Ch. E exercise T10. A is idempotent if $A^2 = A$. Show that the only possible eigenvalues of an idempotent matrix are $\lambda = 0$ and $\lambda = 1$. Then give an example of a matrix that is idempotent and has both these two values as eigenvalues.

Response:

$$\begin{aligned} A\vec{v} &= A^2\vec{v} \\ \lambda\vec{v} &= A\lambda\vec{v} \\ &= \lambda A\vec{v} \\ &= \lambda^2\vec{v} \\ \lambda &= \lambda^2 \\ \lambda = 1 &\text{ or } \lambda = 0 \end{aligned}$$

Consider a 2×2 matrix as a geometric transformation. Then to say that A is idempotent, or that $A = A^2$, is to say that performing the transformation twice is equivalent to performing it once. One example of such a transformation is an orthogonal projection onto $\mathrm{sp}(\vec{w})$. This matrix has the form

$$P = \frac{1}{w_1^2 + w_2^2} \begin{pmatrix} w_1^2 & w_1w_2 \\ w_1w_2 & w_2^2 \end{pmatrix}.$$

So the idempotent matrix for orthogonal projection onto $\mathrm{sp}\begin{pmatrix} 3 \\ 5 \end{pmatrix}$ is $A = \frac{1}{34}\begin{pmatrix} 9 & 15 \\ 15 & 25 \end{pmatrix}$.

Any scalar multiple of $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$, such as $\begin{pmatrix} 6 \\ 10 \end{pmatrix}$, is an eigenvector for A with $\lambda = 1$. And any vector perpendicular to $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$, such as $\begin{pmatrix} -10 \\ 6 \end{pmatrix}$ is an eigenvector for A with $\lambda = 0$.