Solving a system of linear equations via LU decomposition

Given an $m \times n$ matrix A, a "LU decomposition" is an $n \times n$ lower triangular matrix L & an $m \times n$ matrix in upper echelon form U such that

$$A = LU$$

L for lower

U for upper

For example

$$\underset{A}{\begin{pmatrix} 6 & 18 & 3 \\ 2 & 12 & 1 \\ 4 & 15 & 3 \end{pmatrix}} = \underset{L}{\begin{pmatrix} 3 & 0 & 0 \\ 1 & 6 & 0 \\ 2 & 3 & 1 \end{pmatrix}} \underset{U}{\begin{pmatrix} 2 & 6 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

Solving systems of linear equations is easy & quick for upper & lower triangular matrices:

For example

$$\left.\begin{aligned} 2x_1 + 4x_2 + 7x_3 &= 6 \\ x_2 + 2x_3 &= 3 \end{aligned}\right]$$

$$\Leftrightarrow$$

U in upper echelon form

$$\begin{pmatrix} 2 & 4 & 7 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$$

$\shortparallel$

LU

$L = Id_3$

U represents a map from $\mathbb{R}^3 \to \mathbb{R}^2$, there'll be a 1-dimensonal solution set.

Let $x_3 = t$ be the free variable. Then $\underline{x_2 = 3 - 2t}$ & $\underline{x_1 = 6 - 7t - 4(3 - 2t) = t - 6}$.

Thus solution set is $\left\{ \begin{pmatrix} t-6 \\ 3-2t \\ t \end{pmatrix} \,\middle|\, t \in \mathbb{R} \right\} \subseteq \mathbb{R}^3$.

The Method

Assume we have an LU decomp of A

& a system $A\underline{x} = \underline{b}$.

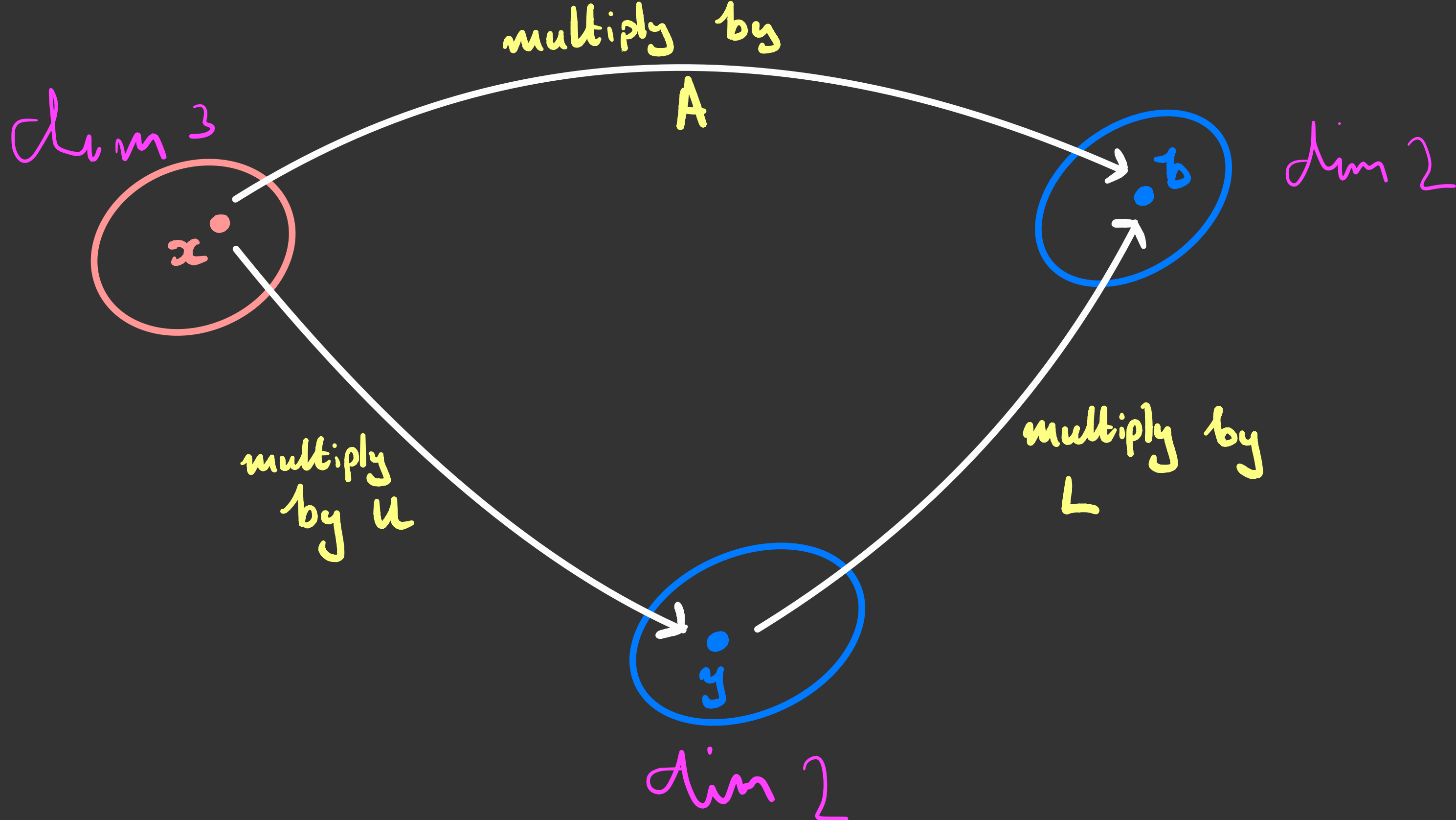
new vars

Step 1: Set $\underline{y} := U\underline{x}$

Step 2: Solve $L\underline{y} = \underline{b}$

Now we have actual values for $\underline{y}$

Step 3: Solve $U\underline{x} = \underline{y}$.



For example

$$\underset{A}{\begin{pmatrix} 6 & 18 & 3 \\ 2 & 12 & 1 \\ 4 & 15 & 3 \end{pmatrix}} \underset{\underline{x}}{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}} \overset{=}{=} \underset{\underline{b}}{\begin{pmatrix} 3 \\ 19 \\ 0 \end{pmatrix}}$$

We use LU decomp from before.

$$\underset{A}{\begin{pmatrix} 6 & 18 & 3 \\ 2 & 12 & 1 \\ 4 & 15 & 3 \end{pmatrix}} = \underset{L}{\begin{pmatrix} 3 & 0 & 0 \\ 1 & 6 & 0 \\ 2 & 3 & 1 \end{pmatrix}} \underset{U}{\begin{pmatrix} 2 & 6 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

Step 1 Set $\underline{U \quad \underline{x}} = \underline{\underline{y}}$

$$\begin{pmatrix} 2 & 6 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

Step 2 Solve $L \quad \underline{y} = \underline{b}$

$$\begin{pmatrix} 3 & 0 & 0 \\ 1 & 6 & 0 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 19 \\ 0 \end{pmatrix} \rightarrow \begin{matrix} 3y_1 = 3 \\ y_1 + 6y_2 = 19 \end{matrix}$$

$$\underline{y_1 = 1} \quad \Rightarrow \quad 1 + 6y_2 = 19 \quad \Rightarrow \quad y_2 = 3$$

$$\Rightarrow \quad 2 + 9 + y_3 = 0 \quad \Rightarrow \quad y_3 = -11 \ .$$

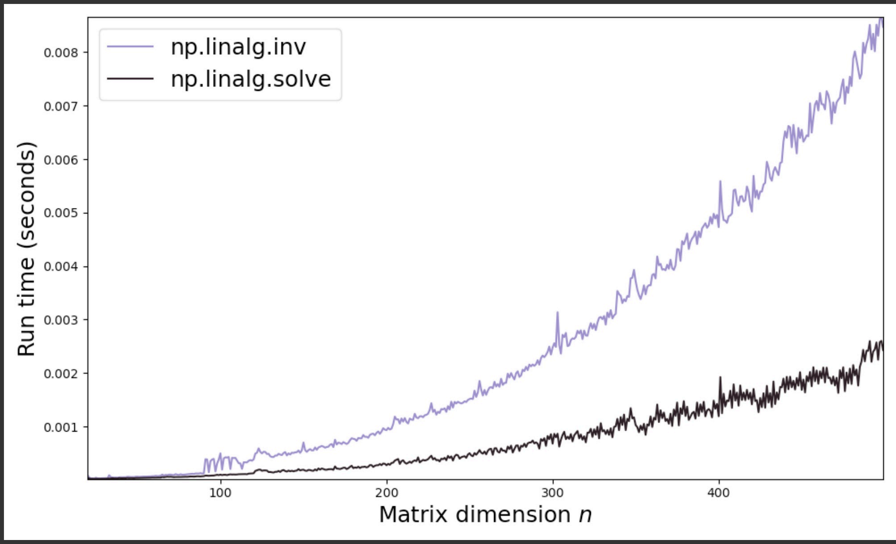
Step 3 Solve $U \underline{x} = \underline{y}$

$$\begin{pmatrix} 2 & 6 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -11 \end{pmatrix} \rightarrow x_3 = -11$$

Thus $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -6 \\ 3 \\ -11 \end{pmatrix}$.

Why LU?

It's faster than inverting a matrix $\longrightarrow$



Careful: - Not every matrix admits an LU decomposition, although using only row operations we can find a form which does. This is called "pivoting". $\det A \neq 0$

- It isn't unique. However, if we insist L is a unit lower triangular matrix, it is.