MA32009 Lecture slides

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Lecture 1

Malthusian model - derivation and qualitative analysis

A general model

$$\frac{dN}{dt} = f(N)N = H(N),$$

Numerical solution

Dimensions and nondimensionalisation

As an example, consider the linear ODE

$$\frac{dN}{dt} = rN.$$

Introducing dimensionless variables

Steady-state analysis

Lecture 10

- Recap
- Techniques for single first order ODE (ctd)
- Example model 1: Logistic growth
- Example model 2: Spruce budworm

Linear stability analysis

Linear stability analysis (ctd)

Graphical solution

Bifurcation diagrams

Example model 1: the logistic growth equation

$$\frac{dN}{dt} = rN(t)\left(1 - \frac{N(t)}{K}\right).$$

Numerical solution

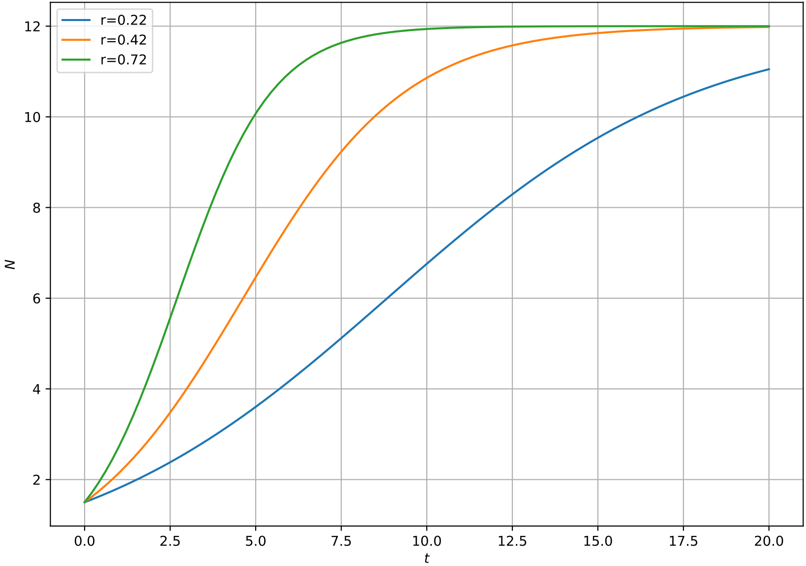


Figure 1: Numerical solution of the logistic growth model

Steady states and linear stability

Graphical analysis

An exact solution of the logistic growth equation

Example model 2: the spruce budworm model

$$\frac{dN}{dt} = r_B N \left(1 - \frac{N}{K_B}\right) - \frac{BN^2}{A^2 + N^2}, \tag{1}$$

Nondimensionalisation

$$\frac{dn}{d\tau} = rn\left(1 - \frac{n}{q}\right) - \frac{n^2}{1+n^2} = H(n),$$

Plotting the RHS

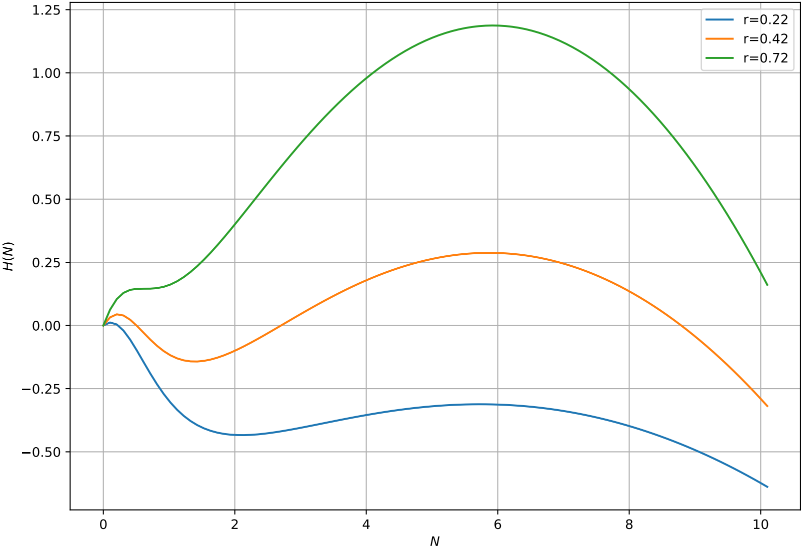
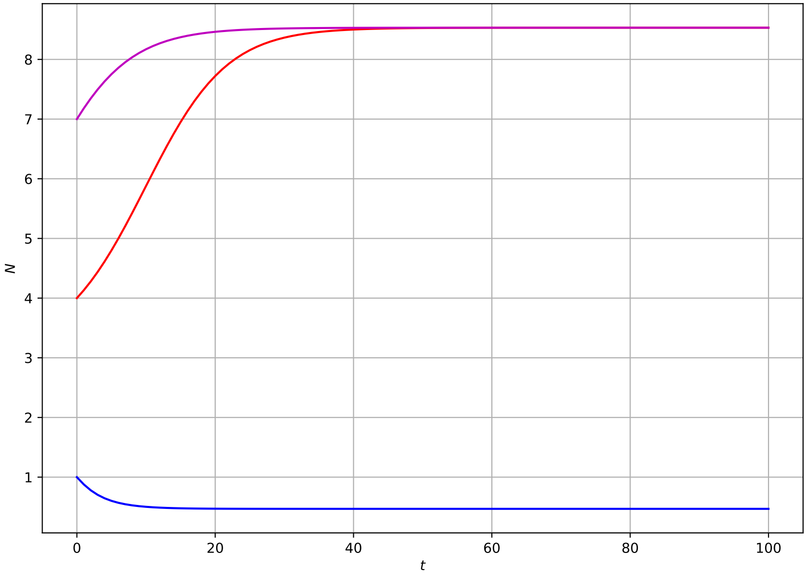


Figure 2: RHS of spr. budworm model

Numerical solution



Steady state analysis

$$rn^*(1 - \frac{n^*}{q}) - \frac{n^{*2}}{1 + n^{*2}} = 0.$$

Linear stability analysis

Lecture 12

Recap - Spruce budworm model

$$\frac{dn}{d\tau} = rn\left(1 - \frac{n}{q}\right) - \frac{n^2}{1+n^2} = H(n),$$

Steady states: $n^* = 0$ or

$$rn^*(1 - \frac{n^*}{q}) - \frac{n^{*2}}{1+n^{*2}} = 0.$$

- r small - one stable steady state
- r large - one stable steady state (outbreak)
- r intermediate - bistability (two stable steady states and one unstable)

Today: bifurcation analysis, hysteresis, harvesting

Tangent bifurcations in rq space

Plotting stability regions in the rq plane

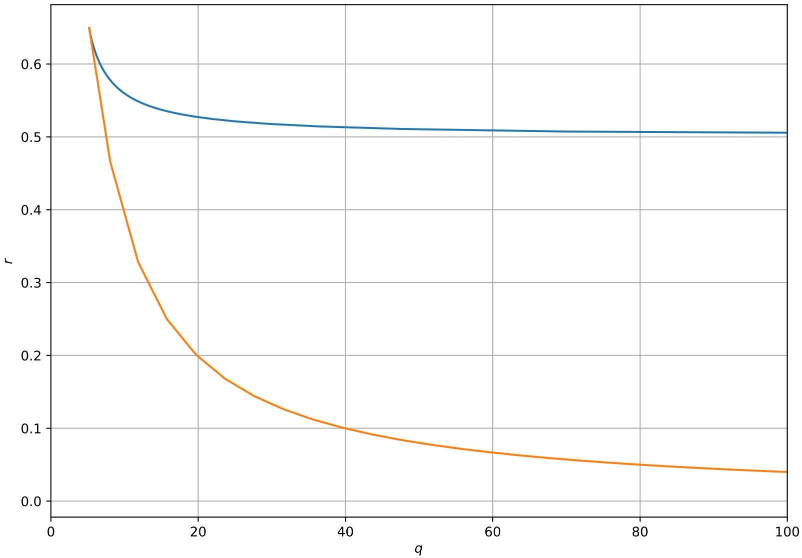


Figure 3: Bifurcations in the rq plane

Hysteresis - irreversible transitions in solution behaviour

Hysteresis

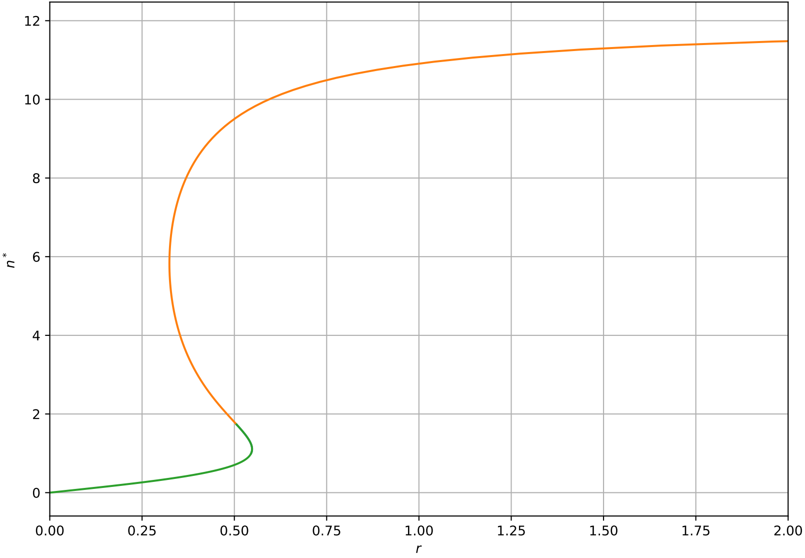


Figure 4: Bifurcations in the rq plane

Harvesting

- use models to simulate how much resource can be extracted?
- approach: take model without harvesting and add in harvesting terms

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - EN.$$

where E is the harvesting rate.

Question: what value of E maximises the long term yield?

Delay differential equation models

$$\frac{dN}{dt} = H(N(t), N(t-T)),$$

A linear delay differential equation model

$$\frac{dN}{dt} = -N(t - T),$$

Linear stability analysis (ctd.)

$$\frac{dN}{dt} = -N(t-T),$$

Two dependent variable ODE models

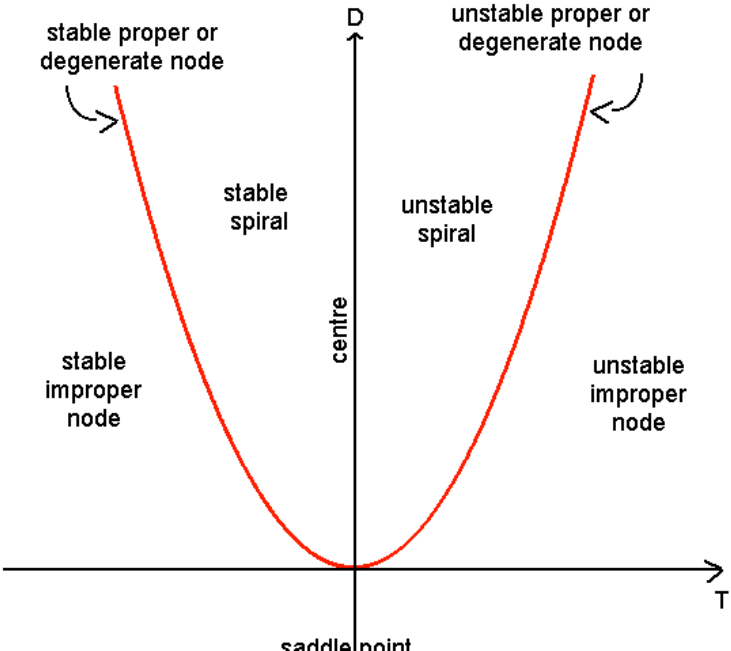
$$\frac{du}{dt} = f(u, v),$$
$$\frac{dv}{dt} = g(u, v).$$

Steady states

Linear stability analysis

Linear stability analysis (ctd.)

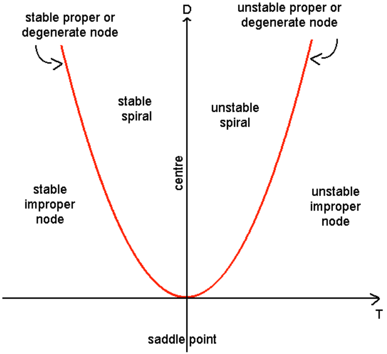
The trace determinant plane



Lecture 14

Recap

$$\frac{du}{dt} = f(u, v),$$
$$\frac{dv}{dt} = g(u, v).$$



$$\lambda = \frac{\mathrm{tr}A \pm \sqrt{\mathrm{tr}A^2 - 4\det A}}{2}.$$

Nullclines

Periodic solutions (Poincaire-Bendixson theorem)

- System of two ODEs
- Confined set containing unstable node or spiral
- as $t \to \infty$, the trajectory will tend towards a limit cycle.

No periodic soltutions - (Dulac criterion)

- D simply connected region in the plane
- $B(x, y)$, continuously differentiable on D, with

$$\frac{\partial}{\partial u}(Bf) + \frac{\partial}{\partial v}(Bg)$$

not identically zero and does not change sign in D.

Lotka Volterra

$$\frac{dN}{dt} = aN - bNP,$$
$$\frac{dP}{dt} = cNP - dP,$$

Nondimensionalisation

$$\frac{dn}{d\tau} = n(1-p) = f(n,p),$$
$$\frac{dp}{d\tau} = \alpha p(n-1) = g(n,p),$$

Lecture 15 - recap

Lotka-Volterra model - predator prey interaction

n - prey p - predator

$$\frac{dn}{d\tau} = n(1-p) = f(n,p),$$
$$\frac{dp}{d\tau} = \alpha p(n-1) = g(n,p),$$

Strategy:

- numerical solution
- steady states
- nullclines
- linear stability

Numerical solutions

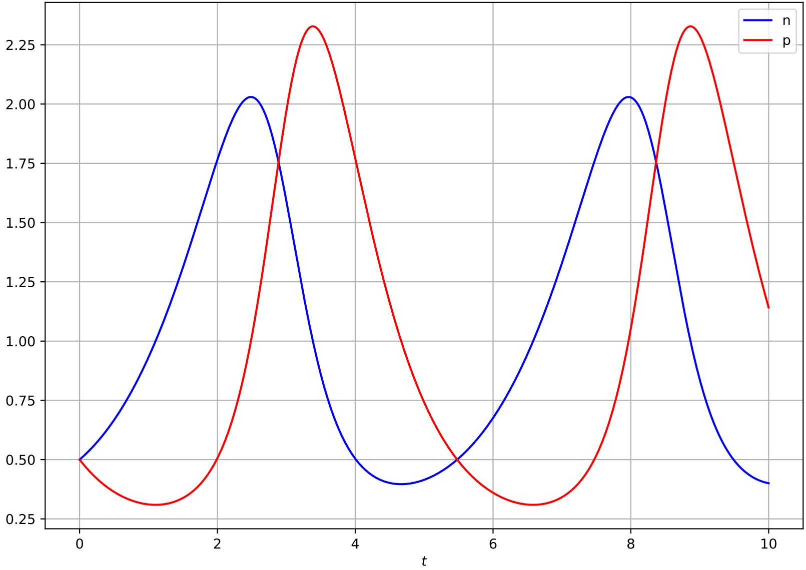


Figure 6: **?(caption)**

Steady states

Nullclines

The phase plane

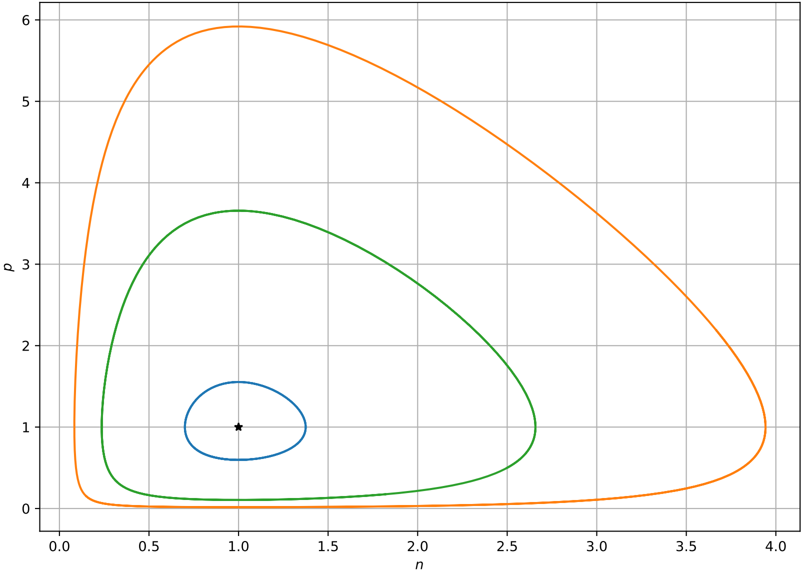


Figure 7: **?(caption)**

Linear stability

Integration

Lecture 16 Competition

Recap: Lotka-Volterra model

- predator prey interaction
- n - prey
- p - predator

$$\frac{dN}{dt} = aN - bNP,$$
$$\frac{dP}{dt} = cNP - dP,$$

Aim - introduce and analyse a model of competition

Competition model

$$\frac{dN_1}{dt} = r_1 N_1 \left(1 - \frac{N_1}{K_1} - b_{12}\frac{N_2}{K_1}\right),$$
$$\frac{dN_2}{dt} = r_2 N_2 \left(1 - \frac{N_2}{K_2} - b_{21}\frac{N_1}{K_2}\right),$$

- Justify why this a is a model for competition
- Define model parameters
- Explain the meaning of each of the terms in the model

Nondimensionalisation

$$\frac{dn_1}{d\tau} = n_1\left(1 - n_1 - a_{12}n_2\right) = f(n_1, n_2),$$
$$\frac{dn_2}{d\tau} = \rho n_2\left(1 - n_2 - a_{21}n_1\right) = g(n_1, n_2),$$

- Define ρ, a_{12} and a_{21}

Nondimensionalisation (ctd.)

Steady states

Steady states (ctd)

Nullclines

Linear stability analysis

$$A_{(n_1^*, n_2^*)} = \begin{pmatrix} 1 - 2n_1 - a_{12}n_2 & -a_{12}n_1 \\ -\rho a_{21}n_2 & \rho(1 - 2n_2 - a_{21}n_1) \end{pmatrix}_{({n_1}^*, {n_2}^*)}.$$

Linear stability analysis (ctd)

Phase portrait (one for each qualitatively distinct case)

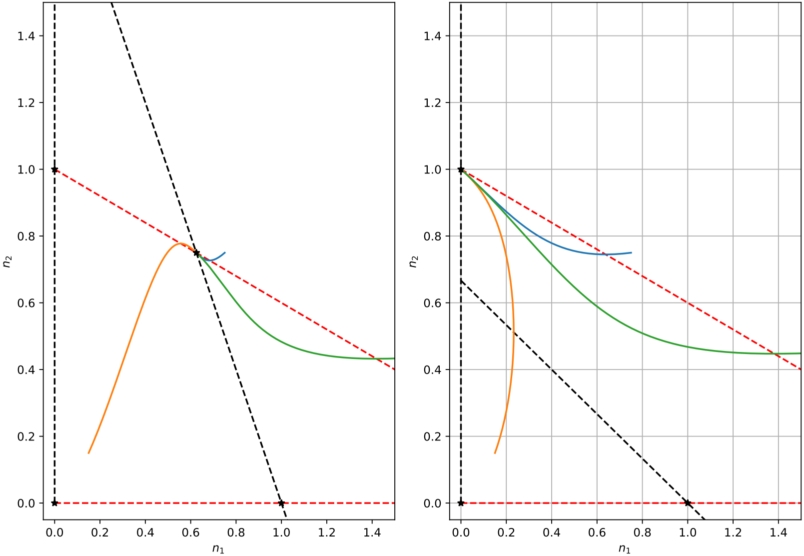


Figure 8: **?(caption)**

Insight