diagram until there are no more.



Natural questions about knot diagrams

Question

What is the average crossing # of a delooped 8-crossing diagram?

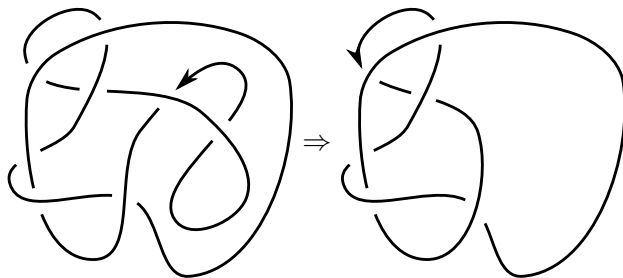


Natural questions about knot diagrams

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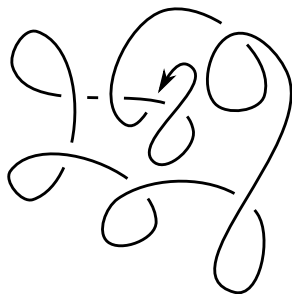
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Natural questions about knot diagrams

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How many 8-crossing diagrams can be delooped to the unknot?

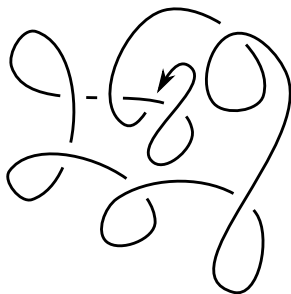


Natural questions about knot diagrams

Question

How many 8-crossing diagrams can be delooped to the unknot?

44.51%



Knotting and Polymers

Figure: Knotting in polymers. DNA must be unlinked during mitosis (left). Enzymes must fold appropriately (right).

Random curve distributions

Classical workflow for understanding knotting in random polymers:
Random distributions on spaces of curves

- Random space polygons. (Fixed edge length, equilateral, confined, etc.)
- Random closed self-avoiding lattice walks.

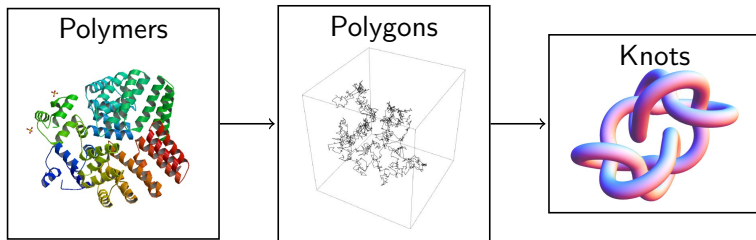


Figure: Typical random curve workflow

Combinatorial knot distributions

Alternative approach: Combinatorial distributions

- Petaluma model (Evan-Zohar, Hass, et al.)
- Random braid words

Combinatorial models are recent.

The Petaluma model

Many satisfying theorems have been proven for the Petaluma model

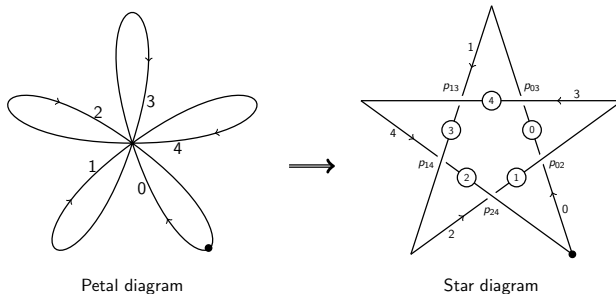


Figure: Petal diagram and corresponding star diagram for the trefoil.
(Diagram from Evan-Zohar, et al.)

The void

There is no clear connection between the two models: E.g. how to produce a star diagram from a random polygon?

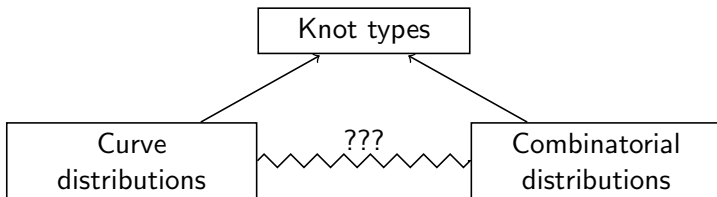


Figure: There is no convenient middle between the two methods

The random diagram model

Every space curve can project to a diagram, **and** diagrams are combinatorial objects.

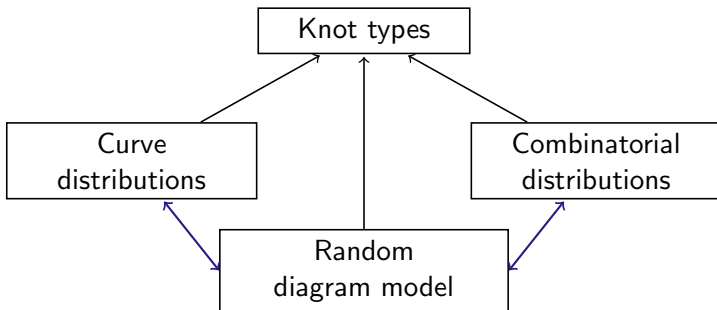


Figure: We're trying to fill the void with the random diagram model

Random diagrams

Definition

In the **random diagram model** of random knotting, a n -crossing diagram is drawn uniformly from the finite set of n -crossing knot diagrams.

Diagrams from shadows

Sample diagrams uniformly through tabulation:

- 1 Enumerate shadows (underlying graph structure behind diagrams).
- 2 Expand shadows into diagrams.

Knot shadows and circle immersions

Knot shadows in n crossings \Leftrightarrow unoriented, generic immersions of the circle into the sphere with n double points, up to unoriented diffeomorphism.

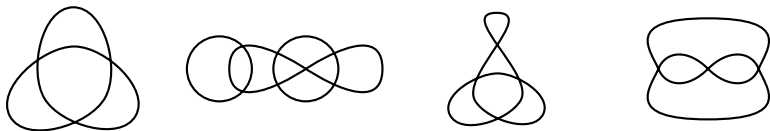


Figure: Knot shadows. The shadow on the left is equivalent to the shadow on the right.

How many shadows?

n	# knot shadows
0	1
1	1
2	2
3	6
4	19
5	76
6	376*
7	2194*
8	14614**
9	106421**

Table: Counts on knot shadows. Numbers are large, but finite.

How many knot shadows?

Counts of knot shadows with n crossings match Arnol'd's counts of immersions of the unoriented circle into the unoriented sphere with n double points (OEIS A008989).

Caveat: Arnol'd's list is for $n = 0$ to $n = 5$; the terms for $n = 6$ and $n = 7$ are attributed to Guy H. Valette with no clear source.

How many shadows?

n	# knot shadows
0	1
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Table: *: Attributed to Guy H. Valette. **: New; values not in OEIS.

Tabulating knot shadows

Generated table of knot shadows two different ways as a check.

Both methods use features from McKay and Brinkmann's `plantri`.

Duals to quadrangulations

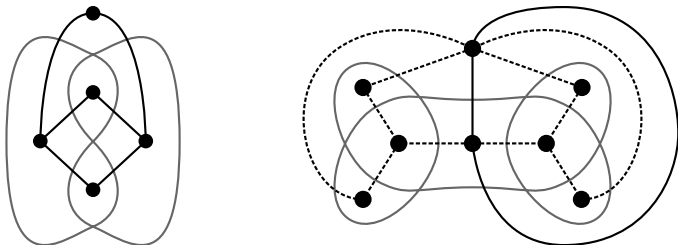


Figure: Find all quadrangulations of the sphere in n faces. Shadows are dual to quadrangulations.

Planar graph expansions

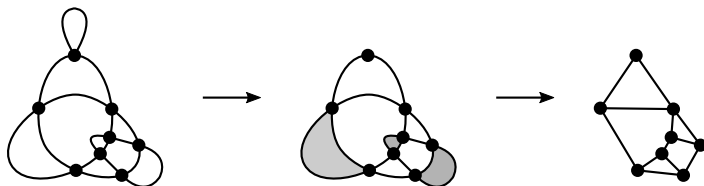


Figure: Reduction to a planar graph of degree ≤ 4 and connectivity ≥ 1 .
Expansion is the inverse.

The space of shadows

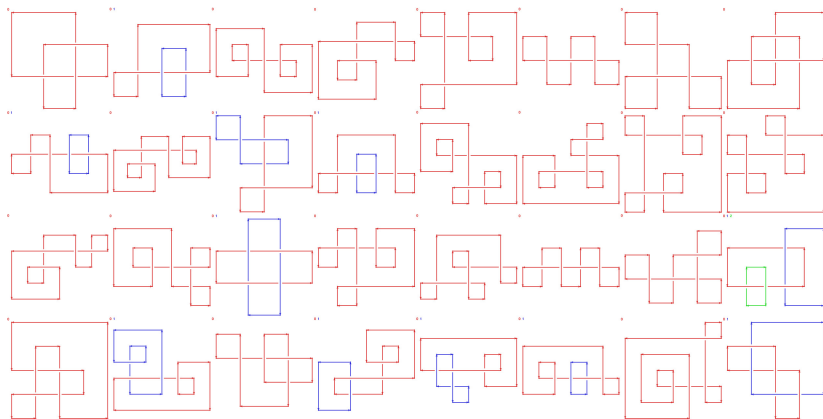


Figure: Link shadows. Pictures generated by Eric Lybrand (UGA) with SnapPy. A map of all shadows with between 3 and 6 crossings is here.

The space of shadows

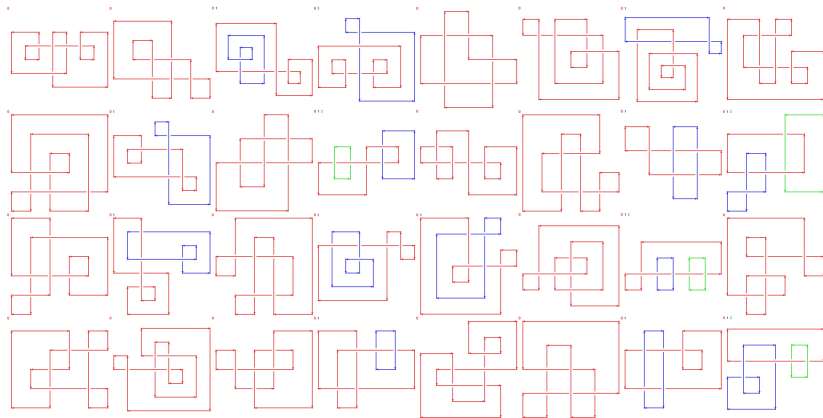


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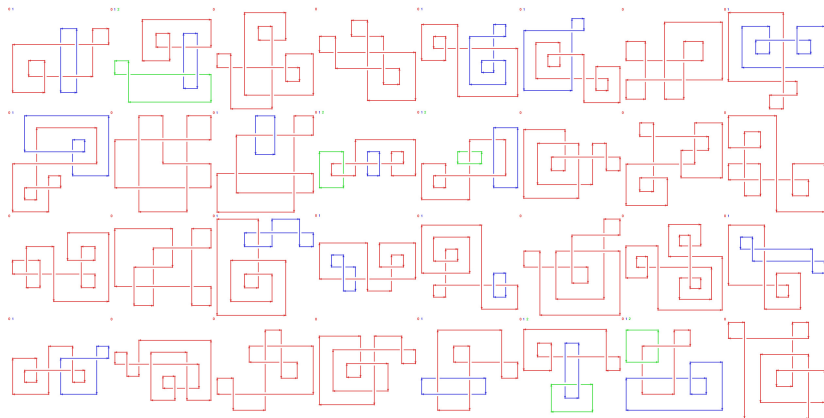


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The space of shadows

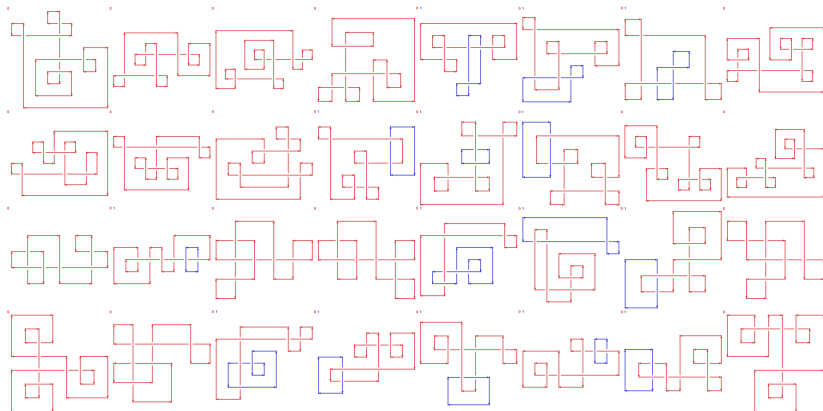


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Tabulation is difficult!

Accounting for symmetry is complicated.

Figure: The trefoil shadow has 12-fold symmetry (Left). This diagram with trefoil shadow has 6-fold symmetry (Right).

Breaking symmetries could make counting easier

Easier to count shadows/diagrams with broken symmetries (rooted diagrams). E.g., is a correspondence:

Figure: Two-leg diagrams (left) correspond to rooted shadows (right).

Diagrams on the left are counted by a generating function (Bouttier, et. al).

From shadows to diagrams

Expansion of shadows to diagrams procedure:

- 1 Orient each component. ($2^{\#\text{components}}$ choices)
- 2 Assign over-under information to each vertex. ($2^{\#\text{crossings}}$ choices)
- 3 Group diagrams by isomorphism.

How many knot diagrams?

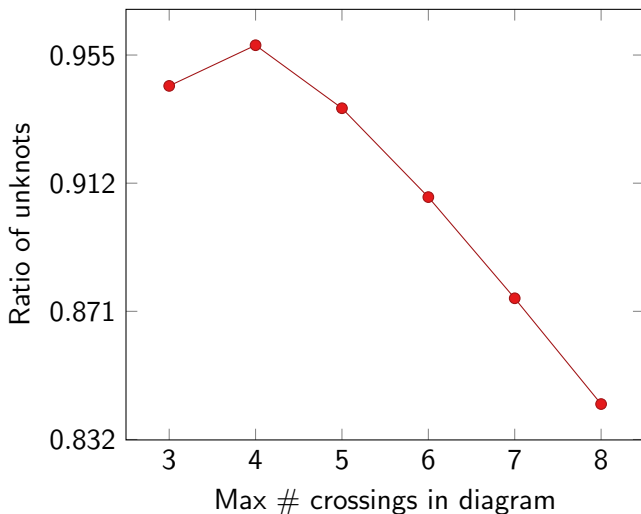
n	# knot shadows	# knot diagrams	# knot iso. classes
3	6	96	36
4	19	608	276
5	76	4,864	2,936
6	376	48,128	35,872
7	2,194	561,664	484,088
8	14,614	7,482,368	6,967,942
9	106,421	108,975,104	in process

Table: Counts of knot shadows and diagrams

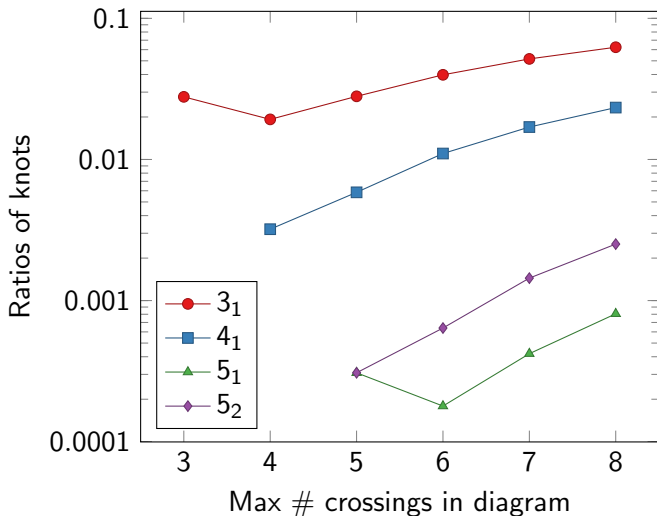
Knotting probabilities

- Advantage of a combinatorial model: Able to run searches across entire space computationally.
- Can check knot type of each diagram (HOMFLY is typically enough for our low crossing number)
- Possible to run many different types of searches

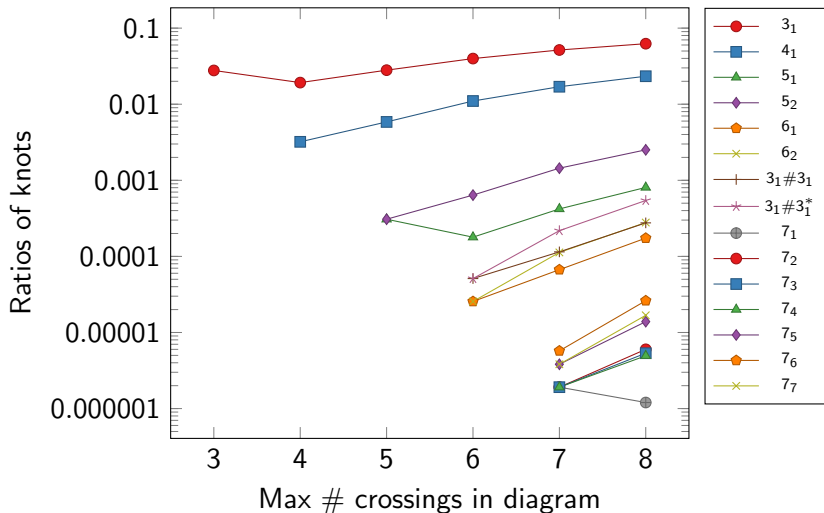
Ratio of unknots in n -crossing diagram iso. classes (log scale)



Ratios of knots in n -crossing diagram iso. classes (log scale)



Ratios of knots in n -crossing diagram iso. classes (log scale)

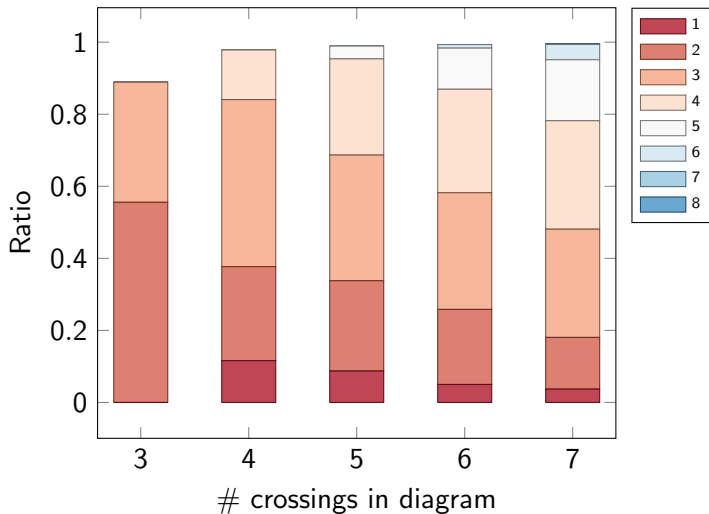


Counting monogons and bigons in knot shadows

n	shadows	1-gon	2-gon	neither
3	6	5 (83.33%)	3 (50%)	0
4	19	18 (94.74%)	11 (57.89%)	0
5	76	74 (97.37%)	52 (68.42%)	0
6	376	371 (98.67%)	275 (73.14%)	0
7	2,194	2,178 (99.27%)	1,714 (78.12%)	0
8	14,614	14,562 (99.64%)	11,892 (81.37%)	1
9	106,421	106,216 (99.81%)	89,627 (84.22%)	1

Table: Counts of knot shadows with monogons, bigons, or neither. 8- and 9- crossing shadows with neither are Conway's 8^* and 9^* .

Monogons in diagrams



Basic polyhedra 8^* and 9^*

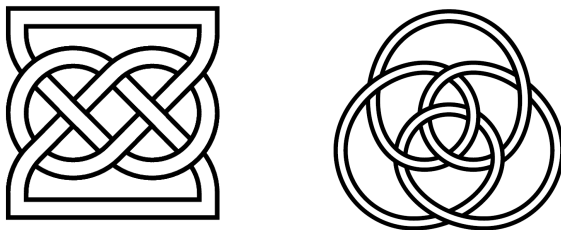


Figure: Knots with the two knot shadows in ≤ 9 crossings which are planar simple graphs: 8_{18} (left), 9_{40} (right).

Tree-like curves

A **tree-like curve** is a knot shadow which can be untwisted to the trivial shadow.

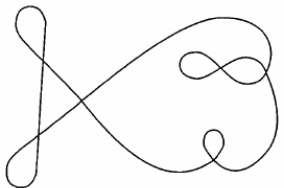


Figure: A tree-like curve.

Tree-like curves \Rightarrow lower bound on unknottedness.

Tree-like curves

n	# knot shadows	# tree-like	% tree-like
1	1	1	100.00%
2	2	2	100.00%
3	6	5	83.33%
4	19	16	84.21%
5	76	55	72.37%
6	376	240	63.83%
7	2,194	1149	52.37%
8	14,614	6,229	42.62%
9	106,421	35,995	33.82%

Table: Counts of knot shadows and tree-like curves

Unknottedness and tree-like shadows

Ratio of unknots, tree-like curves in $\leq n$ -crossing diagram iso. classes (log scale)

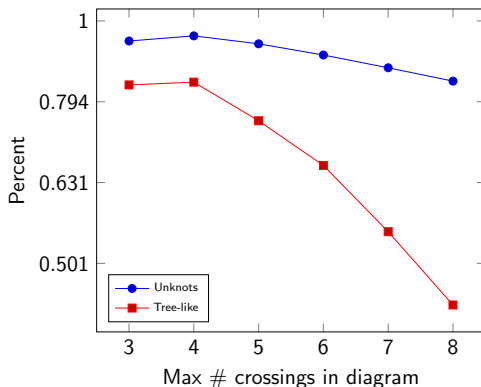


Figure: The number of unknots is bounded by the number of tree-like diagrams.

Delooped crossing number

n	Average delooped crossing #
3	0.50
4	0.53
5	0.92
6	1.25
7	1.72
8	2.19
9	2.70

Table: Average delooped crossing number over shadows with n crossings.

Questions to answer

Conjecture

The ratio of unknots in diagrams tends to zero as n increases.

This would match data from random curve experiments.

Questions to answer

Random curves project to diagrams.

Question

How does the pushforward measure differ from uniform diagram sampling?

Questions to answer

Question

Can we sample diagrams uniformly without enumeration?

Future directions

- Most analysis here is on knot diagrams; what can we say about link diagrams?
- How does the random diagram model compare to other models?
 - *Petaluma* model (Evan-Zohar, Hass, Linial, and Nowik)
 - Random space polygons, random equilateral space polygons, random confined space polygons
 - Random closed self-avoiding lattice walks
- Uniform sampling of diagrams of higher crossing number: Can we avoid outright enumeration?

Link diagrams

Counts for link diagrams

Knot distances

Can study pure knot theoretic things, not just probabilistic things—transitions between knots
bat graph [figure]

Thank you!