On the asymptotics of uniformly random knot diagrams

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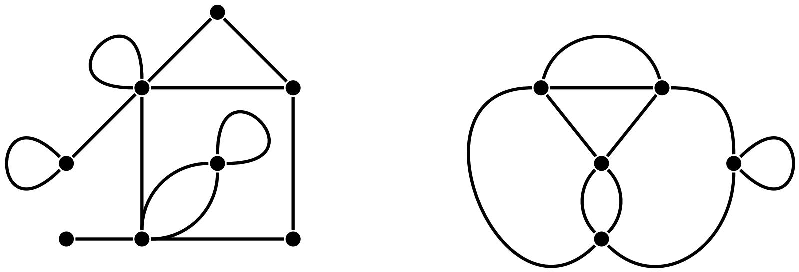

FIG. 1. Two planar maps. The map on the right is in the class of knot shadows.

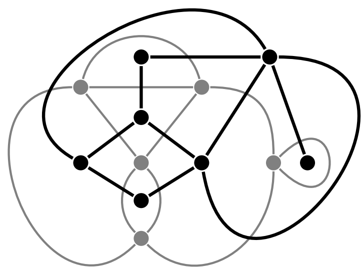

FIG. 2. Planar quadrangulation which is dual to a knot shadow.

I. INTRODUCTION

There is a dearth of models for drawing random knots; self avoiding lattice walks [cite], random space polygons [cite], random braid words [cite], *Petaluma* [cite], et. al. In this paper we will discuss the *random diagram model* under which *knot diagrams* are drawn uniformly from the set of all diagrams with a given number of crossings. Alternating knot and link diagrams have been studied [cite] but little is published about the knottiness of arbitrary random diagrams of large size.

In this paper we begin by considering a slightly different object, *rooted diagrams*, which break symmetries (as opposed to in [1]). We are then able to prove that in the limit, knot diagrams behave similarly to rooted diagrams, so that these results carry over.

II. DEFINITIONS

A *knot* is an isotopy class of embeddings of the circle into S^3. A *link* is an isotopy class of embeddings of one or more circles into S^3. Both of the prior are considered up to *ambient*

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isotopy of the embedded circles. The study of links and knots is well known to be equivalent to the study of *link diagrams* and *knot diagrams* (more formally defined below) up to the so-called *Reidemeister moves*. We wish to examine the underlying *planar map* structure of knot and link diagrams;

Definition 1. *A* planar map with n vertices P *is a graph* $\tilde{P}$ *embedded in the sphere. The planar map* P *is* 4-regular *or* quartic *if every vertex in the underlying graph* $\tilde{P}$ *has degree 4. A* rooted planar map *is a planar map together with a single edge marked with a direction.*

Planar maps (indeed, maps on any surface) have a well defined notion of the *dual map*; a map $M = (V, E, F)$ has dual $M^* = (F, E^*, V)$, where there is an edge $(f_1, f_2) \in E^*$ if f_1 is adjacent to f_2 in M. The dual graph of a 4-regular planar map is a *planar quadrangulation*.

By breaking symmetries with a root, we may study certain classes of planar maps by way of the celebrated bijection with *blossom trees*[2]. We carry this idea to link and knot diagrams:

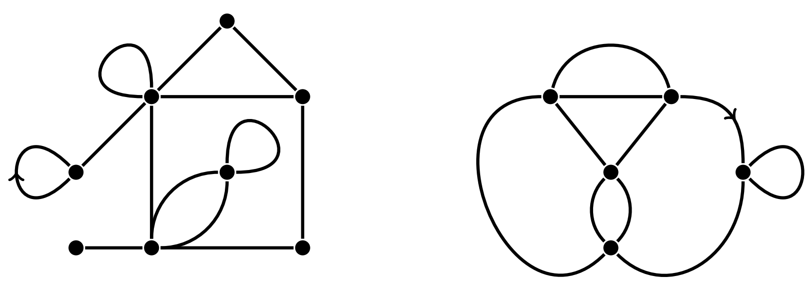

FIG. 3. Two rooted planar maps. The map on the right is in the class of rooted knot shadows.

Definition 2. *A* (rooted) link diagram with n crossings *is a 4-regular (rooted) planar map of* n *vertices together with a choice of over-under strand information at each vertex. The class of rooted link diagrams with* n *crossings is denoted* $\mathcal{L}_n$*.*

The class of maps which represent rooted link shadows *in* n *crossings, i.e. maps which can be found as the underlying map structure of a rooted link diagram are denoted* $\mathscr{L}_n$*.*

It is well understood that the class of rooted link shadows in n-vertices is identical to the class of 4-regular planar maps in n vertices is identical to $\mathscr{L}_n$; furthermore, the class of rooted planar quadrangulations is dual to $\mathscr{L}_n$.

Definition 3. *A* (rooted) knot diagram *is a (rooted) link diagram which consists of only one knot component. The class of rooted knot diagrams with n crossings is denoted by $\mathcal{K}_n$.*

The class of maps which represent rooted knot shadows *in n crossings are denoted $\mathscr{K}_n$.*

Knot shadows $\mathscr{K}_n$ represent a curious, small subclass of $\mathscr{L}_n$.

Rooted (knot or link) diagrams are equivalently viewed as *two-leg diagrams* or *2-tangles* as illustrated below.

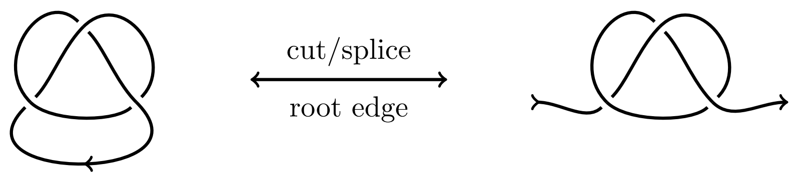

FIG. 4. Rooted diagrams of the trefoil and its mirror image

Additionally, rooted diagrams can be viewed as *four-leg diagrams*, or 4-*tangles* by deleting the root crossing. This identification is not injective for diagrams as it forgets the sign of the removed crossing.

A (rooted) shadow is *prime* if it cannot be disconnected by removing two edges (i.e. it is at least 3-connected); otherwise it is *composite*. A rooted shadow is *two-leg-prime* if it cannot be disconnected by removing two edges, one being the root edge. Diagrams are (two-leg-)prime if their underlying shadow structure is.

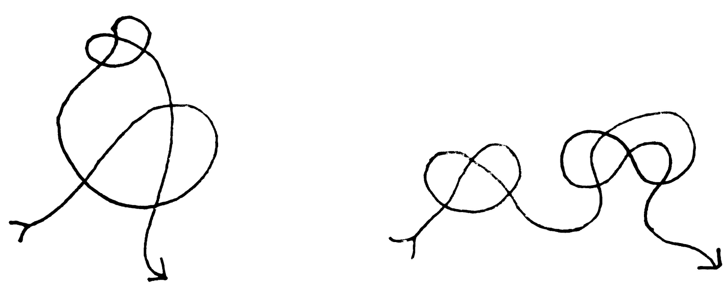

FIG. 5. A shadow which is not prime (left). A composite shadow which is two-leg-prime (right).

III. ASYMPTOTIC STRUCTURE THEOREMS FOR DIAGRAMS

It is believed and numerically evident [3] that the number of link diagrams in a random link diagram grows exponentially, hence a random link diagram is almost certainly not a knot diagram.

III.1. A pattern theorem for classes of link diagrams

We will show that different classes of link diagrams (really, their duals) satisfy the requirements for the following theorem from [4]. We reproduce the theorem below:

Theorem 1 (Bender, Gao, Richmond 1989). *Let $\mathscr{M}$ be some class of maps on a surface of type g and let P be a planar map that can be found as a submap of maps in $\mathscr{M}$. Let $M(x)$ be the generating function by number of edges for $\mathscr{M}$. Let $H(x)$ be the generating function by number of edges for those maps M in $\mathscr{M}$ that contain less than $ce(M)$ pairwise disjoint copies of P. Suppose that we can embed P in a larger rooted map Q and attach copies of Q to each map K counted by $H(x)$ in such a way that*

1. *for some fixed positive integer k, at least $\lfloor e(K)/k \rfloor$ possible non-conflicting places of attachment exist,*
2. *only maps in $\mathscr{M}$ are produced,*
3. *for any map produced as such we can identify the copies of Q that have been added and they are all pairwise disjoint, and*
4. *given the copies that have been added, the original map and associated places of attachment are uniquely determined.*

If $c > 0$ is sufficiently small, then $r(M) < r(H)$. The maps may be rooted or not.

We can define attachment operations for different kinds of tangles:

1. **Connect sum.** Let L be dual to a rooted link shadow, and Q be the dual to a rooted 2-tangle shadow. Orient the remaining edges of L canonically using the bijection with blossom trees. Define the *connect sum* of Q into L at an edge $e \in L$, $Q\#_e L$, by
   (a) Cut and split the edge e, creating a map L' and leaving a distinguished, oriented bigon f. Denote the two edges formed by splitting e by e_1, e_2, so that the loop $e_1(-e_2)$ is a counterclockwise cycle around f. If e was the root of L, make e_1 the new root of L'.

(b) Cut and split the root edge ϵ of Q, creating a map Q' and leaving a distinguished, oriented bigon g. Denote the two edges formed by splitting ϵ by ϵ_1, ϵ_2, so that the loop $\epsilon_1(-\epsilon_2)$ is a counterclockwise cycle around g. Make ϵ_2 the new root of Q'.

(c) Glue the map Q' into the map L''s distinguished face f along the boundary of the distinguished bigon g so that e_1 and ϵ_2 are mapped to the same edge and so that the orientations of the boundaries align.

(d) Forget about all edge orientations except for the root edge of L.

The result is a new rooted link shadow. Any given link shadow in n edges has precisely n different non-conflicting sites for connect summation (i.e. $k = 1$ for this attachment operation). If L^* is a knot shadow and Q^* consists of only one link component, then $Q\#_e L$ will also be a knot shadow.

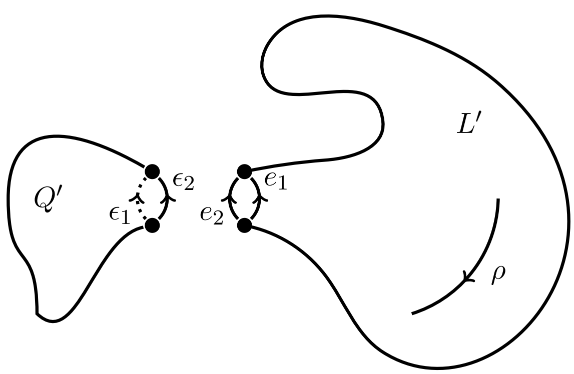

FIG. 6. The connect sum operation. Q' and L' are viewed as CW-complexes, and their boundaries are appropriately identified.

2. **4-tangle replacement.** Let L be a rooted link shadow, and Q be the dual to a rooted link shadow.

Consider the following applications of this theorem to classes of link diagrams:

1. Let $\mathscr{K}$ be the class of all rooted knot shadows (so that $\mathscr{K}^*$ is the class of quadrangulations which are dual to knot shadows), and P be the dual of a prime 2-tangle shadow of only one component. Define the attachment operation of P into $\mathscr{K}^*$ as follows. Let Q be P with a rooted edge and embed P identically, and take the connect

sum attachment operation. Since P was chosen to have one component, only maps in K^* are created (the attachment operation is just the connect sum).

2. Again consider $\mathscr{K}^*$, but let P be the dual of a prime 4-tangle shadow of one component with *alternating* loose strands and let attachment be 4-tangle replacement.

3. If we consider a tangle P^* which consists of more than one component, we obtain another proof that link diagrams almost surely have more than one component.

4. Consider again connect summation. Given a pattern P^* representing a $2k$-tangle, we can obtain a rooted two-leg-prime diagram Q^* as follows. Connect $2(k-1)$ of the loose tangle strands into pairs iteratively if the two strands correspond to different components of the tangle. It is possible that many new crossings are introduced in this step, but the resultant 2-tangle will have only one component. This proves the pattern theorem for arbitrary $2k$-tangles.

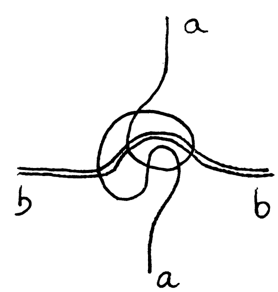

FIG. 7. A 4-tangle with alternating strands

III.1.1. Strategy for proving smooth growth

Bender, et al. [4] give a powerful proof strategy for proving smooth growth of a sequence. Let $\mathscr{C}$ be a class of combinatorial objects with generating function $\sum_{n=0}^{\infty} c_n\ z^n$; let the radius of convergence of the OGF be r, and $\mathscr{D}$ some other class with generating function $\sum_{n=0}^{\infty} d_n\ z^n$ to be specified later. Suppose we have some preliminary, loose bound away from zero on r (there are at most 12^n planar maps, and so in our cases we have $r \geq 1/12$) and that $r \leq 1$ Let $C_i > 0$ and $1 - r > \delta > 0$ be arbitrary. The method involves three steps;

1. *Show that there exists some $n \geq 0$ with $c_n > C_1(r+\delta)^{-n}$.* This step follows from the Cauchy-Hadamard theorem, which says that

$$\limsup_{n\to\infty} c_n^{1/n} = r^{-1}.$$

By the definition of lim sup, we have that if $a < r^{-1}$, then for any M we have that there is some $n \geq M$ with $c_n^{1/n} > a$. For instance, we know that $(r+\delta/2)^{-1} < r^{-1}$, hence for any M we have some $n \geq M$ with $c_n > (r+\delta/2)^{-n}$. Notice now that as

$$\left(\frac{r+\delta}{r+\delta/2}\right) > 1,$$

there must be some M so that for all $m \geq M$

$$\left(\frac{r+\delta}{r+\delta/2}\right) > C_1^{1/m}, \text{ implying that } (r+\delta/2)^{-m} > C_1(r+\delta)^{-m},$$

whence we then have (by lim sup) some $n \geq M$ with $k_n > (r+\delta/2)^{-n} > C_1(r+\delta)^{-n}$.

2. *Show that there exists some $m \geq 0$ with $d_m > C_2(r+\delta)^{-m}$ and $d_{m+1} > C_2(r+\delta)^{-(m+1)}$.* This step will depend on the classes $\mathscr{C}$ and $\mathscr{D}$. Notice that $(r+\delta) < 1$ and so $(r+\delta)^{-m} < (r+\delta)^{-(m+1)}$. If there exist injections ψ_0, ψ_1 from $\mathscr{C}_n$ into $\mathscr{D}_{n+\ell}$ and $\mathscr{D}_{n+\ell+1}$, then setting $m = n+\ell$ and $C_1 = C_2(r+\delta)^{-m+n-1)}$ we have that

$$d_m \geq |\operatorname{im}\psi_0| = c_n > C_1(r+\delta)^{-n} = C_2(r+\delta)^{-(m+1)} > C_2(r+\delta)^{-m}$$

and

$$d_{m+1} \geq |\operatorname{im}\psi_1| = c_n > C_1(r+\delta)^{-n} = C_2(r+\delta)^{-(m+1)}.$$

3. *Show that there exists some N so that for any $n \geq N$, $c_n > (r+\delta)^{-n}$.* To show this, we define a product operation $\times$ on elements $A, B \in \mathscr{C} \cup \mathscr{D}$ so that,

   (a) $A \times B \in \mathscr{C}$,

   (b) there exists some fixed $k \in \mathbb{Z}$ so that $|A \times B| = |A| + |B| + k$, and

   (c) given any $C \in \mathscr{C}$, there is at most one maximal factorization $A_1 \times A_2 \times \cdots \times A_s = C$ with $A_i \in \mathscr{D}_m \cup \mathscr{D}_{m+1}$.

   Let $C_2 = (r+\delta)^{-k}$. Let $N = (m+k)(m+k+1)$. Then if $n \geq N$, we can write n as a linear combination $a(m+k) + b(m+k+1) = am + b(m+1) + (a+b)k$, with

$a, b \geq 0$. Observe that $c_n > d_m^a d_{m+1}^b$ as there exists a subset of objects $S \subset \mathscr{C}_n$ which can be expressed uniquely as a product of a elements of $\mathscr{D}_m$ and b elements of $\mathscr{D}_{m+1}$ (and $|S| > d_m^a d_{m+1}^b$). Then

$$c_n > d_m^a d_{m+1}^b > C_2^a (r+\delta)^{-am} C_2^b (r+\delta)^{-b(m+1)} = (r+\delta)^{-(am+b(m+1)+(a+b)k)} = (r+\delta)^{-n}.$$

To finish the proof we realize that this last statement implies that the lim inf is r^{-1} and hence the limit result follows. Observe that $\liminf_{n\to\infty} c_n^{1/n} = r^{-1}$ if for any $\epsilon > 0$, there exists N so that for all $n \geq N$, $c_n^{1/n} > r^{-1} - \epsilon = \frac{1-r\epsilon}{r}$. We may assume that $\epsilon < 1$ since otherwise the inequality is clear since $c_n \geq 0$ always. So we are done if we can choose δ so that

$$\frac{r^2\epsilon}{1-r\epsilon} > \delta,$$

as then we have from our prior result that $c_n^{1/n} > (r+\delta)^{-1} > r^{-1} - \epsilon$. Indeed, we have $r^2\epsilon > 0$ and $1 > 1 - r\epsilon > 0$ so that the left hand side of the inequality is positive; but we may choose $\delta > 0$ as small as we desire. Hence for $N = (m+k)(m+k+1)$, the result that $\lim_{n\to\infty} c_n^{1/n} = r^{-1}$ follows.

III.1.2. Smooth growth for knot and link diagrams

Now, theorem in the prior section by itself does not sufficiently prove *ubiquity* as required to prove asymmetry. Indeed, we require that the class of maps *grow smoothly*, i.e. that (for $m_n = |\mathscr{M}_n|$) the limit

$$\lim_{n\to\infty} m_n^{1/n}$$

exists. The class $\mathscr{L}$ of rooted link shadows has been counted exactly. Rooted link shadows are in bijection with rooted 4-regular planar maps and the coefficients of the generating function are known [5]. If $l_n = |\mathscr{L}_n|$, then:

$$l_n = \frac{2(3^n)}{(n+2)(n+1)}\binom{2n}{n}.$$

On the other hand, asymptotics of knot shadows are as of yet unknown. We are still however able to prove that they grow smoothly, as to prove our result of asymptotic asymmetry.

Theorem 2. *The class $\mathscr{K}$ of rooted knot shadows grows smoothly. I.e., the limit $\lim_{n\to\infty} k_n^{1/n}$ exists (and is equal to $1/r(K)$).*

Proof. A very loose bound on $r = r(K)$ is $1/12 \leq r \leq 1$ as the number of planar maps in general is bounded by 12^n. Let $C_i > 0$ and $1 - r > \delta > 0$ be arbitrary.

The result will follow from that for a power series (by the Cauchy-Hadamard theorem)

$$\limsup_{n\to\infty} k_n^{1/n} = 1/r(K) = r^{-1}$$

so long as we can find some N so that for all $n \geq N$, $k_n > C(r+\delta)^{-n}$.

By the definition of lim sup, we have that if $a < r^{-1}$, then for any M we have that there is some $n \geq M$ with $k_n^{1/n} > a$. For instance, we know that $(r+\delta/2)^{-1} < r^{-1}$, hence for any M we have some $n \geq M$ with $k_n > (r+\delta/2)^{-n}$. Notice now that as

$$\left(\frac{r+\delta}{r+\delta/2}\right) > 1,$$

there must be some M so that for all $m \geq M$

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whence we then have (by lim sup) some $n \geq M$ with $k_n > (r+\delta/2)^{-n} > C_1(r+\delta)^{-n}$.

Let φ be the map which twists the root edge, making the loop the new root (using the appropriate induced orientation). Then $\varphi : \mathscr{K}_* \hookrightarrow \mathscr{K}_{*+1}$ is an injection into the knot shadows which are at least 2-connected as 2-leg diagrams. Then if p_n counts knot shadows

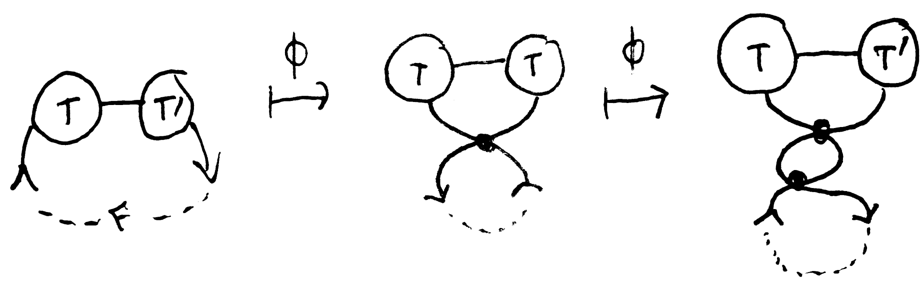

FIG. 8. The map φ adds a vertex and ensures that the new map is 2-leg-prime.

which are at least 2-connected as 2-leg diagrams, and since φ^2 acts similarly while increasing the number of crossings by 2, we have that:

$$p_{n+1} > k_n > C_1(r+\delta)^{-n} \text{ and } p_{n+2} > k_n > C_1(r+\delta)^{-n},$$

whence setting $C_1 = C_2(r+\delta)^{-2}$, $m = n+1$ and observing that since $(r+\delta) < 1$ we know $(r+\delta)^{-\ell} < (r+\delta)^{-(\ell+1)}$, we get

$$p_m > C_2(r+\delta)^{-m-1} > C_2(r+\delta)^{-m} \text{ and } p_{m+1} > C_2(r+\delta)^{-(m+1)}.$$

Now define a concatenation $K = K_1 + K_2$ on shadows counted by p_n by gluing the front leg of K_1 to the hind leg of K_2. Since K_1 and K_2 are "prime" as 2-leg diagrams, any diagram K has at most one (possibly no!) decompositions into the ordered pair (K_1, K_2) as K_1+K_2. Notice that $|K_1 + K_2| = |K_1| + |K_2|$ exactly.

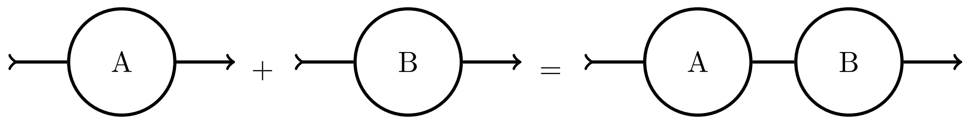

FIG. 9. The two-leg sum operation $+$. If both A and B are two-leg-prime, then $A+B$ has exactly one separating edge.

Now, let C_2 be the maximum of C_3 and 1. By iterating this construction, it follows from that $C_2^k \geq C_2 \geq 1$ that $k_n > C_2(r+\delta)^{-n}$ whenever $n > 0$ can be written as a linear combination of m and $m+1$ with non-negative integer coefficients, as then

$$k_n > p_m^a p_{m+1}^b > (C_2(r+\delta)^{-m})^a (C_2(r+\delta)^{-(m+1)})^b = C_2^{a+b}(r+\delta)^{-(am+b(m+1)} > C_2(r+\delta)^{-n}.$$

Hence, given $N = m(m+1)$ we are assured that for any $n \geq N$, $k_n > C_2(r+\delta)^{-n}$.

To finish the proof we realize that this last statement implies that the lim inf is r^{-1} and hence the limit result follows. Observe that $\liminf_{n\to\infty} k_n^{1/n} = r^{-1}$ if for any $\epsilon > 0$, there exists N so that for all $n \geq N$, $k_n^{1/n} > r^{-1} - \epsilon = \frac{1-r\epsilon}{r}$. We may assume that $\epsilon > 1$ since otherwise the inequality is clear since $k_n \geq 0$ always. So we are done if we can choose δ so that

$$\frac{(C^{1/n}-1)r + r^2\epsilon}{1-r\epsilon} > \delta,$$

as then we have from our prior result that $k_n^{1/n} > C^{1/n}(r+\delta)^{-1} > r^{-1} - \epsilon$. Indeed, if M is sufficiently large then for any $n \geq M$ we have $r(C^{1/n} - 1 + r\epsilon) > 0$ (if $C \geq 1$ it is immediate, otherwise take $M > \frac{\ln C}{\ln(1-r\epsilon)}$) and $1 > 1 - r\epsilon > 0$ so that the left hand side of the inequality is positive; but we may choose $\delta > 0$ as small as we desire. Hence for N being the maximum of M and N from our prior statement about the growth of k_n, the result that $\lim_{n\to\infty} k_n^{1/n} = r^{-1}$ follows. □

Corollary 3. *A random rooted knot diagram is knotted a.a.s. Indeed, given any 2-tangle T and large enough n a random rooted knot diagram with n crossings contains T as a connect summand.*

III.1.3. Smooth growth for prime knot and link diagrams

If, however, we are considering a class $\mathscr{P}$ of prime or reduced rooted diagrams, the method of proof for smoothness does not immediately carry over; it is possible that φ introduces numerous isthmi, in which case our diagrams created in the final step would not even be reduced. If $\mathscr{P}$ is the case of prime rooted link shadows exact counts are known from their bijection with simple quadrangulations;

$$s_n = \frac{4(3n)!}{n!(2n+2)!}.$$

In other cases again smoothness is more complicated to prove, although we can use a similar argument to that in the case of all knot shadows.

Proof. Step i is again immediate, so we begin with step ii. Let ψ, ψ' respectively be maps which take the root vertex to the two 4-tangle shadows:

Observe that neither ψ nor ψ' remove primeness or reducedness. Their images provide an injection into the spaces with 2 and 3 additional crossings, respectively. So take m appropriately.

Define the operation + now by the detour-glom. Notice that primeness is preserved and the process is splittable; given the root edge we can identify the bendy edges and rebuild the old two shadows. Notice that $|A+B| = |A|+|B|+4$. Now let $C_3 \geq 1$ and $C_2 = C_3(r+\delta)^{-4}$. Then if there exist nonnegative integers a, b such that $n = am + b(m+1) + (a+b)4 = a(m+4) + b(m+5)$, i.e. if $n \geq (m+4)(m+5)$, then

$$p_n > p_{m+4}^a p_{m+5}^b > C_2^{a+b}(r+\delta)^{-(am+b(m+1))} > C_3^{a+b}(r+\delta)^{-(a(m+4)+b(m+5))} > C_3(r+\delta)^{-n}.$$

□

III.2. Asymmetry of diagrams and consequences

The following theorem of Richmond and Wormald [6] provides a sufficient set of criteria for almost all elements of $\mathscr{K}$ to have trivial automorphism group.

Theorem 4 (Richmond-Wormald 1996)**.** *Let $\mathscr{C}$ be a class of rooted maps on a surface. Suppose that there is an outer-cyclic rooted planar map M_1 such that in all maps in $\mathscr{C}$, all copies of M_1 are pairwise disjoint, and such that*

1. *M_1 has no reflective symmetry in the plane preserving the unbounded face,*

2. *there exist constants $c > 0$ and $d < 1$ such that the proportion of n-vertex maps in $\mathscr{C}$ that do not contain at least cn pairwise disjoint copies of M_1 is at most d^n for n sufficiently large (M is "ubiquitous"), and*

3. *for any map M in $\mathscr{C}$ containing a copy of M_1, all maps obtainable by removing M_1 and gluing it back in to the same face are in $\mathscr{C}$ (M is "free").*

Then the proportion of n-vertex maps in $\mathscr{C}$ with nontrivial automorphisms is exponentially small.

We will prove this for $\mathscr{K}$ by proving it for its dual $\mathscr{K}^*$, a class of quadrangulations of the sphere. Specifically, we will take M_1 to be the dual of the underlying planar map of the following 2-tangle: Clearly M_1 has no reflective symmetry by inspection, and certainly any

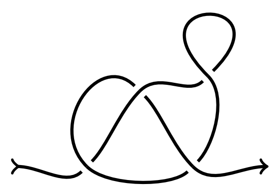

FIG. 10. The dual 2-tangle M_1, and its representation as a 2-tangle.

of the ways of replacing M_1 keep the object in the class of quadrangulations dual to knot maps. Finally, the ubiquity condition is exactly the pattern theorem for 2-tangles proved in the prior section!

Application of the above theorems provides us with the following corollary which enables us to transfer any asymptotic results on rooted diagrams to unrooted diagrams.

Corollary 5. *Let L be a uniform random variable taking values in the space $\mathcal{K}_n$ or $\mathcal{L}_n$. Then there exist constants $C, \alpha > 0$ so that $\mathbb{P}(\operatorname{aut} L \neq 1) < Ce^{-\alpha n}$. Hence, rooted diagrams behave like unrooted diagrams.*

Indeed, link diagrams with n vertices are dual to quadrangulations with $n+2$ faces; there are $n + 2$ ways of choosing the "exterior" root face and then 4 ways of rooting the edges around this chosen face. Hence if $\tilde{\ell}_n$, $\tilde{k}_n$ are the counts of unrooted link or knot diagrams we have that in the limit,

$$\tilde{\ell}_n \underset{n\to\infty}{\sim} \frac{\ell_n}{4(n+2)} \text{ and } \tilde{k}_n \underset{n\to\infty}{\sim} \frac{k_n}{4(n+2)}.$$

Corollary 6. *A random knot or link diagram has the pattern theorem. Namely, a random knot diagram is almost surely composite and almost surely knotted, and a random link diagram is almost surely not a knot diagram.*

IV. NUMERICAL RESULTS ON KNOTTING

One may be concerned that the "asymptotic" behavior proved in the prior section only applies to knot diagrams with an absurd number of crossings (in the sense that no physical knot should be expected to be so complicated). However, exact and numerical results show that this behavior is attained very quickly. For example, almost all 10-crossing knot diagrams have no nontrivial automorphisms!

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