Preliminary: partially ordered set, relation, the concept of formulas positive in a predicate R

1 Foundation

1.1 Fixed points in complete lattice

Notation.

- $\cup S$: the join of a subset S of a poset.
- $\cap S$: the meet of a subset S of a poset.

Definition. (endofunction)
An endofunction on a set L is a function from L to L.

Definition. Let F be an endofunction on a poset L,

- F is **monotone** if $x \leq y$ implies $F(x) \leq F(y)$, for all x, y.
- $x \in L$ is a **pre-fixed point** of F if $F(x) \leq x$; $x \in L$ is a **post-fixed point** of F if $x \leq F(x)$.
- A **fixed point** of F is an element x that is both a pre-fixed and post-fixed point, i.e. $F(x) = x$. In the set of fixed points of F, the least element and the greatest element are called the least fixed point of F (denoted as $\texttt{lfq}(F)$) and greatest fixed point of F (denoted as $\texttt{gfq}(F)$).

Definition. (Complete Lattice)
A complete lattice is a poset s.t. all its subsets have a join.

Example. The power set of any set is a complete lattice with the partial relation as $\subseteq$.

Proposition. All subsets of a complete lattice have a meet.

Proof. Denote the complete lattice as L and consider a subset S of L. Suppose Y is the set of all lower bounds of S, i.e. $Y = \{l \in L : l \leq s, \forall s \in S\}$. we prove that the join of Y is the meet of S. Since S is a set of upper bounds of Y, $\cup Y \subseteq S, \forall s \in S$. Thus $\cup Y$ is a lower bound of S. Since Y is the set of all lower bounds of S and $\cup Y$ is the greatest one, $\cup Y = \cap S$. $\square$

Proposition. A complete lattice always has a greatest and least element.

Denote them as $\top$ and $\bot$ resp.

Proposition. For a complete lattice L and a monotone endofunction F on L,

- if S is an arbitrary subset of post-fixed points, $\cup S$ is also a post-fixed point.

- if S is an arbitrary subset of pre-fixed points, $\cap S$ is also a pre-fixed point.

Proof. Only prove for one side. Consider the set $Y = \{x \in S : F(x)\}$. Since L is a complete lattice, Y must have a join f s.t. $\forall x \in S, x \leq F(x) \leq f$, i.e. f is a upper bound of S, and thus $\cup S \leq f$. Because $F(\cup S)$ is a upper bound of Y, we have $f \leq F(\cup S)$. Thus $\cup S \leq f \leq F(\cup S)$. $\square$

Note that under the same setting, $\cap S$ may not be a post-fixed point.

Example. $L = \{a, b, c, d, e\}$, $a \leq b$, $b \leq c$, $b \leq d$, $c \leq e$, $d \leq e$. Let $F(b) = a$ and $F(x) = x$ for $x \neq b$. Consider the subset $\{c, d\}$, where $\cup\{c, d\} = b$.

Theorem. (Knaster-Tarski Theorem, or Fixed-point Theorem)
On a complete lattice L, a monotone endofunction $F : L \mapsto L$ has a complete lattice of fixed points. In particular the least fixed point of the function is the meet of all its pre-fixed points, and the greatest fixed point is the join of all the post-fixed points.

Proof. First part: Denote S the set of fixed points of L. Consider a subset $X \subseteq S$, and take the set Y of pre-fixed points that are also upper bounds of X:

$$Y \triangleq \{y \in L | F(y) \leq y, \, \forall x \in X, \, x \leq y\}$$

Since X is a set of lower bounds for Y in L, $x \leq \cap Y, \forall x \in X$. Now we need to show that $\cap Y \in S$. According to the above proposition, since Y is a set of pre-fixed points, so is $\cap Y$, i.e. $F(\cap Y) \leq \cap Y$. And for all x, $x \leq \cap Y \implies x = F(x) \leq F(\cap Y) \implies F(\cap Y) \in Y$, which indicates that $\cap Y \leq F(\cap Y)$. Thus $\cap Y \in S$. Since the set of upper bounds of X in S is a subset of Y, $\cap Y$ is the least upper bound of X in S.

Second part: note that $\{l \in L | \, F(l) \subseteq l\}$ is the set of all pre-fixed points, and according to the proposition above,

$$F\left(\cap\{l \in L | \, F(l) \leq l\}\right) \subseteq \cap\{l \in L | \, F(l) \leq l\}$$

since F is monotone,

$$F\left(F\left(\cap\{l \in L | \, F(l) \leq l\}\right)\right) \subseteq F\left(\cap\{l \in L | \, F(l) \leq l\}\right)$$

which means

$$\begin{aligned} & F\left(\cap\{l \in L | \, F(l) \leq l\}\right) \in \{l \in L | \, F(l) \leq l\} \\ & \implies \cap\{l \in L | \, F(l) \leq l\} \subseteq F\left(\cap\{l \in L | \, F(l) \leq l\}\right) \end{aligned}$$

and thus

$$F\left(\cap\{l \in L | \, F(l) \leq l\}\right) = \cap\{l \in L | \, F(l) \leq l\}$$

likewise, we can prove that

$$F\left(\cup\{l \in L | \, l \leq F(l)\}\right) = \cup\{l \in L | \, l \leq F(l)\}$$

and it's easy to prove that $\cap\{l \in L | \, F(l) \leq l\}$ is the least among all fixed points and $\cup\{l \in L | \, S \subseteq F(S)\}$ is the greatest. $\square$

A typical example in model checking is that $L = 2^S$ and $F : 2^S \mapsto 2^S$, where S is the set of states.

1.2 Constructive Solution

First see section 2.8 of [1]. Please pay attention to the recommended exercise.

Example. An example of cocontinuous but not continuous function:

$$F(x) = \begin{cases} 0 & x \in [0,1) \\ 1 & x \in [1,2] \end{cases}$$

note that $F\left(\bigcup_{n\in\mathbb{N}}\{1 - \frac{1}{n+1}\}\right) = F(1) = 1 \neq \bigcup_{n\in\mathbb{N}} F\left(1 - \frac{1}{n+1}\right) = 0$. The case for continuous but not cocontinuous is similar.

Continuity/Cocontinuity Theorem is the key to the approximate semantics of the least and greatest fixed point operator μ and ν in modal mu-calculus.

Proof. Prove F is continuous $\implies \texttt{lfq}(F) = \bigcup_{n\geq 0} F^n(\perp)$: note that

$$\perp \leq F(\perp) \leq F^2(\perp) \leq \dots$$

is a sequence of increasing points, and since F is continuous, we know that

$$F\left(\bigcup_{n\geq 0} F^n(\perp)\right) = \bigcup_{n\geq 1} F^n(\perp) = \bigcup_{n\geq 0} F^n(\perp)$$

i.e. $\bigcup_{n\geq 0} F^n(\perp)$ is a fixed point. Because

$$\perp \leq \texttt{lfq}(F) \implies F^n(\perp) \leq \texttt{lfq}(F), \forall n \in \mathbb{N} \implies \bigcup_{n\geq 0} F^n(\perp) \leq \texttt{lfq}(F)$$

there must be $\texttt{lfq}(F) = \bigcup_{n\geq 0} F^n(\perp)$. The other proof is similar. □

2 Modal μ-calculus

Please refer to [2]. The following is just a note.

2.1 syntax

Syntactically there are 3 sets: the set of action symbols $\texttt{Act} = \{a, b, \dots\}$, the set of propositions $\texttt{Prop} = \{p_i | i \in \mathbb{N}\}$ and the set of variables $\texttt{Var} = \{X, Y, Z, \dots\}$ whose intended meanings are sets of states.

2.2 Denotational Semantics

The denotational semantics of a formula in a transition system is a set of states satisfying the formula(note that a variable is also a formula). Formally, given a transition system $\mathcal{M} = \langle S, \{R_a\}_{a\in Act}, \{P_i\}_{i\in\mathbb{N}}\rangle$ (S is a set of states, $R_a \subseteq S \times S$ defining transitions for action a and P_i is the set of states where p_i holds) and a valuation $\mathcal{V} : \texttt{Var} \to 2^S$, the semantics is as follow(α, β, ϕ are formulas):

$$
\begin{aligned}
&[\![X]\!]^{\mathcal{M}}_{\mathcal{V}} = \mathcal{V}(X),\ \forall X \in \texttt{Var}\\
&[\![p_i]\!]^{\mathcal{M}}_{\mathcal{V}} = P_i,\ [\![\neg p_i]\!]^{\mathcal{M}}_{\mathcal{V}} = S - P_i,\ \forall p_i \in \texttt{Prop}\\
&[\![\alpha \vee \beta]\!]^{\mathcal{M}}_{\mathcal{V}} = [\![\alpha]\!]^{\mathcal{M}}_{\mathcal{V}} \cup [\![\beta]\!]^{\mathcal{M}}_{\mathcal{V}}\\
&\dots
\end{aligned}
$$

For a valuation $\mathcal{V}$, denote $\mathcal{V}[X \mapsto T]$ the valuation that maps X to $T \subseteq S$ while preserving the mappings of $\mathcal{V}$ elsewhere. If ϕ is a formula containing variable X,

$$
\begin{aligned}
[\![\mu X.\phi]\!]^{\mathcal{M}}_{\mathcal{V}} &= \bigcap\{T \subseteq S : [\![\phi]\!]^{\mathcal{M}}_{\mathcal{V}[X\mapsto T]} \subseteq T\}\\
[\![\nu X.\phi]\!]^{\mathcal{M}}_{\mathcal{V}} &= \bigcup\{T \subseteq S : T \subseteq [\![\phi]\!]^{\mathcal{M}}_{\mathcal{V}[X\mapsto T]}\}
\end{aligned}
$$

Note that for any formula ϕ where the variable X occurs positively, $[\![\phi]\!]^{\mathcal{M}}_{\mathcal{V}[X\mapsto T]} = F(T) : \mathcal{P}(S) \mapsto \mathcal{P}(S)$ is a monotone endofunction on the complete lattice $\mathcal{P}(S)$. So actually,

$$
\begin{aligned}
[\![\mu X.\phi]\!]^{\mathcal{M}}_{\mathcal{V}} &= \texttt{lfq}([\![\phi]\!]^{\mathcal{M}}_{\mathcal{V}[X\mapsto T]})\\
[\![\nu X.\phi]\!]^{\mathcal{M}}_{\mathcal{V}} &= \texttt{gfq}([\![\phi]\!]^{\mathcal{M}}_{\mathcal{V}[X\mapsto T]})
\end{aligned}
$$

For convenience, we use $\mathcal{M}, \mathcal{V}, s \vDash \phi$ instead of $s \in [\![\phi]\!]^{\mathcal{M}}_{\mathcal{V}}$. When $S = [\![\phi]\!]^{\mathcal{M}}_{\mathcal{V}}$, we simply omit s and write $\mathcal{M}, \mathcal{V} \vDash \phi$.

Example. $\mathcal{M}, \mathcal{V}, s_0 \vDash \mu X.[a]X$ means that all sequences of a-transitions starting at s_0 are finite. To see this, first note that

$$[\![\mu X.[a]X]\!]^{\mathcal{M}}_{\mathcal{V}} = \bigcap\{T \subseteq S : \{s \in S : \forall s'.(s, s') \in R_a \implies s' \in T\} \subseteq T\}$$

Proof by contradiction. Suppose there are one or more infinite sequences of a-transitions starting with s_0, and consider a set Y obtained by removing all states in these infinite sequences from S. Then it can be showed that

$$Y \in \{T \subseteq S : \{s \in S : \forall s'.(s, s') \in R_a \implies s' \in T\} \subseteq T\}$$

This means $s_0 \in Y$, which contradicts with the construction of Y.

And $\mathcal{M}, \mathcal{V}, s_0 \vDash \nu X.[a]X$ means that there is an infinite sequence of a-transitions starting at s_0.

$$[\![\nu X.[a]X]\!]^{\mathcal{M}}_{\mathcal{V}} = \bigcup\{T \subseteq S : T \subseteq \{s \in S : \exists s'.(s, s') \in R_a \wedge\ s' \in T\}\}$$

Then there exists a set $Y \in \{T \subseteq S : T \subseteq \{s \in S : \exists s'.(s,s') \in R_a \wedge s' \in T\}\}$ s.t. $s_0 \in Y \subset \{s \in S : \exists s'.(s,s') \in R_a \wedge s' \in Y\}$. Thus there is an infinite sequence of a-transitions starting from s_0.

2.3 approximate

By using the fact that

$$\bigcup_{i\in\mathbb{N}}(A_i \cap B_i) = \bigcup_{i\in\mathbb{N}} A_i \cap \bigcup_{i\in\mathbb{N}} B_i$$

for any increasing sequence A_i and B_i, we can show that $F(T) = [\![\phi]\!]^{\mathcal{M}}_{\mathcal{V}[X\mapsto T]}$ is a continuous and cocontinuous function for any formula ϕ where X occurs positively, so

$$\begin{aligned}
\bigcup_{i\in\mathbb{N}} F^i(\bot) &= \mathtt{lfq}(F(T)) = [\![\mu X.\phi]\!]^{\mathcal{M}}_{\mathcal{V}} \\
\bigcap_{i\in\mathbb{N}} F^i(\top) &= \mathtt{gfq}(F(T)) = [\![\nu X.\phi]\!]^{\mathcal{M}}_{\mathcal{V}}
\end{aligned}$$

So for a natural number k, we can let $\mu^k X.\phi$ be formulas, with the semantics

$$\begin{aligned}
[\![\mu^0 X.\phi]\!]^{\mathcal{M}}_{\mathcal{V}} &= F^0(\bot) = \varnothing \\
[\![\mu^k X.\phi]\!]^{\mathcal{M}}_{\mathcal{V}} &= F^k(\bot) = [\![\phi]\!]^{\mathcal{M}}_{\mathcal{V}[X\mapsto [\![\mu^{k-1}X.\phi]\!]^{\mathcal{M}}_{\mathcal{V}}]}
\end{aligned}$$

In a similar way, we can let $\nu^k X.\phi\,(k \in \mathbb{N})$ be formulas:

$$\begin{aligned}
[\![\nu^0 X.\phi]\!]^{\mathcal{M}}_{\mathcal{V}} &= F^0(\top) = S \\
[\![\nu^k X.\phi]\!]^{\mathcal{M}}_{\mathcal{V}} &= F^k(\top) = [\![\phi]\!]^{\mathcal{M}}_{\mathcal{V}[X\mapsto [\![\nu^{k-1}X.\phi]\!]^{\mathcal{M}}_{\mathcal{V}}]}
\end{aligned}$$

and have

$$\begin{aligned}
\mathcal{M},\mathcal{V},s \vDash \mu X.\phi &\iff \mathcal{M},\mathcal{V},s \vDash \mu^k X.\phi \text{ for some } k \in \mathbb{N} \\
\mathcal{M},\mathcal{V},s \vDash \nu X.\phi &\iff \mathcal{M},\mathcal{V},s \vDash \nu^k X.\phi \text{ for any } k \in \mathbb{N}
\end{aligned}$$

Besides, we can prove the substitution theorem; i.e. for a formula $\phi(X)$ with a variable X and a formula Y,

$$[\![\phi(Y/X)]\!]^{\mathcal{M}}_{\mathcal{V}} = [\![\phi]\!]^{\mathcal{M}}_{\mathcal{V}[X\mapsto [\![Y]\!]^{\mathcal{M}}_{\mathcal{V}}]}$$

Also, note that

$$\begin{aligned}
[\![\mathtt{True}]\!]^{\mathcal{M}}_{\mathcal{V}} &= [\![p_i \vee \neg p_i]\!]^{\mathcal{M}}_{\mathcal{V}} = S \\
[\![\mathtt{False}]\!]^{\mathcal{M}}_{\mathcal{V}} &= [\![p_i \wedge \neg p_i]\!]^{\mathcal{M}}_{\mathcal{V}} = \varnothing
\end{aligned}$$

thus we can prove that, syntactically speaking,

$$\mu^0 X.\phi \equiv \texttt{False}$$
$$\mu^k X.\phi \equiv \phi(\mu^{k-1} X.\phi / X)$$

and

$$\nu^0 X.\phi \equiv \texttt{True}$$
$$\nu^k X.\phi \equiv \phi(\nu^{k-1} X.\phi / X)$$

3 Fixed Point Logics

This section is a more friendly overview of the preliminary part in [3]. Fixed point logics are extensions of first-order logic with fixed point operators.

Let $\phi(R, \mathbf{x})$ be a FOL formula containing a relation symbol R of arity m and a tuple of free variables $\mathbf{x}$ whose length is m. Note that $\mathbf{x}$ is unnecessarily the argument tuple of R and there can be free variables other than $\mathbf{x}$. Suppose $M = (\Delta, \sigma, \mathcal{I})$ is a model and v a variable assignment interpreting all symbols in ϕ other than R and $\mathbf{x}$. Consider an endofunction

$$F(P) = \{\mathbf{a} \in \Delta^m | M, R^{\mathcal{I}} = P, v[\mathbf{x} \mapsto \mathbf{a}] \vDash \phi\}$$

where $P \in \Delta^m$ is an interpretation of R. Under the partial order of $\subseteq$, Δ^m is a complete lattice. And it can be proved that, if R occurs in ϕ positively, F is a monotone(or see *Lyndon's Theorem*), continuous and cocontinuous function.

Now we can add a new kind of formulas; if R is a m-ary relation variable, $\mathbf{x}$ is a m-tuple of first order variables, $\mathbf{t}$ is a m-tuple of terms and ϕ is a formula in which R occurs only positively, then

$$[\mu_{R,\mathbf{x}}.\phi](\mathbf{t})$$
$$[\nu_{R,\mathbf{x}}.\phi](\mathbf{t})$$

are formulas, where we say R and $\mathbf{x}$ are bound. The semantics are:

$$M, v \vDash [\mu_{R,\mathbf{x}}.\phi](\mathbf{t}) \iff \mathbf{t}^{\mathcal{I}}[v] \in \texttt{lfq}(F) = \bigcup_{i \in \mathbb{N}} F^i(\bot)$$
$$M, v \vDash [\nu_{R,\mathbf{x}}.\phi](\mathbf{t}) \iff \mathbf{t}^{\mathcal{I}}[v] \in \texttt{gfq}(F) = \bigcap_{i \in \mathbb{N}} F^i(\top)$$

where $\bot \doteq \varnothing$, $\top \doteq \Delta^m$ and F is defined as above. Using similar ideas of approximation in modal μ-calculus, $[\mu^k_{R,\mathbf{x}}.\phi](\mathbf{t})$ and $[\nu^k_{R,\mathbf{x}}.\phi](\mathbf{t})$ ($k \in \mathbb{N}$) are formulas with the semantics

$$M, v \vDash [\mu^k_{R,\mathbf{x}}.\phi](\mathbf{t}) \iff \mathbf{t}^{\mathcal{I}}[v] \in F^k(\bot)$$
$$M, v \vDash [\nu^k_{R,\mathbf{x}}.\phi](\mathbf{t}) \iff \mathbf{t}^{\mathcal{I}}[v] \in F^k(\top)$$

We can regard $[\mu^k_{R,\mathbf{x}}.\phi]$ and $[\nu^k_{R,\mathbf{x}}.\phi]$ as relation symbols with arity m interpreted as $F^k(\bot)$ and $F^k(\top)$, respectively. Since

$$M, v, R^{\mathcal{I}} = T^{\mathcal{I}} \vDash \phi \iff M, v \vDash \phi[T/R]$$

where T is a relation symbol interpreted in M. Thus, we have(similarly for ν)

$$\begin{aligned} M, v \vDash \phi([\mu^k_{R,\mathbf{x}}.\phi]/R, \mathbf{t}) &\iff M, v, R^{\mathcal{I}} = F^k(\bot) \vDash \phi(R, \mathbf{t}) \\ \iff \mathbf{t} \in F(F^k(\bot)) &\iff M, v \vDash [\mu^{k+1}_{R,\mathbf{x}}.\phi](\mathbf{t}) \end{aligned}$$

Finally, syntactically we have

$$\begin{aligned} &[\mu^0_{R,\mathbf{x}}.\phi](\mathbf{t}) \equiv \texttt{False} \\ &[\mu^{k+1}_{R,\mathbf{x}}.\phi](\mathbf{t}) \equiv \phi([\mu^k_{R,\mathbf{x}}.\phi]/R, \mathbf{t}) \end{aligned}$$

and

$$\begin{aligned} &[\nu^0_{R,\mathbf{x}}.\phi](\mathbf{t}) \equiv \texttt{True} \\ &[\nu^{k+1}_{R,\mathbf{x}}.\phi](\mathbf{t}) \equiv \phi([\nu^k_{R,\mathbf{x}}.\phi]/R, \mathbf{t}) \end{aligned}$$

References

[1] D. Sangiorgi, *Introduction to bismulation and coinduction.* Cambridge University Press, 2011.

[2] J. Bradfield and I. Walukiewicz, “The mu-calculus and model-checking.” `https://www.labri.fr/perso/igw/Papers/igw-mu.pdf`. (Accessed on 11/22/2018).

[3] A. Dawar and Y. Gurevich, “Fixed point logics,” *Bulletin of Symbolic Logic*, vol. 8, pp. 65–88, 2002.