```
(%i1)  load(stringproc);
(%o1)  C:/maxima-5.40.0/bin/../share/maxima/5.40.0/share/stringproc/stringproc.mac

(%i2)  declare( trigsimp , evfun),
         declare( trigrat , evfun);
(%o2)  done
```

Depends on operpart
https://github.com/dprodanov/operpart

```
(%i3)  load(operpart);
(%o3)  C:/Dropbox/maxima/operpart.mac

(%i4)  %dir:1;
(%o4)  1
```

Ito-Taylor expansion for Holder 1/2 functions

```
(%i5)  idiff(f, var):=block([t, x,  ret],
         if length(var)#2 then error(var),
       [  t, x]:var,
         f:distrib(f),
         if inop(f)="+" then
           ret:map(lambda([u], idiff(u, var)), f)
         else (
              ret:[ diff(f, t)+ %dir*diff(f,x,2)/2, diff(f, x) ]
       ),
         expand(ret)
       );
(%o5)  idiff(f,var):=block([t,x,ret],if length(var)#2 then error(var) ,[t,x]:var,f:distrib(f),if inop(f)=+ then ret:
map(lambda([u],idiff(u,var)),f) else ret:[d/dt f+(%dir (d^2/dx^2 f))/2,d/dx f],expand(ret))
```

Ito differential, change of varaibles x -> f(x)

```
(%i6)  ito(f, eq, var):=block([t, x, aa, dw],
         [t,x]:var, [aa, dw]: eq,
         [diff(f,t)+aa*diff(f,x)+%dir*dw^2/2*diff(f,x,2), dw*diff(f,x)]
       );
(%o6)  ito(f,eq,var):=block([t,x,aa,dw],[t,x]:var,[aa,dw]:eq,[d/dt f+aa (d/dx f)+(%dir dw^2)/2 (d^2/dx^2 f),dw (d/dx f)])

(%i7)  depends(v,[t,x]);
(%o7)  [v(t,x)]
```

Geodesic equation

```
(%i8)  Q1:ito(v, [v, b], [t,x]);
(Q1)   [(b^2 (d^2/dx^2 v))/2+v (d/dx v)+d/dt v,b (d/dx v)]

(%i9)  depends(F,[t,x]);
(%o9)  [F(t,x)]
```

(%i10)
```
colhop1(expr, v, fn, xx, sp ):=block([l, r,  uu,   sexpr,   dexpr, lv, lsu, kk:[], sol ],
  lv: sublist(listofvars(xx), lambda([uu], freeof(asymbol, uu))),
  lsu: makelist(v= sp[uu]*'diff(log(fn), uu), uu, lv),
[  sexpr, dexpr]:expr,
  sexpr: subst(lsu, sexpr),
  dexpr: subst(lsu, dexpr),
  sexpr: ev (sexpr, diff),
  sexpr: (expand(sexpr)),
  display(sexpr),
  for p in lv do (
[    l, r]:oppart(sexpr, lambda([uu], not freeof ((('diff(fn, p, 1)))^3, uu) )),
    if l # 'nil then  (
      sol: sublist(solve(l, sp[p]), lambda([uu], rhs(uu)#0)),
      /*display(l, sp[p], sol),*/
      push(first(sol), kk)
)
),
  reverse(kk),
  lsu:subst(kk, lsu),
  display(kk, lsu),
  sol:[ expand(subst(kk, sexpr)), expand(subst(kk, dexpr)), lsu]
);
```

(%o10)
```
colhop1(expr,v,fn,xx,sp):=block([l,r,uu,sexpr,dexpr,lv,lsu,kk:[],sol],lv:
sublist(listofvars(xx),lambda([uu],freeof(asymbol,uu))),lsu:makelist(v=sp_uu (d/d uu log(fn)),uu,lv),[sexpr,dexpr]:expr,sexpr:
subst(lsu,sexpr),dexpr:subst(lsu,dexpr),sexpr:ev(sexpr,diff),sexpr:expand(sexpr),display(sexpr),for p in lv do ([l,r]:
oppart(sexpr,lambda([uu],not freeof((d/d p fn)^3,uu))),if l#'nil then
(sol:sublist(solve(l,sp_p),lambda([uu],rhs(uu)#0)),push(first(sol),kk)) ),reverse(kk),lsu:subst(kk,lsu),display(kk,lsu),sol:[
expand(subst(kk,sexpr)),expand(subst(kk,dexpr)),lsu])
```

Homogeneous Cole-Hopf transform

(%i11)
```
Q2:colhop1((Q1), v, F, [x], k);
```

$$sexpr=\frac{\left(\frac{d}{dx}F\right)\left(\frac{d^2}{dx^2}F\right)k_x^2}{F^2}-\frac{\left(\frac{d}{dx}F\right)^3 k_x^2}{F^3}+\frac{\left(\frac{d^3}{dx^3}F\right)b^2 k_x}{2F}-\frac{3\left(\frac{d}{dx}F\right)\left(\frac{d^2}{dx^2}F\right)b^2 k_x}{2F^2}+\frac{\left(\frac{d}{dx}F\right)^3 b^2 k_x}{F^3}-\frac{\left(\frac{d}{dt}F\right)\left(\frac{d}{dx}F\right)k_x}{F^2}+\frac{\left(\frac{d^2}{dt\,dx}F\right)k_x}{F}$$

$$kk=[k_x=b^2]$$

$$lsu=\left[v=\left(\frac{d}{dx}\log(F)\right)b^2\right]$$

(Q2)
$$\left[\frac{\left(\frac{d^3}{dx^3}F\right)b^4}{2F}-\frac{\left(\frac{d}{dx}F\right)\left(\frac{d^2}{dx^2}F\right)b^4}{2F^2}-\frac{\left(\frac{d}{dt}F\right)\left(\frac{d}{dx}F\right)b^2}{F^2}+\frac{\left(\frac{d^2}{dt\,dx}F\right)b^2}{F},b\left(\frac{d}{dx}\left(\left(\frac{d}{dx}\log(F)\right)b^2\right)\right),\left[v=\left(\frac{d}{dx}\log(F)\right)b^2\right]\right]$$

The transformation coefficient is k[x]=b^2
giving v=('diff(log(F),x,1))*b^2

Gradient decomposition

(%i12)
```
colhopcoeff1(expr, fn, var):=block ( [t, x, ret, aa, bb, cc, k:gensym("k"), qq, qp, qp1, f1, f2],
[  t, x]:var,
  f1: first( expr),
  f2: second(expr),
  bb:coeff(f1, 'diff(fn, x, 3))*fn,
  aa:coeff(f1, 'diff(fn, t, 1, x, 1))*fn,
  qp:(aa*'diff(fn,t)+bb*'diff(fn,x,2)),
  qq:  subst(fn=1/sqrt(t)*exp(k*x^2/t), qp ),
  qq: ev(qq, diff),
  sol: sublist(solve(qq, k),  lambda([uu], lfreeof ( [t,x], rhs(uu) ))),
  display(sol),
  qq:subst(sol, 1/sqrt(t)*exp(k*x^2/t)),
  f2: ev(subst(sol, f2), diff),
  f2:expand(f2),
  display(f2),
  cc : coeff(f2, 'diff(fn, x, 2))*fn,
  qp1: cc*'diff(fn,x),
[   ['diff(qp/fn,x),  'diff(qp1/fn, x) ], qq, aa, bb, cc]
);
```

(%o12)
```
colhopcoeff1(expr,fn,var):=block([t,x,ret,aa,bb,cc,k:gensym(k),qq,qp,qp1,f1,f2],[t,x]:var,f1:first(expr),f2:second(expr)
,bb:coeff(f1,d^3/dx^3 fn)fn,aa:coeff(f1,d^2/(dt dx) fn)fn,qp:aa (d/dt fn)+bb (d^2/dx^2 fn),qq:subst(fn=1/sqrt(t)exp(k x^2/t),qp),qq:ev(qq,diff),sol:
sublist(solve(qq,k),lambda([uu],lfreeof([t,x],rhs(uu)))),display(sol),qq:subst(sol,1/sqrt(t)exp(k x^2/t)),f2:ev(subst(sol,f2),diff),f2:
expand(f2),display(f2),cc:coeff(f2,d^2/dx^2 fn)fn,qp1:cc (d/dx fn),[[d/dx qp/fn,d/dx qp1/fn],qq,aa,bb,cc])
```

(%i13) Q31:colhopcoeff1(Q2, F, [t,x]) ;

$$sol=\left[k460=\frac{1}{2\,b^2}\right]$$

$$f2=\frac{\left(\frac{d^2}{d\,x^2}F\right)b^3}{F}-\frac{\left(\frac{d}{d\,x}F\right)^2 b^3}{F^2}$$

(Q31) $$\left[\left[\frac{d}{d\,x}\frac{\frac{\left(\frac{d^2}{d\,x^2}F\right)b^4}{2}+\left(\frac{d}{d\,t}F\right)b^2}{F},\frac{d}{d\,x}\frac{\left(\frac{d}{d\,x}F\right)b^3}{F}\right],\frac{\%e^{\frac{x^2}{2\,b^2\,t}}}{\sqrt{t}},b^2,\frac{b^4}{2},b^3\right]$$

Checks

Drift term

(%i14) first(Q2)-first(Q31)[1];

(%o14) $$-\frac{d}{d\,x}\frac{\frac{\left(\frac{d^2}{d\,x^2}F\right)b^4}{2}+\left(\frac{d}{d\,t}F\right)b^2}{F}+\frac{\left(\frac{d^3}{d\,x^3}F\right)b^4}{2\,F}-\frac{\left(\frac{d}{d\,x}F\right)\left(\frac{d^2}{d\,x^2}F\right)b^4}{2\,F^2}-\frac{\left(\frac{d}{d\,t}F\right)\left(\frac{d}{d\,x}F\right)b^2}{F^2}+\frac{\left(\frac{d^2}{d\,t\,d\,x}F\right)b^2}{F}$$

(%i15) %,diff,expand;

(%o15) 0

Diffusion term

(%i16) second(Q2)-first(Q31)[2];

(%o16) $$b\left(\frac{d}{d\,x}\left(\left(\frac{d}{d\,x}\log(F)\right)b^2\right)\right)-\frac{d}{d\,x}\frac{\left(\frac{d}{d\,x}F\right)b^3}{F}$$

(%i17) %,diff,expand;

(%o17) 0

Extended gradient decomposition, including a drift

```
(%i18) colhopcoeff2(expr, fn, C, var):=block ( [t, x, ret, aa, bb, k:gensym("k"), qq, qp, qq1],
       [  t, x]:var,
         bb:coeff(expr, 'diff(fn, x, 3))*fn,
         aa: coeff(expr, 'diff(fn, t, 1, x, 1))*fn,
         qp:(aa*'diff(fn,t)+bb*'diff(fn,x,2)),
         qq:  subst(fn=1/sqrt(t)*exp(k*x^2/t), qp ),
         qq: ev(qq, diff),
         sol: sublist(solve(qq, k),  lambda([uu], lfreeof ( [t,x], rhs(uu) ))),
         display(sol),
         qq:subst(sol, 1/sqrt(t)*exp(k*x^2/t)),
         qq1: subst(fn=qq *exp(k*t), qp - C*fn ),
         qq1: expand(ev(qq1, diff)),
         qq1: factor(qq1),
         sol: sublist(solve(qq1, k),  lambda([uu], lfreeof ( [t,x], rhs(uu) ))),
         display(sol),
         qq: subst(sol, qq *exp(k*t)),
       [  'diff(qp/fn - C, x),  fn=qq, aa, bb]
       );
```

(%o18) colhopcoeff2(expr,fn,C,var):=block([t,x,ret,aa,bb,k:gensym(k),qq,qp,qq1],[t,x]:var,bb:coeff(expr,$\frac{d^3}{d\,x^3}$fn)fn,aa:coeff(expr,$\frac{d^2}{d\,t\,d\,x}$fn)fn,qp:aa$\left(\frac{d}{d\,t}fn\right)$+bb$\left(\frac{d^2}{d\,x^2}fn\right)$,qq:subst$\left(fn=\frac{1}{\sqrt{t}}\exp\left(\frac{k\,x^2}{t}\right),qp\right)$,qq:ev(qq,diff),sol:sublist(solve(qq,k),lambda([uu],lfreeof([t,x],rhs(uu)))),display(sol),qq:subst$\left(sol,\frac{1}{\sqrt{t}}\exp\left(\frac{k\,x^2}{t}\right)\right)$,qq1:subst(fn=qq exp(k t),qp−C fn),qq1:expand(ev(qq1,diff)),qq1:factor(qq1),sol:sublist(solve(qq1,k),lambda([uu],lfreeof([t,x],rhs(uu)))),display(sol),qq:subst(sol,qq exp(k t)),[$\frac{d}{d\,x}\left(\frac{qp}{fn}-C\right)$,fn=qq,aa,bb])

(%i19) Q3:colhopcoeff2(first(Q2), F, E, [t,x]) ;

$$sol=\left[k478=\frac{1}{2\,b^2}\right]$$

$$sol=\left[k478=\frac{E}{b^2}\right]$$

(Q3) $$\left[\frac{d}{d\,x}\left(\frac{\frac{\left(\frac{d^2}{d\,x^2}F\right)b^4}{2}+\left(\frac{d}{d\,t}F\right)b^2}{F}-E\right),F=\frac{\%e^{\frac{x^2}{2\,b^2\,t}+\frac{E\,t}{b^2}}}{\sqrt{t}},b^2,\frac{b^4}{2}\right]$$

➔ ;