Arc-To-Chord Minimum Curvature

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Abstract

For accurate geologic positioning and reservoir management, prevention of waste and protection of correlative rights, and to mitigate health, safety and environmental issues, a simpler alternative construct of the minimum curvature survey calculation method is presented, the arc-to-curve method, as is a refactored spherical linear interpolation method, SLERP, for interpolating the wellbore at arbitrary positions along its path. Handling of an edge case when converting 3D cubic coordinates to spherical coordinates is also presented. Building on the work of Sawaryn and Thorogood [S&T], emphasis is placed on working in a coordinate-free vector reference frame, and relying on the guarantees of IEEE 754 standard [754] for floating point math to simplify these algorithms while maintaining their accuracy, robustness and repeatability. These simplifications help minimize implementation errors of these safety-critical calculations.

Introduction

The minimum curvature survey method is the modern accepted method for calculating the position of a subsurface wellbore [S&T], such as those that are drilled to extract hydrocarbons from deep rock layers, as the minimum curvature method well models portions of the path of a wellbore as a 3D circular arc, an arc developed on the surface of a sphere in the plane cutting the sphere containing the arc, that plane defined by tangent vectors to the arc at the start and the end points of the arc, **Figure 1**. The geometry of the constructed wellbore path is then well modeled as a sequence of connected consecutive 3D circular arcs where the end of a previous arc shares the same position and direction as the next, a 3D circular arc space curve, **Figure 2**. Such a space curve is described as having G1 continuity because its connected arcs share a common tangent direction vector where they meet, but two such connected arcs need not have the same curvature.

Minimum Curvature Derivation

Given survey readings,

$$\begin{aligned} MD &= measured\ depth \\ \theta &= inclination \\ \emptyset &= azimuth, \end{aligned}$$

at two consecutive survey stations[1], define the unit length vector, $\bar{\boldsymbol{t}}$, Equation 1, tangent to the circular arc at a station, and the course length, S_{12}, Equation 2, the difference in measured depth between two consecutive stations, **Figure 3**,

[1] In this note, the domains of the inclinations are strictly $0 \leq \theta < \pi$, and the azimuths, $0 \leq \emptyset < 2\pi$.

$$\bar{\boldsymbol{t}} = \begin{bmatrix} \Delta\mathrm{N} \\ \Delta\mathrm{E} \\ \Delta\mathrm{V} \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\emptyset \\ \sin\theta\sin\emptyset \\ \cos\theta \end{bmatrix} \tag{1}$$

$$S_{12} = MD_2 - MD_1. \tag{2}$$

The problem to solve is, having the survey readings at the start and end of a wellbore segment and knowing the $[N, E, V]$ position-coordinates of the start of the arc, to find the $[N, E, V]$ position-coordinates at the end of the arc. In developing an algorithm for minimum curvature, Sawaryn and Thorogood present equations [S&T] for the angle, $\propto$, Equation 3, subtended by the tangent unit vectors at consecutive survey stations and a shape factor, $f(\propto)$, Equation 4, as,

$$\propto = 2\sin^{-1}\left\{\left[\sin^2\left(\frac{\theta_2 - \theta_1}{2}\right) + \sin\theta_1\sin\theta_2\sin^2\left(\frac{\emptyset_2 - \emptyset_1}{2}\right)\right]^{\frac{1}{2}}\right\} \tag{3}$$

$$f(\propto) = \frac{\tan\left(\frac{\propto}{2}\right)}{\left(\frac{\propto}{2}\right)}. \tag{4}$$

The equation for $\propto$, attributed to Lubinski [D20], is designed to avoid taking the inverse cosine, arccosine, of the dot product to calculate the angle between the tangent vectors, as is very common in many online references for calculating minimum curvature. It is well known [KHN] that taking the arccosine of the dot product [DOT] when the vectors are near (anti-)parallel, that is, when the angle between the vectors nears (π)zero, suffers serious rounding errors and is unstable because the first derivative of the arccosine is asymptotically vanishing, **Figure 4**. The latter means small errors in the input to the arccosine function are non-linearly exacerbated on output[2] [DMB].

Sawaryn and Thorogood present equation 5 [S&T] for minimum curvature when the angle between the tangent vectors is greater than 0.02 radians,

$$\boldsymbol{P}_2 = \boldsymbol{P}_1 + \frac{S_{12} f(\propto)}{2}\left(\bar{\boldsymbol{t}}_1 + \bar{\boldsymbol{t}}_2\right), \tag{5}$$

where $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ are the start and end points in [N,E,V] coordinates of the arc defined by its tangents and the course length swept out between the points, $f(\propto)$ is a shape factor, and $\left(\bar{\boldsymbol{t}}_1 + \bar{\boldsymbol{t}}_2\right)$ is the bisector of the angle between the tangents, a vector pointing from $\boldsymbol{P}_1$ to $\boldsymbol{P}_2$. The form of Equation 5 is recognized as a scaling of the balanced tangential-survey method, **Figure 5**, where

[2] Given the propensity of drillers to want to drill straight wellbores and the inherent uncertainty of survey measurements, the circular arc model nearing degenerate orientation is quite common so any amplified errors are potentially non-trivial.

the shape factor presented in Equation 4, $f(\propto)$, derived from the well-known Weierstrass tangent half-angle substitution, scales the balanced tangential isosceles triangle such that the arc length of the circular arc fit to the tangents at the end points is equal to the measured course length. When straight-hole conditions are approached, when $\propto$ is less than or equal to 0.02 radians, the scale factor switches to a series expansion, Equation 6, [S&T] to handle the numeric issues with and to smoothly transition to the degenerate case when the vectors are near or at parallel,

$$f(\propto) \approx 1 + \frac{\propto^2}{12}\left\{1 + \frac{\propto^2}{10}\left[1 + \frac{\propto^2}{168}\left(1 + \frac{31 \propto^2}{18}\right)\right]\right\}. \tag{6}$$

Arc-To-Chord Derivation

This note develops a simplified, more accurate and stable algorithm, the arc-to-chord method for minimum curvature. First, the formula for the angle between the tangent vectors is replaced with one advocated by Kahn [KHN] , Equation 7, and evaluated by Baker in [DMB],

$$\propto = 2\tan^{-1}\left(\frac{\left\|\bar{\boldsymbol{t}}_2 - \bar{\boldsymbol{t}}_1\right\|}{\left\|\bar{\boldsymbol{t}}_1 + \bar{\boldsymbol{t}}_2\right\|}\right). \tag{7}$$

In both this and the Sawaryn and Thorogood method, a test must be made to ensure the tangent vectors are not anti-parallel. This orientation is at a singularity, when $\propto = \pi$, and there can be no solution. When the tangent vectors are not parallel or anti-parallel, the degenerate and singular cases respectively, they define a 3D plane that the circular arc is confined to. When the orientation is singular or degenerate, there is a pencil of infinitely many planes developed around the co-oriented vectors. As opposed to the anti-parallel case, when the vectors are parallel, pointing in the same direction, the circular arc degenerates into a line segment. Though no unique plane can be found in this case, a solution to the minimum curvature problem is still available. To avoid the singularity, first test the inequality, Equation 8, that the length of the sum of the tangents is greater than zero,

$$0 < \left\|\bar{\boldsymbol{t}}_1 + \bar{\boldsymbol{t}}_2\right\|^2. \tag{8}$$

In their appendix, Sawaryn and Thorogood develop Equation 5 from the base form, Equation 9 [S&T],

$$\boldsymbol{P}_2 = \boldsymbol{P}_1 + \frac{S_{12}}{\propto}\tan\left(\frac{\propto}{2}\right)\left(\bar{\boldsymbol{t}}_1 + \bar{\boldsymbol{t}}_2\right) \tag{9}$$

$$\boldsymbol{P}_2 = \boldsymbol{P}_1 + \frac{S_{12}}{\propto}\left(\frac{\left\|\bar{\boldsymbol{t}}_2 - \bar{\boldsymbol{t}}_1\right\|}{\left\|\bar{\boldsymbol{t}}_1 + \bar{\boldsymbol{t}}_2\right\|}\right)\left(\bar{\boldsymbol{t}}_1 + \bar{\boldsymbol{t}}_2\right) \tag{10}$$

$$\boldsymbol{P}_2 = \boldsymbol{P}_1 + S_{12}\left(\frac{\|\bar{\boldsymbol{t}}_2 - \bar{\boldsymbol{t}}_1\|}{\propto}\right)\frac{(\bar{\boldsymbol{t}}_1 + \bar{\boldsymbol{t}}_2)}{\|\bar{\boldsymbol{t}}_1 + \bar{\boldsymbol{t}}_2\|}. \quad (11)$$

Equation 7 says the tangent of the half angle is the ratio of the length of the difference between the tangent vectors divided by the length of the sum of the tangent vectors. Substituting this ratio into Equation 9 gives Equation 10. Rearranging the terms into Equation 11 yields the base form of the arc-to-chord method for minimum curvature. Referring to **Figure 6**, Equation 12 employs a simple rational expression for the scaling of the unit length vector pointing at $\boldsymbol{P}_1$ from $\boldsymbol{P}_2$,

$$r(\propto) = \left(\frac{\|\bar{\boldsymbol{t}}_2 - \bar{\boldsymbol{t}}_1\|}{\propto}\right) \quad (12)$$

$$\boldsymbol{P}_2 = \boldsymbol{P}_1 + S_{12}\, r(\propto)\frac{(\bar{\boldsymbol{t}}_1 + \bar{\boldsymbol{t}}_2)}{\|\bar{\boldsymbol{t}}_1 + \bar{\boldsymbol{t}}_2\|}. \quad (13)$$

The simple rational expression, Equation 12, comes from the circle fact that the ratio of the arc to the chord is constant up to a given radius [COX]. The constant ratio of the arc to the chord, regardless of the radius, means the arc-to-chord ratio of unit circular arc, Equation 12, can be scaled by the course length to get the length of the chord between the arc end points, $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$. This emits a numerically stable representation of the circular arc without reference to the center of the circle or its radius, Equation 13, the final form for the arc-to-curve survey method.

Because of the axiom that the arc length of a circular arc segment is strictly greater than its chord length [COX], one need only confirm the inequality that the arc length is greater than the chord length, Equation 14,

$$\propto\, > \|\bar{\boldsymbol{t}}_2 - \bar{\boldsymbol{t}}_1\|, \quad (14)$$

to use Equation 13. Otherwise, simply scale the unit length vector pointing from $\boldsymbol{P}_1$ to $\boldsymbol{P}_2$ by the course length to recover $\boldsymbol{P}_2$,

$$\boldsymbol{P}_2 = \boldsymbol{P}_1 + S_{12}\frac{(\bar{\boldsymbol{t}}_1 + \bar{\boldsymbol{t}}_2)}{\|\bar{\boldsymbol{t}}_1 + \bar{\boldsymbol{t}}_2\|}. \quad (15)$$

In real terms, the rational expression in Equation 11, $\left(\frac{\|\bar{t}_2 - \bar{t}_1\|}{\propto}\right)$, the chord length defined by the difference in the tangent vectors divided by that angle subtended by these vectors tends to one in the limit as the vectors verge to parallel. Given the guarantees afforded by the floating point math standard [754], up to the inequality test in Equation 14[3] within machine precision, the ratio equals

[3] Practically, the inequality test in Equation 14 avoids divide-by-zero errors where the tangents are absolutely parallel, when $\propto = 0$. Otherwise, and when within machine precision, the rational expression equals one. That is, when the test is false, either the denominator is zero or both terms, because we are working with floating point numbers, are equal.

one long before $\propto$ equals zero, eliminating the need to use series expansion[4]. Furthermore, when using floating point math, it can be shown that Equation 6 goes to one before Equation 12 giving equivalent accuracy in these expressions near and at straight-hole orientations.

Spherical Linear Interpolation

In many applications of minimum curvature to model the path of a wellbore, one needs to find the position of a point on the path in between consecutive survey stations. For example, the position of a geologic contact in the wellbore, the position of the start and end of a set of perforations or to find equally spaced points for graphic display. Spherical linear interpolation, SLERP, presented by Sawaryn and Thorogood [S&T] and others [PLK], is a robust method for interpolating points along a circular arc, **Figure 7**. Plunk [PLK] derives a numerically stable implementation of the SLERP algorithm via Equation 18 and Equation 19.

Given $0 \leq x \geq 1$, x being the fraction along the arc between survey stations indexed by measured depth, Equation 16, the formula for SLERP is presented in Equation 17 [S&T],

$$x = \frac{MD_x - MD_1}{S_{12}} \tag{16}$$

$$\bar{\boldsymbol{t}}_x = \frac{\sin\big((1-x)\propto\big)}{\sin(\propto)}\bar{\boldsymbol{t}}_1 + \frac{\sin(x\propto)}{\sin(\propto)}\bar{\boldsymbol{t}}_2. \tag{17}$$

The problem with this form is that x goes to zero in the limit as the tangent vectors verge to parallel, causing divide-by-zero-zero errors. At and near this straight-hole orientation, Sawaryn and Thorogood again employ a complex series expansion (not presented in this note) to handle the transition. See [S&T] for details. To avoid the divide-by-zero errors, Plunk [PLK] offers a simple function, Equation 18[5], for the problematic division, up to machine precision [754], **Figure 8**, then rearranges Equation 15 to recover $\bar{\boldsymbol{t}}_x$ in a robust way, Equation 19,

$$p(\sigma) = 1.0\ if\ 1.0 == 1.0 + (\sigma^2)\ else\ \frac{\sin(\sigma)}{\sigma} \tag{18}$$

$$\bar{\boldsymbol{t}}_x = \frac{p\big((1-x)\propto\big)}{p(\propto)}(1-x)\bar{\boldsymbol{t}}_1 + \frac{p(x\propto)}{p(\propto)}x\bar{\boldsymbol{t}}_2. \tag{19}$$

With the interpolated tangents, the arc-to-chord method can then be used to calculate the positions at the interpolating measured depths.

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[4] When implementing of Equation 13, Equation 12 can be refactored to,

$$r(\propto) = 1.0\ if\ !\left(\propto > \left\|\bar{\boldsymbol{t}}_2 - \bar{\boldsymbol{t}}_1\right\|\right)\ else\left(\frac{\left\|\bar{\boldsymbol{t}}_2 - \bar{\boldsymbol{t}}_1\right\|}{\propto}\right),$$

so there is a smooth transition into the degenerate case and when $\propto$ equals zero.

[5] The same inequality test in Equation 14 can be substituted here as they are just different derivations of the same rational expression, the sine of the angle divided by the angle subtended by the unit vectors, $\sin(\propto) = \left\|\bar{\boldsymbol{t}}_2 - \bar{\boldsymbol{t}}_1\right\|$.

Conclusions

Developed in this note is a simplified and more robust minimum curvature survey method, the arc-to-chord construction, for calculating the positions of the survey stations in a wellbore. Similarly, a refactored spherical linear interpolation method is presented. Both methods are simplified by removing the need to do complex series expansions and by relying on the floating math standard to handle the transition into edge cases without need to refer to specific branch-cut thresholds. As advocated by Sawaryn and Thorogood [S&T], simplifying algorithms and methods yields more stable, predictable and reliable code and is therefore preferred.

Appendix

Spherical Coordinates: When working on wellbore positioning problems, it is often convenient to convert between 3D cubic and spherical coordinates. The spherical coordinates, inclination, θ, and azimuth, $\emptyset$, are recovered from a unit length vector, $\bar{\boldsymbol{t}} = [\Delta\mathrm{N}, \Delta\mathrm{E}, \Delta\mathrm{V}]$, with Equation 20 and Equations 21 through 24 respectively[6],

$$\theta = \mathrm{atan2}\left[(\Delta\mathrm{N}^2 + \Delta\mathrm{E}^2)^{\frac{1}{2}}, \Delta\mathrm{V}\right] \tag{20}$$

$$\emptyset_a = \{\emptyset = \mathrm{atan2}(\Delta\mathrm{E}, \Delta\mathrm{N}) \mid \emptyset \geq 0\} \tag{21}$$

$$\emptyset_b = \{\emptyset = \mathrm{atan2}(\Delta\mathrm{E}, \Delta\mathrm{N}) \mid \emptyset < 0\} \tag{22}$$

$$\emptyset_b = \mathrm{mod}(\emptyset_b + 2\pi, 2\pi) \tag{23}$$

$$\emptyset = \{\emptyset_a \cup \emptyset_b\}. \tag{24}$$

When using floating point math, the expression in Equation 23, the expression, $\emptyset_b + 2\pi$, will equal 2π when $\emptyset_b$ is less than the machine precision [754] putting the result out of the defined domain for azimuths. When recovering the azimuth, the `mod` function[7] assures that the result will remain within the defined domain, $0 \leq \emptyset < 2\pi$.

Dogleg Severity: In the oil field, the curvature of a segment of wellbore is often referred to as the dogleg severity and is expressed as the angle subtended and reported in degrees over a fixed delta of course length, S_Δ, e.g., degrees per 100 feet or 30 meters. When modeling a wellbore segment as a circular arc, the dogleg severity, DLS, can be calculated using Equation 25,

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[6] The version of two argument arctangent function used in this note, $\mathrm{atan2}(y, x)$, takes as a first argument the opposite y component and as the second argument the adjacent x component of the triangle defining the angle.
[7] The two-argument modulo function, $\mathrm{mod}(x1, x2) = x1 - \mathrm{floor}(x1/x2) * x2$, returns the remainder of division [NPY].

$$DLS = \frac{\propto_{degress} S_{\Delta}}{S_{12}}, \tag{25}$$

where $\propto_{degress}$ is the angle in degrees swept out over the course length, S_{12}, between consecutive survey stations.

Nomenclature

MD = measured depth, feet
θ = inclination angle, radians
$\emptyset$ = azimuth angle, radians
S_{12} = course length, feet
$\overline{\boldsymbol{t}}$ = tangent unit length vector in [ΔN, ΔE, ΔV] coordinates, feet
N = northing coordinate, feet
E = easting coordinate, feet
V = vertical coordinate, feet
Δ = delta of unit vector coordinates, feet
$\propto$ = subtended angle between tangent unit vectors, radians
$\propto_{degress}$ = subtended angle between tangent unit vectors, degrees
$\boldsymbol{P}$ = position vector in [N,E,V] coordinates, feet
f = geometric shape factor, function
r = rational chord-to-arc scale factor, function
x = a fraction of course length to interpolate to, scalar
σ = angle, radians
p = stable sine-of-angle divided by angle, function
S_{Δ} = reference curvature ratio, degrees per 100 feet or 30 meters

Acknowledgements
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Figures

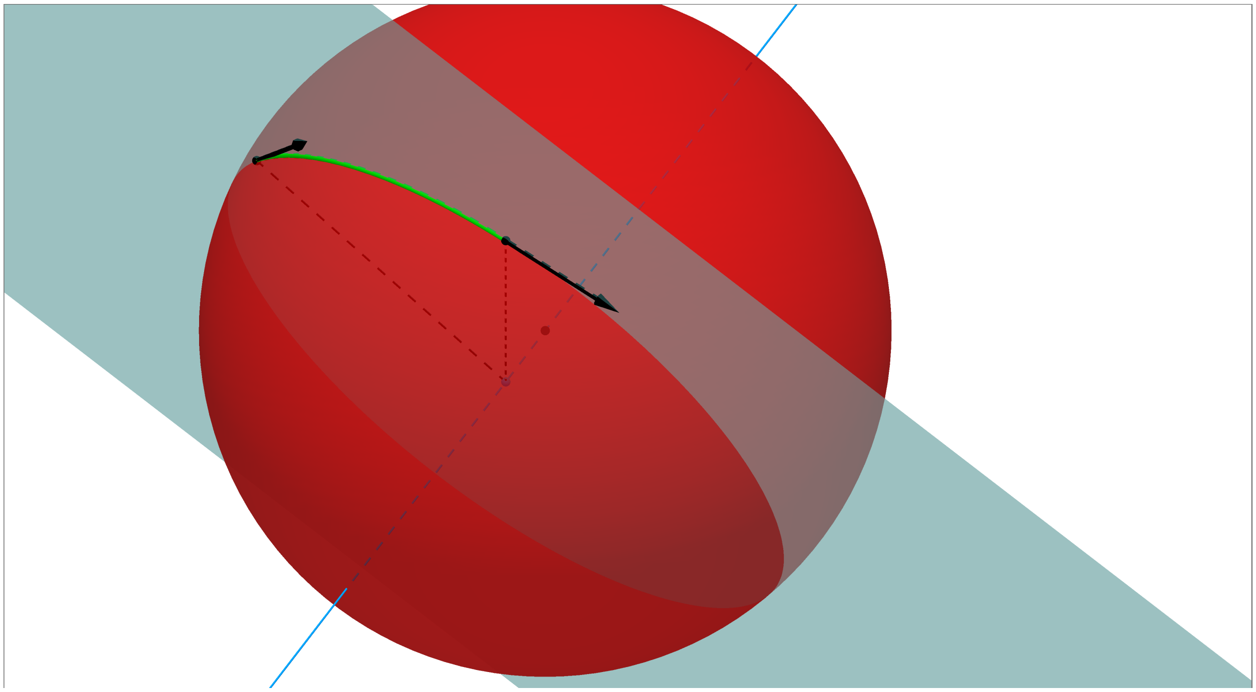

Figure 1 *– Circular Arc In Spherical Cutting Plane.* 3D circular arc (green) developed on the surface of the sphere (red) where the sphere is cut by plane (cyan) defined by two tangent vectors (black).

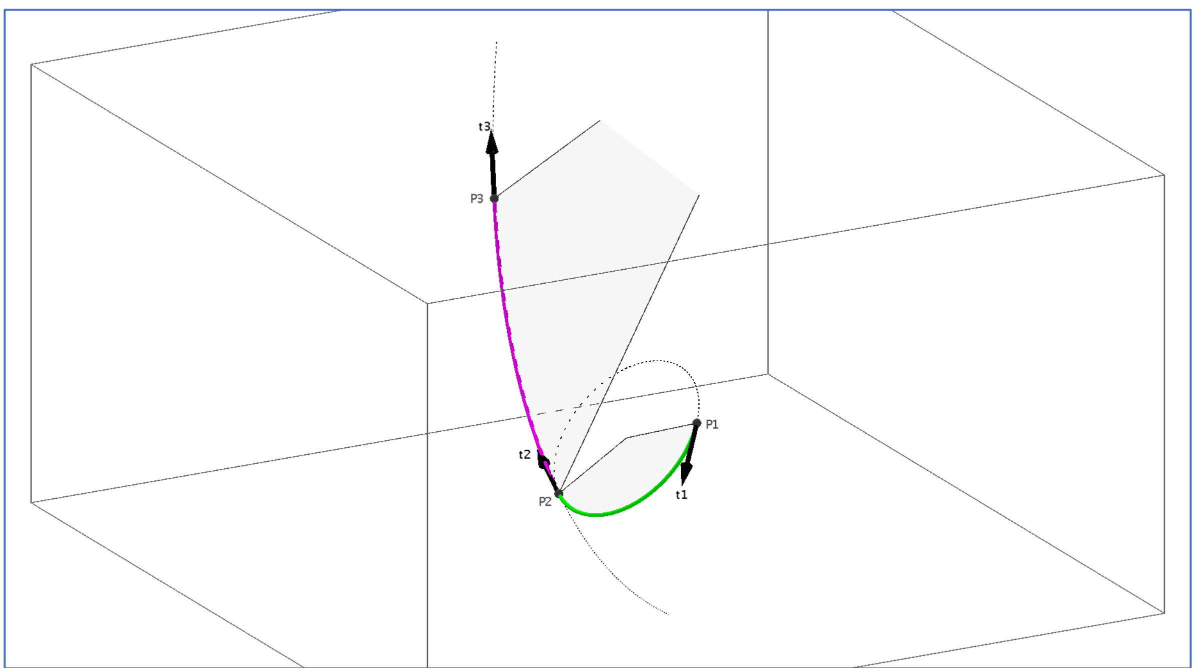

Figure 2 *– Circular Arc Space Curve.* Two 3D circular arcs, $\boldsymbol{P_1}$ to $\boldsymbol{P_2}$ (green) and $\boldsymbol{P_2}$ to $\boldsymbol{P_3}$ (magenta), sharing a common tangent, $\boldsymbol{t_2}$, form a space curve with G1 continuity.

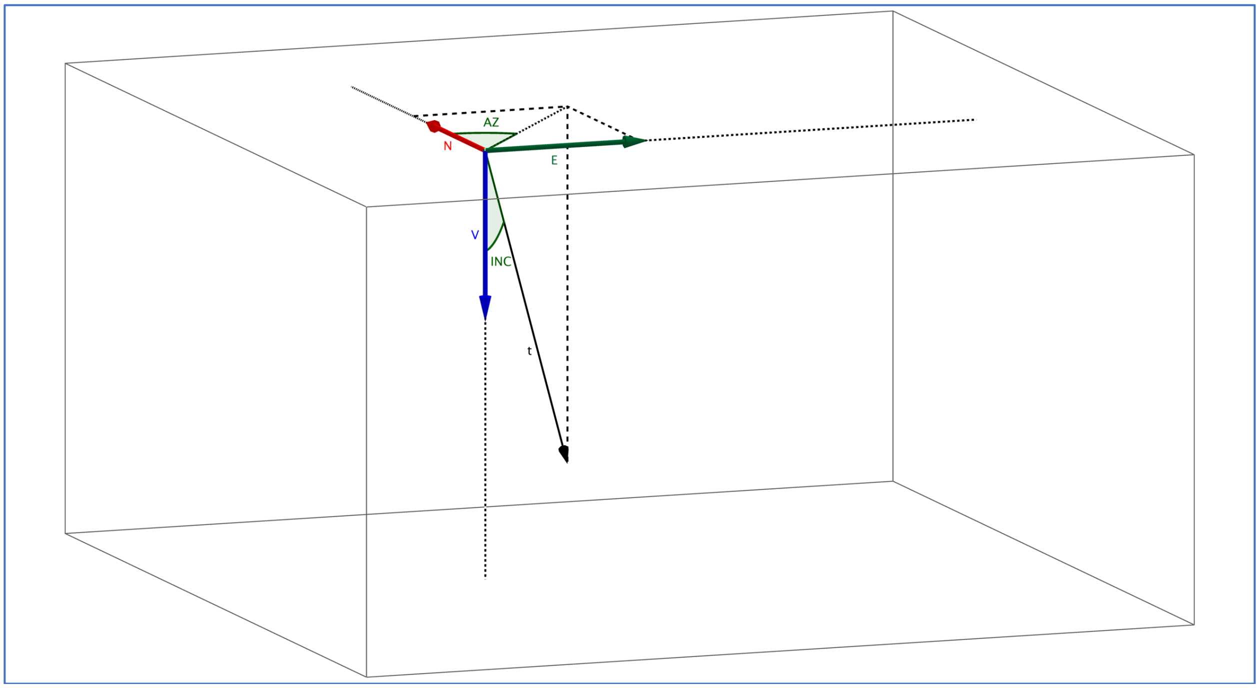

Figure 3 *– North, East, Vertical (NEV) Right Handed Coordinate System.* Vertical is positive down into the earth. Inclination (INC) is measured positive up from Vertical. Azimuth (AZ) is measured positive clockwise from North in the horizontal plane. Unit direction vector, t, from spherical coordinates INC and AZ. The domains of the inclinations are strictly $0 \geq \theta < \pi$, and the azimuths, $0 \geq \emptyset < 2\pi$.

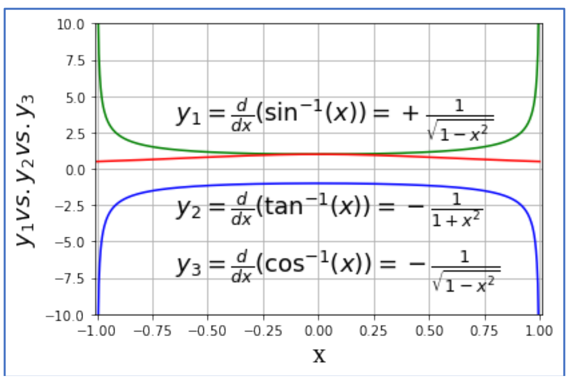

Figure 4 *– Asymptotically Vanishing 1st Derivative Plot.* The plot shows, as the angle approaches zero and π, the 1st derivative of the inverse cosine (blue) goes to negative infinity. Similarly, as the angle approaches, $(\pm)\frac{\pi}{2}$, the 1st derivative of the inverse sine (green) goes to positive infinity. Conversely, the 1st derivative of the inverse tangent (red) remains bounded between zero and including one as the angle ranges between $(\pm)\frac{\pi}{2}$. Off the plot and not shown, as the domain, x, goes to ∞, the tails go to zero in the limit.

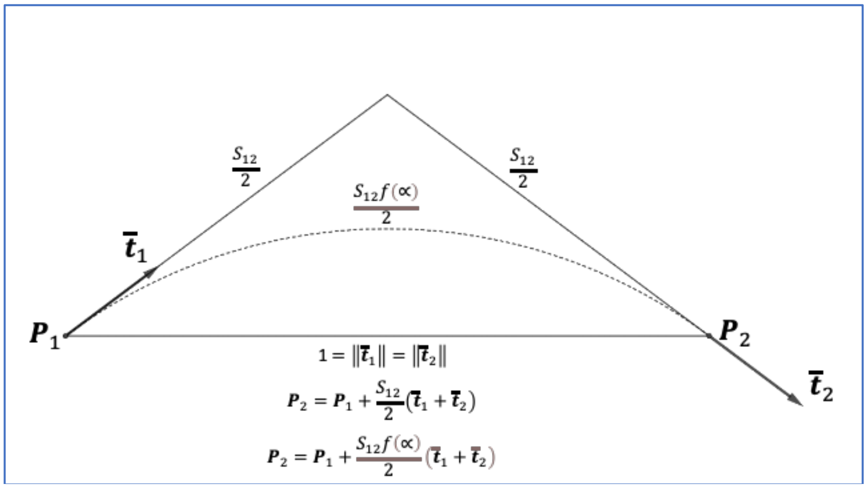

Figure 5 *– Balanced Tangential vs. Minimum Curvature*. The balanced tangential method is the sum of the linearly-scaled tangent vectors, $\boldsymbol{t_1}$ and $\boldsymbol{t_2}$, each by half the course length, $\frac{S_{12}}{2}$, followed between the end points, $\boldsymbol{P_1}$ to $\boldsymbol{P_2}$, geometrically, a scaled isosceles triangle formed by the scaled tangents. Minimum curvature then scales the balanced tangential isosceles triangle by a scale factor, $f(\propto)$, such that a circular arc, defined by the tangents, and by construction the angle between them, $\propto$, at the end points and fit to the end points, has an arc length equal the course length, S_{12}, between the end points.

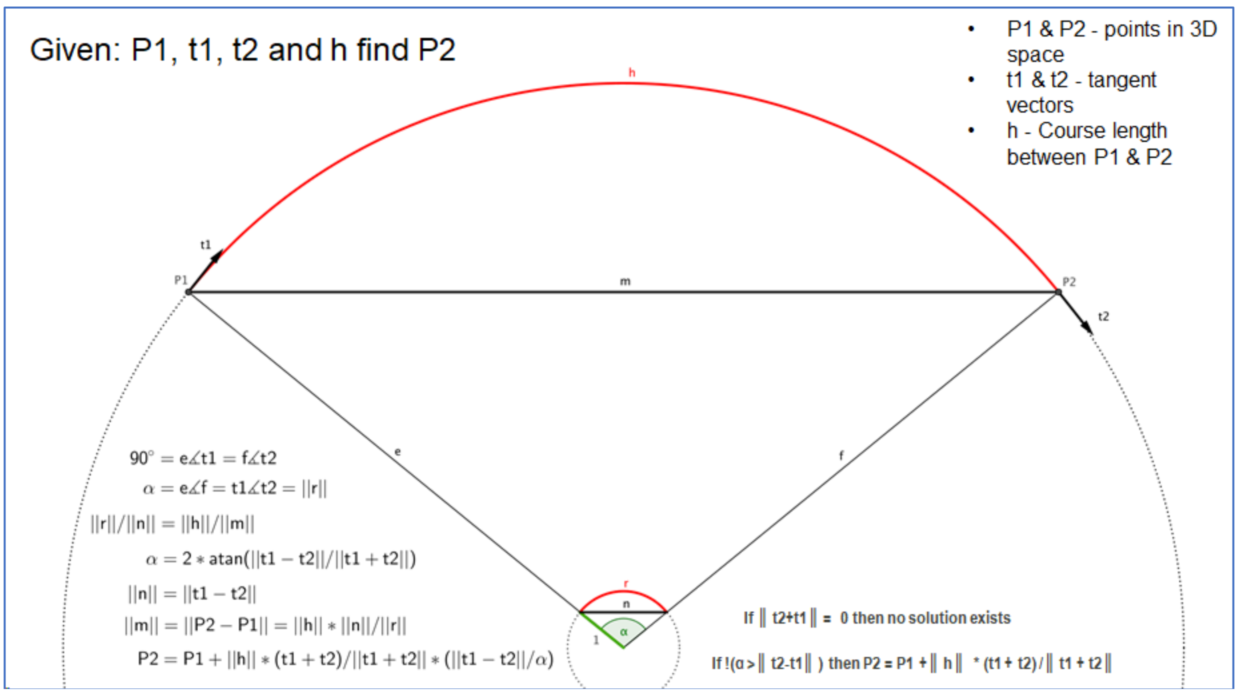

Figure 6 *– Arc-To-Chord Method*. The angle, $\propto$, between the tangent vectors, $\boldsymbol{t_1}$ and $\boldsymbol{t_2}$, equals the length of the arc, r, on the unit circle. The ratio of the length of the difference between the tangent vectors, the length of the chord, n, of the arc on the unit circle, and the length of the arc, r, of the unit circle then equals the ratio of the length of the arc, h, on the primary circle to the length of the chord, m, of the arc on the primary circle.

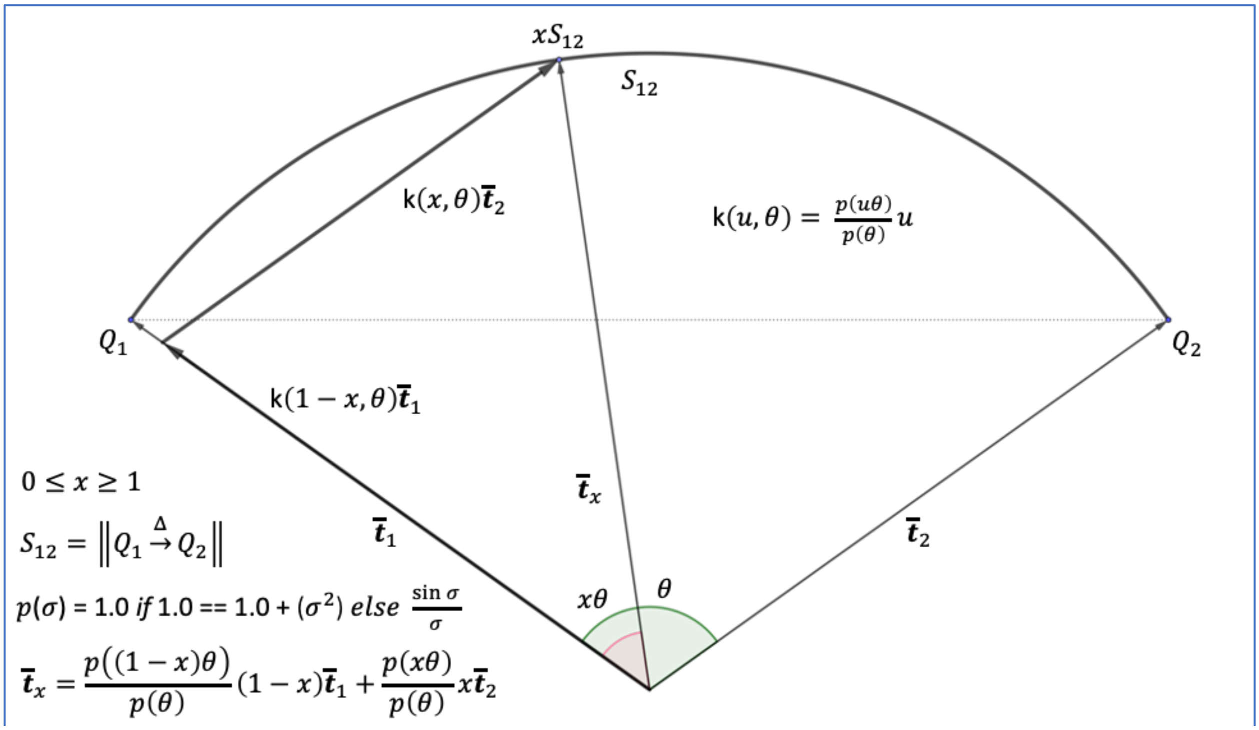

***Figure** 7 – Spherical Linear Interpolation*. The interpolated vector, $\boldsymbol{t}_x$, is rotated by the fraction x, along the arc, by a linear combination of the scaled end point vectors. The bounding vectors are scaled by the complimentary ratios of the fraction of the angle between the vector and the sine of the scaled angle subtending the arc.

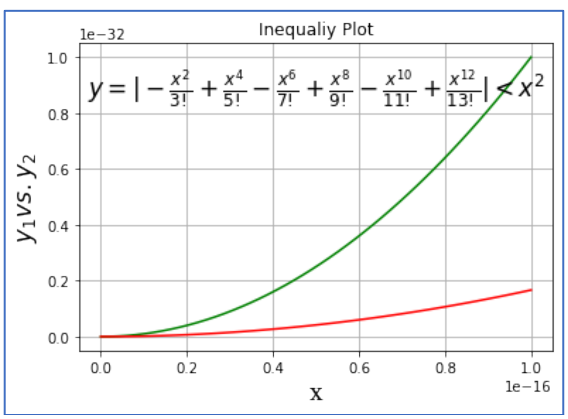

***Figure** 8 – Inequality Plot*. The plot reveals the series expansion of the $\frac{\sin x}{x}$ (red) is always less than x^2 (green). This means the Taylor series expansion of the $\frac{\sin x}{x}$ goes to 1.0 long before expression $1.0 + x^2$ goes to 1.0, a feature of floating point math that can be used in a divide-by-zero test as x goes to zero in the limit.