Computing LU Decomp

We start with an example:

$$\underset{A}{\begin{pmatrix} 3 & 1 \\ -6 & -4 \end{pmatrix}} \overset{?}{=} \underset{L}{\begin{pmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{pmatrix}} \cdot \underset{U}{\begin{pmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{pmatrix}}$$

From this we get equations:

$$\boxed{\begin{array}{ll} u_{11} \cdot l_{11} = 3 & u_{12} \cdot l_{11} = 1 \\ u_{11} \cdot l_{21} = -6 & l_{21} u_{12} + l_{22} \cdot u_{22} = -4 \end{array}}$$

We see here that we have a choice:

l_{11} appears in the two top equations & not again!

We can choose any $l_{11} \neq 0$.

$$\boxed{\text{LU decomp is not unique!}}$$

Let $l_{11} = 1$, then we get $u_{11} = 3$, $u_{12} = 1$

& $l_{21} = -2$.

Again we have a choice for l_{22}, so pick

$l_{22} = 1 \Rightarrow u_{22} = -2$. Hence

$$A = \begin{pmatrix} 3 & 1 \\ -6 & -4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 0 & -2 \end{pmatrix}$$

$$= L \cdot U$$

note if we insist on 1s on diagonal $\Rightarrow$ UNIQUE!

In 3-dim we get the following:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ l_{12} & 1 & 0 \\ l_{13} & l_{23} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$$

$$u_{33} = a_{33} - l_{31} u_{13} - l_{32} u_{23} = \sum_{k=1}^{2} l_{3k} u_{k3}$$

$$= \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ l_{21} u_{11} & l_{21} u_{12} + u_{22} & l_{21} u_{13} + u_{23} \\ l_{31} u_{11} & l_{31} u_{12} + l_{32} u_{22} & \boxed{l_{31} u_{13} + l_{32} u_{23} + u_{33}} \end{pmatrix} \quad 6 \text{ ops}$$

a_{33}

$\| \; a_{33}$

$$l_{21} = \frac{a_{21}}{u_{11}}$$

For general case, see Book 2 p.13 Procedure 2.

$$u_{1j} = a_{1j} \qquad l_{i1} = \frac{a_{i1}}{u_{11}}$$

Note this is just row reduction!

For example:

$$A = \begin{pmatrix} 3 & 1 \\ -6 & -4 \end{pmatrix} \xrightarrow{r_2 \mapsto r_2 + 2r_1} \begin{pmatrix} 3 & 1 \\ 0 & -2 \end{pmatrix} = \underline{U}$$

We express this as matrices:

$$\underbrace{\underset{E}{\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}} \cdot \underset{A}{\begin{pmatrix} 3 & 1 \\ -6 & -4 \end{pmatrix}} = \underset{U}{\begin{pmatrix} 3 & 1 \\ 0 & -2 \end{pmatrix}}}$$

Row reduction matrices

Thus $\underline{L = E^{-1}}$

When is there an LU decomp?

Moral: Whenever a matrix A is non-singular (ie $\det(A)$) there is an LU decomp.

Example: Solve $\begin{pmatrix} 0 & 2 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 9 \end{pmatrix}$

$$\underline{A} \cdot \underline{x} = \underline{b}$$

We can, but our algorithm can't!

Suppose $A = \begin{pmatrix} 1 & 0 \\ l_{21} & 1 \end{pmatrix}\begin{pmatrix} u_{11} & u_{12} \\ 0 & u_{21} \end{pmatrix}$

Then $u_{11} = 0 \Rightarrow \det U = 0$

$$\Rightarrow \det(A) = \det(LU) = \det(L)\det(U) = 0$$

We solve this via (partial) pivoting.

Let $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then

$$PA \begin{pmatrix} x \\ y \end{pmatrix} = P \begin{pmatrix} 8 \\ 9 \end{pmatrix} \Rightarrow \begin{pmatrix} 4 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9 \\ 8 \end{pmatrix}$$

It's just permuting rows so that no u_{ii} is ever 0.

Theorem A a non-singular matrix.
$\exists!$ U upper Δ, L lower Δ w/ 1s on diag & P permutation matrix s.t.

$$PA = LU.$$