In what follows, p will be a positive integer prime.

We are considering $\mathcal{A}$, the set of n such that $n - p^2$ is square-free for all $p \le \sqrt{n}$ and $A(N)$ which counts the elements of $\mathcal{A} \le N$.

We are perturbed that our naive approach grossly underestimated $A(N)$. The first observation is that we have ruled out those $n \equiv 1 \mod 4$ but we should also rule out $n \equiv 0 \mod 4$ since $4|(n - 2^2)$. This actually makes things worse as our naive estimate for $A(2^{23})$ would now be 0.8 in place of 1.2 remembering that the actual figure is 9 384.

The issue is, I think, that we are assuming that the probability the n is square free, say $P(n)$ to be $1/\zeta(2)$ regardless of the size of n. This argument follows from

$$P(n) = \prod_{p \ge 2} \left(1 - \frac{1}{p^2}\right) = \frac{1}{\zeta(2)}.$$

However, this is not tight unless n is large, for some definition of "large". A better estimate is

$$P_1(n) = \prod_{p=2}^{\sqrt{n}} \left(1 - \frac{1}{p^2}\right) = \frac{1}{\zeta(2) \prod_{p > \sqrt{n}} \left(1 - \frac{1}{p^2}\right)}.$$

Now if $n \equiv 2 \mod 4$ or $n \equiv 3 \mod 4$ then $4 \nmid (n - p^2)$ so we need to adjust our probability to

$$P_2(n) = \prod_{p=3}^{\sqrt{n}} \left(1 - \frac{1}{p^2}\right) = \frac{4}{3} P_1(n).$$

Thus the chance of a random $n \not\equiv \{0, 1\} \mod 4$ being in $\mathcal{A}$ is

$$P_A(n) = \prod_{p=2}^{\sqrt{n}} P_2(n - p^2)$$

and the expectation for $A(N)$ is

$$\sum_{n=2}^{N/4} P_A(4n + 2) + P_A(4n + 3).$$

Lemma 0.1.

$$\prod_{p > x} \left(1 - \frac{1}{p^2}\right)^{-1} = \exp\left(\sum_{p > x} \sum_{j \ge 1} \frac{1}{jp^{-2j}}\right).$$

0.1 Some computations

Running a simple single thread program in Pari even my puny desktop can get to $N = 2^{30}$ overnight (actually a little under 10 hours). Using 4 threads I went to 2^{31}. The largest member of $\mathcal{A}$ found was 581 424 623 so by the time we reached 2^{31}, we hadn't seen one for over 1 560 000 000 and I conjecture that there aren't any more.

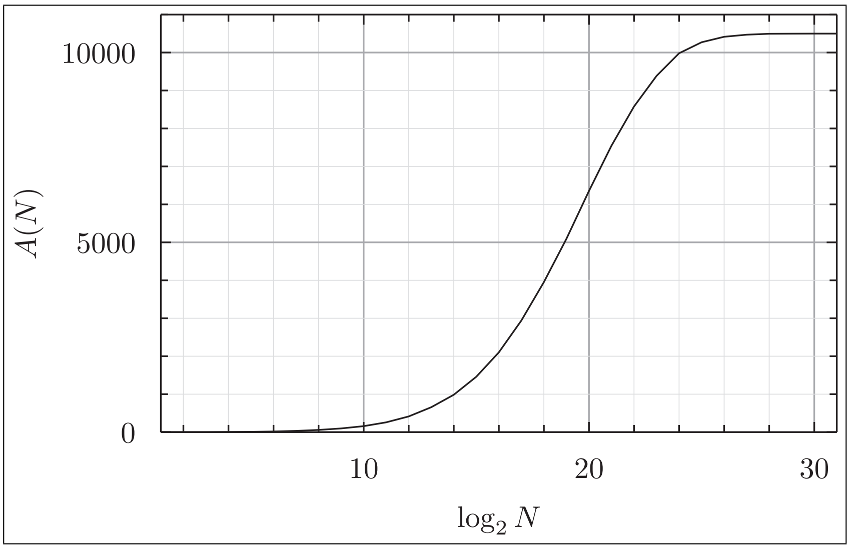

Figure 1: Plot of $A(N)$.