Conventions. “w.r.t.” = with respect to, “i.e.” = in other words, “e.g.” = for example.

1 Introduction

1.1 Overview

Definition. (Description Logic, DL)
Description logics are a family of formal languages designed for knowledge representation and reasoning, and most of these are decidable fragments of FOL.

DLs are more expressive than propositional logic but less expressive than first-order logic. There are general, spatial, temporal, spatiotemporal, and fuzzy descriptions logics, and each description logic features a different balance between DL expressivity and reasoning complexityby supporting different sets of mathematical constructors.

In DLs, constant symbols are called **individuals**, unary relation symbols are called **concepts**, and binary relation symbols are called **roles**. Most description logic starts with one of the following basic description logic:

- $\mathcal{AL}$, called the attributive language
- $\mathcal{FL}$, called the frame based description language
- $\mathcal{EL}$, called the existential language

Followed by any of the semantic extensions labelled by a letter, part of which are listed in wiki:

`https://en.wikipedia.org/wiki/Description_logic#Naming_convention`

There do exist some canonical DLs which don’t exactly fit this convention. Some of them can also be viewed in the above link.

1.2 Components of a DL

syntax including

- alphabet: operators plus logical connectives(logical symbols) and a signature/vocabulary(non-logical symbols plus their arities)
- terms: concept descriptions and role expressions
- well-formed formula: assertions(ABox) and axioms(TBox and RBox)

semantics using the model-theoretic semantics, including a domain and an interpretation function

A set of well-formed formulas of a DL is called a **knowledge base** or an **ontology**[1].

2 $\mathcal{AL}$ and its family

This section is based on [2].

2.1 Syntax

The alphabet includes:

- operators : $\{\neg, \sqcap, \top, \bot\}$
- logical connectives: $\{\forall, \exists, \sqsubseteq, \equiv\}$
- signature : there are 3 sets of symbols, N_C(concept names, unary predicate symbols), N_R(role names, binary predicate symbols) and N_I(individual names, constants). Each element of N_C is called an **atomic concept**.

Definition. (Concept Description/Complex Concept)
A concept description, or simply concept, in $\mathcal{AL}$ is one of the following form:

- $A \in N_C$
- $\neg A$, $A \in N_C$ (atomic negation)
- $\top$ (universal concept or top concept)
- $\bot$ (bottom concept)
- $C \sqcap D$, where C, D are concept descriptions
- $\forall R.C$ (value restriction, like `owl:allValuesFrom`), $R \in N_R$ and C is a concept description
- $\exists R.\top$ (limited existential quantification), $R \in N_R$

Definition. (Well-Formed Formulas of $\mathcal{AL}$)

- ABox assertions: of the form $C(a)$(concept assertions) and $R(a, b)$(role assertions), where C is a concept, $R \in N_R$ and $a, b \in N_I$. A finite set of ABox assertions is called an ABox/world description.
- TBox/terminological axioms: they are of the form $C \sqsubseteq D$ or $C \equiv D$, where C, D are concepts. Axioms of the first form are called **class inclusions**, and axioms of the second form are called **class equalities**. An equality whose left-hand side is an atomic concept is called a **definition**, which are used to introduce symbolic names for complex concept descriptions.

Definition. (TBox)
A set of terminological axioms is called a TBox/T-Box.

TBox serves as a schema, and ABox describes a concrete situation.

2.2 Model-Theoretic Semantics

Definition. (Semantics for $\mathcal{AL}$)
An interpretation $\mathcal{I}$ of $\mathcal{AL}$ consists of a domain $\Delta^{\mathcal{I}}$ and an interpretation function $.^{\mathcal{I}}$ that maps every individual $a \in N_I$ to an element $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$, every concept $A \in N_C$ to a subset $A^{\mathcal{I}} \subset \Delta^{\mathcal{I}}$ and every role $R \in N_R$ to a subset $R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$, such that(here $.^{\mathcal{I}} =$ means the interpretation of left concept is the set on the right)

$$
\begin{aligned}
\top^{\mathcal{I}} &= \Delta^{\mathcal{I}} \\
\bot^{\mathcal{I}} &= \varnothing \\
(\neg A)^{\mathcal{I}} &= \Delta^{\mathcal{I}} \backslash A^{\mathcal{I}} \\
(C \sqcap D)^{\mathcal{I}} &= C^{\mathcal{I}} \cap D^{\mathcal{I}} \\
(\forall R.C)^{\mathcal{I}} &= \{x \in \Delta^{\mathcal{I}} \mid \text{For any } y \in \Delta^{\mathcal{I}},\ \text{if } (x,y) \in R^{\mathcal{I}},\ \text{then } y \in C^{\mathcal{I}}\} \\
(\exists R.\top)^{\mathcal{I}} &= \{x \in \Delta^{\mathcal{I}} \mid \text{There is some } y \in \Delta^{\mathcal{I}} \text{ such that } (x,y) \in R^{\mathcal{I}}\}
\end{aligned}
$$

And

- $\mathcal{I}$ satisfies an ABox assertion $C(a)$, or $C(a)$ is true under $\mathcal{I}$ iff $a^{\mathcal{I}} \in C^{\mathcal{I}}$, denoted as $\mathcal{I} \vDash C(a)$; $\mathcal{I}$ satisfies $R(a,b)$ iff $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in R^{\mathcal{I}}$, denoted as $\mathcal{I} \vDash R(a,b)$
- $\mathcal{I}$ satisfies a TBox axiom $C \sqsubseteq D$, or $C \sqsubseteq D$ is true under $\mathcal{I}$, iff $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$, denoted as $\mathcal{I} \vDash C \sqsubseteq D$. $\mathcal{I}$ satisfies $C \equiv D$ iff $C^{\mathcal{I}} = D^{\mathcal{I}}$.

If an interpretation $\mathcal{I}$ satisfies a set of formula Σ($\mathcal{I}$ satisfies each element of Σ, which is either a TBox axiom or an ABox assertion), then $\mathcal{I}$ is also called a model of Σ.

2.3 Extensions of $\mathcal{AL}$

General complement(indicated by the letter $\mathcal{C}$)
The negation of any concept is added, and interpreted as

$$(\neg C)^{\mathcal{I}} = \Delta^{\mathcal{I}} \backslash C^{\mathcal{I}}$$

Union of concepts(indicated by the letter $\mathcal{U}$)
A new kind of concept description $C \sqcup D$ (C, D are concepts) is added, and interpreted as

$$(C \sqcup D)^{\mathcal{I}} = C^{\mathcal{I}} \cup D^{\mathcal{I}}$$

Full existential quantification(indicated by the letter $\mathcal{E}$)
A new kind of concept description $\exists R.C$ ($R \in N_R, C \in N_C$) is added, and interpreted as

$$(\exists R.C)^{\mathcal{I}} = \{a \in \Delta^{\mathcal{I}} \mid \text{There is some } b \in C^{\mathcal{I}}, (a,b) \in R^{\mathcal{I}}\}$$

This extension corresponds to `owl:someValuesFrom` in OWL.

Number restrictions(indicated by the letter $\mathcal{N}$)
Two new kinds of concept description $\leqslant n\,R$ and $\geqslant n\,R\,(R \in N_R, n \in \mathbb{N})$ are introduced, and interpreted as(# is used to denote the cardinality of a set)

$$(\geqslant n\,R)^{\mathcal{I}} = \{a \in \Delta^{\mathcal{I}}\,|\#\{b \in \Delta^{\mathcal{I}}\,|(a,b) \in R^{\mathcal{I}}\} \geq n\}$$
$$(\leqslant n\,R)^{\mathcal{I}} = \{a \in \Delta^{\mathcal{I}}\,|\#\{b \in \Delta^{\mathcal{I}}\,|(a,b) \in R^{\mathcal{I}}\} \leq n\}$$

This extension corresponds to `owl:maxCardinality` and `owl:minCardinality`.

Qualified Number Restriction(indicated by the letter $\mathcal{Q}$)
Two new kinds of concept $\leqslant n\,R.C$ and $\geqslant n\,R.C\,(n \in \mathbb{N}, R \in N_R, C$ a concept) are included, and interpreted as

$$(\geqslant n\,R.C)^{\mathcal{I}} = \{a \in \Delta^{\mathcal{I}}\,|\#\{b \in C^{\mathcal{I}}|(a,b) \in R^{\mathcal{I}}\} \geq n\}$$

Object Role transitivity(not indicated by a letter; $\mathcal{ALC}$ plus this extension is denoted as $\mathcal{S}$)
A new kind of axiom $\mathsf{Tran}(R)\,(R \in N_R)$ is added(and thus the operator Tran is added into the alphabet), and

$$\mathcal{I} \vDash \mathsf{Tran}(R) \text{ iff } R^{\mathcal{I}} = \left(R^{\mathcal{I}}\right)^{+}$$

here $^{+}$ denotes the transitive closure.

Role Hierarchy(indicated by the letter $\mathcal{H}$)
Formulas like $R \sqsubseteq S\,(R, S \in N_R)$ are included, and $\mathcal{I} \vDash R \sqsubseteq S$ iff $R^{\mathcal{I}} \subseteq S^{\mathcal{I}}$.

Role Constructors(indicated by the letter $\mathcal{R}$)
This is also known as limited complex role inclusion axioms.

- there is a special role name $U \in N_R$ called the **universal role** which relates all pairs of individuals; i.e. $U^{\mathcal{I}} = \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$.
- a new kind of terms called **role expression** is added, of which the definition varies considering the constructors involved.
- new forms of formulas called role axioms are added, used to declare characteristics of a role or disjointness of two roles. A finite set of such formulas is called a **RBox**.
- for decidability purpose, there is a subset of role expressions called **safe** role expressions.

As shown above, $\mathcal{R}$ encompasses $\mathcal{H}$ and role transitivity. For details of this part, see chapter 3 and 5 of [3].

Nominals(indicated by the letter $\mathcal{O}$)
In this extension, special atomic concepts A_a are added, each of which forms a concept, called a **nominal**, and interprted as the set of a instance it names, i.e., $A_a^{\mathcal{I}} = \{a \in \Delta^{\mathcal{I}}\}$(sometimes also written as $\{o\}^{\mathcal{I}} = \{o^{\mathcal{I}}\}$, $o \in N_I$). With the presence of $\mathcal{U}$, a closed class(enumerated class) can be expressed as a union of nominals:

$$\{a_1\} \sqcup \cdots \sqcup \{a_n\},\ a_1, \ldots, a_n \in \Delta^{\mathcal{I}}$$

Also note that with $\mathcal{N}$, we can express individuals are different from each other:

$$\{a_1\} \sqcap \cdots \sqcap \{a_n\} \sqsubseteq \bot$$

Inverse role(indicated by the letter $\mathcal{I}$)
New terms of the form R^- ($R \in N_R$) are added, and interpreted as

$$(R^-)^{\mathcal{I}} = \{(b, a) | (a, b) \in R^{\mathcal{I}}\}$$

Concrete Domain/Datatype(indicated by the letter (**D**) or $^{(\mathcal{D})}$)
As state in [4],

> Concrete domains integrate DLs with concrete sets such as the real numbers, integers, or strings, as well as concrete predicates defined on these sets, such as numerical comparisons (e.g., $\leq$), string comparisons (e.g., isPrefixOf), or comparisons with constants(e.g., $\leq$17).

Role Functionality(indicated by the letter $\mathcal{F}$)
with this extension, a role can be stated to be functional $\mathsf{Func}(R)$ as an axiom. However, this can also be done via $\mathcal{N}$:

$$\top \sqsubseteq \leqslant 1R$$

Extending $\mathcal{AL}$ by any subset of the above constructors yields a particular $\mathcal{AL}$-language. We name each $\mathcal{AL}$-language by a string of the form

$$\mathcal{AL}[\mathcal{U}][\mathcal{E}][\mathcal{N}][\mathcal{C}]\ldots$$

where a letter in the name stands for the presence of the corresponding constructor. For instance, $\mathcal{ALEN}$ is the extension of $\mathcal{AL}$ by full existential quantification and number restrictions. Especially, we have the following table(p167 of [5])

OWL languages	underlying DL
OWL DL	$\mathcal{SHOIN}(D)$
OWL Lite	$\mathcal{SHIF}(D)$
OWL 2 DL	$\mathcal{SROIQ}(D)$
OWL 2 EL	$\mathcal{EL}^{++}$
OWL 2 RL	$\mathcal{DLP}$
OWL 2 QL	DL-Lite

A translation from $\mathcal{AL}$ to first order logic can be seen in Ch 2.2.1.3 of [6].

2.4 Inferences

This section is just a nutshell of ch 2.2.4 in [6].
Suppose $\mathcal{T}, \mathcal{A}, \mathcal{R}$ are a TBox, ABox, RBox, respectively. For a knowledge base $\mathcal{K} = \mathcal{T} \cup \mathcal{A} \cup \mathcal{R}$ and a set of formulas Σ, if every model of $\mathcal{K}$ is also a model of Σ, then we say $\mathcal{K}$ **entails** Σ, denoted as $\mathcal{K} \vDash \Sigma$; otherwise, it's denoted as $\mathcal{K} \nvDash \Sigma$. If $\mathcal{K}$ is empty, then it's simply denoted as $\vDash \Sigma$, meaning Σ is true under any interpretation(so if no interpretation can satisfy Σ, then it's denoted as $\nvDash \Sigma$). Two formulas ϕ, ψ are said to be **semantically equivalent** iff $\phi \vDash \psi$ and $\psi \vDash \phi$, denoted as $\phi \,\Dashv\vDash\, \psi$.

The following are some common entailments:

$$\begin{aligned}
C \sqsubseteq D &\,\Dashv\vDash\, (C \sqcap \neg D) \sqsubseteq \bot \\
C \sqsubseteq D &\,\Dashv\vDash\, \neg D \sqsubseteq \neg C \\
C \sqsubseteq D &\,\Dashv\vDash\, \top \sqsubseteq (\neg C \sqcup D)
\end{aligned}$$

If a formula is true under any interpretation, then it is called valid or a tautology. The following are some common tautologies(C, D are concepts, R is a role):

$$\begin{aligned}
\neg\neg C &\equiv C \\
\neg(C \sqcap D) &\equiv \neg C \sqcup \neg D \\
\neg \forall R.C &\equiv \exists R.\neg C \\
\neg \leqslant nR.C &\equiv \geqslant (n+1)R.C \ (n \in \mathbb{N}) \\
\neg \geqslant 0R.C &\equiv \bot
\end{aligned}$$

2.4.1 Reasoning for concepts

Typical types of inferences for concepts in DL are the following:

Satisfiability A concept C is satisfiable or consistent w.r.t. $\mathcal{T}$ iff there exists a model $\mathcal{I}$ of $\mathcal{T}$ such that $C^{\mathcal{I}} \neq \varnothing$, and $\mathcal{I}$ is called a model of C.

Subsumption A concept C is subsumed by a concept D w.r.t. $\mathcal{T}$ iff $\mathcal{T} \vDash C \sqsubseteq D$.

Equivalence Two concepts C and D are equivalent w.r.t. $\mathcal{T}$ iff $\mathcal{T} \vDash C \equiv D$.

Disjointness Two concepts C and D are disjoint w.r.t. $\mathcal{T}$ iff $\mathcal{T} \vDash C \sqcap D \sqsubseteq \bot$.

The aforementioned inferencing tasks can be reduced to only subsumption in $\mathcal{AL}$:

Proposition. (Reduction to Subsumption)
Suppose C and D are two concepts, then the following statements hold w.r.t. $\mathcal{T}$:

- C is unsatisfiable iff C is subsumed by $\bot$; i.e. the unsatisfiability of C expressed in DL is $\mathcal{T} \vDash C \sqsubseteq \bot$

- C and D are equivalent iff C is subsumed by D and D is subsumed by C; i.e. $\mathcal{T} \vDash C \equiv D$ iff $\mathcal{T} \vDash C \sqsubseteq D$ and $\mathcal{T} \vDash D \sqsubseteq C$.

- C and D are disjoint iff $C \sqcap D$ is subsumed by $\bot$.

In $\mathcal{ALCC}$, with general negation, the other 3 inferencing tasks can be reduced to unsatisfiability:

Proposition. (Reduction to Unsatisfiability)
Suppose C and D are two concepts. Then the following statements hold:

- $\mathcal{T} \vDash C \sqsubseteq D$ iff $\mathcal{T} \cup \{(C \sqcap \neg D)(a)\}$ is unsatisfiable, where a is a new individual not occurring in $\mathcal{K}$.

- C and D are equivalent iff both $(C \sqcap \neg D)$ and $(\neg C \sqcap D)$ are unsatisfiable; i.e. $\mathcal{T} \vDash C \equiv D$ iff $\mathcal{T} \vDash (C \sqcap \neg D) \sqsubseteq \bot$ and $\vDash (\neg C \sqcap D) \sqsubseteq \bot$.

- C and D are disjoint iff $C \sqcap D$ is unsatisfiable.

note the basic fact that $A \subseteq B$ iff $A \cap B^c = \varnothing$.

2.4.2 Reasoning for assertions

After checking the satisfiability of the schema, it's time to depict the concrete situation. An ABox, denoted as $\mathcal{A}$ in this section, is said to be **consistent** w.r.t $\mathcal{T}$ iff there is a model of $\mathcal{A}$ and $\mathcal{T}$. $\mathcal{A}$ is simply called consistent if it's consistent w.r.t. an empty TBox.

Obviously, for a concept C and an individual a, $\mathcal{A} \vDash C(a)$ iff $\mathcal{A} \cup \{\neg C(a)\}$ is inconsistent. This is called **instance chekcing**.

2.5 Natural Deduction

It seems that the Gentzen deduction system is rarely used in description logics; still, such a deduction system for $\mathcal{ALC}$, an extension of $\mathcal{AL}$, can be found in section 2.2.1.3 of [7], and another can be found in [8].

3 OWL

Web ontology language now is developed to its second version, OWL 2. Its proper subset, OWL, is indeed a semantic extension of RDF(just like RDFS). The sublanguages of OWL, OWL DL and OWL Lite, are also description logics $\mathcal{SHOIN}(D)$ and $\mathcal{SHIF}(D)$, respectively. They can be translated into RDF(see W3C's documentation [9]).

The introduction of $\mathcal{SROIQ}(D)$, the underlying DL of OWL 2 DL, can be found in section 3 of [1]. As stated by Dr. Leslie F. Sikos[10],

"($\mathcal{SROIQ}(D)$) It's one of the most expressive **decidable** description logic to date..."

To check the decidability of different DLs, see [11].

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