DATA 605 HW 10

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Smith is in jail and has \$1. He can get out on bail if he has \$8. A guard agrees to make a series of bets with him. If Smith bets \$A, he wins \$A with probability 0.4 *and loses \$A with probability* 0.6*.*

*Find the probability that he wins \$*8 *before losing all of his money if:*
(a) *He bets \$*1 *each time (timid strategy);*
(b) *He bets, each time, as much as possible but not more than necessary to bring his fortune up to \$*8 *(bold strategy).*

Which strategy gives Smith the better chance of getting out of jail?

The series of bets can be modeled as an absorbing Markov chain. The transition matrix below gives the probabilities that the state represented by the row will result in the state represented by the column. For example, if Smith bets while he holds \$4, the probability that the bet will result in him holding \$3 is 0.6. The transition matrix, `tim_A`, is given in canonical form.

```
tim_A <- matrix(0, nrow = 9, ncol = 9)
tim_A[1,2] = 0.4
tim_A[2,3] = 0.4
tim_A[3,4] = 0.4
tim_A[4,5] = 0.4
tim_A[5,6] = 0.4
tim_A[6,7] = 0.4
tim_A[7,9] = 0.4
tim_A[1,8] = 0.6
tim_A[2,1] = 0.6
tim_A[3,2] = 0.6
tim_A[4,3] = 0.6
tim_A[5,4] = 0.6
tim_A[6,5] = 0.6
tim_A[7,6] = 0.6
tim_A[8,8] = 1
tim_A[9,9] = 1
rownames(tim_A) <- c(1,2,3,4,5,6,7,0,8)
colnames(tim_A) <- c(1,2,3,4,5,6,7,0,8)
tim_A
```

```
##     1   2   3   4   5   6   7   0   8
## 1 0.0 0.4 0.0 0.0 0.0 0.0 0.0 0.6 0.0
## 2 0.6 0.0 0.4 0.0 0.0 0.0 0.0 0.0 0.0
## 3 0.0 0.6 0.0 0.4 0.0 0.0 0.0 0.0 0.0
## 4 0.0 0.0 0.6 0.0 0.4 0.0 0.0 0.0 0.0
## 5 0.0 0.0 0.0 0.6 0.0 0.4 0.0 0.0 0.0
## 6 0.0 0.0 0.0 0.0 0.6 0.0 0.4 0.0 0.0
```

```
## 7 0.0 0.0 0.0 0.0 0.0 0.6 0.0 0.0 0.4
## 0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 1.0 0.0
## 8 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 1.0
```

From `tim_A` in canonical form, we can construct the fundamental matrix, `tim_N`:

```
tim_N <- solve(diag(nrow = 7) - tim_A[1:7,1:7])
tim_N
```

```
##           1        2         3         4         5         6          7
## 1 1.6328311 1.054718 0.6693101 0.4123711 0.2410785 0.1268834 0.05075337
## 2 1.5820777 2.636796 1.6732752 1.0309278 0.6026963 0.3172086 0.12688343
## 3 1.5059477 2.509913 3.1792228 1.9587629 1.1451229 0.6026963 0.24107851
## 4 1.3917526 2.319588 2.9381443 3.3505155 1.9587629 1.0309278 0.41237113
## 5 1.2204600 2.034100 2.5765266 2.9381443 3.1792228 1.6732752 0.66931007
## 6 0.9635210 1.605868 2.0340999 2.3195876 2.5099128 2.6367962 1.05471848
## 7 0.5781126 0.963521 1.2204600 1.3917526 1.5059477 1.5820777 1.63283109
```

The fundamental matrix contains the expected number of times the chain is in a particular state, given an initial state. For example, with the initial state of Smith holding \$3, we can expect him to hold \$4 about twice before the betting ends/the chain is absorbed.

Finally, we can construct `tim_B`, the matrix of probabilities of entering either of the absorbing states, given an initial state.

```
tim_B <- tim_N %*% tim_A[1:7,8:9]
tim_B
```

```
##           0          8
## 1 0.9796987 0.02030135
## 2 0.9492466 0.05075337
## 3 0.9035686 0.09643140
## 4 0.8350515 0.16494845
## 5 0.7322760 0.26772403
## 6 0.5781126 0.42188739
## 7 0.3468676 0.65313243
```

The entry in row 1 col 2 of `tim_B` shows that the probability of making bail is very low: only about 2%. Smith would need to start with \$7 in order for his probability of making bail to exceed 1/2.

How is Smith likely to fare with the bold strategy? The matrix below gives the probabilities for how one state might follow another. The transition matrix `bold_A` is given in canonical form.

```
bold_A <- matrix(c(0, 0, 0, 0, 0,
               .4,0, 0, 0, 0,
               0,.4, 0, 0, 0,
               .6,.6,.6,1, 0,
               0, 0,.4, 0, 1), nrow = 5)
rownames(bold_A) <- c(1,2,4,0,8)
colnames(bold_A) <- c(1,2,4,0,8)
bold_A
```

```
##   1   2   4   0   8
## 1 0 0.4 0.0 0.6 0.0
## 2 0 0.0 0.4 0.6 0.0
## 4 0 0.0 0.0 0.6 0.4
## 0 0 0.0 0.0 1.0 0.0
## 8 0 0.0 0.0 0.0 1.0
```

from `bold_A` in canonical form, we can construct the fundamental matrix, `bold_N`:

```
bold_N <- solve(diag(nrow = 3) - bold_A[1:3, 1:3])
bold_N
```

```
##   1   2    4
## 1 1 0.4 0.16
## 2 0 1.0 0.40
## 4 0 0.0 1.00
```

Finally, we construct `bold_B`:

```
bold_B <- bold_N %*% bold_A[1:3, 4:5]
bold_B
```

```
##       0     8
## 1 0.936 0.064
## 2 0.840 0.160
## 4 0.600 0.400
```

The entry in row 1 col 2 of `bold_B` is the probability that, starting with \$1, the Markov chain will be absorbed in the state \$8. The probability that Smith will make bail betting as much as possible is still low: only 6.4%. However, this is much better than the timid strategy of betting only \$1 in each round.