On a Conjecture of Louboutin

David J. Platt
Heilbronn Institute for Mathematical Research
University of Bristol, Bristol, UK
dave.platt@bris.ac.uk

Abstract

Here is a computational challenge that you might be able to tackle. Let χ range over the $(p-1)/2$ odd Dirichlet character mod a large prime p. For $k \geq 1$ a positive integer, set

$$S_{2k}^{-}(p) := \sum_{\chi \in X_p^-} |\theta(1,\chi)|^{2k}$$

where

$$\theta(x,\chi) = \sum_{n \geq 1} n\chi(n) e^{-\pi n^2 x/p}.$$

My student and I proved (see the reprint below):

$$S_2^-(p) \sim \frac{p^{5/2}}{16\pi\sqrt{2}}$$

and

$$S_4^-(p) \sim \frac{3p^4 \log p}{512\pi^3}$$

(asympotics as p goes to infinity).

It seems reasonable to conjecture that for each k there exist $C_k > 0$ and $f(k)$ quadratic in k such that

$$S_{2k}^-(p) \sim C_k p^{1+3k/2} \log^{f(k)} p.$$

We tried some numerical computation to guess what $f(k)$ should be but could not go far enough to be confident that $f(k) = (k-1)^2$ should be the right answer.

Do you think you could do much more extended computations to guess the right answer for $f(k)$?

The idea is that approximating each $\theta(1,\chi)$ by the finite sum $\sum_{1\le n\le p-1} n\chi(n)e^{-\pi n^2/p}$ is much more that enough to compute S_{2k}^{-} accurately.