Definition. (Norm)
For a vector space X over a subfield of complex numbers, if a map $\|.\| : X \mapsto \mathbb{R}$ satisfies the following so-called axioms of norm:
(i) $\|x\| \geq 0, \forall x \in X$, and $\|x\| = 0 \Leftrightarrow x = \overrightarrow{0} \in X$ (positive definite)
(ii) $\|\alpha x\| = |\alpha| . \|x\|, \forall \alpha \in K, \forall x \in X$ (absolutely homogenous)
(iii)$\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality)
Then $\|.\|$ is called a **norm** on X. A vector space on which a norm is defined is called a **normed vector space(线性赋范空间)**.

relationship between metric and norm
Proposition. For a normed vector space $(X, \|.\|)$, a map $d : X \times X \mapsto \mathbb{R}$ defined as $d(x, y) = \|x - y\|$ is a metric on X.

Theorem. For a vector space X over the number field K, a metric d on X is a norm(satisfies the axioms of norm) iff d satisfies the following properties:
$d(x - y, 0) = d(x, y), \forall x, y \in X$
$d(\alpha x, 0) = |\alpha| d(x, 0), \forall x \in X, \forall \alpha \in K$

Lemma. (Young's inequality)
If $1 < p, q < +\infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, then for any $a, b \geq 0$,
$$\frac{a^p}{p} + \frac{b^q}{q} \geq ab$$
It can be proved by considering the monotonicity of the function, by regarding a as a variable and b a constant.

Corollary. If $f, g \in L^p(E)$, then $fg \in L^p(E)$.

Lemma. (Hölder's inequality)
If $1 < p, q < +\infty$ and $\frac{1}{p} + \frac{1}{q} = 1$,
$f \in L^p(E)$, $g \in L^q(E)$, then
$\int_E |fg| \, d\mu \leq \|f\|_{L^p} \|g\|_{L^q}$
By integration or sum of inequalities obtained through Young's.

E.g. when the measure space is $(\mathbb{N}, \sigma(\mathbb{N}), \mu)$ where μ is the counting measure, we obtain that for any $\{a_n\} \in l^p, \{b_n\} \in l^q$,

$$\sum_{i=1}^{\infty} |a_i b_i| \leq \left(\sum_{i=1}^{\infty} |a_i|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{\infty} |b_i|^q \right)^{\frac{1}{q}} = \|\{a_n\}\|_{l^p} \|\{b_n\}\|_{l^q}$$

Theorem. (Minkowski's inequality)
For $p \geq 1$,
$f, g \in L^p(E)$ then

$$\|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$$

Convergence in Normed Vector Space

Definition.
Suppose $(X, \|.\|)$ is a normed vector space, and $\{x_n \mid n \in \mathbb{N}\}$ a sequence of points in it. Then $\{x_n\}$ is said to converge in x_0 iff $\lim\limits_{n\to\infty} \|x_n - x_0\| = 0$, denoted as $x_n \overset{\|.\|}{\to} x_0$ or simply $x_n \to x_0$.

some properties:
1. $x_n \to x_0$ and $x_n \to y_0$ implies $x_0 = y_0$.
2. $x_n \to x_0$ implies that $\exists M \geq 0, \|x_n\| \leq M$
3. $x_n \to x_0, y_n \to y_0 \Rightarrow x_n + y_n \to y_0 + x_0$
4. $x_n \to x, \alpha_n \to \alpha \Rightarrow \alpha_n x_n \to \alpha x, \alpha, \alpha_n \in K$

Examples:
For $(C[a,b], \|.\|_{L^\infty}), f_n \to f \Leftrightarrow \|f_n - f\|_{L^\infty} \to 0 \Leftrightarrow \max\limits_{x\in[a,b]} |f_n(x) - f(x)| \to 0$, that is, f_n converges to f uniformly.

Proposition. Closure of a subspace is still a subspace.

Definition.
Two norms on a vector space X are said equivalent iff $\|x_n\|_1 \to 0 \Leftrightarrow \|x_n\|_2 \to 0$ for any $\{x_n\} \subset X$.

Lemma.
For a vector space X and two norms $\|.\|_1, \|.\|_2$ defined on it,

$\|x_n\|_1 \to 0 \Rightarrow \|x_n\|_2 \to 0$ for any $\{x_n\} \subset X$ iff $\exists c > 0, \forall x \in X,\ \|x\|_2 \le c\|x\|_1$.

Note that if $\forall c > 0,\ \exists x \in X, \|x\|_2 > c\|x\|_1$, then for $c = 1,2,\dots,\ \exists x_1, x_2, \dots,$ s.t. $\dfrac{\|x_n\|_1}{\|x_n\|_2} \le \dfrac{1}{n}$. $y_n \triangleq \dfrac{x_n}{\|x_2\|_2}$, then $\|y_n\|_1 \to 0$ but $\|y_n\|_2 = 1$.

Theorem.
Two norms on a vector space X are said equivalent iff $\exists c_1, c_2 > 0, \forall x \in X$,
$c_1\|xVert_1 \le \|x\|_2 \le c_2\|x\|_1$.

Examples of non-equivalent norms:
see 张海樟课件 week12a

Definition. (Continuous Map between normed vector spaces)
For two normed vector spaces $(X,\|.\|_X)$ and $(Y,\|.\|_Y)$, a map $T : X \mapsto Y$ is called **continuous at** $x_0 \in X$, if for any $\{x_n\} \in X \overset{\|.\|_X}{\to} x_0$, $\{Tx_n\} \overset{\|.\|_Y}{\to} Tx_0$. If T is continuous at every point of X, then T is said to be continuous on X.

Based on the observation of $(\mathbb{R}, |\,.\,|)$ which is also a normed vector space, it's natural to have the following:

Proposition.
a map $T : X \mapsto Y$ is continuous at $x_0 \in X$ iff $\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x \in X$ s.t.
$\|x - x_0\|_X < \delta$, $\|Tx - Tx_0\|_Y < \varepsilon$.

Theorem.
For any normed vector space $(X,\|.\|)$, $\|.\| : X \mapsto \mathbb{R}$ is a continuous map between $(X,\|.\|)$ and $(\mathbb{R}, |\,.\,|)$.

Since for any norm $\|.\|$, the map $d(x, y) = \|x - y\|$ is always a metric, we have the following definition of dense set in normed vector space (dense set is originally defined on metric space)

Definition. (Dense Set)
For a normed vector space $(X,\|.\|)$, $M \subset N \subset X$, if $N \subset \overline{M} = M \cup M'$, where M' is the set of all limit points of M, then M is a dense set for N.

When $N = X$, M is called a dense set of X.

Definition. (Separable Space)
A normed vector space is called separable iff it has a **countable** dense set.

E.g. of separable spaces:
1. $\mathbb{R}^n$: $\mathbb{Q}^n$ is a countable dense set for $\mathbb{R}^n$

2. $C[a, b]$: The set of polynomials with rational cofactors is a countable dense set for $C[a, b]$. Note that any function in $C[a, b] \subset L^2[a, b]$ can be expanded as its Fourier series($\sin nx, \cos nx$), and $\sin nx, \cos nx$ can be rewritten as Taylor series which are composed of polynomials with rational cofactors.

3. $L^p[a, b]$: $C[a, b]$ is a dense set of $L^p[a, b]$
For some other examples, see 张海樟's lecture notes.

Definition. (Homeomorphic Map between Normed Vector Spaces)
For two normed vector space $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$, a continuous bijection $T : X \mapsto Y$ of which the inverse T^{-1} is also continuous is called a **homeomorphic** map between $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$.