Black Hole Evaporation Due to the Hawking Effect

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1 Introduction

Extensions of the no-hair theorem has led to a puzzling situation within the realms of building a theory of quantum gravity. This situation is referred to the "information loss paradox". In classical General Relativity (GR), we can take a complicated collection of matter and collapse it in to a black hole, where it's configuration can be completely determined by mass, change, and spin - we think of this information as "hidden" behind the event horizon. Once we move to quantum field theory (QTF), we find black holes evaporate, due to Hawking Radiation, and eventually disappear. The outgoing Hawking Radiation either has encoded information regarding the original state used to create the black hole or the information seems to disappear. Understanding this step will draw us much closer to unifying GR with quantum mechanics for a theory of quantum gravity.

2 QFT in Curved Spacetime

To begin discussing the effects of QFT on black holes, we must first establish what QFT looks like in a curved spacetime. We approach the situation as we have with any physical theory in flat spacetime - we express the theory in a coordinate-invariant form, asserting that it's valid in a curved spacetime. There are minor subtitles with establishing QTF in curved spacetime, however we will discuss these later on.

Let's begin with the Lagrange density of an arbitrary scalar field ϕ, in curved spacetime,

$$\mathcal{L} = \sqrt{-g}(-\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}m^2\phi^2 - \xi R\phi^2)$$

where ξ is the coupling constant to the curvature scalar R. Now we wish to quantize our Lagrangian in order to find field $\phi(t, \mathbf{x})$ and momentum π. Thus,

$$\pi = \frac{\partial\mathcal{L}}{\partial(\nabla_0\phi)} = -\frac{g^{\mu\nu}\sqrt{-g}}{2}\frac{\partial}{\partial(\nabla_0\phi)}[\nabla_\mu\phi\nabla_\nu\phi]$$

where $g^{00} = -1$ and hence $\pi = \sqrt{-g}\nabla_0\phi$. Now, similar to the quantization of the classical harmonic oscillator, we wish to establish canonical commutation relations, specifically noting:

$$[\phi(t, \mathbf{x}), \pi(t, \mathbf{x}^{'})] = \frac{i}{\sqrt{-g}}\delta^{(n-1)}(\mathbf{x} - \mathbf{x}^{'})$$

To find the equation of motion for this scalar, we must consider the Heisenberg picture, where we see our operators satisfy the Heisenberg Equation of motion,

$$\frac{dA(t)}{dt} = i[H, A(t)]$$

yielding our equation of motion for the scalar field ϕ,

$$\Box\phi - m^2\phi - \xi R\phi = 0$$

where $\Box$ is the d'Alembert operator defined as $\Box = \partial^\mu \partial_\mu$.

From here, we define our creation and annihilation operators a and $a^\dagger$, where the typical commutation relation holds, $[a_i, a_j^\dagger] = \delta_{ij}$. We can now start with a vacuum state, i.e. $a_i|0_f\rangle = 0$, and define an entire Fock basis for the Hilbert Space, thus completing our quantum field theory for a curved spacetime. However an issues arises, in a curved spacetime, there is no one better choice of basis for our theory. If one observer defines particles with respect to one basis, f_i and another observer with another basis g_i, they will generally not agree on the number of particles observed, if any at all. To see this more clearly, as well as to form a transition from one basis to another, we utilize the Bogolubov transformation,

$$g_i = \sum_j (\alpha_{ij} f_j + \beta_{ij} f_j^*) f_i = \sum_j (\alpha_{ji}^* g_j + \beta_{ji} g_j^*)$$

where α_{ij}, β_{ij} are Bogolubov coefficients. The issue of basis choice distinguishes QFT in flat spacetime to QFT in curved spacetime. Thus the states that seem appropriate for QFT in curved spacetime are Hadamard states, where any phyisically reasonable quantum state in curved spacetime obeys,

$$G(x_1, x_2) = \frac{U(x_1, x_2)}{4\pi^2 \sigma_\epsilon} + V(x_1, x_2) ln(\sigma_\epsilon) + W(x_1, x_2)$$

where function U,V,and W, are all regular at $x_1 = x_2$ and $U(x, x) = 1$.

3 Hawking Effect

While the original motivation for Hawking radiation began with it's discovery with stars collapsing into black holes, we shall consider an eternal black hole. This way, we can relate the Hawking effect to the Unruh effect, which we will briefly review below.

3.1 Unruh Effect

Consider the metric of Rindler space, $ds^2 = a^2 e^{2\xi}(-d\eta^2 + d\xi^2)$, for $|t| < x$. The Rindler coordinates (η, ξ) can be found via the usual Minkowski transformation of coordinates,

$$\begin{aligned} \eta &= \frac{1}{a} arctanh(\frac{t}{x}) \\ \xi &= \frac{1}{2a} ln(x^2 - t^2) \end{aligned} \tag{1}$$

where by rearranging to solve for a via both equations, we get the uniform acceleration,

$$a = \frac{1}{x^2 - t^2} \tag{2}$$

The Unruh effect states that an accelerating observer in the traditional Minkowski vacuum state will observe a thermal spectrum of particles. Consider the following two observers in Minkowski spacetime,

- Inertial observers - not influenced by external forces with straight world lines in standard Minkowski spacetime
- Uniformly accelerating observers - have a uniform acceleration (2), as seen by an inertial observer, where world lines follow the curve of constant acceleration

With regards to the Unruh Effect, the inertial observer does not have any thermal behaviour in their two-point function $\omega(x, t, x', t')$. The uniformly accelerating observer, however, have a thermal behaviour $\omega_{acc}(\xi', \eta' + i\beta, \xi, \eta)$ at a temperature $T = \frac{1}{\beta} = \frac{a}{2\pi}$.

Applying our new intuition to the eternal black hole, we can draw an analogous solution for QFT in curved spacetime. Comparing the two spacetimes, we see that for curved space, the region out of the horizon of a black hole corresponds to $|T| < X$ in Kruskal-Szekers spacetime, analogous to $|t| < x$ in Rindler spacetime. The analogous coordinate transformations from (1) are,

$$\begin{aligned} t &= 4Marctanh(\frac{T}{X}) \\ r_* &= 2Mln(X^2 - T^2) \end{aligned} \tag{3}$$

Similar to (1), the equations suggest to define two observers, in curved spacetime and flat spacetime.

- free falling observer - not influenced by external forces, freely falling in a gravitational field using the Kruskal-Szekeres coordinates (T, X).
- static observers - kept at a constant distance $r > 2M$, using coordinates (t, r_*)

By comparing this case to that of the Unruh effect, we except the static observer to see a thermal behaviour as a KMS state of temperature $T = \frac{1}{\beta} = \frac{a}{2\pi} = \frac{1}{8\pi M}$ between the horizon and the region outside the black hole. The Hawking effect essentially states (analogous to the Unruh effect), the static observer "sees" a thermal behaviour at $T = \frac{a}{2\pi}$, while the free falling observer does not experience any thermal behaviour.

3.2 Derivation

Let's begin with a field satisfying the equation, say a massless free Klein-Gordon field on a globally hyperbolic spacetime,

$$\Box_g \phi = 0 \tag{4}$$

where

$$\Box_g \equiv \nabla_\mu \nabla^\mu = |g|^{\frac{1}{2}} \partial_\mu g^{\mu\nu} |g|^{\frac{1}{2}} \partial_\nu$$

rewriting this in Eddington–Finkelstein coordinates,

$$\Box_g = \left(1 - \frac{2M}{r}\right)^{-1} (\partial_t^2 - \frac{1}{r}\partial_{r_*}^2 r) + \left(\frac{2M}{r^3} - \frac{\Delta_{s^2}}{r^2}\right)$$

where we used Δ_{s^2} is the Laplacian over a two sphere. Solving for the spatial part of $\Box_g$,

$$\alpha = -\frac{1}{r}\partial_{r_*}^2 r + \left(1 - \frac{2M}{r}\right)^{-1} \left(\frac{2M}{r^3} - \frac{\Delta_{s^2}}{r^2}\right)$$

Now we must solve for the eigenfunctions of Δ_{s^2} in the form

$$\Phi(r_*, \theta, \phi) = \sum_{lm} f(r_*) Y_{lm}(\theta, \phi)$$

This yields α to take the form of,

$$\alpha = -\frac{1}{r}\partial_{r_*}^2 r + \left(1 - \frac{2M}{r}\right)^{-1} \left(\frac{l(l+1)}{r^2} + \frac{2M}{r^3}\right) = -\frac{1}{r}\partial_{r_*}^2 r + V_l(r)$$

Thus, our new operator α describes the movement of a particle subject to potential $V_l(r)$. However this is not as useful as we move forward - we are only interested in the thermal effects near the horizon, so we can simply neglect the potential term. The two-point function then takes the form of

$$\begin{aligned} \omega(t, r_*; t', r'_*) &= \frac{1}{2\sqrt{\alpha}} \left(\frac{exp(-i\sqrt{\alpha}(t-t'))}{1-exp(\beta\sqrt{\alpha})} + \frac{exp(i\sqrt{\alpha}(t-t'))}{exp(\beta\sqrt{\alpha})-1} \right) (r_*, r'^*) \\ &:= f(\sqrt{\alpha})(r_*, r'^*) \end{aligned} \tag{5}$$

We now take a quick aside to verify this two-point function satisfies (4).

$$\begin{aligned}\Box_g\omega(f,g) &= \int dr_* d^4 r'_* \Box_g \omega(t,r_*;t',r_{*'}) f(r_*) g(r'_*) \\ &= \int dr_* d^4 r'_* \Big(1-\tfrac{2M}{r}\Big)^{-1} (\partial_t^2 - \alpha)\omega(t,r_*;t',r_{*'}) f(r_*) g(r'_*) \\ &= \int dr_* d^4 r'_* \Big(1-\tfrac{2M}{r}\Big)^{-1} (\alpha - \alpha)\omega(t,r_*;t',r_{*'}) f(r_*) g(r'_*) \\ &= 0\end{aligned}$$

To be a state, ω must also be positive, which we will check after a quick change of variables.

Let's take a look at how $\hat{\alpha}$ acts on a function $f \,\epsilon\, C(\mathbb{R}\times S^2)$, $f \geq 0$, where S^2 is the two dimensional unit disk $S^2 = \{(z_1,z_2)\,\epsilon\,\mathbb{C}^2 : |z_1 - z_2|^2 \leq 1\}$ and $\mathbb{R}$, a time direction. Hence f is a element of the subgroup of continuous functions of the Cartesian product between the group of all real numbers and the group of all two dimensional rotations in Complex space.

$$\hat{\alpha}f = -\frac{1}{r}\partial^2_{r_*}(rf) = -\partial^2_{r_*}f - \frac{f}{r}\partial^2_{r_*}r - \frac{2}{r}\partial_{r_*}r\partial_{r_*}f \tag{6}$$

We notice that by changing the basis, we can make a change a variables. Note first that

$$C(\mathbb{R}\times S^2)\ \epsilon\ L^2(\mathbb{R}\times\ S^2, r^2 dr_* d\Omega)$$

where the L^2 space is all functions ψ satisfying $\langle\phi|\psi\rangle = \int \overline{\psi}\phi dx$. We now can take a mapping, call it U, stipulated by,

$$U : L^2(\mathbb{R}\times S^2, r^2 dr_* d\Omega) \to L^2(\mathbb{R}\times S^2, dr_* d\Omega)$$

$$\begin{gathered} U : f \to fr \\ U^{-1} : fr \to f \end{gathered}$$

Let's apply α to fr now, similar to (6)

$$\hat{\alpha}fr = -\frac{1}{r}\partial^2_{r_*}(r^2 f) = -r\partial^2_{r_*}f - \frac{f}{r}\partial^2_{r_*}r - \frac{2}{r}\partial_{r_*}r\partial_{r_*}f \tag{7}$$

Thus our new mapping is unitary. So let's apply this unitary mapping to our operator α. Since α is an operator, the way that U is applied matters. Let $\hat{\alpha}' = U\hat{\alpha}U^{-1}$.

$$\begin{aligned}\hat{\alpha}' &= U\hat{\alpha}U^{-1} = U(\hat{\alpha}U^{-1}) \\ &= U(\hat{\alpha}\tfrac{1}{r}) = -r\hat{\alpha}\tfrac{1}{r} \\ &= -r\Big(-\tfrac{1}{r}\partial^2_{r_*}\Big)\tfrac{1}{r} \\ &= \partial^2_{r_*}\end{aligned}$$

Hence, $\alpha' = U\alpha U^\dagger = \partial^2_{r_*}$ where $U^\dagger U = UU^\dagger = 1$. Since α' is much simpler of an operator, we will work with it as opposed to α. This is valid since we know that a unitary transformation preserves the eigenvalues of the operator it is transforming.

Let's apply α' to some eigenvector in momentum space $\phi_{\vec{p}}\ \epsilon\ \mathcal{H}$

$$\alpha'\phi_{\vec{p}} = |\vec{p}|^2\phi_{\vec{p}} = |\vec{p}|^2 e^{i\vec{p}\cdot\vec{r_*}}$$

We see then that $\alpha' > 0$ and diagonal on $\phi_{\vec{p}}$, so we can define a new operator $\sqrt{\alpha'}$. Since α' is self-adjoint, and thus $\sqrt{\alpha'}$, we can write our two points function, $\omega(t, r_*; t', r_{*'})$ as a mode decomposition via the spectral theorem, which states that for a function ξ of a self-adjoint operator $\hat{O}$ acting on an element of the Hamiltonian space, $h \in \mathcal{H}$

$$\xi(\hat{O})h = \int f(o(k))\Phi_k(\Phi_k, h)d\mu_k$$

where $\hat{O}\Phi_k = o\Phi_k$.

Substituting $\hat{O}$ with $\sqrt{\alpha'}$ we can now solve for $f(\sqrt{\alpha'})$ in (5).

$$\begin{aligned}
f(\sqrt{\alpha'})h(r_*) &= \int f(p)\phi_{\vec{p}}(\phi_{\vec{p}}, h)d\mu_{\vec{p}} \\
&= \int d\vec{p}dr'_* f(p)\phi_{\vec{p}}(r_*)\phi_{\vec{p}}(r'_*)h(r'_*) \\
&= \int d\vec{p}dr'_* f(p)e^{-i\vec{p}\cdot r'_*}e^{i\vec{p}\cdot r'_*}h(r'_*) \\
&= \int d\vec{p}dr'_* f(p)e^{i\vec{p}\cdot(r'_*-r_*)}h(r'_*)
\end{aligned}$$

$$\Rightarrow \quad f(\sqrt{\alpha'})(r_*, r'_*) = \int d\vec{p} f(p)e^{i\vec{p}\cdot(r'_*-r_*)} \tag{8}$$

So we can treat (8) as the kernel of a momentum distribution. Now, we can express our original two-point function (5), in terms of (8).

$$\omega(t, r_*; t', r'_*) \; = (2\pi)^{-1}\int \frac{dp}{2p}\left(\frac{e^{-ip(t-t')}}{1-e^{-\beta p}} + \frac{e^{ip(t-t')}}{e^{\beta p}-1}\right)e^{-ip(r'_*-r_*)} \tag{9}$$

Now that we have changed our operator from α to α' and simplified out two-point function, we must prove that $\omega(t, r_*; t', r'_*) > 0$ and that it satisfies the KMS conditions in order to be a KMS state.

$$\begin{aligned}
\omega(t, r_*; t', r'_*) = \omega(f^*, f) &= (2\pi)^{-1}\int \tfrac{dp}{2p}\left(\tfrac{e^{-ip(t-t')}}{1-e^{-\beta p}} + \tfrac{e^{ip(t-t')}}{e^{\beta p}-1}\right)e^{-ip(r'_*-r_*)} \\
&= (2\pi)^{-1}\int d^4x d^4x' \tfrac{dp}{2p}\left(\tfrac{e^{-ip(t-t')}}{1-e^{-\beta p}} + \tfrac{e^{ip(t-t')}}{e^{\beta p}-1}\right) \times e^{-ip(r'_*-r_*)}f^*(x)f^*(x') \\
&= (2\pi)^{-1}\int \tfrac{dp}{2p}c_p c_p^* \;, \quad c_p = \int d^4x\left(\tfrac{e^{-ip_\mu(t,r_*)^\mu}}{1-e^{-\beta p}} + \tfrac{e^{ip_\mu(t,r_*)^\mu}}{e^{\beta p}-1}\right)f(x) \\
&= (2\pi)^{-1}\int \tfrac{dp}{2p}|c_p|^2 > 0
\end{aligned}$$

Thus $\omega(t, r_*; t', r'_*)$ is always positive. Finally we check the KMS condition,

$$\omega^\beta(t+i\beta, r_*; t', r'_*) = (2\pi)^{-1} \int \frac{dp}{2p} \left(\frac{e^{-ip(t+i\beta-t')}}{1-e^{-\beta p}} + \frac{e^{ip(t+i\beta-t')}}{e^{\beta p}-1} \right) e^{-ip(r'_*-r_*)}$$

$$= (2\pi)^{-1} \int \frac{dp}{2p} \left(\frac{e^{\beta p} e^{-ip(t-t')}}{1-e^{-\beta p}} + \frac{e^{-\beta p} e^{ip(t-t')}}{e^{\beta p}-1} \right) e^{-ip(r'_*-r_*)}$$

$$= (2\pi)^{-1} \int \frac{dp}{2p} \left(\frac{e^{ip(t-t')}}{1-e^{-\beta p}} + \frac{e^{-ip(t-t')}}{e^{\beta p}-1} \right) e^{ip(r'_*-r_*)}$$

$$= \omega^\beta(t', r'_*; t, r_*)$$

So our two-point correlation function is a KMS state, hence it describes the properties of our system in thermal equilibrium. Our new function has a singularity in momentum space at $p = 0$, so let's take a derivative with respect to time and see if the singularity still exits.

$$\partial_t \partial_{t'} [\omega(t+i\beta, r_*; t', r'_*)] \mid_{t=t'=0} = (4\pi)^{-1} \int \frac{dp\ p^2}{2p} \left(\frac{e^{-ip(t-t')}}{1-e^{-\beta p}} + \frac{e^{ip(t-t')}}{e^{\beta p}-1} \right) \times e^{-ip(r'_*-r_*)}$$

$$= (4\pi)^{-1} \int dp\ p \left(\frac{1}{1-e^{-\beta p}} + \frac{1}{e^{\beta p}-1} \right) e^{-ip(r'_*-r_*)}$$

$$= (4\pi)^{-1} \int dp\ p\ coth \frac{\beta p}{2}\ e^{-ip(r'_*-r_*)}$$

But now we have a pole for $\beta p = 2\pi n i$, where $n \, \epsilon \, \mathbb{N}$. However, we can use the Cauchy Residue Theorem,

$$\partial_t \partial_{t'} [\omega(t+i\beta, r_*; t', r'_*)] \mid_{t=t'=0} = (4\pi)^{-1} 2\pi i \sum_{n \epsilon \mathbb{N}} Res(\frac{2\pi i}{\beta}) \tag{10}$$

where the residue is

$$Res\Big(p\ coth \frac{\beta p}{2}\ e^{-ip(r'_*-r_*)}\Big) \mid_{p=\frac{2\pi n}{\beta}} = \frac{2\pi n i}{\beta} e^{\frac{2\pi n}{\beta}(r'_*-r_*)}$$

Thus (10) becomes,

$$\begin{aligned} \partial_t \partial_{t'} [\omega(t+i\beta, r_*; t', r'_*)] \mid_{t=t'=0} &= (4\pi)^{-1} 2\pi i \sum_{n \epsilon \mathbb{N}} Res(\frac{2\pi i}{\beta}) \\ &= \frac{\pi}{\beta^2} \sum_{n \epsilon \mathbb{N}} n e^{\frac{2\pi n}{\beta}(r'_*-r_*)} \\ &= \frac{\pi}{\beta^2} sinh^{-2} \Big[\frac{\pi}{\beta}(r'_* - r_*) \Big] \end{aligned} \tag{11}$$

This is the correlation between the horizon and beyond. To make this more transparent, let's express (11) in terms of Kruskal-Szekeres time coordinates T, where we define $T = 0$ to be on the horizon. First, we recall the that by the chain rule,

$$\partial_T \partial_{T'} = \Big(\frac{\partial t}{\partial T} \frac{\partial}{\partial t} \Big) \Big(\frac{\partial t'}{\partial T'} \frac{\partial}{\partial t'} \Big)$$

where from (3), we get

$$\begin{aligned}\frac{\partial t}{\partial T} &= (4M)\ \partial_T\ arctanh\Big(\frac{T}{X}\Big)\\ &= 4M\bigg(\Big[1-\Big(\frac{T}{X}\Big)^2\Big]X\bigg)^{-1}\\ &= 4M\frac{X}{X^2-T^2}\end{aligned}$$

$$\begin{aligned}\frac{\partial t'}{\partial T'} &= (4M)\ \partial_{T'}\ arctanh\Big(\frac{T'}{X'}\Big)\\ &= 4M\bigg(\Big[1-\Big(\frac{T'}{X'}\Big)^2\Big]X'\bigg)^{-1}\\ &= 4M\frac{X'}{X'^2-T'^2}\end{aligned}$$

Now, setting $T = T' = 0$,

$$\begin{aligned}\partial_T\partial_{T'}\ \Big|_{T=T'=0} &= \Big(\frac{\partial t}{\partial T}\frac{\partial}{\partial t}\Big)\Big(\frac{\partial t'}{\partial T'}\frac{\partial}{\partial t'}\Big)\ \Big|_{T=T'=0}\\ &= \Big(\frac{\partial t}{\partial T}\frac{\partial t'}{\partial T'}\Big)\Big(\frac{\partial}{\partial t}\frac{\partial}{\partial t'}\Big)\ \Big|_{T=T'=0}\\ &= \Big[(4M)^2\frac{X}{X^2-T^2}\frac{X'}{X'^2-T'^2}\Big]\Big(\frac{\partial}{\partial t}\frac{\partial}{\partial t'}\Big)\ \Big|_{T=T'=0}\\ &= (4M)^2\frac{1}{XX'}\partial_t\partial_{t'}\ \Big|_{t=t'=0}\end{aligned}$$

Consider $X = X(r_*)$, obtained via the transformation that $X = e^{r_*/4M}$.

$$\begin{aligned}\partial_T\partial_{T'}\ \Big|_{T=T'=0} &= (4M)^2\frac{1}{XX'}\partial_t\partial_{t'}\ \Big|_{t=t'=0}\\ &= (4M)^2 e^{-\frac{r_*}{4M}}e^{-\frac{r'_*}{4M}}\partial_t\partial_{t'}\ \Big|_{t=t'=0}\\ &= (4M)^2 e^{-\frac{(r_*+r'_*)}{4M}}\partial_t\partial_{t'}\ \Big|_{t=t'=0}\end{aligned}$$

We now can apply the time derivative to our correlation function (11) in terms of Kruskal-Szekeres time coordinates.

$$
\begin{aligned}
\partial_T \partial_{T'}\ \omega^{\beta}(r_*, r'_*) &= (4M)^2 e^{-\frac{(r_*+r'_*)}{4M}}\, \partial_t \partial_{t'}\ \omega^{\beta}(r_*, r'_*) \\
&= (4M)^2 e^{-\frac{(r_*+r'_*)}{4M}}\ \frac{\pi}{\beta^2} sinh^{-2}\left[\frac{\pi}{\beta}(r'_* - r_*)\right] \\
&= \frac{(4M)^2\pi}{\beta^2} \frac{e^{-\frac{(r_*+r'_*)}{4M}}}{\left(e^{\frac{\pi}{\beta}(r'_*-r_*)} - e^{-\frac{\pi}{\beta}(r'_*-r_*)}\right)^2} \\
&= \gamma\ \frac{e^{-\frac{(r_*+r'_*)}{4M}}}{e^{\frac{2\pi}{\beta}(r'_*-r_*)} + e^{-\frac{2\pi}{\beta}(r'_*-r_*)} - 2}\ , \qquad \gamma \equiv \frac{(4M)^2\pi}{\beta^2} \\
&= \gamma\ \left[e^{r'_*(\frac{2\pi}{\beta}-\frac{1}{4M})} e^{r_*(-\frac{2\pi}{\beta}-\frac{1}{4M})} + e^{r'_*(-\frac{2\pi}{\beta}-\frac{1}{4M})} e^{r_*(\frac{2\pi}{\beta}-\frac{1}{4M})} - 2e^{-\frac{(r_*+r'_*)}{4M}}\right]^{-1} \\
&= \gamma\ \left[e^{r'_*(\frac{8\pi M-\beta}{4M\beta})} e^{r_*(-\frac{8\pi M+\beta}{4M\beta})} + e^{r'_*(-\frac{8\pi M+\beta}{4M\beta})} e^{r_*(\frac{8\pi M-\beta}{4M\beta})} - 2e^{-\frac{(r_*+r'_*)}{4M}}\right]^{-1} \\
&= \gamma\ \left[e^{\zeta(r'_*-r_*)} + e^{-\zeta(r'_*-r_*)} - 2e^{-\frac{(r_*+r'_*)}{4M}}\right]^{-1}\ , \qquad \zeta \equiv \frac{8\pi M - \beta}{4M\beta}
\end{aligned} \tag{12}
$$

We are interested in correlations between the horizon ($r_* \to -\infty$) and an arbitrary r'_* beyond the horizon. So we are left with figuring out under what conditions the derivative $\partial_T \partial_{T'}\ \omega^{\beta}(-\infty, r'_*)$ is well defined. For convenience, let's rearrange (12) to take a form where we can more easily see the effects of taking $r_* \to -\infty$,

$$
\begin{aligned}
\partial_T \partial_{T'}\ \omega^{\beta}(r_*, r'_*) &= \gamma\ \left[e^{\zeta(r'_*-r_*)} + e^{-\zeta(r'_*-r_*)} - 2e^{-\frac{(r_*+r'_*)}{4M}}\right]^{-1} \\
&= \gamma\ \left[e^{\zeta r_*}\left(e^{\zeta(r'_*-2r_*)} + e^{-\zeta r'_*} - 2e^{-\frac{(2r_*+r'_*)}{4M}}\right)\right]^{-1} \\
&= \gamma\ \frac{e^{-\zeta r_*}}{e^{\zeta(r'_*-2r_*)} + e^{-\zeta r'_*} - 2e^{-\frac{(2r_*+r'_*)}{4M}}}
\end{aligned}
$$

Clearly, we see that for $r_* \to -\infty$, the function blows up. Thus we need let $\beta = 8\pi M$ so the numerator will vanish. Recall that $\zeta \equiv \frac{8\pi M-\beta}{4M\beta}$ and $\gamma \equiv \frac{(4M)^2\pi}{\beta^2}$,

$$
\begin{aligned}
\partial_T \partial_{T'}\ \omega^{\beta=8\pi M}(r_*, r'_*) &= \gamma\ \left.\frac{e^{-\zeta r_*}}{e^{\zeta(r'_*-2r_*)} + e^{-\zeta r'_*} - 2e^{-\frac{(2r_*+r'_*)}{4M}}}\right|_{r_*\to-\infty} \\
&= \frac{(4M)^2\pi}{\beta^2}\ \left.\frac{e^{-(\frac{8\pi M-\beta}{4M\beta})r_*}}{e^{(\frac{8\pi M-\beta}{4M\beta})(r'_*-2r_*)} + e^{-(\frac{8\pi M-\beta}{4M\beta})r'_*} - 2e^{-\frac{(2r_*+r'_*)}{4M}}}\right|_{r_*\to-\infty} \\
&= \frac{1}{4M}\ e^{\frac{-r'_*}{2M}}
\end{aligned} \tag{13}
$$

where we now have our well-defined function as $r_* \to -\infty$. Note that as $r'_* \to \infty$, the region beyond the black hole vanishes asymptotically, while for $r'_* \to -\infty$, the correlation diverges, since the correlation function between the same set of points is ill-defined.

Hence for a static observer, the correlation function (13) is a KMS state, i.e. a thermal state, well-defined for temperature $T_H = \frac{1}{\beta} = \frac{1}{8\pi M}$. Thus for an observer a constant distance r away from the black hole, thermal radiation is "seen" from the horizon at T_H. As the black hole radiates away particles, it's temperature increases. This is called the Hawking Effect.

4 Black Hole Evaporation

The notion of Hawking Radiation causes a predicament. As particles radiate away from the black hole, the black hole itself must lose a small portion of it's energy, and therefore mass. So we ask, at some point in time, will a black hole radiate enough particles away such that it itself will vanish. To find a relation corresponding time for a black hole to evaporate and the thermal radiation it emits, we must first find the power emitted by such a black hole. Note that for all following equations, we use natural units: $c = G = k_b = \hbar = 1$.

Let's take a simple example, a non-rotating, non-charged Schwarzchild black hole of mass M. Recall the usual Schwarzchild metric,

$$ds^2 = -\Big(1 - \frac{2M}{r}\Big)dt^2 + \Big(1 - \frac{2M}{r}\Big)^{-1}dr^2 + r^2 d\Omega^2$$

The power emitted can be easily estimated for this type of black hole. With the Schwarzchild radius, $r_s = 2M$, we can use the Stefan-Boltzman power law,

$$P = A_s \epsilon \sigma T_H^4$$

where $\epsilon = 1$, as a black hole is a perfect blackbody, A_s is the Schwarzchild surface area via the Schwarzchild radius, and σ is the Stefan-Boltzman Constant, $\sigma = \frac{\pi^2}{60}$

$$\begin{aligned} P &= A_s \epsilon \sigma T_H^4 \\ &= 4\pi r_s^2 \frac{\pi^2}{60} \frac{1}{(8\pi M)^4} \\ &= \frac{16\pi^3 M^2}{60(8\pi M)^4} \\ &= \frac{1}{15360\pi M^2} \end{aligned}$$

So as the mass of the black hole decreases, the power emitted increases quadratically.

Assume that no matter of any form falls into our black hole, we wish to find how the mass of our black hole corresponds to the time for complete evaporation.

$$P = \frac{1}{15360\pi M^2} = -\frac{dE}{dt} = -\frac{dM}{dt}$$

Integrating to solve for the time it take for complete evaporation, t_{evap},

$$\begin{aligned} M^2 dM &= -\frac{dt}{15360\pi} \\ \int_{M_0}^{0} M^2 dM &= \frac{t_{evap}}{15360\pi} \\ \therefore t_{evap} &= 5120 M_0^3 \pi \end{aligned} \tag{14}$$

Thus the time for total evaporation is proportional to the cube of the initial mass of the black hole.

4.1 Information Loss Paradox

Once the black hole evaporates, what becomes of the information that resided in it, such as the state of matter that collapsed into the black hole. It appears that once a black hole completely evaporates, we cannot appeal to the notion that states are "hidden" behind the horizon. Since the Hawking radiation is supposedly purely thermal, there is no way of conveying the vast amount of information needed to specify the states implied by entropy. Given two different original states, if we collapse them both into black holes, the cloud of emitted Hawking particles will be indistinguishable. So it seems as if the information that went into the specification of each system prior to it's collapse has disappeared. We refer to this as the Information Loss Paradox. The information regarding the state at an earlier time hails from the unitary evolution that we apply in both QFT and GR, but this information appears to fall past the horizon into a singularity. Hence, somewhere along the process of establishing our theory of quantum gravity by combining QFT and GR, we have violated unitarity. Another way to conceptualize this loss of information is to think of the process of black hole evaporation as being a time-irreversible, despite the notion that the laws used to predict it are invariant under time reversal.

While this paradox lies within the realm of our hybrid theory quantum gravity, we may still attempt to address this loss of information in the terms of the real world. Many ideas have risen in order to address this paradox, however no true answer exists. String theory does present some evidence against this loss of information. Supersymmetry configurations of strings and branes can be assembled to describe black hole geometries in various dimensions. Let's consider a configuration describing a black hole at a specific value of the string coupling. As we decrease the string coupling, the Schwarzchild radius will shrink, eventually becoming smaller than the configuration at hand, thus turning into a series of weakly-coupled strings and branes. We expect our degrees of freedom, and in turn the entropy, must remain unchanged, due to the high degree of supersymmetry. Since entropy does not change, we, ideally, should not lose any information about the states. Strominger and Vafa considered this for a particular five-dimensional supersymmetric black hole with charge. Paraphrasing, their results were that the number of degrees of freedom of the system at weak coupling matches precisely to that which would be theoretically calculated based on the entropy of the black hole at strong coupling. This result is remarkable; the black hole's entropy non-trivially depends on the charges of the configuration, thus an agreement like this appears unlikely to be a coincidence. Hence from the string theory point of view, the information seems to be conveyed via the Hawking radiation, however the theory doesn't explain how this occurs. One "drastic", as Carroll states, approach to this explaination is assuming locality in QFT is false when coupled with GR. Thus the information that is "lost" is actually spread across the horizon, non-locally. It is a possibility that quantum gravity is different that all other field theories, where information would normally be described as being located in some region on space. One particular, and actively researched, realization of non-locality hails from the holographic principle, i.e. the number of degrees of freedom in a region of space is proportional to the area of the boundary, as opposed to the volume of the region, as one would see in a local field theory. These attempts to address the paradox, as with any others, are still actively and heavily researched, in hope to continue the assembly of the missing piece of GR, quantum gravity.

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