ALGEBRA 2
ÜBUNGSBLATT 2

(1) (a) There is a ring isomorphism $\mathbb{Z}/(30) \xrightarrow{\cong} \mathbb{Z}/(5) \times \mathbb{Z}/(6)$. Give an explicit inverse to this map.
(b) Find a number x such that $x \equiv 2 \mod 5$ and $x \equiv 3 \mod 7$.

(2) Let $A[X]$ denote the ring of polynomials in the variable X and coefficients in a ring A. Let $f = a_0 + a_1X + \cdots + a_nX^n$. Prove:
(a) f is a unit iff a_0 is a unit in A and $a_1, \ldots, a_n$ are nilpotent.
(b) f is nilpotent iff $a_0, \ldots, a_n$ are nilpotent.
(c) f is a zero divisor iff there exists a non-zero $a \in A$ such that $af = 0$.
(d) The Jacobson radical of $A[X]$ equals the nilradical of $A[X]$.

(3) Let $\mathfrak{a} \leq A$ be an ideal. Prove that $\mathfrak{a}$ is radical (that is $r(\mathfrak{a}) = \mathfrak{a}$) iff $\mathfrak{a}$ is an intersection of prime ideals.

(4) (a) Let p be a prime number and $e \geq 1$ a positive integer. Prove that $\mathbb{Z}_{p^e}$ is a local ring. Prove that it is a field if and only if $e = 1$.
(b) Let k be a field and $A = k[[x]]$ be the power series ring in one variable. Prove that A is a local ring.

(5) Show that for ideals $\mathfrak{a}, \mathfrak{b}$ of a ring A we have for the radicals

$$r(\mathfrak{a} + \mathfrak{b}) = r(r(\mathfrak{a}) + r(\mathfrak{b})).$$

The exercises on this page, and any marked with a $(*)$, are bonus exercises. They are optional and are intended as helpful foreshadowing for things which come up later and / or offering some of the applications of commutative algebra.

(6*) Let A be a reduced ring (that is, A ia a ring with no nilpotent elements) with finitely many minimal primes $\mathfrak{p}_i$. Prove that the direct sum of the canonical homomorphisms

$$A \longrightarrow \bigoplus_{i=1}^{n} A/\mathfrak{p}_i$$

is injective. Moreover, show that the image intersects each direct summand.

Note: A prime ideal $\mathfrak{p}$ is minimal over an ideal I if $I \subset \mathfrak{p}$ and there is no other prime ideal $\mathfrak{q}$ such that $\mathfrak{q} \subsetneq \mathfrak{p}$. A prime ideal is called minimal if it is minimal over the zero ideal.

(7*) Let $a = (a_1, \ldots, a_n) \in k^n$ where k is a field. Consider the map

$$e_a : k[x_1, \ldots, x_n] \to k, \quad f \mapsto f(a).$$

In other words, e_a is the map given by evaluating polynomials at a point $a \in k^n$. Prove that $\ker e_a = (x_1 - a_1, \ldots, x_n - a_n)$ and conclude that this ideal is maximal.

Hint: First prove this for $a = (0, \ldots, 0)$ and then consider a 'change of coordindate' homomorphism (isomorphism) defined by $x_i \mapsto x_i - a_i$.