DATA 609 HW 1

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1. Find the minimum of $f(x, y) = x^2 + xy + y^2$ in $(x, y) \in \mathbb{R}^2$.

Response. The minimum is $f(0, 0) = 0$.

Stationary conditions:

$\frac{\partial f}{\partial x} = 2x + y = 0$

$\frac{\partial f}{\partial y} = 2y + x = 0$

Solving the pair of equations, we obtain $(x, y) = (0, 0)$.

$\Delta = \det(\boldsymbol{H}) = f_{xx}f_{yy} - f_{xy}^2 = 3$.

Since $\Delta > 0$, $(0, 0)$ is the location of a local minimum. And since there are no other stationary points, $(0, 0)$ is also the location of the global minimum for $f(x, y)$.

2. For $f(x) = x^4$ in $\mathbb{R}$, it has the global minimum at $x = 0$. Find its new minimum if a constraint $x^2 \geq 1$ is added.

Response. The minima of the function under the constraint are $f(-1) = 1$ and $f(1) = 1$.

Rearranging the constraint: $-x^2 + 1 \leq 0$.

$\Pi(x, \mu) = x^4 + \mu(-x^2 + 1)^2$.

$\Pi'(x) = 0$:

$$
\begin{aligned}
4x^3 + 2\mu(-x^2 + 1)(-2x) &= 0 \\
x^3 + \mu x^3 - \mu x &= 0 \\
x(x^2 + \mu x^2 - \mu) &= 0 \to x = 0, \text{or} \\
x^2 + \mu x^2 - \mu &= 0 \\
x^2(1 + \mu) &= \mu \\
x^2 &= \frac{\mu}{1 + \mu} \\
x &= \pm\sqrt{\frac{\mu}{1 + \mu}}
\end{aligned}
$$

As μ increases, x approaches positive 1 or negative 1, the locations of the minima of the constrained function.

We can confirm that both these points are minima by checking the second derivative at these points.

$f''(x)|_{x=-1} = f''(x)|_{x=1} = 12$.

Since the second derivative is positive at both these points, they represent minima.

3. Use a Lagrange multiplier to solve the optimization problem
   $\min f(x, y) = x^2 + 2xy + y^2$, subject to $y = x^2 - 2$.

Response. The minima of the function under the constraint are $f(-2, 2) = 0$ and $f(1, -1) = 0$.

Form the Lagrangian: $\Pi = x^2 + 2xy + y^2 + \lambda(x^2 - y - 2)$.

Stationary conditions:

$\frac{\partial \Pi}{\partial x} = 2x + 2y + 2\lambda x = 0$

$\frac{\partial \Pi}{\partial y} = 2x + 2y - \lambda = 0$

$\frac{\partial \Pi}{\partial \lambda} = x^2 - y - 2 = 0$

$$\begin{aligned} x + y + \lambda x &= 0 \\ (\lambda + 1)x + y &= 0 \\ y &= -(\lambda + 1)x \end{aligned}$$

$$\begin{aligned} 2x - 2x(\lambda + 1) - \lambda &= 0 \\ 2x - 2\lambda x - 2x - \lambda &= 0 \\ -\lambda(1 + 2x) &= 0 \\ \lambda &= 0 \text{ or } x = -1/2 \end{aligned}$$

If $\lambda = 0$:

$$\begin{aligned} 2x + 2y &= 0 \\ y &= -x \end{aligned}$$

$$\begin{aligned} x^2 + x - 2 &= 0 \\ (x + 2)(x - 1) &= 0 \\ x = -2 \text{ or } x &= 1 \end{aligned}$$

Therefore $(-2, 2)$ and $(1, -1)$ are stationary points.

If $x = -1/2$ then $\left(\frac{-1}{2}, -1\frac{3}{4}\right)$ is a stationary point.

$f\left(\frac{-1}{2}, -1\frac{3}{4}\right) = 5\frac{1}{16}$.

$f(-2, 2) = f(1, -1) = 0$. So the minima of the function under the constraint occur at $(-2, 2)$ and $(1, -1)$.