Definition. (Orthogonal)

For an inner product space H with an inner product $\langle \,.\,,.\, \rangle$,

1. for $x, y \in H$, x is orthogonal to y, denoted as $x \perp y$ iff $\langle x, y \rangle = 0$.
2. for $M \subseteq H$, $x \perp M$ iff $x \perp y, \forall y \in M$.
3. $M, N \subseteq H, M \perp N \Leftrightarrow x \perp y, \forall x \in M, \forall y \in N$

Properties.

1. $x \perp y_n, y_n \to y \implies x \perp y$
2. $x \perp y_i \, (i = 1,2,\ldots,n) \implies x \perp \overline{\text{span}\{y_1, \ldots, y_n\}}$
3. $x \perp M \iff x \perp \overline{\text{span} M}$

Definition. (Orthogonal complement)

For $M \subseteq H$, the orthogonal complement of M is

$$M^{\perp} = \{x \in H : x \perp M\}$$

Properties.

For an inner product space H, and $M \subset H$,

1. $M^{\perp}$ is a closed subspace.
2. $M \subseteq N \Rightarrow N^{\perp} \subseteq M^{\perp}$
3. $M^{\perp} = (\overline{M})^{\perp} = (\overline{\text{span} M})^{\perp}$
4. $M \cap M^{\perp} \subseteq \{\theta\}$
5. $H^{\perp} = \{\theta\}$
6. $M \subseteq (M^{\perp})^{\perp}$

Note.

The closeness of $M^{\perp}$ is due to the continuity of the inner product.

Theorem. (Pythagorean Theorem, 勾股定理)

For an inner product space H, if $x \perp y$, then

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2$$

Generalization. $h_i \, (i = 1,2,\ldots,m)$ are pairwise orthogonal, then

$\|h_1 + h_2 + \ldots + h_m\|^2 = \displaystyle\sum_{i=1}^{m} \|h_i\|^2.$

Definition. (Element of best approximation)
For a metric space (X, d), $A \subset X$, $x \in X$, $x_0 \in A$ is called an element of best approximation of x in A if

$$d(x, x_0) = d(x, A)$$

***Theorem.**
For a Hilbert space $\mathscr{H}$, $M \subseteq \mathscr{H}$ a non-empty **closed convex** set, $\forall x \in \mathscr{H}$, there exists a unique element of best approximation of x in M.

The proof of this theorem best illustrate the application of parallelogram law:
$\|x-y\|^2 = 2(\|x-z\|^2 + \|y-z\|^2) - \|2z-(x+y)\|^2$

Theorem. (variational principle, 变分原理)
For a Hilbert space $\mathscr{H}$, $M \subseteq \mathscr{H}$ a closed convex set, $\forall x \in \mathscr{H}$, $x_0 \in M$ is the element of best approximation for x in M iff

$$\mathrm{Re}\langle x - x_0, y - x_0 \rangle \leq 0,\ \forall y \in M$$

Hint: consider the function
$F(t) = \|x - [(1-t)x_0 + ty]\|^2 = \|x - x_0\|^2 + t^2\|x_0 - y\|^2 - 2t\mathfrak{R}\langle x - x_0, y - x_0 \rangle$,
$t \in [0,1]$, $F(1) = \|x - y\|^2$, $F(0) = \|x - x_0\|^2$

***Corollary 1.**
For a Hilbert space $\mathscr{H}$, $M \subseteq \mathscr{H}$ a closed subspace, $\forall x \in H$, $x_0 \in M$ is the element of best approximation for x in M iff

$$x - x_0 \perp M$$

When $\dim M = n < +\infty$,
if a **orthogonal** basis(Hamel basis) of M is $\{e_1, e_2, \ldots, e_n\}$,
then the element of best approximation for x is $\displaystyle\sum_{i=1}^{n} \frac{\langle x, e_i \rangle}{\langle e_i, e_i \rangle} e_i$

when $\dim M = +\infty$,
if $\{e_n | n \in \mathbb{N}\}$ is an orthogonal Schauder basis of M,
then the element of best approximation for x is $\displaystyle\sum_{i=1}^{\infty} \frac{\langle x, e_i \rangle}{\langle e_i, e_i \rangle} e_i$

Definition. (orthogonal projection)
For a Hilbert space $\mathscr{H}$, $M \subseteq \mathscr{H}$ a **closed subspace**, the element of best approximation for $x \in H$ in M is called the orthogonal projection of x on M, denoted as Px.
Properties.
1. $P = P^2$
2. $\|Px\| \leq \|x\|$ (note that $x - Px \perp Px$)
3. $P(x + y) = Px + Py, Pax = aPx, \forall a \in \mathbb{K}$

***Theorem. (Orthogonal Decomposition)**
For a Hilbert space $\mathscr{H}$, $M \subseteq \mathscr{H}$ a **closed subspace**,
$$H = M \oplus M^\perp$$
i.e. $\forall x \in H$, there exist a unique $x_1 \in M$ and a unique $x_2 \in M^\perp$ such that $x = x_1 + x_2$. Actually, $x_1 = Px, x_2 = x - Px$.

Corollary 1.
For a Hilbert space $\mathscr{H}$ and a closed subspace $M \subseteq \mathscr{H}$,
$$M = M^{\perp\perp}$$
With this corollary it's easy to see that $M^{\perp\perp} = \overline{\text{span} M}$.
Corollary 2.
For a Hilbert space $\mathscr{H}$, $M \subseteq \mathscr{H}$ a **subspace**,
$$M^\perp = \{\theta\} \iff \overline{M} = \mathscr{H}$$
Note. This corollary provides a method to prove denseness of a subspace.
Corollary 3.
For a Hilbert space $\mathscr{H}$ and a subset $M \subset \mathscr{H}$,
$$M^\perp = \{\theta\} \iff \overline{\text{span} M} = \mathscr{H}$$
Proof.
$M^\perp = (\overline{\text{span} M})^\perp = \{\theta\}$, and $\overline{\text{span} M}$ is a closed subspace of $\mathscr{H}$.

Generalized Fourier Series in the inner product space

Definition. (Orthogonal system, 正交系)
For an inner product space H, a set of vectors $\{e_i \mid i \in I\}$ is called an orthogonal system for H, if $\langle e_i, e_j \rangle = 0\,(\forall i \neq j, i, j \in I)$.

A orthogonal system s.t. $\langle e_i, e_i \rangle = 1\,(\forall i \in I)$ is called a **orthonormal system**(标准正交系).
Note. I can be uncountable.

Proposition.
For an inner product space H, the vectors of an at most countable orthogonal system for H are linearly independent.

Theorem. (Gram-Schmidt Process)
For an inner product space H and a set of **finitely many** vectors $\{x_i \mid i = 1,2\ldots,n\}$, there exists a orthonormal system $\{e_i \mid i = 1,2,\ldots,n\}$ s.t.

$$\operatorname{span}\{e_1, e_2, \ldots, e_n\} = \operatorname{span}\{x_1, x_2, \ldots, x_n\}$$

Definition. (Generalized Fourier Series)
Suppose $\{e_n \mid n \in \mathbb{N}\}$ is a countable orthonormal system for an inner product space H, then for $\forall f \in H$,

$$c_n = \langle f, e_n \rangle$$

are called the **Fourier coefficients** of f, and

$$\sum_{i=1}^{+\infty} \langle f, e_n \rangle e_n$$

is called the **Fourier series** of f.

Lemma.
For a Hilbert space $\mathscr{H}$, the linear span of a finite orthogonal subset $S \subseteq \mathscr{H}$ is closed.

Theorem.
Suppose $\{e_n \mid n \in \mathbb{N}\}$ is a orthonormal system for a inner product space H, and define $H_n \triangleq \operatorname{span}\{e_1, e_2, \ldots, e_n\}$
then for $\forall f \in H$,

$$\sum_{k=1}^{n} c_k e_k$$

where $c_k = \langle f, e_k \rangle$ $(k = 1,2,\ldots,n)$, is the element of best approximation of f on H_n.

Note that H_n is a closed subspace, and thus the corollary 1 can be applied here.

Theorem. (Bessel's inequality)
Suppose $\{e_n \mid n \in \mathbb{N}\}$ is a orthonormal system for an inner product space H, then for $\forall f \in H$,

$$\sum_{n=1}^{+\infty} |c_n|^2 \leq \|f\|^2$$

where $c_n = \langle f, e_n \rangle$ are the Fourier coefficients of f.
Proof. By applying the Pythagorean theorem.

Corollary. (Riemann-Lebesgue Lemma)
Suppose $\{e_n | n \in \mathbb{N}\}$ is a orthonormal system for a **inner product space** H, then for $\forall f \in H$,

$$\lim_{n\to\infty} \langle f, e_n \rangle = \lim_{n\to\infty} c_n = 0$$

Theorem. (Convergence of Fourier Series)
For any **orthonormal** system $\{e_n | n \in \mathbb{N}\}$ of a **Hilbert space** $\mathscr{H}$, the Fourier series of any $f \in H$

$$\sum_{n=1}^{\infty} \langle f, e_n \rangle$$

converges.

Note. it's suffice to show that $\left\{ \sum_{i=1}^{n} c_i | n \in \mathbb{N} \right\}$ is a Cauchy sequence. And

$$\sum_{k=n}^{n+p} |c_k|^2 \leq \sum_{k=n}^{\infty} |c_k|^2, \sum_{n=1}^{\infty} |c_n|^2 < +\infty \implies \lim_{n\to\infty} \sum_{k=n}^{\infty} |c_k|^2 = 0$$

Definition. (complete orthonormal system, 完全标准正交系)
For an inner product space H, an orthonormal system $\{e_i | i \in I\}$ is called complete iff $\{e_i | i \in I\}^{\perp} = \{\theta\} \Leftrightarrow \overline{\text{span}\{e_i | i \in I\}} = H$.

Note
1. If an inner product space has a complete orthonormal system, then it's a Hilbert space.(proof by definition, consider a Cauchy sequence)

2. Obviously, if a Hilbert space has a countable complete orthonormal system, then it must be separable.

3. For an inner product space H, consider the set of all its orthonormal system, then the set equipped with $\subsetneq$ is a partially ordered set P. **It can be seen that an orthonormal system for H is complete iff it's the maximal element of P.**

Review. An element $x \in S$ is a maximal in a partially ordered set $(S, \leq)$ iff there doesn't exist $y \in S$ s.t. $x \leq y$. An element $x \in S$ is a upper bound for $C \subseteq S$ iff $\forall y \in C, y \leq x$.

Definition. (Chain)
A totally ordered subset of a partially ordered set is called a chain.

Lemma.(Zorn's lemma)
Suppose a partially ordered set P has the property that every chain in P has an upper bound in P. Then the set P contains at least one maximal element.

https://en.wikipedia.org/wiki/Zorn%27s_lemma#Statement_of_the_lemma

Theorem.
Every Hilbert space $\mathscr{H}$ has a complete orthonormal system.

Proof. Suppose $\Omega = \{A_\alpha \mid \alpha \in I\}$ is the set of all orthonormal system of $\mathscr{H}$.
Then $(\Omega, \subsetneq)$ is a partially ordered set. For every chain $\{B_\alpha \mid \alpha \in I' \subseteq I\}$,

$$\bigcup_{\alpha \in I'} B_\alpha$$

is an upper bound of the chain. So $(\Omega, \subsetneq)$ has a maximal element.
(why Hilbert space? see the first comment of the complete orthonormal system)

Theorem.
For a Hilbert space $\mathscr{H}$ and one of its countable orthonormal system $\{e_n \mid n \in \mathbb{N}\}$, the following statements are equivalent:
(i). $\{e_n \mid n \in \mathbb{N}\}$ is complete
(ii). (Parseval's identity) $\sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2 = \|x\|^2, \forall x \in \mathscr{H}$
(iii). $x = \sum_{n=1}^{\infty} \langle x, e_n\rangle e_n$
(iv). $\langle x, y\rangle = \sum_{n=1}^{\infty} \langle x, e_n\rangle\langle e_n, y\rangle$

(i) $\Longrightarrow$ (ii), (ii) $\Longleftrightarrow$ (iii), (iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (i).

A complete orthonormal system of $\mathscr{H}$ is also called a **Hilbert basis** or **orthonormal basis** of $\mathscr{H}$.

E.g. $\{e^{inx} \mid n \in \mathbb{N}\}$ is an orthonormal basis for $L^2[-\pi, \pi]$ with inner product

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\,\overline{g}\,d\mu$$

The theorem show that every complete countable orthonormal system is a Schauder basis, and vice versa.

Note. Since $[a,b]$ is Lebesgue measurable for all $a,b \in \mathbb{R}$, all functions in $C[a,b]$ are Lebesgue measurable functions. And it can be proved that $C[a,b] \subset L^2[a,b]$. So any continuous function in a finite closed interval has a Fourier series.

Some further statements:

1. Any orthonormal system of a separable Hilbert space is at most countable.

2. For n-dimension vector space X and a basis $\{e_1, e_2, \ldots, e_n\}$, $\langle x, y\rangle$ is an inner product iff there exists a positive definite hermitian matrix A s.t.

$$\langle x, y\rangle = \left\langle \sum_{i=1}^{n} x_i e_i, \sum_{i=1}^{n} y_i e_i \right\rangle = x^T A \overline{y} = \sum_{i,j=1}^{n} a_{ij} x_i \overline{y}_j$$

actually, $a_{ij} = \langle e_i, e_j\rangle$.

Extension of Fourier series. (from wiki)

We can also define the Fourier series for functions of two variables x and y in the square $[-\pi, \pi] \times [-\pi, \pi]$:

$$f(x,y) = \sum_{j,k \in \mathbf{Z} \text{ (integers)}} c_{j,k} e^{ijx} e^{iky}$$

$$c_{j,k} = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x,y) e^{-ijx} e^{-iky} \, dx \, dy$$