Homework #1

Due on February 11th, 2020
Machine Learning
CS559WS−Spring 2020
Professor In Suk Jang

Daniel Kadyrov
10455680

Problem 1

By using a change of variables, verify that the univariate Gaussian distribution given by:

$$N(x \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\{-\frac{1}{2\sigma^2}(x-\mu)^2\}$$

satisfies $E(x) = \mu$. Next, by differentiating both sides of normalization condition

$$\int_{-\infty}^{-\infty} N(x \mid \mu, \sigma^2) dx = 1$$

with respect to σ^2, verify that the Gaussian satisfies $E(x^2) = \mu^2 + \sigma^2$.

Solution

$$E(x) = \int_{-\infty}^{\infty} p(x|\mu, \sigma^2) x dx = E(x) = \int_{-\infty}^{\infty} \frac{x}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx$$

$$E(x) = \int_{-\infty}^{\infty} \frac{x-\mu}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2\sigma^2}\right\} dx = \int_{-\infty}^{\infty} \frac{x}{\sigma\sqrt{2\pi}} \exp\{-\frac{x^2}{2\sigma^2}\} dx - \int_{-\infty}^{\infty} \frac{\mu}{\sigma\sqrt{2\pi}} \exp\{-\frac{x^2}{2\sigma^2}\} dx$$

The first part of the integral:

$$-\int_{0}^{-\infty} x \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2\sigma^2}\right\} dx + \int_{0}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2\sigma^2}\right\} dx =$$

$$\int_{0}^{\infty} (-x) \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(-x)^2}{2\sigma^2}\right\} dx + \int_{0}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2\sigma^2}\right\} dx =$$

$$-\int_{0}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2\sigma^2}\right\} dx + \int_{0}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2\sigma^2}\right\} dx = 0$$

So now the integral is:

$$E(x) = \int_{-\infty}^{\infty} \mu \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2\sigma^2}\right\} dx = \frac{2\mu}{\sqrt{\pi}} \int_{0}^{\infty} e^{-x^2} dx$$

$$E(x) = \lim_{i\to\infty} \frac{2\mu}{\sqrt{\pi}} \int_{0}^{i} e^{-x^2} dx = \mu \lim_{i\to\infty} \operatorname{erf}(i) = \mu \Leftarrow$$

$$E(x) = \mu \rightarrow E(x)^2 = \mu^2$$

$$E(x^2) - E(x)^2 = E(x^2) - \mu^2 = \sigma^2$$

$$E(x^2) = \mu^2 + \sigma^2 \Leftarrow$$

Problem 2

Use $E(x) = \mu$ to prove $E(xx^T) = \mu\mu^T + \Sigma$. Now, using the results two definitions, show that:

$$E[x_n x_m] = \mu\mu^T + I_{nm}\Sigma$$

where x_n denotes a data point same from a Gaussian distribution with mean μ and covariance Σ,and I_{nm} denotes the (n, m) element of the identity matrix. Hence prove the result (2.124)

Solution

$$\begin{aligned}
\Sigma &= E[(x - E[x])(x - E[x])^T] \\
&= E[xx^T - xE[x]^T - E[x]x^T + E[x]E[x]^T] \\
&= E[xx^T - x\mu^T - ^T + \mu\mu^T] \\
&= E[xx^T] - \mu\mu^T - \mu\mu^T + \mu\mu^T \\
&= E[xx^T] - \mu\mu^T \\
E[xx^T] &= \mu\mu^T + \Sigma \Leftarrow
\end{aligned}$$

Since $I_{nm}\Sigma = \Sigma I_{nm} = \Sigma$:

$$E[xx^T] = \mu\mu^T + I_{nm}\Sigma$$

Problem 3

Consider a linear model of the form:

$$f(x, w) = w_0 + \sum_{i=1}^{D} w_i x_i$$

together with a sum of squares/loss function of the form:

$$L_D(w) = \frac{1}{2} \sum_{n=1}^{N} (f(x_n, w) - y_n)^2$$

Now suppose that Gaussian noise ϵ_i with zero mean and variance σ^2 is added independently to each of the input variables x_i. By making use of $E[\epsilon_i] = 0$ and $E[\epsilon_i \epsilon_j] = \delta_{ij}\sigma^2$, show that minimizing L_D averaged over the noise distribution is equivalent to minimizing the sum of square error for noise-free input variables with the addition of a weight-decay regularization term, in which the bias parameters w_0 is omitted from the regularizer.

Solution

$$y_n(x_n, w) = w_0 + \sum_{i=1}^{D} w_i(x_{ni} + \epsilon_{ni}) = w_0 + \sum_{i=1}^{D} w_i x_{ni} + \sum_{i=1}^{D} w_i \epsilon_{ni} = y(x_n, w) + \sum_{i=1}^{D} w_i$$

$$E_{dn}(w) = \frac{1}{2} \sum_{n=1}^{N} (y_n(x_n, w) - t_n)^2 = \frac{1}{2} (y(x_n, w) + \sum_{i=1}^{D} w_d \epsilon_n i - t_n)^2$$

$$= \frac{1}{2} \sum_{n=1}^{N} ((y(x_n, w) - t_n)^2 + 2(y(x_n, w) - t_n)(\sum_{i=1}^{D} w_d \epsilon_{ni}) + (\sum_{i=1}^{D} w_d \epsilon_{ni})^2)$$

$$\mathbb{E}(L_{Dn}(w)) = \frac{1}{2} \sum_{n=1}^{N} ((y(x_n, w) - t_n)^2 + 2(y(x_n, w) - t_n)(\sum_{i=1}^{D} w_d \mathbb{E}(\epsilon_{ni})) + \mathbb{E}((\sum_{i=1}^{D} w_d \epsilon_{ni})^2))$$

$\mathbb{E}(\epsilon_{ni}) = 0$ so:

$$\mathbb{E}((\sum_{i=1}^{D} w_d \epsilon_{ni})^2) = \mathbb{E}(\sum_{i_1=1}^{D} \sum_{i_2=1}^{D} w_{i_1} w_{i_2} \epsilon_{ni_1} \epsilon_{ni_2})$$

$$= \sum_{i_1=1}^{D} \sum_{i_2=1}^{D} w_{i_1} w_{i_2} \mathbb{E}(\epsilon_{ni_1} \epsilon_{ni_2})$$

$$= \sum_{i_1=1}^{D} \sum_{i_2=1}^{D} w_{i_1} w_{i_2} \delta_{i_1 i_2}$$

$$= \sum_{i=1}^{D} w_d^2$$

$$\mathbb{E}(L_{Dn}(w)) = \frac{1}{2} \sum_{n=1}^{N} ((y(x_n, w) - t_n)^2 + \sum_{i=1}^{D} w_d^2) = L_D(w) + \frac{N}{2} \sum_{i=1}^{D} w_d^2 \Leftarrow$$

Problem 4

UCI Machine Learning: Bike Sharing Data Set
Build at least four regression models (e.g., linear, polynomial, non-linear) to predict the count of total rental bikes including both casual and registered. Explore data to reduce the number of features. Use K-fold cross validation and report the mean squared error (MSE) on the testing data. You need to write down every step in your experiment.

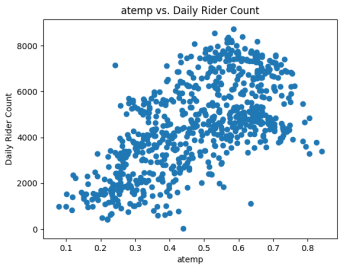
Solution

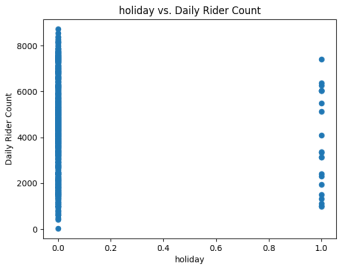
The necessary packages and files are imported into the Python script:

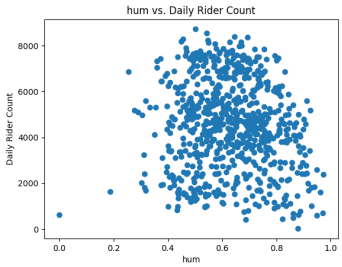
```
10  import pandas as pd
11  from matplotlib import pyplot as plt
12  from sklearn.linear_model import LinearRegression, Ridge, Lasso, BayesianRidge
13  from sklearn.model_selection import KFold
14  from sklearn.metrics import mean_squared_error
15  import numpy as np
16
17  hour = pd.read_csv("hour.csv")
18  day = pd.read_csv("day.csv")
19
20  cnt_hour = hour["cnt"].values.reshape(-1, 1)
21  cnt_day = day["cnt"].values.reshape(-1, 1)
```

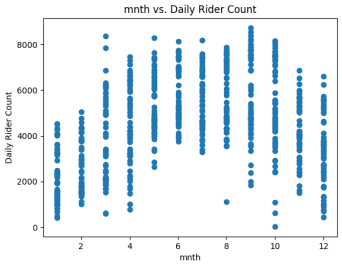
The data is explored to examine it's features. Both hourly and daily rider data is examined. The graphs outputted for the daily and hourly ridership are shown on the next pages, respectively.

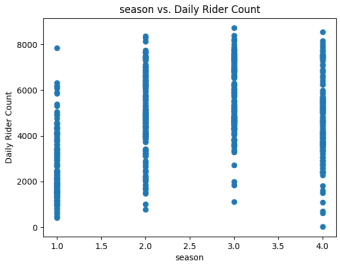
```
1   for col in range(len(hour.columns)):
2       if hour.columns[col] not in ["instant", "cnt", "dteday", "registered", "casual"
            ]:
3           plt.figure()
4           plt.scatter(hour[hour.columns[col]].values.reshape(-1, 1), cnt_hour)
5           plt.title("{} vs. Total Rider Hourly Count".format(hour.columns[col]))
6           plt.xlabel("{}".format(hour.columns[col]))
7           plt.ylabel("{}".format("Total Rider Count"))
8           plt.savefig("images/hour_{}.png".format(hour.columns[col]))
9           plt.clf()
10
11  for col in range(len(day.columns)):
12      if day.columns[col] not in ["instant", "cnt", "dteday", "registered", "casual"
            ]:
13          plt.figure()
14          plt.scatter(day[day.columns[col]].values.reshape(-1, 1), cnt_day)
15          plt.title("{} vs. Total Rider Daily Count".format(day.columns[col]))
16          plt.xlabel("{}".format(day.columns[col]))
17          plt.ylabel("{}".format("Total Rider Count"))
18          plt.legend()
19          plt.savefig("images/day_{}.png".format(day.columns[col]))
20          plt.clf()
```

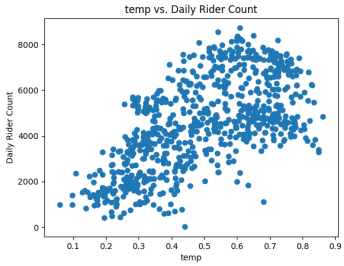

atemp vs. Daily Rider Count
Daily Rider Count
atemp

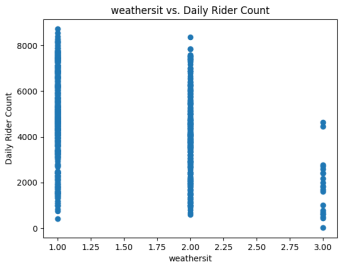

holiday vs. Daily Rider Count
Daily Rider Count
holiday

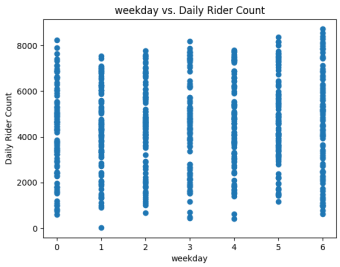

hum vs. Daily Rider Count
Daily Rider Count
hum

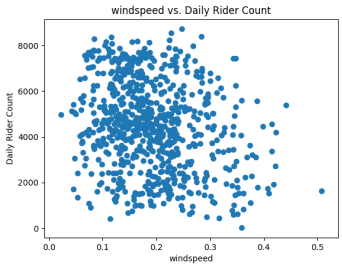

mnth vs. Daily Rider Count
Daily Rider Count
mnth

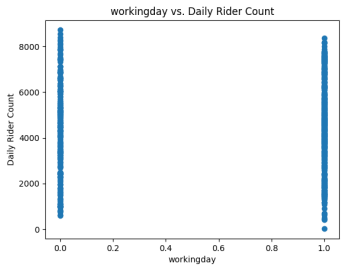

season vs. Daily Rider Count
Daily Rider Count
season

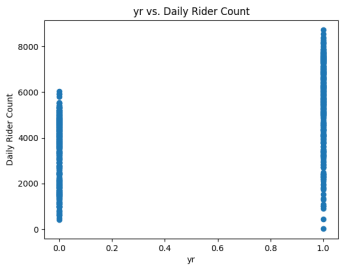

temp vs. Daily Rider Count
Daily Rider Count
temp

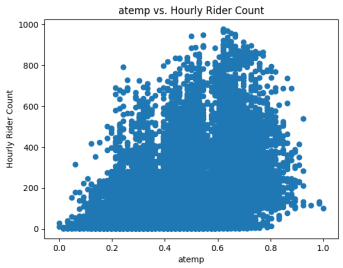

weathersit vs. Daily Rider Count
Daily Rider Count
weathersit

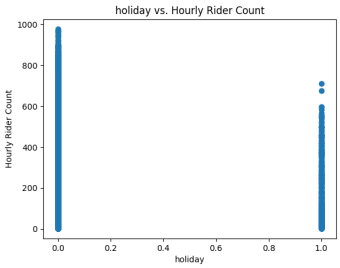

weekday vs. Daily Rider Count
Daily Rider Count
weekday

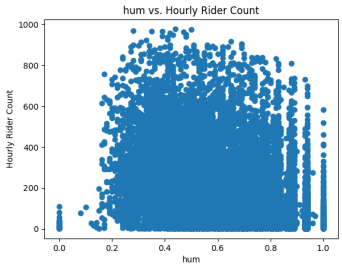

windspeed vs. Daily Rider Count
Daily Rider Count
windspeed

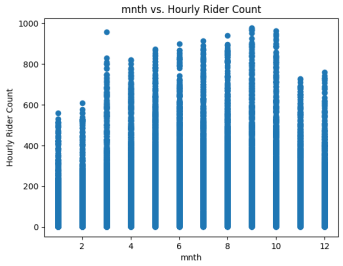

workingday vs. Daily Rider Count
Daily Rider Count
workingday

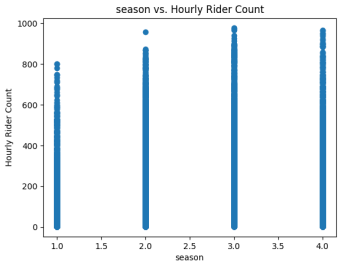

yr vs. Daily Rider Count
Daily Rider Count
yr

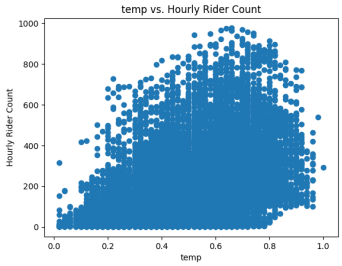

atemp vs. Hourly Rider Count
Hourly Rider Count
atemp

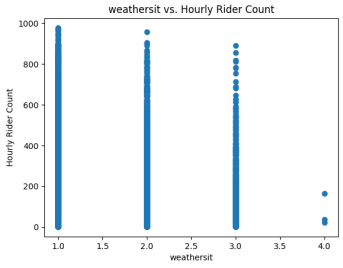

holiday vs. Hourly Rider Count
Hourly Rider Count
holiday

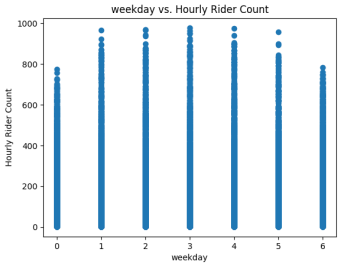

hum vs. Hourly Rider Count
Hourly Rider Count
hum

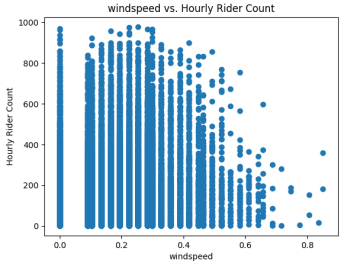

mnth vs. Hourly Rider Count
Hourly Rider Count
mnth

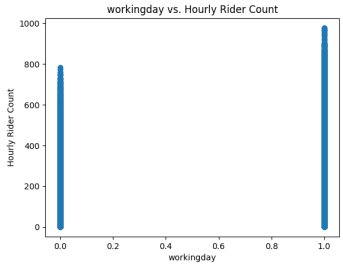

season vs. Hourly Rider Count
Hourly Rider Count
season

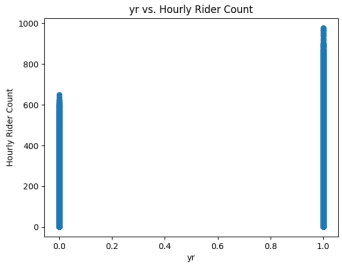

temp vs. Hourly Rider Count
Hourly Rider Count
temp


weathersit vs. Hourly Rider Count
Hourly Rider Count
weathersit


weekday vs. Hourly Rider Count
Hourly Rider Count
weekday


windspeed vs. Hourly Rider Count
Hourly Rider Count
windspeed


workingday vs. Hourly Rider Count
Hourly Rider Count
workingday


yr vs. Hourly Rider Count
Hourly Rider Count
yr

Based on the feature graphs, temperature was reduced to the best feature to model. According to the README the temperature was normalized with a 41 maximum so the temperature array is multiplied by 41 for human readability.

```
day_temp = day["temp"].values.reshape(-1, 1) * 41
```

KFold cross-validation was used with a 10-split. The regression models applied were linear, ridge, lasso, and bayesian through their respective `sklearn` packages. For each k-fold, the models are fitted using the separated training data.

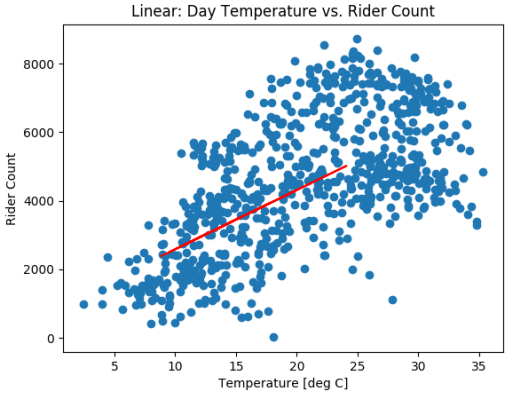
```
crossvalidation = KFold(n_splits=10)
linear = LinearRegression()
ridge = Ridge()
lasso = Lasso()
bayesian = BayesianRidge()

for train_index, test_index in crossvalidation.split(day_temp):
    X_train, X_test = day_temp[train_index], day_temp[test_index]
    y_train, y_test = cnt_day[train_index], cnt_day[test_index]
    linear.fit(X_train, y_train)
    ridge.fit(X_train, y_train)
    lasso.fit(X_train, y_train)
```

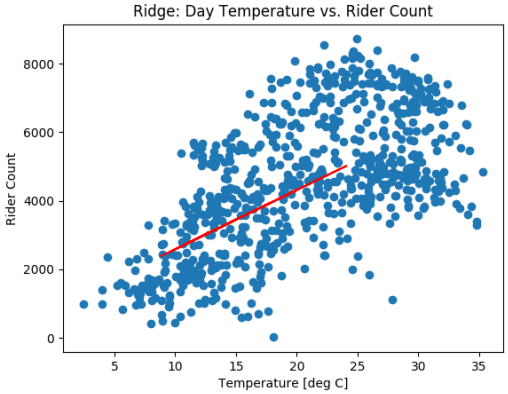
The test portion of the data was then used to test the prediction quality of the models. The mean square error was calculated between the test portion data and the values predicted by the models.

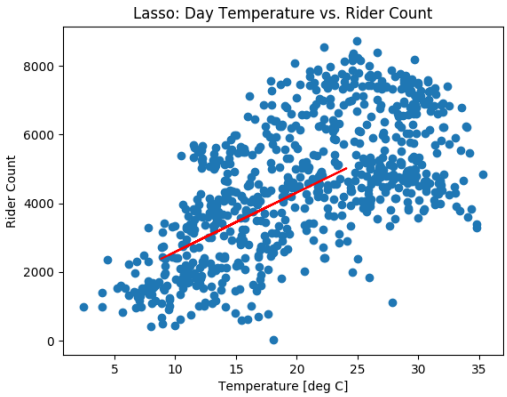
```
linear_pred = linear.predict(X_test)
ridge_pred = ridge.predict(X_test)
lasso_pred = lasso.predict(X_test)
bayesian_pred = bayesian.predict(X_test)

mse_linear = mean_squared_error(y_test, linear_pred)
mse_ridge = mean_squared_error(y_test, ridge_pred)
mse_lasso = mean_squared_error(y_test, lasso_pred)
mse_bayesian = mean_squared_error(y_test, bayesian_pred)
```

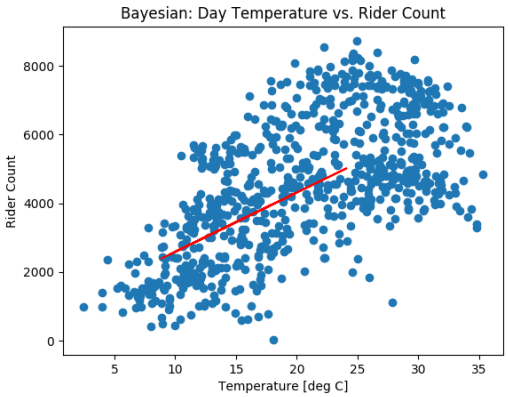
The following a table showing the different mean squared error values of the models.

Model	Mean Squared Error
Linear	4306257.419012716
Ridge	4306194.805960745
Lasso	4306018.563140404
Bayesian	4301889.80709847


Linear: Day Temperature vs. Rider Count
Rider Count
Temperature [deg C]


Ridge: Day Temperature vs. Rider Count
Rider Count
Temperature [deg C]


Lasso: Day Temperature vs. Rider Count
Rider Count
Temperature [deg C]


Bayesian: Day Temperature vs. Rider Count
Rider Count
Temperature [deg C]

Problem

UCI Machine Learning: Iris Data Set

a. Implement Linear Discriminant Analysis for each pair of the classes and report your results. Note that there are three class labels in the data set. Write down each step of your solution.

b. Perform the kNN classification for each k value from 1 to 50 to predict the species. For each k value, compute the percentage of misclassified values on the testing set. Print out your results as a table showing the values of k and the misclassification percentages. Then plot the misclassification rates on the testing set versus the k values.

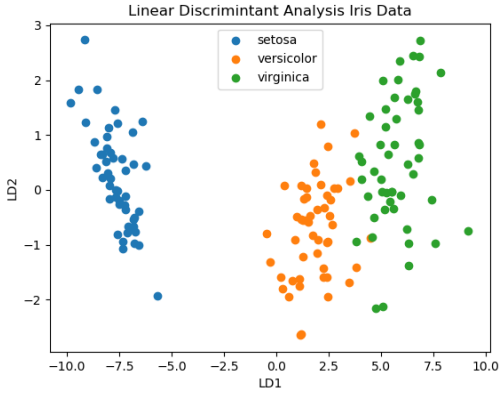
Solution

a. The necessary packages and data were imported.

```
import matplotlib.pyplot as plt

from sklearn import datasets
from sklearn.discriminant_analysis import LinearDiscriminantAnalysis as LDA

iris = datasets.load_iris()

X = iris.data
y = iris.target
names = iris.target_names
```

The linear discriminant analysis function from the Sklearn package was used to implement the analysis.

```
lda = LDA(n_components=2)
model = lda.fit(X, y).transform(X)

plt.figure()
for i, name in zip([0, 1, 2], names):
    plt.scatter(model[y == i, 0], model[y == i, 1], label=name)
plt.legend()
plt.xlabel("LD1")
plt.ylabel("LD2")
plt.show()
```



b. The data was split to accommodate for a 20% testing set.

```
14  X_train, X_test, y_train, y_test = train_test_split(X, y, test_size=0.2)
15
16  k_range = range(1,51)
17
18  scores = {}
19  scores_list = []
20  print("K Value | Accuracy Score")
21  for k in k_range:
22      knn = KNeighborsClassifier(n_neighbors=k)
23      knn.fit(X_train, y_train)
24      pred = knn.predict(X_test)
25      scores[k] = metrics.accuracy_score(y_test, pred)
26      scores_list.append(scores[k])
27      print("%i | %f"%(k, scores[k]))
28
29  plt.figure()
30  plt.plot(k_range, scores_list)
31  plt.title("Misclassification Rate")
32  plt.xlabel("KNN K Value")
33  plt.ylabel("Accuracy")
34  plt.show()
```

K Value	Accuracy Score								
1	0.966667	11	0.966667	21	0.966667	31	0.966667	41	0.966667
2	0.933333	12	0.966667	22	0.966667	32	0.966667	42	0.933333
3	0.966667	13	0.966667	23	0.966667	33	0.966667	43	0.966667
4	0.966667	14	0.966667	24	0.966667	34	0.966667	44	0.933333
5	0.966667	15	0.966667	25	0.966667	35	0.966667	45	0.933333
6	0.933333	16	0.966667	26	0.966667	36	0.966667	46	0.933333
7	0.933333	17	0.966667	27	0.966667	37	0.966667	47	0.966667
8	0.966667	18	0.966667	28	0.966667	38	0.933333	48	0.933333
9	0.966667	19	0.966667	29	0.966667	39	0.966667	49	0.966667
10	0.966667	20	0.966667	30	0.966667	40	0.933333	50	0.933333