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HW 4
Problem 3

Let e_i be the edge we choose in the ith iteration during the greedy algorithm. Now look at the algorithm at the ith iteration, when $i \leq n-1$. At the beginning of the ith iteration, we have already visited i node, thus there are $n-i$ edges to choose. We choose e_i because it has the minimal weight among this $n-i-1$ edges. Since the weight of the edges are assigned uniformly and independently from the interval $[0,1]$,

$$\begin{aligned} Pr(\mathrm{e}_i = x) &= Pr(\text{other n-i-1 edges has weight greater than x}) \\ &= (1-x)^{n-i-1} \end{aligned}$$

Thus, for $i \leq n-1$,

$$\begin{aligned} E[w(e_i)] &= \int_0^1 x(1-x)^{n-i-1} \\ &= \frac{1}{(n-i-1)^2+3(n-i-1)+2} \\ &= \frac{1}{n-i} - \frac{1}{n-i+1} \end{aligned}$$

At the last iteration, we have to choose the edge back to the first node. The weight of the the edge is random, thus $E[w(e_n)] = \frac{1}{2}$

Therefore , the expected weight of the whole Hamiltonian path is

$$\begin{aligned} E[path] &= E[\textstyle\sum_{i=1}^{n} w(e_i)] \\ &= \textstyle\sum_{i=1}^{n} E[w(e_i)] \\ &= \textstyle\sum_{i=1}^{n-1} \frac{1}{n-i} - \frac{1}{n-i+1} + \frac{1}{2} \end{aligned}$$

$$= \frac{3}{2} - \frac{1}{n}$$