CS 310, Assignment 1

Answers

1. Let $\Sigma = \{a, b\}$ and consider languages $A = \{ba, bb, b\}$ and $B = \{ba, aa, a\}$.

   (a) Write down all strings in Σ^* that have length at most two.

   ANSWER:

   ε, a, b, aa, ab, ba, bb

   (b) How many strings are in $A \cdot B$? Write down all of them.

   ANSWER:

   $A \cdot B = \{baba, baaa, baa, bbba, bbaa, bba, bba, baa, ba\}$. The language consists of 7 strings rather than 9. Indeed, both baa and bba have two different decompositions as a concatenation of strings from A and B; however a language is a set, and duplicates are ignored in a set. In other words, actually $A \cdot B = \{baba, baaa, baa, bbba, bbaa, bba, ba\}$.

   (c) How many strings are in $B \cdot A$? Write down all of them.

   ANSWER:

   $B \cdot A = \{baba, babb, bab, aaba, aabb, aab, aba, abb, ab\}$. The language consists of 9 (distinct) strings.

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2. Let $R = (10^*1 + 010^*10)^*$ and $S = (0^*10^*10^*)^*$, both over $\Sigma = \{0, 1\}$.

   (a) Give an example of a string z that is both in R and in S (that is, $z \in R \cap S$).

   ANSWER:

   Many such strings exist including ε, 11, 1001, etc.

(b) Is it possible to find a string x that is in R and is not in S (that is, $x \in R \cap \overline{S}$)? If yes, write it down; if not explain briefly why.

ANSWER:

No. Every string in R is also in S, meaning that $R \cap \overline{S} = \emptyset$. Indeed, note first that S contains all the strings with an even number of 1s. In turn, both 10^*1 and 010^*10 contain exactly two 1s, so any string in the closure of the two will certainly contain an even number of 1s.

(c) Is it possible to find a string y that is in S and is not in R (that is, $y \in S \cap \overline{R}$)? If yes, write it down; if not explain briefly why.

ANSWER:

Note that if a string from R that contains only two occurrences of 1 must start and end with the same symbol (either 0 or 1). Any string containing two 1s and starting and ending with different symbols will thus be a possible value for y (meaning a member of $S \cap \overline{R}$). Possible such strings include 1010, 0101, 010001, etc.

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3. Show how to define the following languages over $\Sigma = \{0, 1\}$ using only ε, the alphabet symbols 0 and 1, and the operations of union, concatenation, and closure.

(a) All strings that begin with 1 *and* end with 01.

ANSWER:

$1(0+1)^*01$

(b) All strings that have both 01 and 10 as substrings. Note that the substrings can occur in either order and possibly overlap.

ANSWER:

The possible variants are: 01 before 10, 10 before 01, one occurrence of 101 (the two substrings are overlapped), and one occurrence of 010 (ditto, but in a different order). The most straightforward answer is therefore the following regular expression: $(0+1)^*01(0+1)^*10(0+1)^* + (0+1)^*10(0+1)^*01(0+1)^* + (0+1)^*101(0+1)^* + (0+1)^*010(0+1)^*$

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4. Describe the language L defined by each of the following equations. Explain briefly.

(a) $L = 1 + La + aL$

ANSWER:

We can apply the method presented in class:

$$\begin{aligned}
L_0 &= \emptyset \\
L_1 &= 1 \\
L_2 &= \{1, 1a, a1\} \\
L_3 &= \{1, a1, 1a, 1aa, a1a, aa1\} \\
L_4 &= \{1, a1, 1a, 1aa, a1a, aa1, 1aaa, a1aa, a1aa, aa1a, aa1a, aaa1\} \\
&\dots
\end{aligned}$$

We note that all the strings consists of a's and exactly one 1. This is consistent with the recursive definition, that introduces one 1 (in the base case) and any number of a's (in one of the two recursive terms). Furthermore, we note that the 1 symbol can appear anywhere in the string. Again, this is consistent with the recursive definition which can introduce a's equally on both sides of the one introduced by the base case. Overall the language appears to be a string containing exactly one 1 and any number of a's in any combination. This is equivalent to the regular expression a^*1a^*.

We now proceed to show that a^*1a^* is indeed a solution for the recursive equation. We note that a string in a^*1a^* can have either no a's, at least one a to the left of the 1, or at least one a to the right of the 1. No other variant is possible. That is,

$$a^*1a^* = 1 + a^*1a^*a + aa^*1a^*$$

This proves that $L = a^*1a^*$.

(b) $L = L + L$

ANSWER:

Union is an idempotent operation, so the equation is just a fancy way of writing $L = L$. It is quite obvious that any set is equal to itself, so *any* set is a possible solution to this equation.

Note that the method presented in class reports that $L = \emptyset$, which is normal since the said algorithm will give the *smallest* solution (and $\emptyset$ is the smallest set of them all).
