This note is an intro to first order logic with equality.

Definition 3. *The first-order logic is a formal system such that*

1 *The alphabet is divided into 2 types of symbols, one is logical symbols, which always have the same meaning, and non-logical symbols, whose meaning varies by interpretation.*
The logical symbols usually include

- *quantifiers(* $\forall$ *and* $\exists$*)*
- *the logical connectives(*$\wedge, \vee, \rightarrow, \leftrightarrow, \neg$*)*
- *an infinite set of variables*
- *an equality symbol*
- *parentheses, brackets and other punctuation symbols*

And the non-logical symbols include predicates(relations), functions and constants on the domain of discourse. The non-logical symbols are always given by a signature.

2 *The grammer:*

Formation rules: define the terms and formulas of first-order logic. In mathematical logic, terms denote mathematical objects and formulas denote facts.

The set of terms is inductively defined by the following rules:

1) *Any variable is a term.*

2) *Functions. Any expression $f(t_1, \ldots, t_n)$ of n arguments (where each argument t_i is a term and f is a function symbol of valence n) is a term.* ***In particular, symbols denoting individual constants are nullary function symbols, and are thus terms.***

Only expressions which can be obtained by finitely many applications of rules 1 and 2 are terms.

The set of formulas is inductively defined by the following rules:

1) *Predicate symbols. If P is an n-ary predicate symbol and $t_1, \ldots, t_n$ are terms then $P(t_1, \ldots, t_n)$ is a formula.*

2) *Negation. If φ is a formula, then $\neg\varphi$ is a formula.*

3) *Binary connectives. If φ and ψ are formulas, then for any binary logical connectives B, $\varphi \, B \, \psi$ is a formula.*

4) *quantifiers. If φ is a formula and x is a variable, then $\forall x \varphi$(for all x, φ holds) and $\exists x \varphi$(there exists x such that φ) are formulas.*

Only expressions which can be obtained by finitely many applications of rules 1–5 are formulas.

The role of the parentheses is to ensure that any formula can only be obtained in one way by following the inductive definition(so there is a unique parse tree for each formula). This is known as unique readability of formulas.

Variables and constants are **atomic terms**.

The formulas $t_1 = t_2$ and $P(t_1, \ldots, t_n)$ where $t_i \, (i = 1,2,\ldots, n)$ are terms and P is a predicate symbol, are called **atomic formulas(atoms)**.

Binding Priorities: $\{\neg, \forall, \exists\} > \{\vee, \wedge\} > \{\rightarrow\}$, and $\rightarrow$ is **right-associative**.

some further terminologies：

- terms without variables are called **variable-free terms** or ground terms
- A **subformula** of a formula φ is a consecutive sequence of symbols from φ which is itself a formula
- An occurrence of a variable v in a formula φ is **bound** iff there is a subformula ψ of φ containing that occurrence of v such that ψ begins with a quantifier $\forall v$ or $\exists v$. Otherwise the occurrence of v is **free**.
- A variable is said to **occur free** if it has at least one free occurrence(w.r.t a formula)
- A **sentence** of FOL is a formula without free occurrences of any variable(i.e. all variables are bound); e.g. In $\varphi \triangleq x \wedge \forall x P(x)$ (P is a predicate), the first occurrence of x is free, and the second is bound. Since not all occurrence of x is bound, it is not a sentence.
- An **open** formula is a formula without quantifiers
- An **atomic sentence** is a sentence without quantifiers or logical connectives(i.e. a predicate applied on constants)

semantics[1]

The semantics of first order logic is represented by a structure:

1. a structure $\mathscr{M} = (|\mathscr{M}|, \sigma, I)$, where $|\mathscr{M}|$ is called the domain or universe of $\mathscr{M}$, of which the elements are called **individuals**. σ is the function signature, and I is the interpretation function[2] which maps each n-ary function symbol to an n-ary function from $|\mathscr{M}|$ to $|\mathscr{M}|$, and each n-ary relation symbol P to an n-ary relation $P^{\mathscr{M}}$ on $|\mathscr{M}|$($=$ is mapped to the so-called identity relation).
2. each individual is associated to a constant symbol, called the name of the individual.
3. Truth value. The meaning of a formula with free variable is not clear, so an assignment from free variables to individuals is needed:

Definition 2.17 A look-up table or environment for a universe A of concrete values is a function $l\colon \mathsf{var} \to A$ from the set of variables var to A. For such an l, we denote by $l[x \mapsto a]$ the look-up table which maps x to a and any other variable y to $l(y)$.

look-up table is also called a **object assignment or environment or valuation**.

Lemma. If l and l' agree on the variables of term t, then $t^{\mathscr{M}}[l] = t^{\mathscr{M}}[l']$.

Proof by structural induction:
base: t is a null-ary function, or a variable
inductive: suppose $t_1, t_2, \ldots, t_n$ are terms such that $t_i^{\mathscr{M}}[l] = t_i^{\mathscr{M}}[l']\,(i = 1,2,\ldots,n)$.

Let $\mathcal{M}$ be an interpretation for L, l an object assignment for $\mathcal{M}$, and t a term. The denotation of t in $\mathcal{M}$ under l, denoted $t^{\mathcal{M}}[l]$, is defined as follows:

a) if t is a variable x, then $t^{\mathcal{M}}[l] = l(x)$

b) if $t = f(t_1, \ldots, t_n)$, then $t^{\mathcal{M}}[l] = f^{\mathcal{M}}(t_1^{\mathcal{M}}[l], \ldots, t_n^{\mathcal{M}}[l])$

e.g., let $l(x) = 1$, then $(0''')^{\mathcal{N}^*}[l] = 3$, $(x + 0'')^{\mathcal{N}^*}[l] = 3$

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[1] this part is based on the book 《mathematical logic》, Joseph.R.Shoenfield and the lecture note of 2018 Spring Mathematical logic course taught by YongMei Liu.

[2] see the note 《formal logic》

Corresponding to each structure/model $\mathscr{M}(D \triangleq |\mathscr{M}|, \sigma, I)$ and an object assignment l is a **unique truth assignment** for all sentences (formulas with no free variables) in the language.

1. $P(t_1, \ldots, t_n)$ is associated with true iff $(t_1^{\mathscr{M}}[l], \ldots, t_n^{\mathscr{M}}[l]) \in P^{\mathscr{M}}$
2. $t_1 = t_2$ is associated with true iff $t_1^{\mathscr{M}}[l]$ is the same as $t_2^{\mathscr{M}}[l]$
3. logical connectives: the same as propositional logic
4. $\exists x \phi(x)$ is associated with true iff $\exists d \in D$, $\phi(x)$ is true under $\mathscr{M}$ and $l[x \mapsto d]$.
5. $\forall x \phi(x)$ is associated with true iff $\forall d \in D$, $\phi(x)$ is true under $\mathscr{M}$ and $l[x \mapsto d]$.

it can be also denoted as

For A an L-formula, the notion $\mathcal{M} \models_l A$ ($\mathcal{M}$ *satisfies* A *under* l) is defined by structural induction on formulas A as follows:

a) $\mathcal{M} \models_l P(t_1, \cdots, t_n)$ iff $\langle t_1^{\mathcal{M}}[l], \ldots, t_n^{\mathcal{M}}[l] \rangle \in P^{\mathcal{M}}$

b) $\mathcal{M} \models_l (s = t)$ iff $s^{\mathcal{M}}[l] = t^{\mathcal{M}}[l]$

c) $\mathcal{M} \models_l \neg A$ iff $\mathcal{M} \not\models_l A$, *i.e.*, not $\mathcal{M} \models_l A$.

d) $\mathcal{M} \models_l (A \vee B)$ iff $\mathcal{M} \models_l A$ or $\mathcal{M} \models_l B$.

e) $\mathcal{M} \models_l (A \wedge B)$ iff $\mathcal{M} \models_l A$ and $\mathcal{M} \models_l B$.

f) $\mathcal{M} \models_l \forall x A$ iff $\mathcal{M} \models_{l[x \mapsto a]} A$ for all $a \in |\mathcal{M}|$

g) $\mathcal{M} \models_l \exists x A$ iff $\mathcal{M} \models_{l[x \mapsto a]} A$ for some $a \in |\mathcal{M}|$

reference:
https://en.wikipedia.org/wiki/First-order_logic#Evaluation_of_truth_values

With substitution[3]: if $y \neq x$ and t is free for y, then
$\mathscr{M} \models_l \forall x A[t/y]$ iff $\mathscr{M} \models_{l[x \mapsto a]} A[t/y]$ for all $a \in |\mathscr{M}|$.

[3] see substitution theorem in http://www.cs.toronto.edu/~toni/Courses/438/Mynotes/page18.pdf

Entailment, Satisfiability and Validity

Definition 2.20 Let Γ be a (possibly infinite) set of formulas in predicate logic and ψ a formula of predicate logic.

1. Semantic entailment $\Gamma \vDash \psi$ holds iff for all models $\mathcal{M}$ and look-up tables l, whenever $\mathcal{M} \vDash_l \phi$ holds for all $\phi \in \Gamma$, then $\mathcal{M} \vDash_l \psi$ holds as well.
2. Formula ψ is satisfiable iff there is some model $\mathcal{M}$ and some environment l such that $\mathcal{M} \vDash_l \psi$ holds.
3. Formula ψ is valid iff $\mathcal{M} \vDash_l \psi$ holds for all models $\mathcal{M}$ and environments l in which we can check ψ.
4. The set Γ is consistent or satisfiable iff there is a model $\mathcal{M}$ and a look-up table l such that $\mathcal{M} \vDash_l \phi$ holds for all $\phi \in \Gamma$.

If a formula ϕ is valid, it's denoted as $\vDash \phi$. A valid formula is called a **tautology**, denoted as $\top$.

Lemma: If l and l' agree on the free variables of A, then $\mathcal{M} \models_l A$ iff $\mathcal{M} \models_{l'} A$.

Proof: Structural induction on formulas A.

Corollary: If A is a sentence, then for any object assignments l, l', $\mathcal{M} \models_l A$ iff $\mathcal{M} \models_{l'} A$.

base case: for two kinds of atomic formula: identity and relation
inductive case: logical connectives, quantifiers

From the corollary, if A is a sentence, then l is irrelevant, so we omit mention of l and simply write $\mathscr{M} \vDash A$.

Definition. (logically equivalent)
Two formulas A and B are said **logically equivalent**, denoted as $A \iff B$ iff for all model M and environment l, $\mathscr{M} \vDash_l A$ iff $\mathscr{M} \vDash_l B$; i.e., $A \vDash B$ and $B \vDash A$.

For two formulas A and B, $A \iff B$ iff $A \leftrightarrow B$ is valid (iff $\vdash A \leftrightarrow B$ iff $A \dashv\vdash B$).
Note. the symbol $\iff$ is different from $\Leftrightarrow$, which is just an alternative of $\leftrightarrow$.

e.g. $\forall x(P(x) \wedge Q(x)) \iff \forall x P(x) \wedge \forall x Q(x)$
$\exists x(P(x) \vee Q(x)) \iff \exists x P(x) \vee \exists x Q(x)$
$\exists x(P(x) \wedge Q(x)) \vDash \exists x P(x) \wedge \exists x Q(x)$ but the reversed is false.
$\forall x P(x) \vee \forall x Q(x) \vDash \forall x(P(x) \vee Q(x))$ but the reversed is false.

Proposition. For a finite set of sentences ϕ and another set of sentences ψ, $\phi \vDash \psi$ iff $\vDash \phi \to \psi$; i.e. we can solve one for the rest : entailment, validity and satisfiability.

substitution

Definition. (substitution1)
Given a variable x, a term t, and a formula ϕ, $\phi[t/x]$ is defined to be the formula obtained by replacing each free occurrence of variable x in ϕ with t.

In this note, **the default definition of substitution is substitution1.**

Definition. (substitution2)
A substituion of terms for variables is a set:
$\{x_1 \leftarrow t_1, \ldots, x_n \leftarrow t_n\}$
where each x_i is a distinct variable and each t_i is an arbitrary term. The empty substitution is the empty set.

Definition. (substitution instance)
An expression is a term, a literal, a clause or a set of clauses. Let E be an expression and let $\theta = \{x_1 \leftarrow t_1, \ldots, x_n \leftarrow t_n\}$ be a substitution. An instance $E\theta$ of E is obtained by simultaneously replacing each ocurrence of x_i in E by t_i.

Note.
Since a null-ary function is not a variable, there is no quantifiers of it. Thus we can always substitute a variable with a null-ary function (constant).

Lemma. Suppose t is a term, then for any model(structure) $\mathscr{M}$ and environment l,

$$(t[s/x])^{\mathscr{M}}[l] = t^{\mathscr{M}}[l[x \mapsto s^{\mathscr{M}}[l]]]$$

This can be proved by structural induction on terms.

> Question: Does the above lemma apply to formulas A? I.e. can we say $\mathcal{M} \models_l A(t/x)$ iff $\mathcal{M} \models_{l[x \mapsto a]} A$, where $a = t^{\mathcal{M}}[l]$? Something can go wrong.
>
> Example: Suppose A is $\forall y \neg(x = y + y)$. This says "$x$ is odd". But $A(x + y/x)$ is $\forall y \neg(x + y = y + y)$, which does not say "$x + y$ is odd" as desired, but instead it is always false. The problem is that y in the term $x + y$ got "caught" by the quantifier $\forall y$.

Thus we have such a definition:
Definition. (substitutable)
For a formula ϕ with free variable v and a term t, t is called **substitutable** for ϕ of v iff free variable in t remain free in $\phi(t/v)$. t is also said to be free for v in ϕ.

Substitution Theorem:

Substitution Theorem: If t is free for x in A then for all interpretations $\mathcal{M}$ and all object assignments l, $\mathcal{M} \models_l A(t/x)$ iff $\mathcal{M} \models_{l[x \mapsto a]} A$, where $a = t^{\mathcal{M}}[l]$.

The base case can be proved using the lemma above. (relation and identity)

inductive step:
here we only consider constructed formulas of the form $\forall yB$, and **now we have** if t is free for x in A, $\mathscr{M} \vDash_l B[t/x] \iff \mathscr{M} \vDash_{l[x \mapsto a]} B \ (a = t^{\mathscr{M}}[l])$.
for all interpretations and object assignments.

There are two cases(if x does not occur free, then the proof is done. So we assume that some occurrences of x are free):
1.$y = x$
$\mathscr{M} \vDash_l (\forall xB)[t/x] \iff \mathscr{M} \vDash_l \forall xB$ (all occurrences of x are bound)
Since l and $l[x \mapsto a]$ agree on all free variables on B (x is bound), from the **lemma** that for all formula A, if l and l' agree on the free variables of A, then
$\mathscr{M} \vDash_l A \iff \mathscr{M} \vDash_{l'} A$, we can conclude that
$\mathscr{M} \vDash_l (\forall xB)[t/x] \iff \mathscr{M} \vDash_l \forall xB \iff \mathscr{M} \vDash_{l[x \mapsto a]} \forall xB$

2. $y \neq x$

$$\begin{aligned}
\mathscr{M} \vDash_l (\forall yB)[t/x] &\iff \mathscr{M} \vDash_{l[y \mapsto b]} B[t/x] \text{ for all } b \\
&\iff \mathscr{M} \vDash_{l[y \mapsto b, x \mapsto a]} B \text{ for all } b \text{ (inductive hypothesis)} \\
&\iff \mathscr{M} \vDash_{l[x \mapsto a]} \forall yB
\end{aligned}$$

Change of bound variable. For a variable z,

Definition: $\forall zA(z/y)$ results from $\forall yA$ by *change of bound variable* provided z does not occur in A. Similarly for $\exists zA(z/y)$.

Lemma: If z does not occur in A, then $\forall zA(z/y)$ and $\forall yA$ are logically equivalent. Also $\exists zA(z/y)$ and $\exists yA$ are equivalent.

The proof is easy using the substitution theorem.

For variant and replacement theorem, see the 18 Spring ML lecture note 5.

Natural Deduction

Including the rules of inference of propositional logic, plus

$$\frac{\forall x \phi}{\phi[t/x]}\forall e, \qquad \frac{\phi[t/x]}{\exists x \phi}\exists i$$

$$\frac{\boxed{\begin{array}{l} x_0 \\ \quad\vdots \\ \phi[x_0/x] \end{array}}}{\forall x\, \phi}\ \forall x\,\mathrm{i}. \qquad \frac{\exists x\, \phi \quad \boxed{\begin{array}{l} x_0\ \phi[x_0/x] \\ \quad\vdots \\ \quad\chi \end{array}}}{\chi}\ \exists \mathrm{e}.$$

and the rules of inferences for equality:

$\dfrac{}{t = t} =_i$ (reflexitivity) $\dfrac{t_1 = t_2}{t_2 = t_1}$ (symmetry) $\dfrac{t_1 = t_2, t_2 = t_3}{t_1 = t_3}$ (transitivity)

$\dfrac{t_1 = t_2}{f(t_1, \ldots) = f(t_2, \ldots)}$ (substitution for functions)

$\dfrac{t_1 = t_2}{P(t_1, \ldots) \leftrightarrow P(t_2, \ldots)}$ (substitution for predicates)

from above we can obatin $\dfrac{t_1 = t_2, \phi[t_1/x]}{\phi[t_2/x]} =_e$

Note that the x_0 in assumption should not appear anywhere outside the box.

$\exists e$ should be $\exists xe$. (to indicate which variable is eliminated)

the conclusion of the elimination rule for $\forall$ can also be written as $\phi(t), \forall_{x\,e}$

Some basic argument:

$\forall x\phi \vdash \exists x\phi$ (see 《logic in computer science》 page 114)

syntactic equivalence:

$\exists x P(x) \dashv\vdash \exists y P(y)$

$\neg\forall x\phi \dashv\vdash \exists x \neg\phi$

$\neg\exists x\phi \dashv\vdash \forall x \neg\phi$

$\forall x\phi \wedge \forall x\psi \dashv\vdash \forall x(\phi \wedge \psi)$

$\exists x\phi \vee \exists x\psi \dashv\vdash \exists x(\phi \vee \psi)$

Strong Soundness and Completeness

The Strong soundness and completeness theorem also holds for first-order logic. That is, for any premise Γ,

$$\Gamma \vDash \phi \text{ iff } \Gamma \vdash \phi$$

In other words, if $\Gamma \vdash \phi$, then $\mathscr{M} \vDash_l \Gamma$ implies $\mathscr{M} \vDash_l \phi$ (for any model and object assignment).

Hence $\vDash \phi$ iff $\vdash \phi$. Since the validity is not solvable, $\vdash \phi$ is also not solvable (that is, given any formula, we can not proof it by natural deduction without any premises).

https://en.wikipedia.org/wiki/G%C3%B6del%27s_completeness_theorem#Statement_of_the_theorem

Solvability

The problem of deciding if a formula is valid is an example of a decision problem.
A solution to a decision problem is a program that takes an instance of the problem as input and always terminates(for any instance of the problem), producing a correct 'yes' or 'no' output.

If there is a solution to a decision problem, then we say this decision problem is **solvable**.

Theorem. For first order logic, the validity of a formula is not solvable.

Proof.
We first find a already known unsolvable problem, and show that if the validity is solvable, then this unsolvable problem is also solvable (Proof by Contraposition).

PCP(post correspondence problem):

- Given a finite sequence of pairs (s_1, t_1), (s_2, t_2), $\dots, (s_k, t_k)$ such that all s_i and t_i are binary strings of positive length, is there a sequence of indices $i_1, i_2, \dots, i_n$ with $n \geq 1$ such that the concatenation of strings $s_{i_1} s_{i_2} \dots s_{i_n}$ equals $t_{i_1} t_{i_2} \dots t_{i_n}$?
- Note: An index can appear multiple times in the sequence.

The proof that PCP is unsolvable can be found online.

Now suppose the validity is solvable. The consider a formula ϕ such that:

Given $C = (s_1, t_1), (s_2, t_2), \dots, (s_k, t_k)$

Our ϕ is $\phi_1 \wedge \phi_2 \to \exists z P(z, z)$, where

$$\phi_1 = \bigwedge_{i=1}^{k} P(f_{s_i}(e), f_{t_i}(e)),$$

$$\phi_2 = \forall v \forall w [P(v, w) \to \bigwedge_{i=1}^{k} P(f_{s_i}(v), f_{t_i}(w))]$$

- A constant symbol e with intended meaning: the empty string
- Two unary function symbols f_0 and f_1: $f_b(x)$ means the string xb
  - so the binary string $b_1 b_2 \dots b_l$ can be represented as $f_{b_l}(\dots(f_{b_2}(f_{b_1}(e)))\dots)$
  - we abbreviate $f_{b_l}(\dots(f_{b_2}(f_{b_1}(t)))\dots)$ as $f_{b_1 b_2 \dots b_l}(t)$
- A binary predicate symbol P
  - $P(s,t)$ intends to mean: there is a sequence of indices $i_1, i_2, \dots, i_n$ such that $s = s_{i_1} s_{i_2} \dots s_{i_n}$ and $t = t_{i_1} t_{i_2} \dots t_{i_n}$

Since the validity of such kind of formula is solvable, for any instance of PCP we can always say "yes" or "no".

Expressiveness of First order logic

- Question: can we express reachability in predicate logic?
- *i.e.*, can we find a formula $\phi(u, v)$ such that it holds in a directed graph iff there is a path in the graph from the node associated to u to the node associated to v?
- For each $k \geq 0$, we can find a formula $\phi_k(u, v)$ such that it holds in a directed graph iff there is a path of k transitions ...
- However, the answer to the question is 'no'.

e.g.
there is a 3 transition from u to v:
$\exists w_1 \exists w_2 R(u, w_1) \wedge R(w_1, w_2) \wedge R(w_2, v)$

Theorem. (Compactness theorem)
For any set of sentences of predicate logic, Γ, if any finite subsets of Γ is satisfiable, then so is Γ.

Proof.
We use proof by contradiction: Assume that Γ is not satisfiable. Then the semantic entailment $\Gamma \vDash \perp$ holds as there is no model in which all $\varphi \in \Gamma$ are true. By

completeness, this means that the sequent $\Gamma \vdash \perp$ is valid. (Note that this uses a slightly more general notion of sequent in which we may have infinitely many premises at our disposal. Soundness and completeness remain true for that reading.) Thus, this sequent has a proof in natural deduction; this proof – being a finite piece of text – can use only finitely many premises Δ from Γ. But then $\Delta \vdash \perp$ is valid, too, and so $\Delta \vDash \perp$ follows by soundness. But the latter contradicts the fact that all finite subsets of Γ are consistent.

Theorem. (**Löwenheim-Skolem Theorem**)
Suppose ψ is a sentence of predicate logic such that for any natural number $n \geq 1$, there is a model(structure) with at least n elements under which ψ is true. Then there is a model with infinitely many elements under which ψ is true.

Proof.
$\phi_n \triangleq \exists w_1 \ldots \exists w_n \bigwedge_{1 \leq i < j \leq n} \neg(w_i = w_j)$ (i.e. there are at least n elements in the model)

Consider a set of sentence $\Gamma = \{\psi\} \cup \{\phi_n | n \geq 1\}$ and apply the above two theorems (Any finite subset of it is satisfiable)

Theorem. Reachability is not expressible in predicate logic.

Proof:

- Assume that there is such a formula $\phi(u, v)$.
- Let c and c' be two constants.
- Let $\phi_n(u, v)$ be the formula stating that there is a path of length n from u to v.
- Let $\Gamma = \{\phi[c/u][c'/v]\} \cup \{\neg\phi_n[c/u][c'/v] \mid n \geq 1\}$.