Week 9 Assignment

Daniel Moscoe

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Page 363 Number 11. The price of one share of stock in the Pilsdorff Beer Company (see Exercise 8.2.12) is given by Y_n on the nth day of the year. Finn observes that the differences $X_n = Y_{n+1} - Y_n$ appear to be independent random variables with a common distribution having mean $\mu = 0$ and variance $\sigma^2 = 1/4$. If $Y_1 = 100$, estimate the probability that Y_{365} is
(a) ≥ 100.
(b) ≥ 110.
(c) ≥ 120.

Response. $Y_{365} = X_{364} + X_{363} + ... + X_1 + 100$. So Y_{365} is the sum of 100, and 364 random variables with a common distribution, mean, and variance.

$E(Y_{365}) = 100 + \sum_{i=1}^{364} E(X_i)$
$E(Y_{365}) = 100.$

$V(Y_{365}) = 0 + \sum_{i=1}^{364} V(X_i)$
$V(Y_{365}) = 364/4 = 91.$

Since Y_{365} is a sum of mutually independent random variables, the central limit theorem lets us approximate the distribution of Y_{365} by the normal distribution with $\mu = 100$ and $\sigma^2 = 91$.

(a) $P(Y_{365} \geq 100) = P\Big(Z \geq \frac{100-100}{\sqrt{91}}\Big) = P(Z \geq 0) = \frac{1}{2}$.

(b) $P(Y_{365} \geq 100) = P\Big(Z \geq \frac{110-100}{\sqrt{91}}\Big) = P\Big(Z \geq \frac{10}{\sqrt{91}}\Big) \approx 0.1473$.

(c) $P(Y_{365} \geq 100) = P\Big(Z \geq \frac{120-100}{\sqrt{91}}\Big) = P\Big(Z \geq \frac{20}{\sqrt{91}}\Big) \approx 0.0180$.

The moment-generating function for the binomial distribution. The moment-generating function, $g(t)$, for a random variable X is $g(t) = \mathrm{E}\Big(e^{tX}\Big)$. By repeatedly differentiating this function with respect to t at $t = 0$, we can calculate the moments of X. The first moment of X is its mean, μ. The second moment, μ_2, can be used to compute the variance of X.

If X is a binomial random variable with probability of success p and number of trials n, then the moment-generating function for X is:

$$g(t) = \Sigma_{i=0}^{n} e^{ti}\, \mathrm{P}(X = i)$$

$$g(t) = \Sigma_{i=0}^{n} e^{ti} \binom{n}{i} p^i (1-p)^{n-i}$$

$$g(t) = \Sigma_{i=0}^{n} (pe^t)^i \binom{n}{i} (1-p)^{n-i}.$$

This is the binomial expansion of the right side of the equation below:

$$g(t) = (pe^t + (1-p))^n.$$

We can use $g(t)$ to find $\mathrm{E}(X)$ and $\mathrm{E}(X^2)$:

$$\mu_1 = g'(0)\big|_{t=0} = n(pe^t + (1-p))^{n-1}(pe^t)\Big|_{t=0}$$

$$\mu_1 = np.$$

$$\mu_2 = g''(0)\big|_{t=0} = n(n-1)(pe^t + (1-p))^{n-2}(pe^t)^2 + n(pe^t + (1-p))^{n-1}(pe^t)\Big|_{t=0}$$

$$\mu_2 = n^2p^2 - np^2 + np.$$

With $\mu_1 = \mathrm{E}(X)$ and $\mu_2 = \mathrm{E}(X^2)$, we can find $\sigma^2(X)$:

$$\sigma^2(X) = \mu_2 - \mu_1^2$$

$$\sigma^2(X) = n^2p^2 - np^2 + np - n^2p^2$$

$$\sigma^2(X) = np(1-p).$$

The moment-generating function for the exponential distribution. If X is an exponential random variable with parameter λ, then the moment-generating function for X is:

$$g(t) = \int_{-\infty}^{\infty} e^{tX} f_X(x)\, dx$$

$$= \int_0^{\infty} e^{tX} \lambda e^{-\lambda x}\, dx$$

$$= \lambda \lim_{b\to\infty} \int_0^b e^{x(t-\lambda)}\, dx$$

$= \lambda \lim_{b\to\infty} \frac{1}{t-\lambda}\, e^{x(t-\lambda)}\Big|_{x=0}^{x=b}$

$= \lambda \Big[\lim_{b\to\infty} \frac{1}{t-\lambda} e^{b(t-\lambda)} - \frac{1}{t-\lambda}(1) \Big].$

If $t > \lambda$, then $g(t)$ diverges.
If $t < \lambda$:

$g(t) = \frac{\lambda}{\lambda - t}.$

Differentiating at $t = 0$ gives μ_1:

$\frac{d}{dt}\, \frac{\lambda}{\lambda-t}\Big|_{t=0} = \frac{1}{\lambda}.$

Finding the second derivative at $t = 0$ gives the second moment, μ_2, which can be used to calculate $\sigma^2(X)$:

$\frac{d^2}{dt^2} \frac{\lambda}{\lambda-t}\Big|_{t=0} = \frac{2}{\lambda^2}.$

$\sigma^2(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2$

$= \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2$

$= \frac{1}{\lambda^2}.$