



University  
of Dundee

# MA32009

**MA32009**

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# Preface

Welcome to the module MA32009 Mathematical Biology I.

My name is Philip Murray and I am the module lead.

## How to contact me?

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- office: G11, Fulton Building
- Teams: PM me

## Lecture notes

You can find lecture notes for the module on this page. If you would like a pdf this can be easily generated by clicking on the pdf link of the webpage. I will occasionally edit/update the notes as we proceed through lectures. If you spot any errors, typos or omissions please Raise an Issue

## Reading

Mathematical Biology I, Murray (2002)

Essential Mathematical Biology Britton and Britton (2003)

Mathematical models in Biology Edelstein-Keshet (2005)

## Python codes

I have provided Python codes for most of the figures in the notes (you can unfold code section by clicking ‘Code’). Note that the Python code does not appear in the pdf.

Many of you have taken the Introduction to Programming module at Level 2 and have therefore some experience using Python. I strongly encourage you to use the provided codes as a tool to play around with numerical solutions of the various models that we will be working on. The codes should run as standalone Python codes.

I have also provided some examples of how to use Python as a symbolic calculator. This uses a Python library called *sympy* and is quite similar to Maple.

## Assessment

- Final exam (80 %)
- 2 class tests (8.5 % each), Week 6 and 10
- 3 quizzes (1 % each), Week 2,4 and 8

## Plan

Table 1: Projected delivery

Week	Up to Section	Tutorial sheet	Assessment
1	1.4	1	
2	1.6	1	Quiz 1
3	2.2	2	
4	2.3.7	2	Quiz 2 (Up to 2.2.1)
5	3.3	3	
6	3.5	3	Test 1 (Chapters 1 and 2)
7	4.1	4	
8	4	4	Quiz 3
9	4	5	
10	5	5	Test 2
11	5		

## References



## **Part I**

# **Discrete time**

# 1 Single species population dynamics

Let us firstly consider models of a single population defined at discrete times. Discrete (rather than continuous) time is used in population models where, for example, there are strong annual trends in the underlying animal behaviour (e.g. fish eggs hatching at a fixed time in the year), there are not overlapping generations and the population size at the next time step can be written as a function of the current population size.

## 1.1 A general model for a single population in discrete time

Let  $t$  be an independent variable representing time and  $N_t$  be a dependent variable representing the size of the population of a given species at time  $t$ .

Consider the first order difference equation

$$N_{t+1} = N_t f(N_t) = H(N_t), \quad (1.1)$$

where  $f(N_t)$  is a function that defines the *per capita* growth rate. The function  $H(N_t)$  describes the total (net) growth rate.

Given some initial condition  $N_0$  defined such that

$$N_{t=0} = N_0,$$

the goal of this section is to develop a range of techniques that allow us to analyse the behaviour of Equation 1.1 for a given  $H$ .

Finally, note that many of you were introduced to difference equations in the module MA21003 Discrete Mathematics. It is recommended that you revisit the section on Discrete Difference Equations.

## 1.2 The Malthusian model

Perhaps the simplest form for equation Equation 1.1 is when the net per capita growth rate is a constant.

Let  $b$  be the per capita growth rate and  $d$  the per capita death rate with  $b$  and  $d$  both positive real constants.

Hence, a population of size  $N_t$  at time  $t$  will grow by

$$bN_t$$

over the course of the next iteration and decay by

$$dN_t.$$

Therefore the population size at time  $t + 1$  is

$$N_{t+1} = N_t + bN_t - dN_t = rN_t, \quad (1.2)$$

where the parameter  $r = 1 + b - d$  is known as the net growth rate.

Given the initial condition  $N_0$ , equation Equation 1.2 can be solved recursively as follows:

$$N_1 = rN_0, N_2 = r(N_1) = r^2N_0, N_3 = rN_2 = r^3N_0, \dots, N_t = r^tN_0.$$

Hence for the case of Malthusian growth, the size of the population can be explicitly calculated for all  $t$ .

In Figure 1.1 a numerical solution of the model is computed for different values of the parameter  $r$ .

Can you describe the behaviour of the Malthusian model? How does it depend on the value of the parameter  $r$ ?

Qualitative analyses allows us to categorise solution behaviours into different cases. For example, note that the behaviour of the population at large  $t$  is governed by the magnitude of the parameter  $r$ .

As  $t \rightarrow \infty$

$$N \rightarrow \begin{cases} \infty & r > 1 \\ N_0 & r = 1 \\ 0 & r < 1. \end{cases}$$

In Figure 1.1 solutions in each of the (nontrivial) parameter regimes are plotted. When  $r < 1$  the population tends to zero and when  $r > 1$  the population explodes.

Whilst the simplicity of the Malthusian model allows us to calculate explicit solutions that yield insight into how the processes of birth and death combine to give either net growth or net reduction in population size, an obvious flaw with this model is that it displays unbounded



Figure 1.1: A plot of the Malthusian model solution when (a)  $r=0.5$  and (b)  $r=1.5$ .

growth for  $r > 1$ . In most biological systems, other effects, such as competition for resources or predation, will tend to limit a population's size. In single species population models, these effects are accounted for phenomenologically by introducing nonlinear terms into the function  $f$ .

### 1.3 A motivating example of a nonlinear model - The Ricker model

Many models of population dynamics in discrete time were initially developed to study fisheries. Well-studied examples include the Beverton Holt model

$$N_{t+1} = \frac{rN_t}{1 + \frac{N_t}{K}},$$

the Hassell model

$$N_{t+1} = \frac{rN_t}{(1 + \frac{N_t}{K})^b},$$

and the Ricker model

$$N_{t+1} = N_t e^{r(1 - \frac{N_t}{K})}.$$

where  $r$ ,  $K$  and  $b$  are positive parameters. Below we consider the Ricker model which was initially used to study Canadian sockeye salmon population dynamics.

### 1.3.1 Model development

One way to capture a decreased growth rate at high densities, is to assume that the per capita growth rate is an exponentially decreasing function of population density, i.e.

$$f(N_t) = e^{r(1-\frac{N_t}{K})},$$

where  $r$  is a growth rate and  $K$  a carrying capacity (both positive constants). Hence the governing equation is

$$N_{t+1} = N_t e^{r(1-\frac{N_t}{K})}. \quad (1.3)$$

Note that as the population size gets large ( $N_t \gg K$ ), the growth rate tends to zero but that for small populations  $N_t \ll K$  the growth rate is approximately constant. This is the Ricker model.

Note that the model is nonlinear and does not have an explicit solution.

### 1.3.2 Computational solutions

After directly computing numerical solutions of the Ricker model for different values of the parameter  $r$ , the data in Figure 1.2 were obtained.

Note that model behaviour changes drastically as the parameter  $r$  varies. For instance,

- when  $r = 0.5$  (Figure 1.2 (a)), the solution monotonically approaches the value 2,
- when  $r = 1.5$  (Figure 1.2 (b)) it approaches two in an oscillatory manner; and
- when  $r = 2.5$  (Figure 1.2 (c)) it oscillates around two.

The above numerical results raise the following questions:

- can we develop tools that allow us to qualitatively describe how the model solutions depend on model parameters?
- are there certain families of solutions with qualitatively similar behaviours?
- how do such families of solutions depend on model parameters?

We will return to analysis of the Ricker model in Worksheet 1.

## 1.4 General techniques for analysing nonlinear difference equations

In this section we develop techniques that will be used to analyse first order, discrete-time models of the form

$$N_{t+1} = N_t f(N_t) = H(N_t). \quad (1.4)$$



Figure 1.2: A plot of numerical solutions of the Ricker model. (a)  $r=0.5$ . (b)  $r=1.5$ . (c)  $r=2.5$ .

### 1.4.1 Computational solutions

Perhaps the most obvious way to investigate an equation of the form of Equation 1.4 is to iteratively compute  $N_t$  over a prescribed time range.

Hence given some initial population  $N_0$ , the solution set  $\{N_0, N_1, N_2, \dots, N_T\}$  is computed up to some end time  $T$ . Such an approach provides a numerical solution for a given parameter set and initial condition. However, it does not provide much insight into the general behaviour of model.

### 1.4.2 Fixed points

We define  $N^*$  to be a fixed point of Equation 1.4 if and only if

$$N^* = N^* f(N^*) = H(N^*).$$

Hence the fixed points can be identified by solving an algebraic equation.

```

#| '!!! shinylive warning !!!': |
#|   shinylive does not work in self-contained HTML documents.
#|   Please set `embed-resources: false` in your metadata.
#| standalone: true
#| components: [viewer]
#| viewerHeight: 500

from shiny import App, Inputs, Outputs, Session, render, ui
from shiny import reactive

import numpy as np
from pathlib import Path
import matplotlib.pyplot as plt
from scipy.integrate import odeint

app_ui = ui.page_fluid(
    ui.layout_sidebar(
        ui.panel_sidebar(
            ui.input_slider(id="r",label="r",min=0.0,max=5.0,value=0.1,step=0.001),
            ui.input_slider(id="u0",label="u_0",min=0.0,max=1.0,value=0.5,step=0.01),
            ui.input_slider(id="T",label="Number of iterations",min=0.0,max=60.0,value=20.0,step=1.0),
            ui.input_slider(id="r_range",label="r zoomed",min=0.0,max=4.0,value=[0.0,4.0],step=0.001),
            ui.input_slider(id="u_range",label="u zoomed",min=0.0,max=1.0,value=[0.0,1.0],step=0.01)

        ),

        ui.panel_main(ui.output_plot("plot"),),
    ),
)

def server(input, output, session):

    @render.plot
    def plot():
        fig, ax = plt.subplots(3,1)
        #ax.set_ylim([-2, 2])
        # Filter fata

        r=float(input.r())
        u0=float(input.u0())
        T=int(input.T())
        r_min=float(input.r_range()[0])
        r_max=float(input.r_range()[1])
        u_min=float(input.u_range()[0])
        u_max=float(input.u_range()[1])14

        # Define rhs of logistic map
        def logistic_map(y,t,r):
            K=1.0
            rhs=y*np.exp(r*(1-y/K))

```

### 1.4.3 Linear stability

By linearising about the identified fixed points using a Taylor series expansion, it can be determined whether or not small perturbations around the fixed point grow or decay in subsequent iterations.

Making a change of dependent variables

$$N_t = N^* + \hat{N}_t,$$

Equation 1.4 transforms upon substitution to

$$N^* + \hat{N}_{t+1} = H(N^* + \hat{N}_t).$$

Taylor expanding the right-hand side yields

$$N^* + \hat{N}_{t+1} = H(N^*) + H'(N^*)\hat{N}_t + h.o.t..$$

Noting that at the fixed point

$$N^* = H(N^*),$$

cancellation yields

$$\hat{N}_{t+1} = H'(N^*)\hat{N}_t + h.o.t..$$

Hence considering only small perturbations about the fixed point, the leading order behaviour of the perturbation is governed by

$$\hat{N}_t = (H'(N^*))^t \hat{N}_0.$$

Hence the linear stability of the fixed point is governed by the sign and magnitude of the term  $H'(N^*)$ . If

$$|H'(N^*)| > 1,$$

the fixed point is unstable. Otherwise it is stable.

### 1.4.4 Cobwebbing and linear stability

A cobweb diagram allows solutions of a first order difference equation to be sketched without explicitly solving.



### 1.4.4.1 An algorithm for generating a cobweb diagram

The cobweb diagram is created as follows:

- Plot the  $t^{th}$  iterate,  $N_t$ , on the  $x$  axis and the  $t + 1^{st}$ ,  $N_{t+1}$  on the  $y$  axis.
- Sketch the net growth rate function  $H(N_t)$ .
- Sketch the line  $N_{t+1} = N_t$ .
- Note any intersections between the straight line and the graph of  $H(N_t)$  are fixed points.
- Plot the first point  $(N_0, H(N_0))$  on the cobweb diagram.
- Plot a horizontal line between  $(N_0, H(N_0))$  and  $(H(N_0), H(N_0))$ . Note  $N_1 = H(N_0)$ .
- Plot a vertical line between  $(N_1, N_1)$  and  $(N_1, H(N_1))$ . \*Repeat steps 6 and 7.

Plotting cobweb diagrams in the vicinity of a fixed point  $N^*$  for different values of  $H'(N^*)$  allows us to see graphically how the derivative of the right-hand side affects linear stability of a fixed point (see Figure 1.4). In particular,

- when  $H'(N^*) > 1$ , the fixed point is monotonically unstable;
- when  $0 < H'(N^*) < 1$ , the fixed point is monotonically stable;
- when  $-1 < H'(N^*) < 0$ , the fixed point is oscillatory stable;
- when  $H'(N^*) = -1$ , the fixed point is oscillatory; and
- when  $H'(N^*) < -1$ , the fixed point is oscillatory unstable.

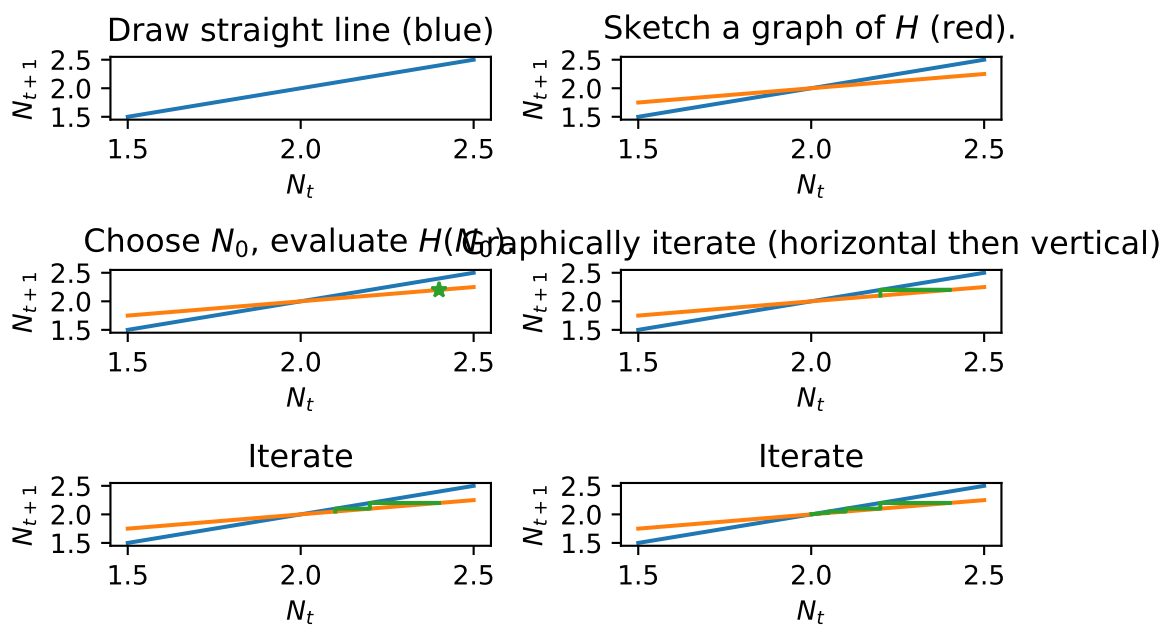


Figure 1.4: Generating a cobweb plot.

### 1.4.5 Bifurcation diagrams

A bifurcation is a qualitative change in solution behaviour (either a change in the number of fixed points or their stability). A bifurcation diagram is a compact means of describing bifurcations as a function of model parameters. Usually, the fixed point value of the population (i.e.  $N^*$ ) is plotted against a model parameter of interest. From this diagram one can immediately see how the number of fixed points and their stability changes as a given parameter increases.

### 1.4.6 Perform a qualitative analysis on the Malthusian model

Let's work through the various concepts introduced above using the Malthusian model (Equation 1.2).

The fixed points satisfy

$$N^* = rN^*.$$

As  $r > 0$  the only solution is  $N^* = 0$ .

In this case

$$H'(N_t) = r.$$

Hence the linear stability of the solution depends on the value of the parameter  $r$ . For  $r > 1$  the fixed point  $N^* = 0$  is monotonically unstable. For  $r < 1$  the fixed point  $N^* = 0$  is monotonically stable. As expected this result is consistent with the simulation results in Figure 1.1.

For cobweb diagram:

- Sketch axes and label with  $N_t$  and  $N_{t+1}$ .
- From linear stability analysis note there are two qualitatively distinct cases ( $r < 1$  and  $r > 1$ ). We will need a cobweb diagram for each case.
- Case 1: graph the function  $H(N_t)$ . In this case it is the straight line  $N_{t+1} = rN_t$  with  $r < 1$ .
- Case 2: graph the function  $H(N_t)$ . In this case it is the straight line  $N_{t+1} = rN_t$  with  $r > 1$ .
- Fill in the cobwebbed trajectories with some arbitrarily chosen initial condition.
- Establish that behaviour of the cobweb is consistent with linear stability analysis.

For bifurcation diagram:

- We will plot the value of fixed point against a parameter of interest. In this case  $N^*$  against  $r$ .
- Deduce from fixed point and linear stability analysis whether there are any changes in solution behaviour as  $r$  varies (e.g. number of solution or their linear stability).

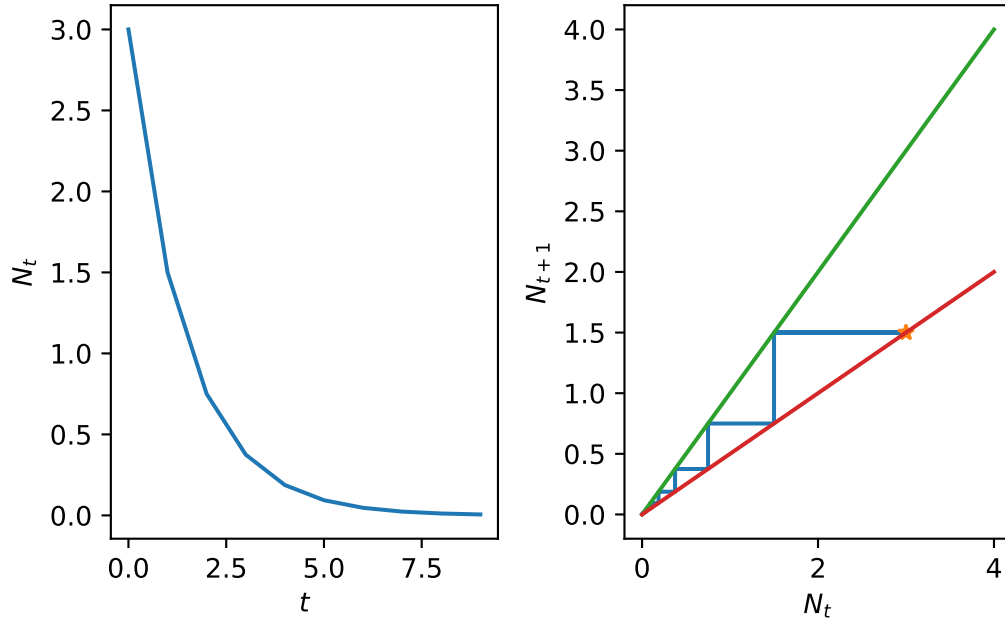


Figure 1.5: A cobweb plot of the Malthusian model solution when  $r=0.5$ . (a) Time series and (b) Cobweb diagrams.

- In this case the value of the fixed point is independent of  $r$  but the linear stability changes at  $r = 1$ . This is a bifurcation.
- Use annotation to represent qualitative solution behaviour.

## 1.5 A nonlinear model of population dynamics in discrete time

Consider a model of population growth given by

$$N_{t+1} = \frac{\gamma N_t}{1 + N_t^2}, \quad (1.5)$$

where  $\gamma \in \mathfrak{R}^+$  is the linear growth rate.

### 1.5.1 The per capita growth rate

Note that the per capita growth rate is described by the function

$$f(N_t) = \frac{\gamma}{1 + N_t^2}.$$

Hence at large populations the per capita growth rate tends to zero whilst for small populations it tends to  $\gamma$ . Hence so long as  $\gamma > 1$  we will have net growth for small populations and net removal for large populations. The net growth rate is given by

$$H(N_t) = \frac{\gamma N_t}{1 + N_t^2}.$$

### 1.5.2 Numerical solution

In Figure 1.6 time series solution of the model are plotted for parameter values  $\gamma = 0.5$ ,  $\gamma = 1.5$ ,  $\gamma = 2.5$ . Note qualitative changes in model behaviour in the different parameter regimes. Can you identify the fixed points? Are the solutions oscillatory?

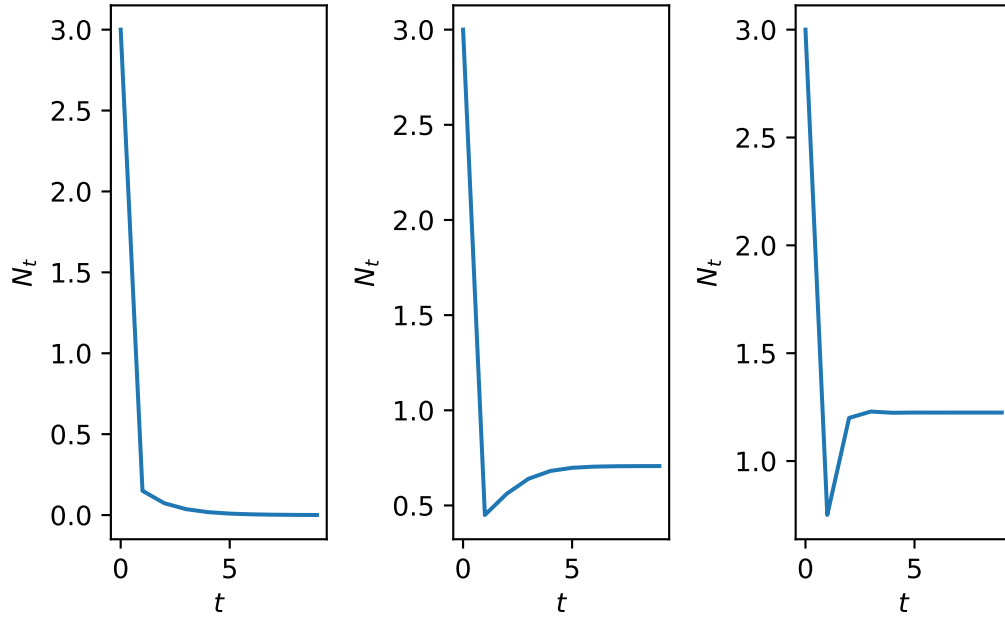


Figure 1.6: Time series solution for (a)  $\gamma=0.5$  and (b)  $\gamma=1.5$  and (c)  $\gamma=10.5$ .

### 1.5.3 Fixed points

The fixed points of Equation 1.5 satisfy the algebraic equation

$$N^* = \frac{\gamma N^*}{1 + N^{*2}}$$

which has solutions

$$N^* = 0 \quad \text{and} \quad N^* = \sqrt{\gamma - 1}.$$

### 1.5.3.1 Parameter restrictions for biological validity

If  $\gamma < 1$ , there is only one biologically relevant fixed point. If  $\gamma > 1$  there are two fixed-points,  $N^* = 0$  and  $N^* = \sqrt{\gamma - 1}$ . Hence the value of the model parameter  $\gamma$  qualitatively affects the behaviour and number of solutions.

### 1.5.4 Linear stability analysis

For the given model we compute

$$H'(N_t) = \frac{\gamma}{1 + N_t^2} + \frac{-2\gamma N_t^2}{(1 + N_t^2)^2}.$$

#### 1.5.4.1 $N^* = 0$

The stability of the fixed point  $N^* = 0$  is determined by

$$H'(0) = \gamma.$$

Hence if  $\gamma < 1$  the first fixed point is monotonically stable as  $0 < H'(0) < 1$ .

When  $\gamma > 1$  the second fixed point becomes biologically relevant and the first fixed point becomes monotonically unstable.

#### 1.5.4.2 $N^* = \sqrt{\gamma - 1}$

The stability of the second fixed point,  $N^* = \sqrt{\gamma - 1}$ , is determined by

$$\begin{aligned} H'(\sqrt{\gamma - 1}) &= \frac{\gamma}{1 + \gamma - 1} + \frac{-2\gamma(\gamma - 1)}{(1 + (\gamma - 1))^2}, \\ &= \frac{2}{\gamma} - 1. \end{aligned}$$

Hence if  $N^*$  were monotonically unstable,

$$H'(\sqrt{\gamma - 1}) = \frac{2}{\gamma} - 1 > 1,$$

and

$$\gamma < 1.$$

As a necessary condition for the biological relevance of  $N^*$  is that  $\gamma > 1$  we obtain a contradiction. Hence  $N^*$ , if it is biologically relevant, cannot be monotonically unstable.

Suppose  $N^*$  is monotonically stable. Then

$$0 < H'(N^*) < 1$$

which upon substitution yields

$$0 < \frac{2}{\gamma} - 1 < 1.$$

Hence  $1 < \gamma < 2$ . Suppose  $N^*$  is oscillatory stable. Then

$$-1 < H'(N^*) < 0$$

which upon substitution yields

$$-1 < \frac{2}{\gamma} - 1 < 0.$$

Hence  $\gamma > 2$ . Hence the linear stability of the fixed point  $N^* = \sqrt{\gamma - 1}$  depends on whether  $\gamma$  lies in the range  $[0, 1]$ ,  $[1, 2]$  or  $[2, \infty]$ .

### 1.5.5 Cobweb diagrams

In Figure 1.7 cobweb diagrams illustrate model behaviour in the three different parameter regimes. Note that the cobweb diagrams can be sketched by hand, given an accurate enough sketch of the right-hand side function  $H(N_t)$ . Note also that the cobweb diagrams are consistent with the linear stability analysis but that the linear approximation is only valid close to the equilibrium point.

### 1.5.6 Bifurcation diagram

Bifurcations at the critical values  $\gamma = 1$  and  $\gamma = 2$  are highlighted in the bifurcation diagram presented in Figure 1.8. Note that the bifurcation diagram allows classification of the different model behaviours in a single plot without explicitly calculating the solution to the model.

### 1.5.7 Symbolic computation

We can use symbolic calculators to aid many of computations performed above. Below is an example using the Python library sympi.

Many of the calculations can be aided by using a symbolic calculator.

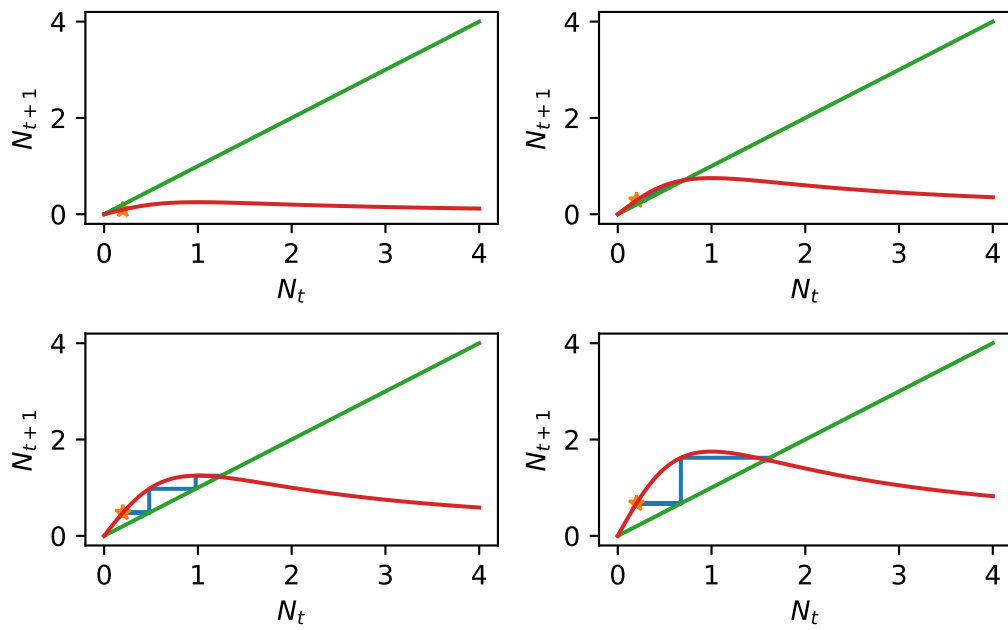


Figure 1.7: Cobweb diagrams for different values of  $\gamma$ .

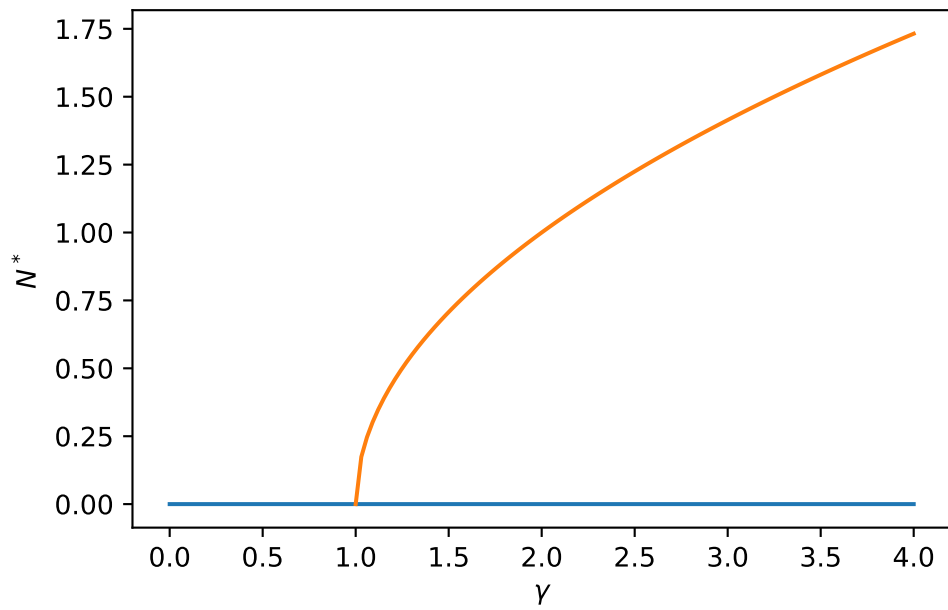


Figure 1.8: A bifurcation diagram.

The FPs are:  
 $\{N: 0\}, \{N: -\sqrt{\gamma - 1}\}, \{N: \sqrt{\gamma - 1}\}$   
The derivative of H is:  
 $\gamma(1 - N^2)/(N^2 + 1)^2$   
The derivative evaluated at FP 1 is:  
 $\gamma$   
The derivative evaluated at FP 2 is:  
 $(2 - \gamma)/\gamma$

## 1.6 An application - how much harvesting can a population sustain?

Models of population dynamics can be used to study how interventions will affect population dynamics. Extending from the model developed in the previous example, a valid question might be what is the maximal rate of harvesting a fish stock can sustain without becoming extinct?   
### Including a harvesting term  
Introducing a harvesting term at per capita harvesting rate  $h$ , the governing model equation becomes

$$N_{t+1} = \frac{\gamma N_t}{1 + N_t^2} - hN_t, \quad (1.6)$$

and the questions we want to ask are: (i) does the introduction of harvesting change the dynamics of the system?; and (ii) what rate of harvesting such a population could withstand?

### 1.6.1 Direct simulation

Based on the previous analysis we consider the system in a parameter regime where without harvesting there is a stable fixed point ( $\gamma > 1$ ).

In Figure 1.9 time series solutions at increasing harvesting rates ( $h = 0.1, h = 1, h = 3$ ) are presented. For low harvesting rates the system behaves almost identically to the no harvesting case but with an expected reduction in the fixed point value. However for intermediate harvesting the population undergoes oscillations. However, for further increased harvesting rates there appears again to be a stable fixed point.

Can analysis of the model help us understand how/why changes in solutions occur?

### 1.6.2 Fixed points

The fixed points of the system satisfy

$$N^* = \frac{\gamma N^*}{1 + N^{*2}} - hN^*.$$



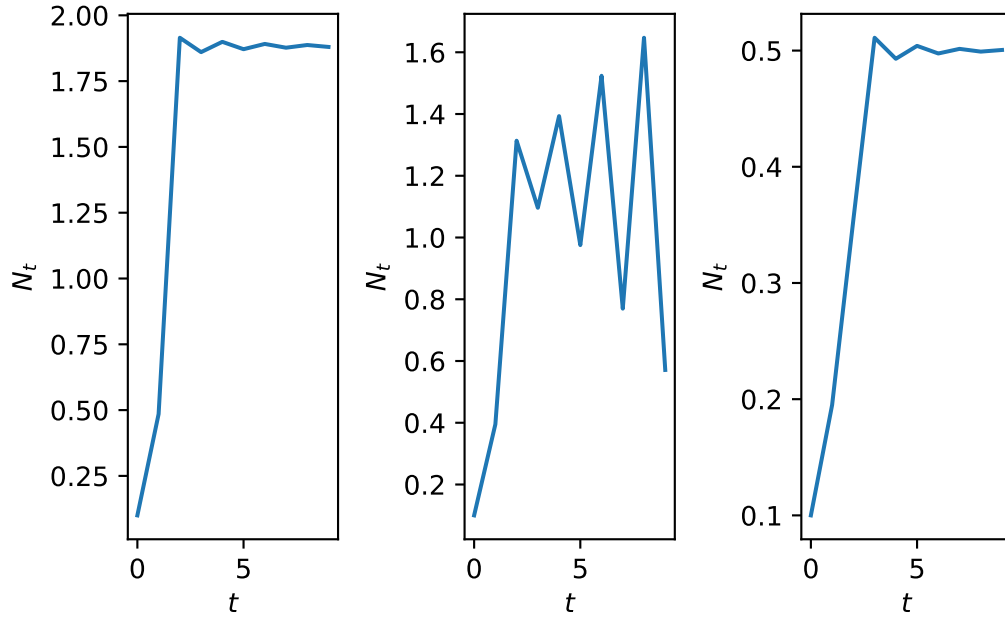


Figure 1.9: Time series solution for different values of  $h$ .

Hence the fixed points are

$$N^* = 0$$

and

$$N^* = \sqrt{\frac{\gamma}{1+h} - 1}.$$

Note that harvesting lowers the population size of the non-zero fixed point (when it exists).

Can we deduce a condition that must hold on the parameters  $\gamma$  and  $h$  in order that there is a non trivial biologically relevant fixed point?

Requiring

$$N^* > 0$$

implies

$$\frac{\gamma}{1+h} - 1 > 0.$$

### 1.6.3 Linear stability

The linear stability is determined by

$$H'(N_t) = \frac{\gamma}{1 + N_t^2} - \frac{2\gamma N_t^2}{(1 + N_t^2)^2} - h.$$

At  $N^* = 0$  we obtain

$$H'(0) = \gamma - h.$$

Hence if  $\gamma > 1 + h$ ,  $N^* = 0$  is unstable. Note that this is the condition that determines whether the non-zero fixed point exists or not.

At

$$N^* = \sqrt{\frac{\gamma}{1 + h} - 1},$$

$$H' \left( \sqrt{\frac{\gamma}{1 + h} - 1} \right) = \frac{\gamma}{1 + \frac{\gamma}{1 + h} - 1} - \frac{2\gamma \frac{\gamma}{1 + h} - 1}{(1 + \frac{\gamma}{1 + h} - 1)^2} - h.$$

Hence linear stability is a function of two parameters  $\gamma$  and  $h$ .

We can show that

$$H'(N^*) = h^2 \frac{2}{\gamma} + h \frac{2}{\gamma} (2 - \gamma) + \frac{2}{\gamma} - 1, \quad (1.7)$$

and hence that when  $h = 0$  we retrieve the stability condition from the previous model.

### 1.6.4 Stability boundaries in the $h\gamma$ plane

The contour at  $H'(N^*) = 1$  can be identified by solving

$$h^2 \frac{2}{\gamma} + h \frac{2}{\gamma} (2 - \gamma) + \frac{2}{\gamma} - 1 = 1,$$

yielding

$$\gamma = 1 + h.$$

The contour at  $H'(N^*) = -1$  can be represented by

$$h^2 \frac{2}{\gamma} + h \frac{2}{\gamma} (2 - \gamma) + \frac{2}{\gamma} - 1 = -1,$$

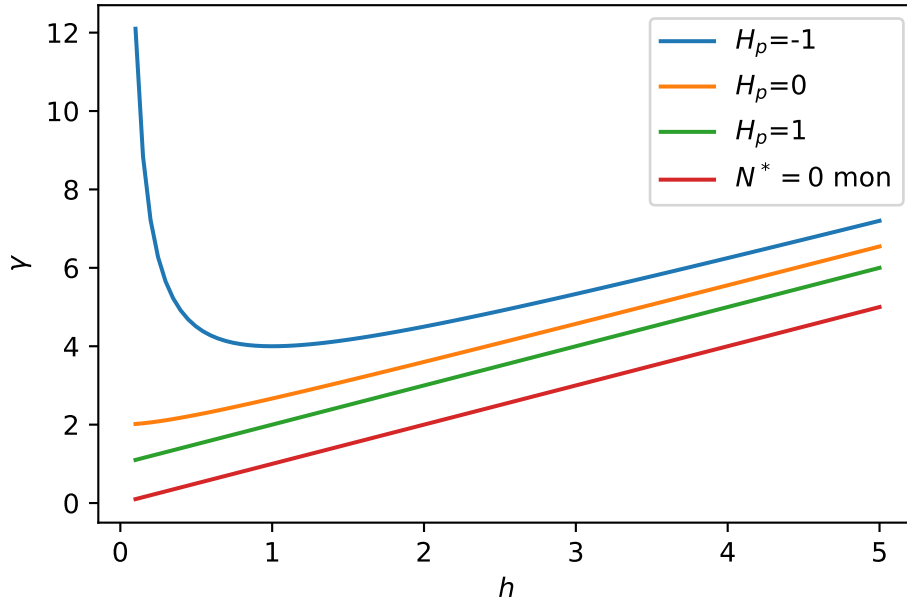


Figure 1.10: Stability regions for the harvesting model.

Solving for  $\gamma$  yields

$$\gamma = \frac{(1+h)^2}{h}.$$

In Figure 1.10 we plot the contours of the hypersurface  $H'(N^*)$  at the critical values of  $-1$ ,  $0$  and  $1$ . Note that the points in parameter space used to generate the simulation results in Figure 1.9 are  $(h, \gamma) = (0.1, 5)$ ,  $(h, \gamma) = (1, 5)$  and  $(h, \gamma) = (3, 5)$ . Cobweb diagrams in different regions of parameter space are presented in Figure 1.11. The plot in Figure 1.10 can be used to explain why the model transfers from a stable fixed point, through oscillatory fixed point and back to a stable fixed point as the harvesting rate increases from  $0.1$ , to  $1$  and then  $5$ ?

Many of the above calculations can be aided using a symbolic calculator:

The FPs are:

$\{N: 0\}$

$\{N: -\sqrt{\gamma - h - 1}/\sqrt{h + 1}\}$

$\{N: \sqrt{\gamma - h - 1}/\sqrt{h + 1}\}$

The derivative of  $H$  is:

$-N^2\gamma/(N^2 + 1)^2 + \gamma/(N^2 + 1)^2 - h$

The derivative evaluated at FP 0 is:

$\gamma - h$

The derivative evaluated at FP 2 is:

$(\gamma + 2(h + 1)(-\gamma + h + 1))/\gamma$

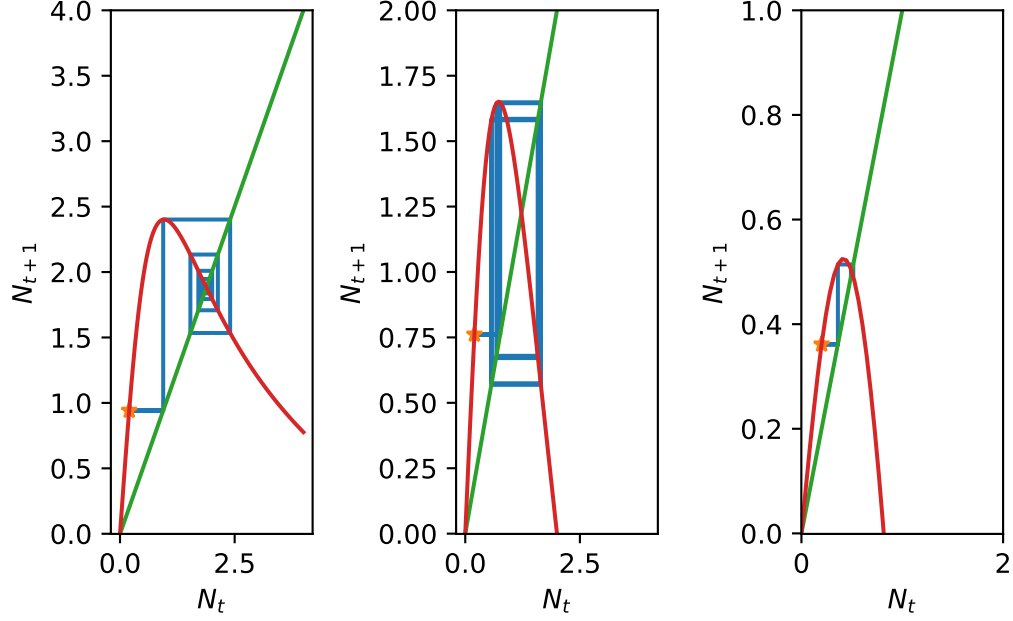


Figure 1.11: Cobweb diagrams for different values of  $h$ .

## 1.7 Oscillations

Linear stability analysis describes the evolution of small perturbations close to a fixed point. But far from a fixed point the nonlinear terms that were dropped in a Taylor expansion are no longer negligible. In particular, in the case where  $H'(N^*) < -1$ , it can be the case that regions of parameter space in which a fixed point is oscillatory unstable can give rise to periodic and chaotic solutions.

### Defining a periodic solution Consider the general form

$$N_{t+1} = H(N_t) \tag{1.8}$$

A solution to Equation 1.8 is defined to be periodic with period  $T$  if

$$N_{t+T} = N_t \quad \forall t, N_{t+\tau} \neq N_t \quad \forall t, \quad \tau < T.$$

Period 2 solutions can be identified by looking for solutions that repeat after two iterations. Suppose  $\bar{N}$  is a period 2 solution of Equation 1.8 and let  $\bar{N} = N_1$ . Then

$$N_2 = H(\bar{N}).$$

However,

$$N_3 = H(N_2) = H(H(\bar{N}))$$

and if  $\bar{N}$  is a period 2 solution  $N_3 = N_1 = \bar{N}$ . Hence period 2 solutions can be calculated by solving the algebraic equation

$$\bar{N} = H(H(\bar{N})).$$

Note that period two solutions are fixed points of the problem

$$N_{t+2} = H^2(N_t) = g(N_t).$$

Using the tools we have developed, period solutions and their stability can be determined. Furthermore, longer period solutions can be identified by generalising the argument but at the expense of ever increasingly complicated right-hand side functions. In many systems, such as the harvesting model and the logistic equation, increasing the growth rate parameter leads initially to the fixed point becoming unstable and the emergence of period two solutions, then period 4 solutions and so on until eventually there is a transition to chaotic solutions (see, for example, Figure 1.12).

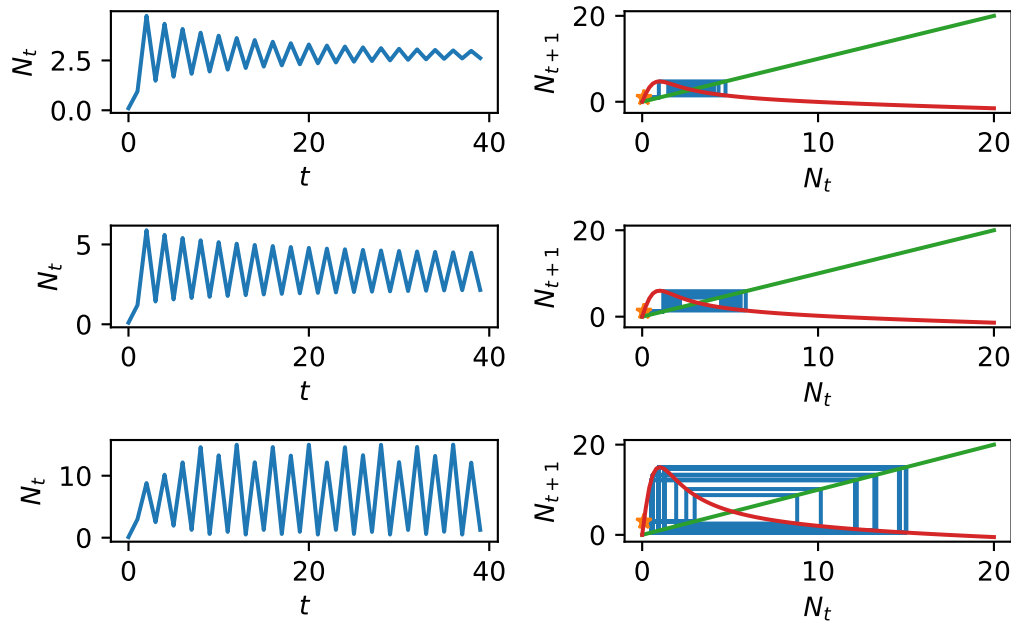


Figure 1.12: Transition from period 2 to period 4 solutions in the harvesting model

Exercise: identify an equation satisfied by period 2 solutions of the logistic map

$$N_{t+1} = rN_t(1 - N_t).$$

## 1.8 A note on the modelling of real-world fish stocks

You can find reports on fish stocks (measurements and modelling work) from the International Council for the Exploration of the Sea at the [link](#). Here's a link to the data for [Atlantic salmon](#).

Note that although the models we have worked on are not detailed enough to accurately model real fish population dynamics, many of the principles we have covered arise in the cutting edge models.

## 2 Multi species population dynamics in discrete time

In this chapter we generalise the previous approach by considering the population dynamics of two interacting populations.

### 2.1 A general model of two interacting species in discrete time

Let  $N_t$  and  $P_t$  represent population densities at time  $t$  where  $t$  is a discrete variable.

We consider governing equations of the form

$$\begin{aligned}N_{t+1} &= f(N_t, P_t), \\P_{t+1} &= g(N_t, P_t).\end{aligned}$$

where the population dynamics of the species are coupled to one another via the functions  $f$  and  $g$ .

The precise form for  $f$  and  $g$  will be defined by the biological system under study. Typical interpopulation interactions that are studied are: predator prey models, competition and cooperation.

### 2.2 General techniques for analysing coupled first order difference equations

#### 2.2.1 Fixed points

The fixed points of Equation 2.2  $(N^*, P^*)$  satisfy

$$\begin{aligned}N^* &= g(N^*, P^*) \\P^* &= f(N^*, P^*).\end{aligned}$$

### 2.2.2 Linear stability analysis

To consider linear stability of a steady state we consider the change of variable

$$\begin{aligned} N_t &= N^* + \hat{N}_t, \\ P_t &= P^* + \hat{P}_t. \end{aligned}$$

After substitution in Equation 2.2 and making Taylor expansions about  $(N^*, P^*)$  we obtain at leading order

$$\begin{pmatrix} \hat{N}_{t+1} \\ \hat{P}_{t+1} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial N} & \frac{\partial f}{\partial P} \\ \frac{\partial g}{\partial N} & \frac{\partial g}{\partial P} \end{pmatrix}_{(N^*, P^*)} \begin{pmatrix} \hat{N}_t \\ \hat{P}_t \end{pmatrix}. \quad (2.1)$$

Note the appearance of the Jacobian matrix

$$A = \begin{pmatrix} \frac{\partial f}{\partial N} & \frac{\partial f}{\partial P} \\ \frac{\partial g}{\partial N} & \frac{\partial g}{\partial P} \end{pmatrix}_{(N^*, P^*)} \begin{pmatrix} \hat{N}_t \\ \hat{P}_t \end{pmatrix}.$$

### 2.2.3 Solving the linearised problem

Defining

$$\mathbf{w}_t = \begin{pmatrix} \hat{N}_t \\ \hat{P}_t \end{pmatrix},$$

$$\mathbf{w}_{t+1} = A\mathbf{w}_t$$

By solving this linear system we can investigate whether small perturbation about the fixed point grow or decay in magnitude as time evolves and hence determine the linear stability of the fixed point.

The solution of Equation 2.1 takes the form

$$\mathbf{w}_t = \sum_{i=1}^2 C_i \lambda_i^t \mathbf{c}_i,$$

where the  $C_i$ 's are constants determined by initial conditions and the  $\lambda_i$ 's and  $\mathbf{c}_i$ 's are the eigenvalues and eigenvectors of  $A$ , respectively. From this form we can see that if the magnitude of all eigenvalues is less than one, i.e.

$$|\lambda_i| < 1, \quad \forall i,$$

the fixed point is linearly stable. If at least one of the eigenvalues has  $|\lambda_i| > 1$  then the fixed point is unstable to linear perturbations.



### 2.2.4 Jury conditions

In many cases it is not very useful to explicitly compute the eigenvalues of the Jacobian matrix. In such cases we can employ the Jury conditions in order to determine when a fixed point is linearly stable.

Recall that for a two dimensional matrix, the eigenvalues satisfy the quadratic characteristic equation

$$\lambda^2 - \text{tr}A\lambda + \det A = 0.$$

The stability of the fixed is guaranteed if  $|\lambda_i| < 1 \forall i$ .

Consider the characteristic equation

$$P(\lambda) = \lambda^2 + a\lambda + b = 0,$$

where  $a, b \in \mathfrak{R}$ . Note that  $a = -\text{tr}A$  and  $b = \det A$ .

The Jury conditions state that  $|\lambda_i| < 1 \forall i$  if, and only if,  $b < 1$ ,  $1+a+b > 0$ ,  $1-a+b > 0$ . See Figure 2.1 for schematic diagram.

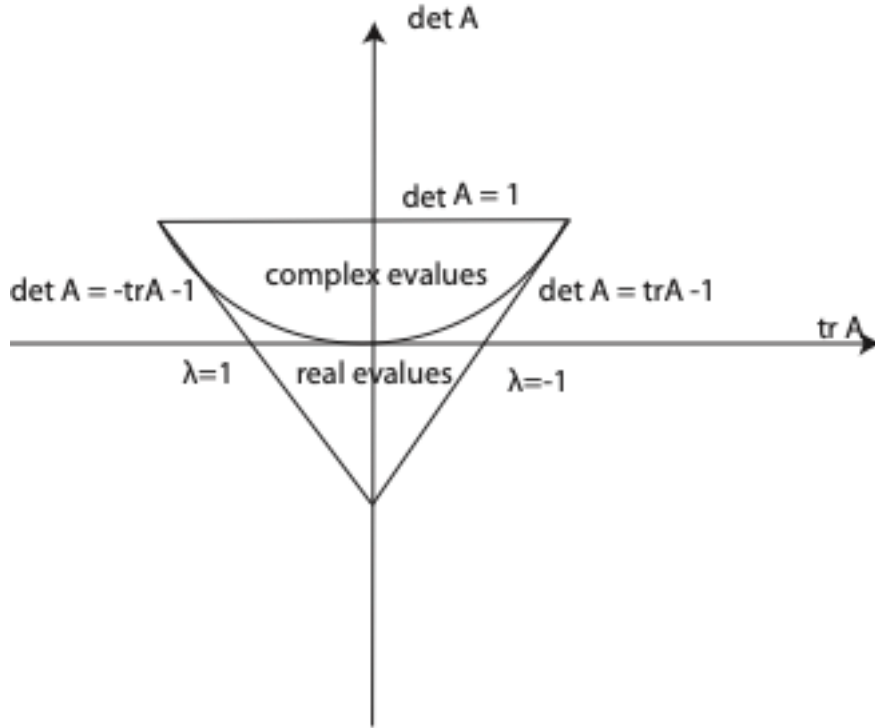


Figure 2.1: Jury conditions

### 2.2.5 Proof of the Jury conditions

The roots of  $P(\lambda)$  are

$$\lambda_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}.$$

### 2.2.6 Complex roots

Suppose  $a^2 - 4b < 0$ . The roots are complex. Since  $b$  is equal to the product of the roots, we find that

$$b = \lambda_1 \lambda_2 = |\lambda_1|^2 = |\lambda_2|^2.$$

Hence  $\lambda_i < 1 \forall i$  if and only if  $b < 1$ .

For the other conditions we introduce the identity:

$$a^2 - 4b = (|a| - 2)^2 - 4(1 + b - |a|).$$

Therefore, when  $a^2 - 4b < 0$ , the inequality requires

$$(|a| - 2)^2 - 4(1 + b - |a|) < 0.$$

This can only occur if

$$1 + b - |a| > 0 \implies 1 + b - a > 0 \text{ and } 1 + b + a > 0.$$

### 2.2.7 Real roots

Suppose  $a^2 - 4b \geq 0$

The largest magnitude of the roots is

$$R = \max\{|\lambda_1|, |\lambda_2|\} = \frac{|a| + \sqrt{a^2 - 4b}}{2}.$$

This is an increasing function of  $|a|$ .

$$R = 1$$

$$\implies \frac{|a| + \sqrt{a^2 - 4b}}{2} = 1 \implies |a| - 2 = -\sqrt{a^2 - 4b} \implies |a|^2 - 4|a| + 4 = |a|^2 - 4b \implies |a| = b + 1.$$

Hence  $0 \leq R < 1$  if and only if  $0 \leq |a| < 1 + b$ . Also, since  $|\lambda_i| < 1 \forall i$  implies that  $|\lambda_1 \lambda_2| < 1$ , it follows that  $|b| < 1$  must hold.

## 2.3 Host Parasitoid infection

Parasitoids are creatures that have a free living and parasitic stage. The free-living adult lays eggs in a host that later hatch and develop after eating the host. The discrete stages of the parasitoids life cycle and the dependence on its reproduction on the availability of host suggest a discrete time, multi species models. Let  $N_t$  and  $P_t$  represent the number of hosts and parasitoids at time  $t$ , respectively. Let  $R_0$  represent the reproductive ratio of host in the absence of parasites and  $C$  the average number of viable eggs laid by each parasite on a host. ### Model equations

We consider equations of the form

$$\begin{aligned}N_{t+1} &= R_0 N_t f(N_t, P_t), \\P_{t+1} &= C N_t (1 - f(N_t, P_t)).\end{aligned}$$

The justification for this form is that at a given time  $t$  there are  $N_t$  hosts. A total of  $N_t f(N_t, P_t)$  escape the parasite and are able to reproduce whilst a total of  $N_t (1 - f(N_t, P_t))$  do not escape the parasite and lead to parasitic reproduction at the next time step.

Choosing

$$f(N_t, P_t) = e^{-aP_t},$$

yields the Nicholson Bailey model

$$\begin{aligned}N_{t+1} &= R_0 N_t e^{-aP_t}, \\P_{t+1} &= C N_t (1 - e^{-aP_t}).\end{aligned}$$

### 2.3.1 Numerical solution

### 2.3.2 Fixed points

The fixed points satisfy

$$\begin{aligned}N^* &= R_0 N^* e^{-aP^*}, \\P^* &= C N^* (1 - e^{-aP^*}).\end{aligned}$$

The first equation yields

$$N^* = 0.$$

Suppose  $N^* \neq 0$

$$1 = R_0 e^{-aP^*}.$$

Hence

$$P^* = \frac{1}{a} \ln R_0.$$

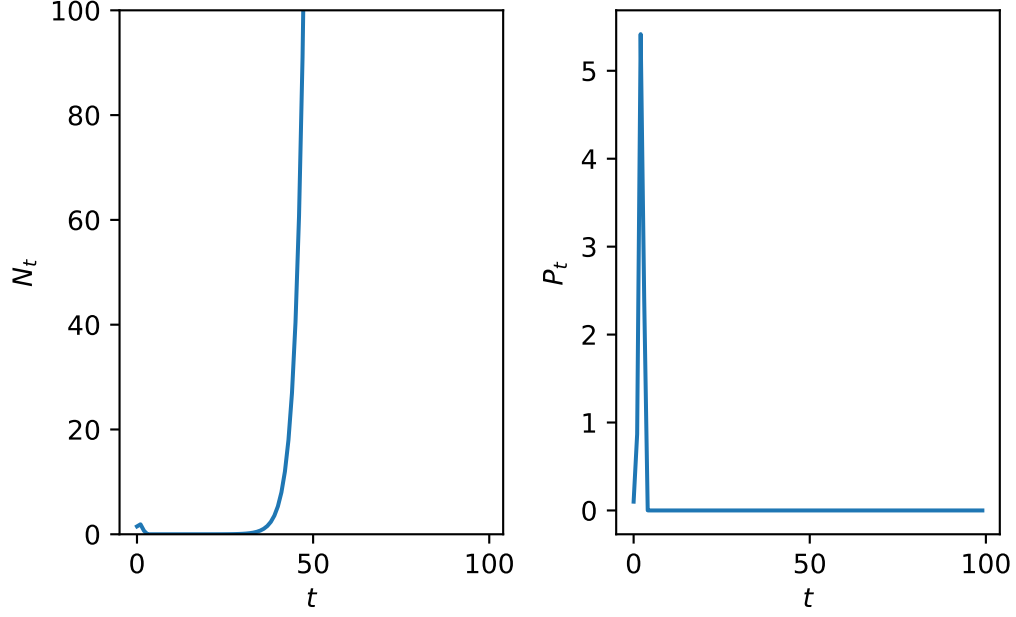


Figure 2.2: A plot of the Nicholson Bailey model solution.

Consider the second equation. Suppose  $N^* = 0$ . We obtain

$$P^* = 0.$$

Hence one fixed point is  $(0, 0)$ .

Suppose  $P^* = \frac{1}{a} \ln R_0$ .

$$\frac{1}{a} \ln R_0 = CN^*(1 - \frac{1}{R_0}).$$

Hence

$$N^* = \frac{\frac{1}{a} \ln R_0}{C(1 - \frac{1}{R_0})} = \frac{R_0 \ln R_0}{aC(R_0 - 1)}.$$

Hence the second fixed point is

$$\left( \frac{R_0 \ln R_0}{aC(R_0 - 1)}, \frac{1}{a} \ln R_0 \right).$$

We can verify by substitution that

$$\left( \frac{R_0 \ln R_0}{aC(R_0 - 1)}, \frac{1}{a} \ln R_0 \right).$$

is a fixed point. We can then deduce a condition on the model parameters that must hold in order that the fixed point is biologically relevant.

To verify by substitution we substitute the proposed solution into the governing equations. Note that

$$e^{-aP^*} = \frac{1}{R_0}.$$

In this case

$$\begin{aligned} \frac{R_0 \ln R_0}{aC(R_0 - 1)} &= R_0 \frac{R_0 \ln R_0}{aC(R_0 - 1)} \frac{1}{R_0}, \\ \frac{1}{a} \ln R_0 &= C \frac{R_0 \ln R_0}{aC(R_0 - 1)} \left(1 - \frac{1}{R_0}\right). \end{aligned}$$

Cancellation shows that both equations hold. Hence the fixed point is a valid fixed point. To be biologically relevant we require that both components of the solution are real and positive. In this case this leads to the condition  $R_0 > 1$ .

### 2.3.3 Linear stability

The Jacobian matrix is given by

$$A_{(N_t, P_t)} = \begin{pmatrix} R_0 e^{-aP_t} & -R_0 a N_t e^{-aP_t} \\ c(1 - e^{-aP_t}) & aC N_t e^{-aP_t} \end{pmatrix}.$$

### 2.3.4 Linear stability of the trivial fixed point

Evaluating at (0,0) yields

$$A = \begin{pmatrix} R_0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence the eigenvalues are  $R_0$  and 0. If  $0 < R_0 < 1$  (0,0) is stable whilst if  $R_0 > 1$  (0,0) is unstable.

### 2.3.5 Linear stability of the nontrivial fixed point

We can show that the Jacobian matrix evaluated at the nontrivial fixed point can be written as

$$A = \begin{pmatrix} 1 & -\frac{R_0 \ln R_0}{c(R_0 - 1)} \\ c(1 - \frac{1}{R_0}) & \frac{\ln R_0}{R_0 - 1} \end{pmatrix}.$$

and deduce that the eigenvalues of A satisfy the characteristic polynomial

$$\lambda^2 - \lambda \left(1 + \frac{\ln R_0}{R_0 - 1}\right) + \frac{R_0 \ln(R_0)}{R_0 - 1} = 0.$$

### 2.3.6 Employing the Jury conditions

To proceed with linear stability analysis we employ the Jury conditions. Consider the polynomial

$$\lambda^2 + a\lambda + b = 0,$$

In our case

$$a = -(1 + \frac{\ln R_0}{R_0 - 1})$$

and

$$b = \frac{R_0 \ln(R_0)}{R_0 - 1}.$$

The third Jury condition ( $b < 1$ ) implies that for linear stability

$$\ln R_0 < 1 - \frac{1}{R_0}.$$

To demonstrate that this inequality is not true for  $R_0 > 1$ , let  $f_1 = \ln R_0$  and  $f_2 = 1 - 1/R_0$ . When  $R_0 = 1$ ,  $f_1 = f_2 = 0$ . However,

$$f'_1 = \frac{1}{R_0} \quad \text{and} \quad f'_2 = \frac{1}{R_0^2},$$

implies

$$f'_1 > f'_2 \quad \forall R_0 > 1.$$

Hence

$$\ln R_0 > 1 - \frac{1}{R_0}.$$

Hence the fixed point is unstable if  $R_0 > 1$ .

### 2.3.7 Symbolic calculations in Python

Below sympy is used to identify the fixed points. The Jacobian matrix is then computed and evaluated at the fixed points.

The FPs are:

```
{N: 0, P: 0}
```

```
{N: R_0*log(R_0)/(C*a*(R_0 - 1)), P: log(R_0)/a}
```

The Jacobian of H is:

```
[[R_0*exp(-P*a), -N*R_0*a*exp(-P*a)], [C - C*exp(-P*a), C*N*a*exp(-P*a)]]
```

The Jacobian evaluated at FP 0 is:

```
[[R_0, 0], [0, 0]]
```

The Jacobian evaluated at FP 1 is:

```
[[1, -R_0*log(R_0)/(C*(R_0 - 1))], [C*(1 - 1/R_0), log(R_0)/(R_0 - 1)]]
```

**Part II**

**Continuous time**

## 3 Single species population dynamics

In this chapter we define time,  $t$ , to be a continuous variable and the population density,  $N(t)$ , to represent the size of a given population.

### 3.1 The Malthusian (linear) model

#### 3.1.1 Derivation

As time is now a continuous variable, we consider an interval of time  $\delta t$ . Consider a population of size  $N(t)$ . If the per capita production rate is  $b$ , then in time  $\delta t$  the increase in population size will be

$$bN(t)\delta t.$$

Similarly, if the per capita death rate is  $d$ , the decrease in population size in time  $\delta t$  will be

$$dN(t)\delta t.$$

Hence

$$N(t + \delta t) = N(t) + (bN(t) - dN(t))\delta t.$$

Rearranging and taking the limit as  $\delta t \rightarrow 0$  yields

$$\frac{dN}{dt} = (b - d)N = rN(t).$$

#### 3.1.2 Solution

The solution is given by

$$N(t) = N_0 \exp(rt).$$

We can describe qualitatively different solutions behaviours.

If  $r > 0$  the solution increases exponentially with time. If  $r < 0$  it decreases exponentially. When  $r = 0$  the solution is constant.

Malthusian growth in the continuous time model leads to either a constant population, exponentially increasing or exponentially decreasing growth. As was the case for the discrete



time model, such a model could account for biological system only in particularly limited circumstances.

The Malthusian model (constant per capita growth rate) can exhibit a limited range of behaviour. To account for limitation of population growth at large population densities due to, for example, limited resources, we consider nonlinear per capita growth rates.

## 3.2 Tools for analysing the dynamics of a single population in continuous time

Let  $N = N(t)$ . We consider the general form for a single species model defined in continuous time

$$\frac{dN}{dt} = f(N)N = H(N), \quad (3.1)$$

where  $f$  is the per capita growth rate and  $H$  is the net growth rate.

### 3.2.1 Numerical solution

Although an equation of the form Equation 3.1 may not be explicitly integrable, so long as the right-hand side is sufficiently well behaved, we can numerically integrate the problem (e.g. using packages such as Python's `odeint`). Such a technique provides a numerical approximation to the exact solution, given specific values of model parameters and initial conditions (i.e. we can graph solutions).

### 3.2.2 Nondimensionalisation

We nondimensionalise a model by changing from variables that have dimensions (both dependent and independent) to variables that are dimensionless.

#### 3.2.2.1 Dimensional analysis

The dimensions/units of each term in a model must be consistent. This observation helps to determine the units of different parameters in a model.

For example, we can deduce that the units of the parameter  $r$  in the Malthusian model

$$\frac{dN}{dt} = rN.$$

The units of the left-hand term are

$$\frac{\#popdensity}{\#time}.$$

The units of the term on the right-hand side are

$$\#r\#popdensity.$$

For dimensional consistency

$$\#r = \frac{1}{\#time}.$$

### 3.2.2.2 Dimensionless variables

We could nondimensionalise Equation 3.1 by defining the dimensionless variables

$$n = \frac{N}{\tilde{N}}$$

$$\tau = \frac{t}{\tilde{T}}.$$

Making the proposed change of variables in Equation 3.1 we obtain the dimensionless model

$$\frac{dn}{d\tau} = \frac{\tilde{T}}{\tilde{N}} H(\tilde{N}n(\tau)).$$

The choice for scalings is in general not unique (i.e.  $N^*$  and  $T^*$ ). Appropriate choices can result in fewer dimensionless parameters in the dimensionless model.

Introducing dimensionless variable

$$n = \frac{N}{\tilde{N}} \quad \tau = \frac{t}{\tilde{T}}$$

yields

$$\frac{dn}{d\tau} = r\tilde{T}n$$

Choosing

$$\tilde{T} = \frac{1}{r}$$

yields

$$\frac{dn}{d\tau} = n.$$

### 3.2.3 Steady-state

We denote steady-state solutions of Equation 3.1 using  $N = N^*$ . At a steady-state,

$$\frac{dN}{dt} = 0$$

and hence steady-states can be obtained by solving the algebraic equation

$$H(N^*) = 0.$$

### 3.2.4 Linear stability analysis

#### 3.2.4.1 A change of dependent variable

To perform a linear stability analysis we make the change of variables

$$N(t) = N^* + \hat{N}(t)$$

where the new dependent variable,  $\hat{N}(t)$ , is a perturbation about the steady state.

The time derivative on the left-hand side of Equation 3.1 transforms to

$$\frac{dN}{dt} = \frac{d}{dt}(N^*) + \frac{d}{dt}(\hat{N}(t)) = \frac{d\hat{N}(t)}{dt}.$$

Hence Equation 3.1 transforms to

$$\frac{d\hat{N}(t)}{dt} = H(N^* + \hat{N}(t)).$$

#### 3.2.4.2 Taylor expansion and a linear system

Employing the Taylor expansion on the right-hand side of Equation 3.1 and making the assumption that perturbations are small

$$\frac{d\hat{N}(t)}{dt} = H(N^*) + H'(N^*)\hat{N}(t) + H''(N^*)\hat{N}^2(t) + h.o.t.$$

Noting that

$$H(N^*) = 0$$

and retaining linear terms yields

$$\frac{d\hat{N}(t)}{dt} = H'(N^*)\hat{N}(t)$$

with solution

$$\hat{N}(t) = \eta e^{H'(N^*)t}$$

where  $\eta$  is some initial perturbation about the steady-state.

### 3.2.4.3 A condition for linear stability

When  $H'(N^*) > 0$  the perturbation grows exponentially fast and the steady-state is unstable. When  $H'(N^*) < 0$  the perturbation decays exponentially fast and the steady-state is stable.

### 3.2.5 Graphical solution

We can graphically identify steady-states and their stability by plotting  $dN/dt$  against  $N(t)$ . Steady-states arise at the roots of  $H(N)$ . The sign of the derivative at a root determines its linear stability.

### 3.2.6 Bifurcation diagrams

Bifurcations arise when the number of solutions or their stability changes at a given value of a model parameter. By plotting the steady state solutions against a model parameter and using annotation to represent stability of solutions we can obtain a bifurcation diagram.

As an exercise perform a qualitative analysis of the Malthusian model.

## 3.3 The logistic growth model

### 3.3.1 Model development

Let  $N = N(t)$ . The logistic model, due to Verhulst, takes the form

$$\frac{dN}{dt} = rN(t) \left(1 - \frac{N(t)}{K}\right), \quad (3.2)$$

where  $r$  is the linear growth rate and  $K$  is carrying capacity. We consider both  $r, K \in \mathfrak{R}^+$ .

Questions to ask of such a model are: what type of biologically realistic solutions does it possess? Are there steady-states? If so, are they stable or unstable? Are there bifurcations in solutions?

#### 3.3.1.1 Numerical solutions

In Figure 3.1 we present numerical solutions of equation using different initial conditions. Note the limiting behaviour of solutions as  $t \rightarrow \infty$ . In Figure 3.1 it is clear that even though some solutions are initialised at  $N_0 = 0.1$ , much closer to  $N^* = 0$  than  $N^* = K$ , they tend to the limit  $N = K$ . Why do solutions not tend to  $N^* = 0$ ?

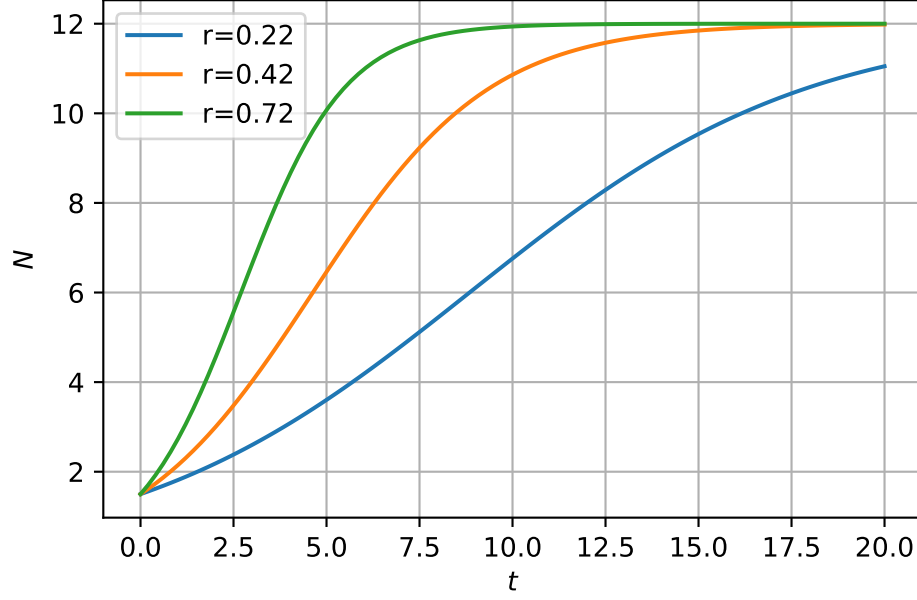


Figure 3.1: Numerical solution of the logistic growth model

### 3.3.1.2 Dimensional analysis and nondimensionalisation

$N$  represents the population density and has units of one over area (say  $1/m^2$ ) and  $t$  has units of time (say, seconds,  $s$ ). Hence the left-hand side of Equation 3.2 has units of  $1/(m^2s)$ . The first term on the right-hand side of Equation 3.2 is  $rN$ .  $N$  has units  $1/m^2$  hence the parameter  $r$  must have units of  $1/s$  for dimensional consistency. This is consistent as  $r$  represents the linear growth rate.

The second term has the form  $rN^2/K$ . Given the chosen units for  $r$  and  $N$ , the parameter  $K$  must have dimensions  $1/m^2$ . Again, this is consistent as  $K$  is a carrying capacity (i.e. it has units of population density).

We define the nondimensionalised variables

$$n = \frac{N}{\tilde{N}} \quad \tau = \frac{t}{\tilde{T}}$$

where  $\tilde{N}$  and  $\tilde{T}$  are constants that have units of population density and time, respectively. Hence Equation 3.2 transforms, upon change of variables, to

$$\frac{\tilde{N}}{\tilde{T}} \frac{dn}{d\tau} = r\tilde{N}n\left(1 - \frac{n\tilde{N}}{K}\right).$$

In the case of the logistic equation there is only one time scale and density scale in the problem, hence we choose

$$\tilde{T} = \frac{1}{r} \quad \text{and} \quad \tilde{N} = K$$

and the dimensionless model is

$$\frac{dn}{d\tau} = n(1 - n) \tag{3.3}$$

Note that we can retrieve the original equation by rescaling and calculating  $N = \tilde{N}n$  and  $t = \tilde{T}\tau$ .

### 3.3.2 Steady states and linear stability

Steady states satisfy

$$n^*(1 - n^*) = 0.$$

Hence

$$n^* = 0, \quad n^* = 1.$$

To determine linear stability we compute

$$H'(n) = (1 - 2n).$$

When  $n = n^* = 0$  we obtain

$$H'(n) = 1.$$

Hence the origin is an unstable steady-state.

At the steady-state  $n^* = 1$

$$H'(n^*) = -1$$

hence  $n^* = 1$  is linearly stable.

Note that the linear stability analysis can explain the observations regarding the numeric solutions presented in Figure [3.1](#).

### 3.3.3 Graphical analysis

In Figure [3.2](#) we plot the right-hand side of Equation [3.3](#). We can qualitatively describe model solutions by considering the arrow along the x axis. Suppose we consider an initial condition with  $0 < n_0 < 1$ . Using the graph of  $H(n)$ ,  $dn/d\tau$  is positive, hence  $n$  increases as a function of time until  $n(\tau) \rightarrow 1$ .

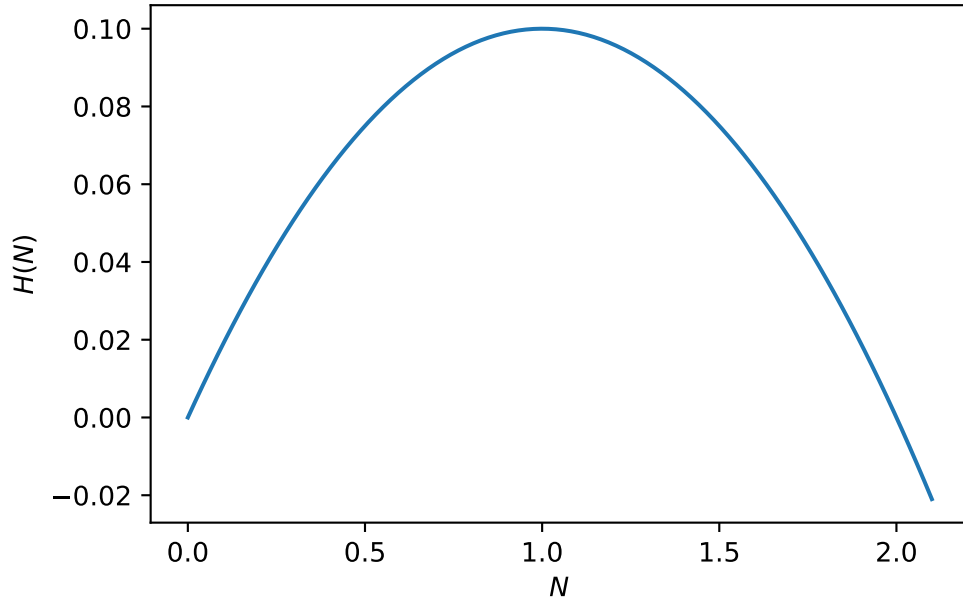


Figure 3.2: Right-hand side of the logistic ODE

### 3.3.4 An exact solution

Separation of variables yields

$$\int \frac{dN}{N(1 - \frac{N}{K})} = r \int dt.$$

Using partial fractions

$$\int \frac{dN}{N} + \frac{1}{K} \int \frac{dN}{1 - \frac{N}{K}} = r \int dt.$$

Integration yields

$$\ln N - \ln \left(1 - \frac{N}{K}\right) = \ln \frac{N}{1 - \frac{N}{K}} = rt + C.$$

Hence

$$N = \frac{De^{rt}}{1 + \frac{D}{K}e^{rt}}$$

Given an initial condition  $N(0) = N_0$ , we obtain

$$N(t) = \frac{N_0 K e^{rt}}{K + N_0(e^{rt} - 1)}$$

### 3.3.4.1 Qualitative analysis of the exact solution

As  $t \rightarrow \infty$ ,  $N \rightarrow K$ . At  $t = 0$ ,  $N = N_0$  and that for small  $N_0 \ll K$  the initial growth phase is exponential, i.e.

$$N(t) \sim N_0 e^{rt} \quad N_0 \ll K, t \ll \frac{1}{r}.$$

Note that in almost all the models that we will consider the above method is not an usually an option as the ODE is not explicitly integrable.

## 3.4 The spruce budworm model

The spruce budworm is a destructive and widely distributed forest defoliator in North America. Massive outbreaks occur periodically and can destroy large quantities of valuable spruce and fir. To understand the outbreak behaviour and develop and management strategies, a series of mathematical models have been developed, beginning with Ludwig et al. (1978). The goal of the models is to explain the qualitative pattern of sudden outbreaks and then a sudden collapse.

### 3.4.1 Model development

Letting  $N(t)$  represent the population size at time  $t$ , it is assumed that budworm exhibits logistic growth and is subject to predation at rate  $p(N)$ . A governing ordinary differential equation is given by

$$\frac{dN}{dt} = r_B N \left( 1 - \frac{N}{K_B} \right) - p(N), \quad (3.4)$$

where

$$p(N) = \frac{BN^2}{A^2 + N^2},$$

$r_B$  is the linear growth rate,  $K_B$  is the carrying capacity,  $B$  is the maximum rate of predation and  $A$  is a measure of budworm population where predation switches on (specifically,  $A$  represents the budworm density at which predation is half its maximum value).

It is informative to graph the predation term

$$p(N) = \frac{BN^2}{A^2 + N^2},$$

and annotate the parameters  $A$  and  $B$ .



There is a root at  $N = 0$ . In the limit  $N \rightarrow \infty$ ,  $p \rightarrow B$ . Note that  $N = A$ ,  $p = A/2$ . The derivative is

$$p'(N) = \frac{2BN}{A^2 + N^2} - \frac{2BN^3}{(A^2 + N^2)^2} = \frac{2BA^2N}{(A^2 + N^2)^2}$$

Thus there is a turning point at  $N = 0$  and  $p' > 0 \forall N > 0$ .

### 3.4.2 Nondimensionalisation

Introducing the (as yet unspecified) dimensional scalings  $\tilde{N}$  and  $\tilde{T}$ , the model is nondimensionalised as follows

$$n = \frac{N}{\tilde{N}} \quad \tau = \frac{t}{\tilde{T}}.$$

Changing variables in Equation 3.4 yields

$$\frac{\tilde{N}}{\tilde{T}} \frac{dn}{d\tau} = r_B \tilde{N} n \left( 1 - \frac{\tilde{N}n}{K_B} \right) - \frac{B \tilde{N}^2 n^2}{A^2 + \tilde{N}^2 n^2}.$$

After some tidying

$$\frac{dn}{d\tau} = r_B \tilde{T} n \left( 1 - \frac{\tilde{N}n}{K_B} \right) - \frac{B \tilde{N} \tilde{T}}{A^2} \frac{n^2}{1 + \frac{\tilde{N}^2}{A^2} n^2}.$$

A natural scale for cell density in the model is given by the parameter  $A$ , as it determines the density of budworm at which predation is half its maximal value. Hence we choose the scaling on the budworm density

$$\tilde{N} = A.$$

Similarly, a natural time scale for the model is given by

$$\tilde{T} = A/B.$$

Substituting for  $\tilde{N}$  and  $\tilde{T}$  yields

$$\frac{dn}{d\tau} = rn \left( 1 - \frac{n}{q} \right) - \frac{n^2}{1 + n^2} = H(n), \quad (3.5)$$

where we define the nondimensional parameters

$$r = \frac{r_B A}{B} \quad \text{and} \quad q = \frac{K_B}{A}.$$

Note that Equation 3.5 has two nondimensional parameters and all variables are dimensionless. See Figure 3.3 for a plot of right-hand side of equation Equation 3.5. What kind of behaviours do you expect to see from the model?

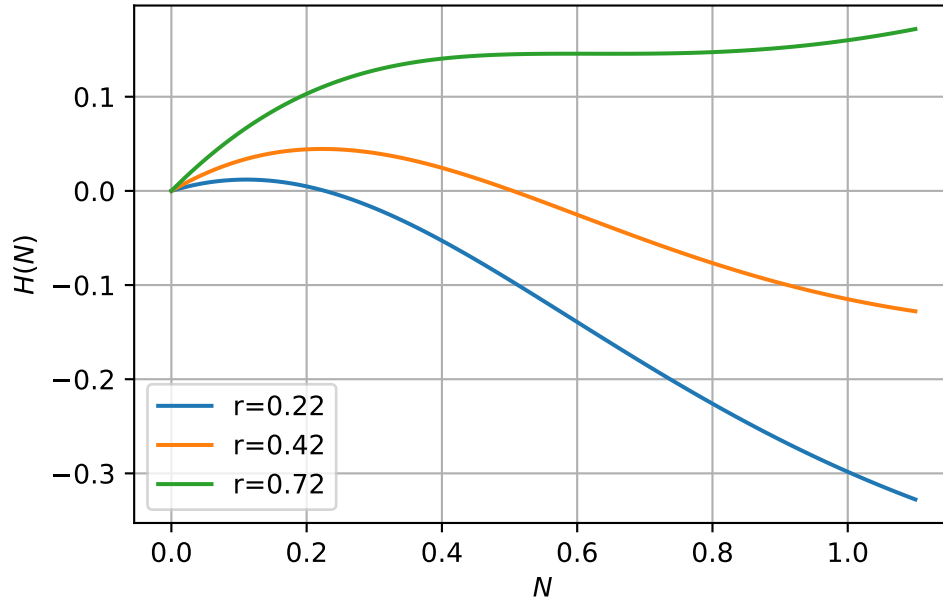


Figure 3.3: RHS of spr. budworm model

### 3.4.3 Numerical solutions

In Figure 3.4 we plot some numerical solutions of Equation 3.5 at different values of the parameter  $r$ .

Numerical solutions of Equation 3.5 indicate that there is a single stable steady state when  $r$  is both small and large but for intermediate values of  $r$  there are two stable steady states.

Our goal is to analyse the model and understand why different parameter values yield these strikingly different model behaviours.

### 3.4.4 Steady state analysis

Letting  $n^*$  represent steady states of Equation 3.5 yields

$$rn^*(1 - \frac{n^*}{q}) - \frac{n^{*2}}{1 + n^{*2}} = 0.$$

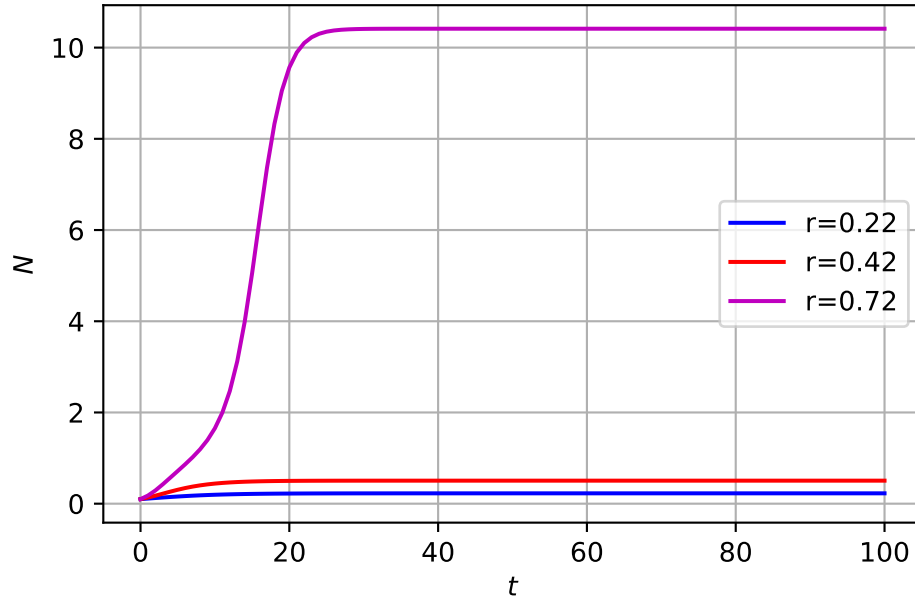


Figure 3.4: Numerical solution of spr. budworm model

Hence either

$$n^* = 0,$$

or  $n^*$  satisfies the cubic equation

$$r \left( 1 - \frac{n^*}{q} \right) - \frac{n^*}{1 + n^{*2}} = 0.$$

Explicit solutions to such a cubic can be immediately written down but they are cumbersome to work with. We proceed using a graphical/qualitative approach.

Define

$$f(n^*) = r \left( 1 - \frac{n^*}{q} \right) \quad \text{and} \quad g(n^*) = \frac{n^*}{1 + n^{*2}}. \quad (3.6)$$

Roots occur for values of  $n^*$  that satisfy  $f = g$ .

In Figure 3.5 (a) we fix the parameter  $q = 10$  and consider model behaviour as a function of the parameter  $r$ . When  $r \gg 1$  there is a nonzero steady-state corresponding to  $n^* \gg 1$ . When  $r \ll 1$  there is a nonzero steady-state corresponding to  $n^* \ll 1$ . In the intermediate case there can be three intersection points.

We can use the curve sketching techniques from Tutorial Sheet 1 to sketch  $f$  and  $g$ .

$f$  is linear. There is a root at  $n^* = q$ . The derivative is  $-r/q$ .  $f(0)=r$ .  $g$  has a unique root at  $n^* = 0$ . The derivative is

$$g' = \frac{1 - n^{*2}}{(1 + n^*)^2}.$$

There is a turning point at  $n^* = 1$ . Here  $g = 1/2$ . As  $n^* \rightarrow \infty$ ,  $g \rightarrow 0$ .  $f'(0) = 1$ .

### 3.4.5 Linear stability analysis

The linear stability of the model is determined by the quantity

$$H'(n) = r(1 - \frac{2n}{q}) - \frac{2n}{1 + n^2} + \frac{2n^3}{(1 + n^2)^2}.$$

Hence at the steady state  $n^* = 0$

$$H'(0) = r$$

and the steady state is linearly unstable.

Given the nonzero steady states have not been calculated explicitly, we proceed using graphical analysis of stability. In Figure 3.5 (b) we plot the right-hand side of Equation 3.5 against  $n$  and examine the cases of large, small and intermediate  $r$  for a given value of  $q$ .

When  $r$  is both large and small the nonzero steady state is stable (the derivative at the roots is negative). In the case where three biologically relevant roots exist, the intermediate root is unstable.

### 3.4.6 Bifurcation analysis

The goal is to identify boundaries of  $rq$  parameter space where the stability changes occur and/or the number of steady states changes.

We can define points in  $rq$  parameter space where bifurcations arise by seeking values of  $n^*$  that satisfy

$$f(n^*) = g(n^*) \quad f'(n^*) = g'(n^*).$$

The first of these equations yields

$$r(1 - \frac{n^*}{q}) = \frac{n^*}{1 + n^{*2}}, \tag{3.7}$$

and the latter yields

$$-\frac{r}{q} = \frac{1}{1 + n^{*2}} - \frac{2n^{*2}}{(1 + n^*)^2} = \frac{1 - n^{*2}}{(1 + n^{*2})^2}. \tag{3.8}$$

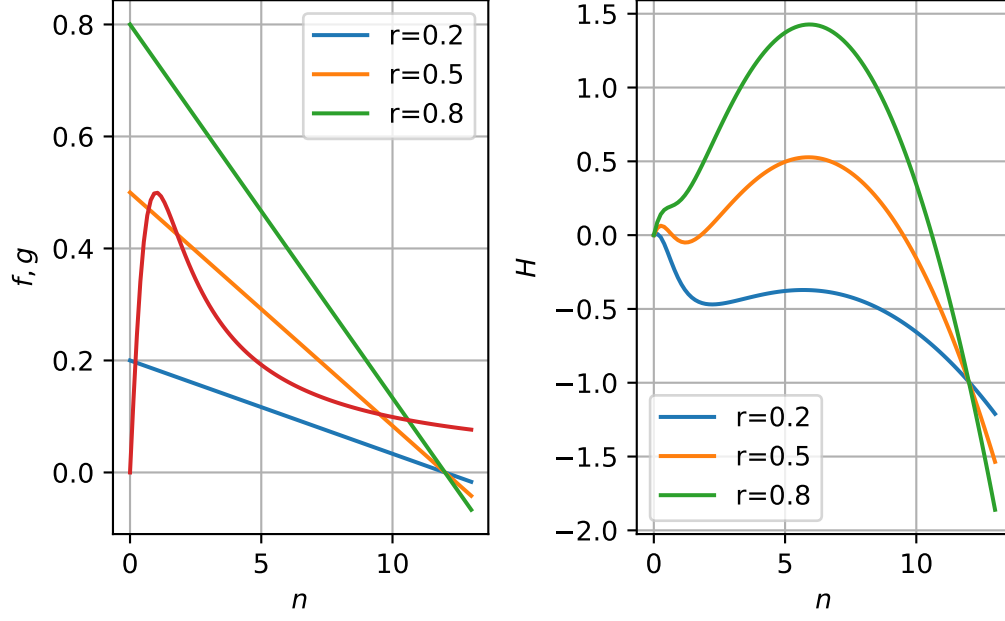


Figure 3.5: RHS of spr. budworm model

Hence

$$\frac{r}{q} = \frac{n^{*2} - 1}{(1 + n^{*2})^2}.$$

Substituting for  $r/q$  in the first equation yields

$$r - \frac{n^{*2} - 1}{(1 + n^{*2})^2} n^* = \frac{n^*}{1 + n^{*2}},$$

which can be written in the form

$$r = \frac{2n^{*3}}{(1 + n^{*2})^2}.$$

Substituting for  $r$  in Equation 3.7 yields

$$\frac{2n^{*3}}{(1 + n^{*2})^2} = \frac{2n^{*3}}{(1 + n^{*2})^2} \frac{n^*}{q} + \frac{n^*}{1 + n^{*2}}$$

which, after some algebra, yields

$$q = \frac{2n^{*3}}{n^{*2} - 1}. \quad (3.9)$$

Hence a set of points that define bifurcations where three steady states transform to a single steady state are given in parametric form in  $qr$  parameter space by

$$\left( \frac{2n^{*3}}{n^{*2} - 1}, \frac{2n^{*3}}{(1 + n^{*2})^2} \right) \quad n^* > 1.$$

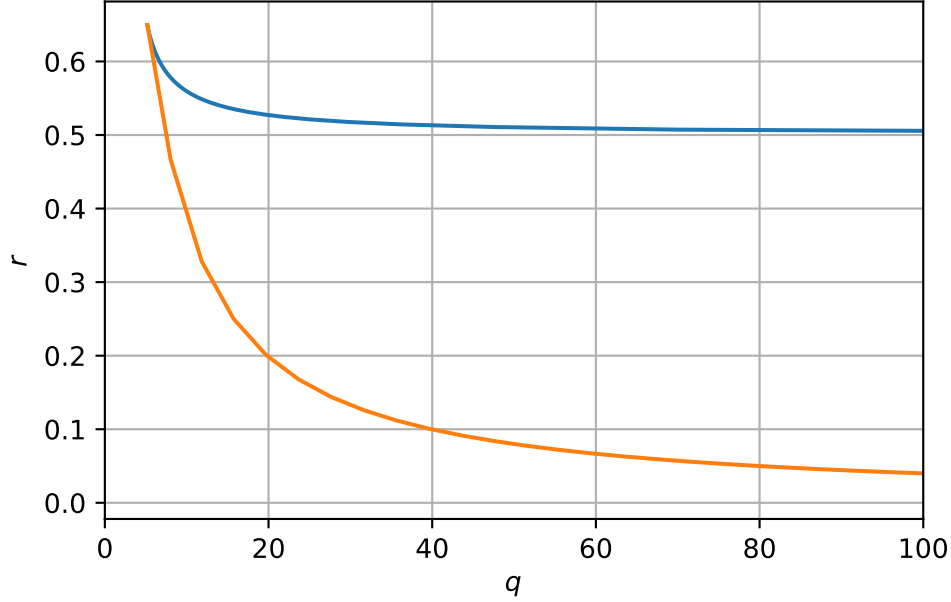


Figure 3.6: Bifurcations in the  $rq$  plane

By varying values of  $n^*$  in Figure 3.6 we plot a region of instability. Note for example that  $q \rightarrow \infty$  as  $n^* \rightarrow 1$  and as  $n^* \rightarrow \infty$ . Note also that in these limits  $r$  must take the value  $1/2$  and  $0$ , respectively.

We can show that the cusp in Figure 3.6 is given by

$$(q, r) = \left( 3^{\frac{3}{2}}, \left( \frac{\sqrt{3}}{2} \right)^3 \right).$$

Note that  $r$  is a decreasing function of  $q$  for  $1 < n^* < \sqrt{3}$  but that  $r$  is an increasing function of  $n^*$  for  $\sqrt{3} < n^* < \infty$ . This can be shown by finding the turning points of  $r$  w.r.t  $n^*$ , i.e. Given that

$$r = \frac{2n^{*3}}{(1 + n^{*2})^2},$$

differentiation with respect to  $n^*$  yields

$$\frac{dr}{dn^*} = \frac{6n^{*2}}{(1+n^*)^2} - \frac{8n^{*4}}{(1+n^*)^3}.$$

The turning point satisfies

$$\frac{6n^{*2}}{(1+n^*)^2} - \frac{8n^{*4}}{(1+n^*)^3} = 0$$

Solving for  $n^*$  yields

$$6(1+n^{*2}) - 8n^{*2} = 0$$

Hence

$$n^* = \sqrt{3}.$$

Thus there is a turning point that minimises  $r$  at  $n^* = \sqrt{3}$ .

Substitution for this value of  $n^*$  yields

$$(q, r) = \left( 3^{\frac{3}{2}}, \left( \frac{\sqrt{3}}{2} \right)^3 \right).$$

See Figure 3.6.

### 3.4.7 Hysteresis

Finally, rearranging the steady-state equation

$$r\left(1 - \frac{n^*}{q}\right) = \frac{n^*}{1+n^{*2}},$$

we obtain

$$r = \frac{n^*}{(1+n^{*2})(1 - \frac{n^*}{q})}. \quad (3.10)$$

Considering  $n^* < q$  we compute  $r$ ; plotting  $n^*$  against  $r$  yields the bifurcation curve presented in Figure 3.7.

A system exhibiting hysteresis shows a response to an increases in a control parameter that is not exactly reversed when the parameter is decreased. We can show that the spruce budworm model exhibits hysteresis by considering the argument below.

Suppose  $r$  is initially small ( $r < r_1$ ). There is only one steady-state and any initial condition will converge towards it.

Suppose we increase the value of the parameter  $r$ . There will be a critical value of  $r$  ( $r = r_1$ ) where a second stable steady-state arises and the model enters the bistable regime, where

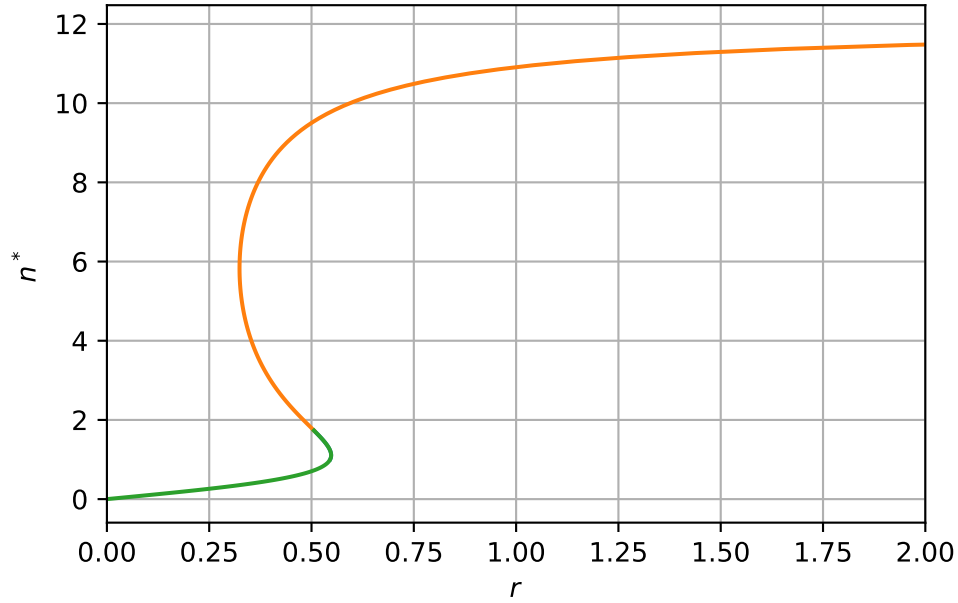


Figure 3.7: Bifurcations in the  $rq$  plane

there are two possible stable steady-states. Given the system was originally in the first stable steady state, it will remain there.

Suppose we continue to increase  $r$ . Eventually we reach critical value of  $r$  ( $r = r_2$ ) where the first stable steady state is lost and there is again only one stable steady-state.

Suppose we now decrease the parameter  $r$  below the threshold  $r = r_2$ . The system again enters the bistable regime but as the second solution is stable, it remains the solution.

Suppose we continue to decrease  $r$  until eventually  $r < r_1$ . We return to the case where the system has only a single stable steady state.

### 3.5 Harvesting

By introducing terms that represent harvesting, population models can be used to investigate management strategies for resource management. The modelling problem is to maximise the sustained yield.

Consider a model for logistic growth supplemented with a harvesting term

$$\frac{dN}{dt} = rN \left( 1 - \frac{N}{K} \right) - EN.$$



The term  $EN$  is the harvesting yield per unit time and the constant  $E$  represents the harvesting effort.

Steady states satisfy

$$rN^* \left(1 - \frac{N^*}{K}\right) - EN^* = 0$$

Thus either  $N^* = 0$  or

$$\left(1 - \frac{N^*}{K}\right) - E = 0$$

Hence

$$N^* = K\left(1 - \frac{E}{r}\right).$$

Note that this is positive only if

$$E < r.$$

Thus if the harvesting rate exceeds the linear growth rate the only steady state corresponds to extinction.

The yield is

$$Y(E) = EN^* = EK\left(1 - \frac{E}{r}\right)$$

To maximise the yields we differentiate with respect to  $E$ .

$$\frac{dY}{dE} = K - 2\frac{EK}{r}.$$

Thus the maximum yield is identified by setting

$$\frac{dY}{dE} = K - 2\frac{EK}{r} = 0.$$

The maximum occurs at harvesting rate

$$E_c = \frac{r}{2}.$$

The maximum yield is

$$E^* = \frac{rK}{4}.$$

We identify the time scale over which stocks return to steady state by linearising about the steady state  $N^* = K(1 - E/r)$ , hence

$$\frac{d\hat{N}}{dt} = (E - r)\hat{N}.$$

Hence the recovery time scale is

$$T_R(E) = O\left(\frac{1}{r - E}\right).$$

As the harvesting rate approaches the linear growth rate  $r$ , not only does the steady state population tend to zero but the timescale for stocks to recover tends to infinity.

### 3.6 Delay differential equation models

Consider a model of the form

$$\frac{dN}{dt} = H(N(t), N(t-T)),$$

such that the right-hand side now depends on the value  $N$  not just at time  $t$  but also at some delay time  $t - T$ . We now need to prescribe initial conditions of the form

$$N(t) = f(t), \quad -T < t \leq 0.$$

Consider a simple case of a linear model

$$\frac{dN}{dt} = -N(t-T).$$

A steady state,  $N^*$ , is defined such that  $N(t) = N(t-T) = N^*$ . Hence

$$0 = -kN^*.$$

The only steady state is  $N^* = 0$ .

To compute the linear stability we consider solution of the linearised equation of the form

$$N(t) = e^{\lambda t}.$$

Hence

$$N(t-T) = e^{\lambda(t-T)}.$$

Substitution yields

$$\lambda e^{\lambda t} = -e^{\lambda(t-T)}.$$

Hence the eigenvalue,  $\lambda$ , satisfies a transcendental equation

$$\lambda = -e^{-\lambda T}.$$

This equation has no solutions for  $\lambda \in \Re^+$ . Hence any steady-states with real eigenvalues are linearly stable.

Now consider solutions  $\lambda \in \mathbb{C}$ . Let  $\lambda = \mu + i\omega$ . Substitution yields

$$\mu + i\omega = -e^{-\mu T}(\cos(\omega T) - i \sin(\omega T)).$$

Hence

$$\mu = -\cos(\omega T)e^{-\mu T} \tag{3.11}$$

and

$$\omega = \sin(\omega T)e^{-\mu T}. \tag{3.12}$$

For linear stability of the steady state it is required that  $\mu < 0$ .

If  $\omega$  is a solution of Equation 3.11 and Equation 3.12 then so is  $-\omega$ . Hence w.l.o.g we consider the case  $\omega > 0$ .

Consider  $\mu = 0$  in Equation 3.11. This implies

$$\omega T = \frac{\pi}{2} + 2n\pi, \quad n \in \mathbb{Z}.$$

We consider the smallest positive root, i.e.  $n = 0$ , as we want to consider the smallest value of delay time,  $T$ , at which instability arises.

Note  $\mu < 0$  for

$$\omega T < \frac{\pi}{2}.$$

On the stability boundary,  $\mu = 0$ . Upon substitution in Equation 3.12

$$\omega = \sin\left(\frac{\pi}{2}\right) = 1.$$

Hence the steady state is linearly stable for

$$T < \frac{\pi}{2}.$$

## 3.7 References

## 4 Multi species population dynamics

In this chapter we consider models of interacting populations.

Let  $u = u(t)$  and  $v = v(t)$  represent populations of two different species at continuous time  $t$ . Suppose that  $u(t)$  and  $v(t)$  satisfy the ordinary differential equations

$$\begin{aligned}\frac{du}{dt} &= f(u, v), \\ \frac{dv}{dt} &= g(u, v).\end{aligned}$$

In this section we will consider particular forms for  $f$  and  $g$  that represent different types of inter species interaction. Before considering particular examples, we develop some general techniques for analysing equations of the form Equation 4.1.

### 4.1 General tools

#### 4.1.1 Steady states

$(u^*, v^*)$  is defined to be a steady state of Equation 4.1 if

$$f(u^*, v^*) = g(u^*, v^*) = 0.$$

Hence, by definition, the time derivatives of  $u(t)$  and  $v(t)$  are both zero at  $(u^*, v^*)$ . As was the case for single species models, steady states are obtained by solving algebraic equations.

#### 4.1.2 Linear stability analysis

Suppose that  $(u^*, v^*)$  is a steady state of equations Equation 4.1.

We consider a change of dependent variables such that

$$u(t) = u^* + \hat{u}(t) \quad \text{and} \quad v(t) = v^* + \hat{v}(t),$$

where  $\hat{u}(t)$  and  $\hat{v}(t)$  are perturbations about the steady state.

Rewriting equation Equation 4.1 in the transformed variables yields

$$\begin{aligned}\frac{d\hat{u}}{dt} &= f(u^* + \hat{u}, v^* + \hat{v}), \\ \frac{d\hat{v}}{dt} &= g(u^* + \hat{u}, v^* + \hat{v}).\end{aligned}$$

Making Taylor expansions about  $(u^*, v^*)$  yields the linearised equations

$$\begin{aligned}\frac{d\hat{u}}{dt} &= \frac{\partial f}{\partial u|_{(u^*, v^*)}} \hat{u} + \frac{\partial f}{\partial v|_{(u^*, v^*)}} \hat{v} + h.o.t., \\ \frac{d\hat{v}}{dt} &= \frac{\partial g}{\partial u|_{(u^*, v^*)}} \hat{u} + \frac{\partial g}{\partial v|_{(u^*, v^*)}} \hat{v} + h.o.t.\end{aligned}$$

Upon making the assumption that the perturbations about the steady state are small, the leading order terms are linear and higher order terms are neglected. Defining

$$\mathbf{w} = \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix},$$

yields the system of linear ODEs given by

$$\frac{d\mathbf{w}}{dt} = A\mathbf{w}, \tag{4.1}$$

where the matrix  $A$ , known as the Jacobian matrix, takes the form

$$A = \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{pmatrix}_{(u^*, v^*)}.$$

The linear system arrived upon in Equation 4.1 was encountered previously in Differential Equations (MA31002). We summarise the important results here. Seeking a solution of Equation 4.1 of the form

$$\mathbf{w} = \mathbf{v}e^{\lambda t}$$

one obtains the characteristic equation

$$\lambda^2 - \lambda \text{tr}(A) + \det(A) = 0,$$

which has solutions

$$\lambda = \frac{\text{tr}A \pm \sqrt{\text{tr}A^2 - 4 \det A}}{2}.$$

Whilst a complete classification of the linear stability of a steady state can be obtained by explicitly calculating the eigenvalues, in many cases it is sufficient to deduce whether or not a steady state is stable or unstable. This can be achieved by calculating the determinant and trace of the Jacobian matrix ( $\det(A)$  and  $\text{tr}(A)$ , respectively) and referring to Figure 4.1.

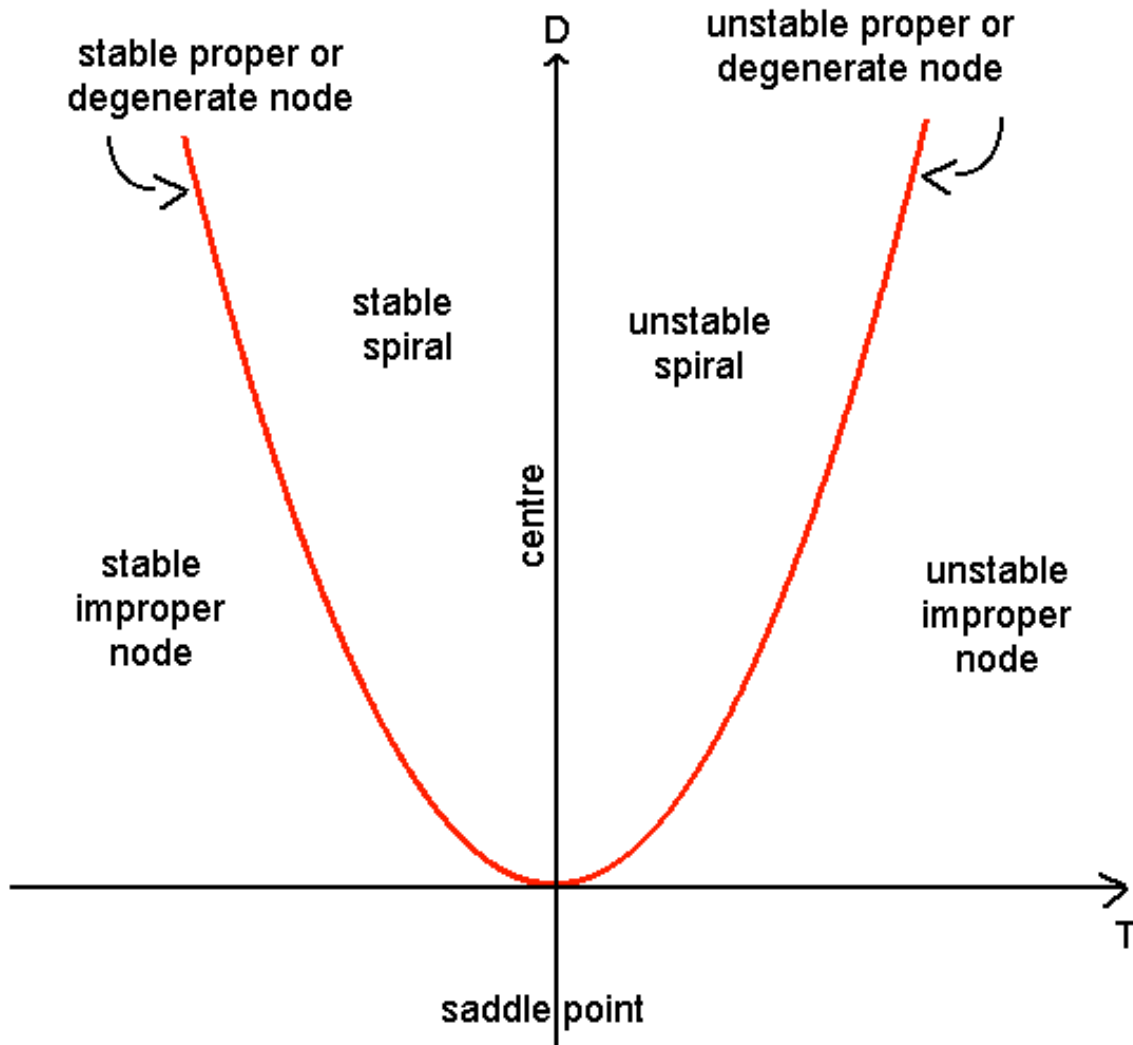


Figure 4.1: Stability in the trace determinant plane.

The different cases can be categorised as follows:

- $\det(A) < 0$  There is one positive and one negative real eigenvalue. Hence the steady state is a saddle which is unstable.
- $\det(A) > 0$  The steady state can be either stable or unstable, depending on  $\text{tr}(A)$  (and the real part of the eigenvalues).
  - If  $\text{tr}(A) > 0$ , the steady state is unstable.
    - \* If  $\text{tr}(A)^2 > 4 \det(A)$ , it is an unstable node.
    - \* If  $\text{tr}(A)^2 < 4 \det(A)$  it is an unstable spiral.
  - If  $\text{tr}(A) < 0$ , the steady state is stable.
    - \* If  $\text{tr}(A)^2 > 4 \det(A)$ , it is a stable node.
    - \* If  $\text{tr}(A)^2 < 4 \det(A)$ , it is a stable spiral.

These different cases can be distinguished in the trace-determinant plane plotted in Figure 4.1.

Note that it can be shown rigorously that the linear stability of the nonlinear system is equivalent to that of the linearised system in all cases except when the steady state is a centre. In this case, nonlinear stability analysis is required to determine the stability of the nonlinear system.

The condition  $\det(A) > 0$  excludes the case that the eigenvalues are real but have opposite signs (i.e. it cannot be a saddle point).

The condition  $\text{tr}(A) > 0$  implies that the real part of both eigenvalues are positive (i.e. the steady state is unstable).

The condition  $\text{tr}(A)^2 < 4 \det(A)$  implies that the eigenvalue are complex. Hence

$$\lambda_{\pm} = \mu \pm i\omega$$

and the solution of the system can be written

$$e^{\lambda t} = e^{(\mu+i\omega)t} = e^{\mu t} e^{i\omega t}.$$

Thus the magnitude of perturbation grows but it oscillates about the steady state. Hence the steady state is an unstable spiral.

### 4.1.3 Nullclines

Nullclines of the equations Equation 4.1 are given by the curves

$$f(u, v) = 0$$

and

$$g(u, v) = 0,$$

respectively. Note that steady states arise at the intersection of nullclines.

Using the  $u$  nullclines we identify domains of the phase plane where  $du/dt > 0$  and  $du/dt < 0$ .

Similarly, using the  $v$  nullcline we identify domains of the phase plane where  $dv/dt > 0$  and  $dv/dt < 0$ .

The nullclines can be used to help identify confined sets and hence apply the Poincare-Bendixson theorem.

### 4.1.4 Poincare-Bendixson theorem

Suppose that the system of equations

$$\frac{du}{dt} = f(u, v), \quad \frac{dv}{dt} = g(u, v) \quad (4.2)$$

possesses a confined set (i.e. a bounded domain in the phase plane upon which the derivative field points into the domain) that contains an unstable node or spiral. Any trajectory cannot leave the confined set, nor can it tend to the unstable steady state. The Poincare-Bendixson theorem states that as  $t \rightarrow \infty$ , the trajectory will tend towards a limit cycle.

### 4.1.5 Dulac criterion

Suppose  $D$  is a simply connected region in the plane and that there exists a function  $B(x, y)$ , continuously differentiable on  $D$ , such that the expression

$$\frac{\partial}{\partial u}(Bf) + \frac{\partial}{\partial v}(Bg)$$

is not identically zero and does not change sign in  $D$ . Then there are no closed orbits in  $D$ .



## 4.2 The Lotka-Volterra model

Suppose that  $N(t)$  and  $P(t)$  represent the prey and predator population densities at time  $t$ , respectively. Consider a model of the form

$$\begin{aligned}\frac{dN}{dt} &= aN - bNP, \\ \frac{dP}{dt} &= cNP - dP,\end{aligned}$$

where  $a$ ,  $b$ ,  $c$  and  $d$  are positive constants.

### 4.2.1 Nondimensionalisation

Nondimensionalising with

$$n = \frac{N}{\tilde{N}}, \quad p = \frac{P}{\tilde{P}} \quad \text{and} \quad \tau = \frac{t}{\tilde{T}},$$

yields, after changing variables,

$$\begin{aligned}\frac{dn}{d\tau} &= \tilde{T}an - b\tilde{T}\tilde{P}np, \\ \frac{dp}{d\tau} &= c\tilde{N}\tilde{T}np - d\tilde{T}p.\end{aligned}$$

Choosing the dimensional scalings

$$\tilde{T} = 1/a, \quad \tilde{P} = a/b \quad \text{and} \quad \tilde{N} = d/c,$$

yields

$$\begin{aligned}\frac{dn}{d\tau} &= n(1 - p) = f(n, p), \\ \frac{dp}{d\tau} &= \alpha p(n - 1) = g(n, p),\end{aligned}$$

where  $\alpha = d/a$ . Note that all variables and parameter are without dimensions.

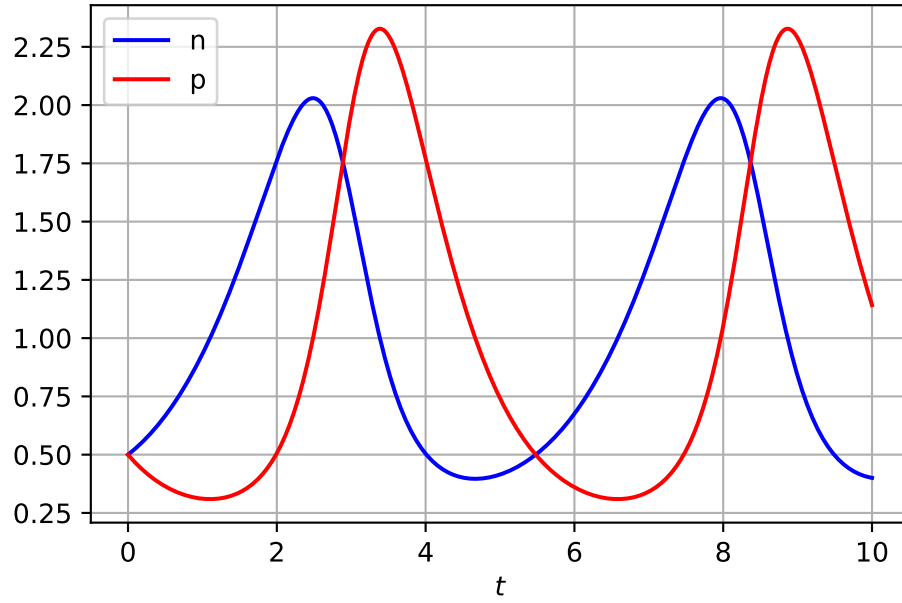


Figure 4.2: Numerical solution of Lotka-Volterra model

#### 4.2.2 Numerical solutions

#### 4.2.3 Steady states

The steady-states of equation Equation 4.3 are identified by seeking solutions of

$$f(n^*, p^*) = g(n^*, p^*) = 0.$$

Hence

$$n^*(1 - p^*) = 0 \quad \alpha p^*(n^* - 1).$$

The first of these equations has solutions either

$$n^* = 0 \quad \text{or} \quad p^* = 1.$$

Substituting for  $n^* = 0$  in the second equation yields  $p^* = 0$ . Hence one steady state is  $(0, 0)$ . Substituting for  $p^* = 1$  in the second equation yields  $n^* = 1$ . Hence a second steady state is  $(1, 1)$ .

#### 4.2.4 Linear stability

The linear stability of the steady states is described by the Jacobian matrix

$$A = \left( \begin{array}{cc} \frac{\partial f}{\partial n} & \frac{\partial f}{\partial p} \\ \frac{\partial g}{\partial n} & \frac{\partial g}{\partial p} \end{array} \right)_{(n^*, p^*)} = \left( \begin{array}{cc} 1-p & -n \\ \alpha p & \alpha(n-1) \end{array} \right)_{(n^*, p^*)}$$

Evaluating at (0,0) yields

$$A = \left( \begin{array}{cc} 1 & 0 \\ 0 & -\alpha \end{array} \right),$$

Hence the eigenvalues of  $A$  are 1 and  $-\alpha$ . As  $\alpha > 0$  the origin is a saddle. The eigenvectors are  $[1, 0]$  and  $[0, 1]$ .

At (1,1)

$$A = \left( \begin{array}{cc} 0 & -1 \\ \alpha & 0 \end{array} \right),$$

and the eigenvalues are  $\pm i\sqrt{\alpha}$ . Therefore the steady state at (1,1) is a centre.

#### 4.2.5 Solutions in the phase plane

See Figure 4.3 for solution trajectories plotted in the phase plane using different initial conditions. Note the expected saddle like behaviour when trajectories that are close to the origin. Furthermore, note that the different initial conditions result in distinct closed loops and that the inner-most loop, i.e. that closest to the steady-state (1,1), behaves like a centre, as expected given the linear stability analysis. However, far from the steady-state the solution trajectory deviates from the linearised model.

#### 4.2.6 Direct integration

The Lotka-Volterra equations are a special case as they are integrable. Trajectories in the phase plane satisfy the differential equation

$$\frac{dp}{dn} = \frac{\alpha p(n-1)}{n(1-p)},$$

Using separation of variables

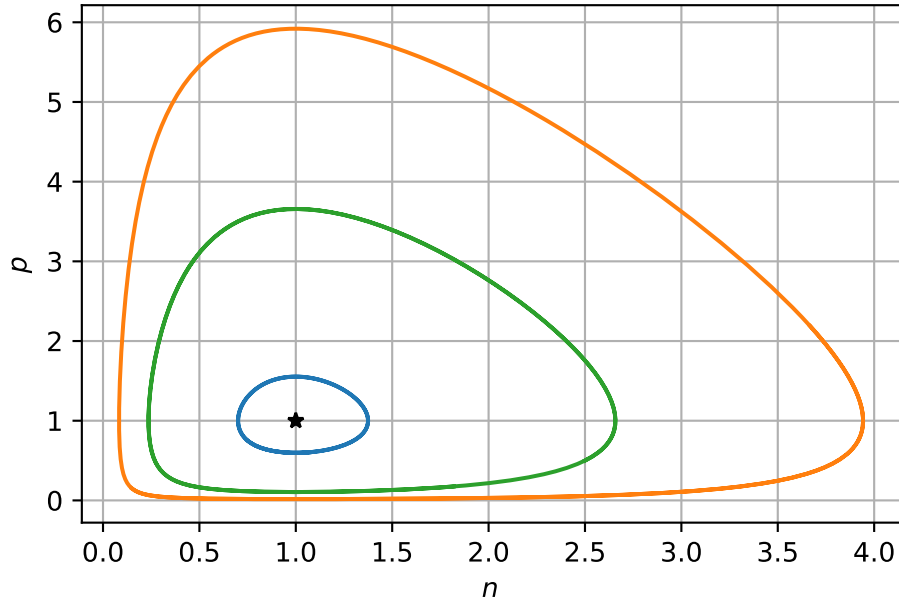


Figure 4.3: Numerical solution of Lotka-Volterra model

$$\int \frac{1-p}{p} dp = \alpha \int \frac{n-1}{n} dn.$$

Integration yields

$$\ln p - p = \alpha(n - \ln n) + H,$$

where  $H$  is a conserved quantity that this is determined by the initial conditions. As the equations take conservative form, the Lotka-Volterra model is said to be structurally unstable, as a small perturbation to the solution at a given point in the oscillatory cycle can result in large changes elsewhere in the cycle. For example, suppose the outermost limit cycle in Figure 4.3 is perturbed by a small amount at the point (1,0.1) onto its nearest limit cycle. Later in the cycle these two trajectories deviate by a large amount.

### 4.3 Competition

In models of competition, two or more species compete for the same resource or in some way inhibit each other's growth. Letting  $N_1(t)$  and  $N_2(t)$  represent the population density of two

species, we consider the ODEs

$$\begin{aligned}\frac{dN_1}{dt} &= r_1 N_1 \left( 1 - \frac{N_1}{K_1} - b_{12} \frac{N_2}{K_1} \right), \\ \frac{dN_2}{dt} &= r_2 N_2 \left( 1 - \frac{N_2}{K_2} - b_{21} \frac{N_1}{K_2} \right),\end{aligned}$$

where  $r_1, r_2, K_1$  and  $K_2$  are positive constants. As before, the  $r$ 's are linear growth rates and the  $K$ 's are carrying capacities. The parameters  $b_{12}$  and  $b_{21}$  measure the competitive effect of  $N_2$  on  $N_1$  and  $N_1$  on  $N_2$ , respectively.

### 4.3.1 Nondimensionalisation

After nondimensionalising using the change of variables

$$n_1 = \frac{N_1}{K_1} \quad n_2 = \frac{N_2}{K_2} \quad \tau = \frac{t}{\frac{1}{r_1}},$$

we obtain the equations

$$\begin{aligned}\frac{dn_1}{d\tau} &= n_1 (1 - n_1 - a_{12} n_2) = f(n_1, n_2), \\ \frac{dn_2}{d\tau} &= \rho n_2 (1 - n_2 - a_{21} n_1) = g(n_1, n_2),\end{aligned}$$

where

$$\rho = \frac{r_2}{r_1}, \quad a_{12} = b_{12} \frac{K_2}{K_1}, \quad a_{21} = b_{21} \frac{K_1}{K_2}.$$

### 4.3.2 Steady states

The steady states of Equation 4.3 are identified in the usual manner, i.e. by seeking  $(n_1^*, n_2^*)$  such that

$$f(n_1^*, n_2^*) = g(n_1^*, n_2^*) = 0.$$

The steady state equations are

$$n_1^* (1 - n_1^* - a_{12} n_2^*) = 0 \quad n_2^* (1 - n_2^* - a_{21} n_1^*) = 0.$$

The first equation has solution

$$n_1^* = 0$$

or

$$(1 - n_1^* - a_{12}n_2^*) \implies n_2 = \frac{1}{a_{12}}(1 - n_1^*).$$

Consider  $n_1^* = 0$ . Substitution in the second equation yields

$$n_2^*(1 - n_2^*) = 0.$$

Hence either  $n_2^* = 0$  or  $n_2^* = 1$ . Hence two steady states are  $(0, 0)$  and  $(0, 1)$ .

Now consider  $n_2^* = \frac{1}{a_{12}}(1 - n_1^*)$  with  $n_1^* \neq 0$ .

Substitution in the second steady state equation yields

$$\frac{1}{a_{12}}(1 - n_1^*) \left( 1 - \frac{1}{a_{12}}(1 - n_1^*) - a_{21}n_1^* \right)$$

Hence either  $n_1^* = 1$  or

$$\left( 1 - \frac{1}{a_{12}}(1 - n_1^*) - a_{21}n_1^* \right) = 0 \implies n_1^* = \frac{1 - a_{12}}{1 - a_{12}a_{21}}.$$

In the case where  $n_1^* = 1$ , we find that  $n_2^* = 0$ . Hence the steady state is  $(1, 0)$ .

In the case where

$$n_1^* = \frac{1 - a_{12}}{1 - a_{12}a_{21}}$$

we find that

$$n_2^* = \frac{1 - a_{21}}{1 - a_{12}a_{21}}$$

Hence the steady state is

$$\left( \frac{1 - a_{12}}{1 - a_{12}a_{21}}, \frac{1 - a_{21}}{1 - a_{12}a_{21}} \right).$$

### 4.3.3 Nullclines

The nullclines for Equation 4.3 are straight lines given by

$$n_1 = 0 \quad n_2 = \frac{1 - n_1}{a_{12}},$$

and

$$n_2 = 0 \quad n_2 = 1 - a_{21}n_1.$$

Note that the steady states  $(0,0)$ ,  $(1,0)$  and  $(0,1)$  are always biologically relevant (i.e. independently of the parameter values for  $a_{12}$  and  $a_{21}$ ).

However, the coexistence steady state is only biologically relevant if the nullclines intersect in the positive quadrant and this occurs only in certain regions of the model's parameter space.

In the cases where  $a_{12}, a_{21} < 1$  and  $a_{12}, a_{21} > 1$  there is a coexistence steady state (i.e. the nullclines intersect in the positive quadrant).

However, if  $a_{21} < 1$  and  $a_{12} > 1$  or  $a_{12} < 1$  and  $a_{21} > 1$  there is not a biologically relevant, coexistence steady state (i.e. the nullclines do not intersect in the positive quadrant).

Hence there are four qualitatively different types of solution to consider.

#### 4.3.4 Linear stability

The linear stability of the different steady states is determined by calculating the Jacobian matrix

$$A = \left( \begin{array}{cc} \frac{\partial f}{\partial n_1} & \frac{\partial f}{\partial n_2} \\ \frac{\partial g}{\partial n_1} & \frac{\partial g}{\partial n_2} \end{array} \right)_{(n_1^*, n_2^*)} = \left( \begin{array}{cc} 1 - 2n_1 - a_{12}n_2 & -a_{12}n_1 \\ -\rho a_{21}n_2 & \rho(1 - 2n_2 - a_{21}n_1) \end{array} \right)_{(n_1^*, n_2^*)}.$$

At  $(0,0)$

$$A = \left( \begin{array}{cc} 1 & 0 \\ 0 & \rho \end{array} \right).$$

Hence the eigenvalues of the Jacobian are 1 and  $\rho$ . As  $\rho > 0$ , the origin is therefore an unstable node (there are two real positive eigenvalues).

At  $(1,0)$

$$A = \left( \begin{array}{cc} -1 & -a_{12} \\ 0 & \rho(1 - a_{21}) \end{array} \right).$$

The trace and determinant are given by

$$\det A = \rho(a_{21} - 1) \quad \text{and} \quad \text{tr} A = -1 + \rho(1 - a_{21}).$$

Hence if  $a_{21} < 1$ ,  $\det A < 0$  and  $(1,0)$  is a saddle point and thus unstable (see Figure 4.1).

If  $a_{21} > 1$ ,  $\det A > 0$  and  $\text{tr} A < 0$ . Hence  $(1,0)$  is a stable node.

Hence the parameter  $a_{21}$ , which describes how strongly Population 1 inhibits the growth rate of Population 2, determines whether or not the steady state representing extinction of Population 2 but not Population 1 is stable or not.

At (0,1)

$$A = \begin{pmatrix} 1 - a_{12} & 0 \\ -\rho a_{21} & -\rho \end{pmatrix}.$$

In this case if  $a_{12} < 1$ ,  $\det A < 0$  and (0,1) is a saddle point. If  $a_{12} > 1$ ,  $\det A > 0$  and  $\text{tr} A < 0$  and (0,1) is a stable node. Hence the parameter  $a_{12}$ , which describes how strongly Population 2 inhibits the growth rate of Population 1, determines whether or not the steady state representing extinction of Population 1 but not Population 2 is stable or not.

At the coexistence steady state, recall the steady state is

$$\left( \frac{1 - a_{12}}{1 - a_{12}a_{21}}, \frac{1 - a_{21}}{1 - a_{12}a_{21}} \right).$$

Note that this steady state is only biologically relevant in the cases  $a_{21} < 1, a_{12} < 1$  or  $a_{21} > 1, a_{12} > 1$ . Evaluating the Jacobian yields

$$A = \frac{1}{1 - a_{12}a_{21}} \begin{pmatrix} a_{12} - 1 & -a_{12}(1 - a_{12}) \\ -\rho a_{21}(1 - a_{21}) & \rho(a_{21} - 1) \end{pmatrix}.$$

The determinant and trace of the Jacobian are given by

$$\begin{aligned} \det A &= \rho((a_{12} - 1)(a_{21} - 1) - a_{12}a_{21}(1 - a_{12})(1 - a_{21})) \frac{1}{(1 - a_{12}a_{21})^2}, \\ &= \rho \frac{(a_{12} - 1)(a_{21} - 1)}{(1 - a_{12}a_{21})}, \end{aligned}$$

and

$$\text{tr} A = (a_{12} - 1 + \rho(a_{21} - 1)) \frac{1}{1 - a_{12}a_{21}},$$

respectively.

Let's firstly consider the case where  $a_{21} < 1$  and  $a_{21} < 1$ . This implies that  $a_{21} - 1 < 0$  and  $a_{12} - 1 < 0$ , hence evaluating the signs of the different products yields

$$\det A = \rho(-)(-)(+) > 0$$



and

$$\text{tr} A = \rho(-) + (-) < 0.$$

Therefore the coexistence steady state is a stable node or spiral.

In the case where  $a_{21} > 1$  and  $a_{12} > 1$

$$\det A = \rho(+)(+)(-) < 0.$$

Hence the coexistence steady state is a saddle.

### 4.3.5 Phase portrait

See Figure 4.5 for phase portraits of three of the four cases that we have considered. It is expected that you can sketch phase portraits. Key details to consider are the steady states and their linear stability. You should also sketch the nullclines and depict the sign of the derivatives in the phase plane on either side of the nullclines. You should also sketch one or more sample trajectories.

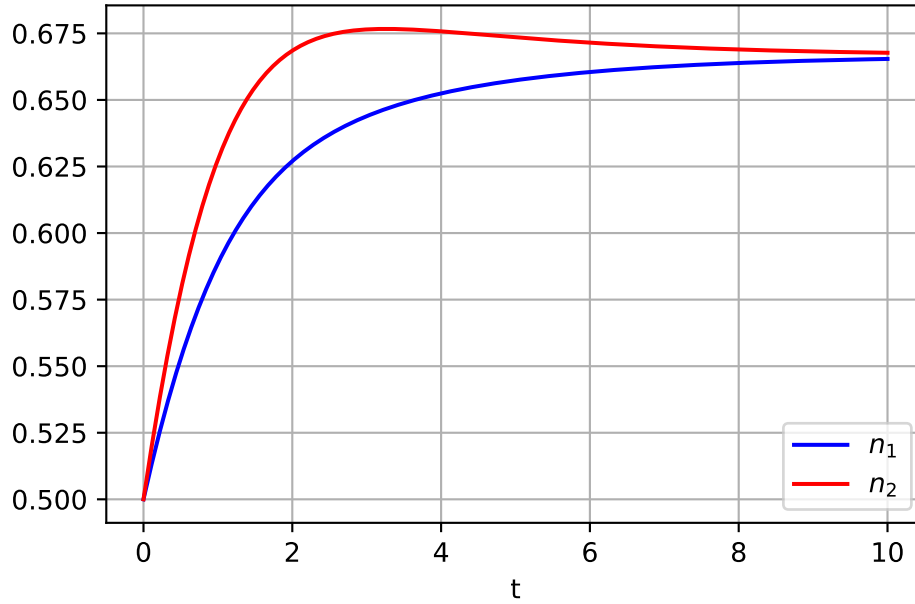


Figure 4.4: Numerical solutions of competition model

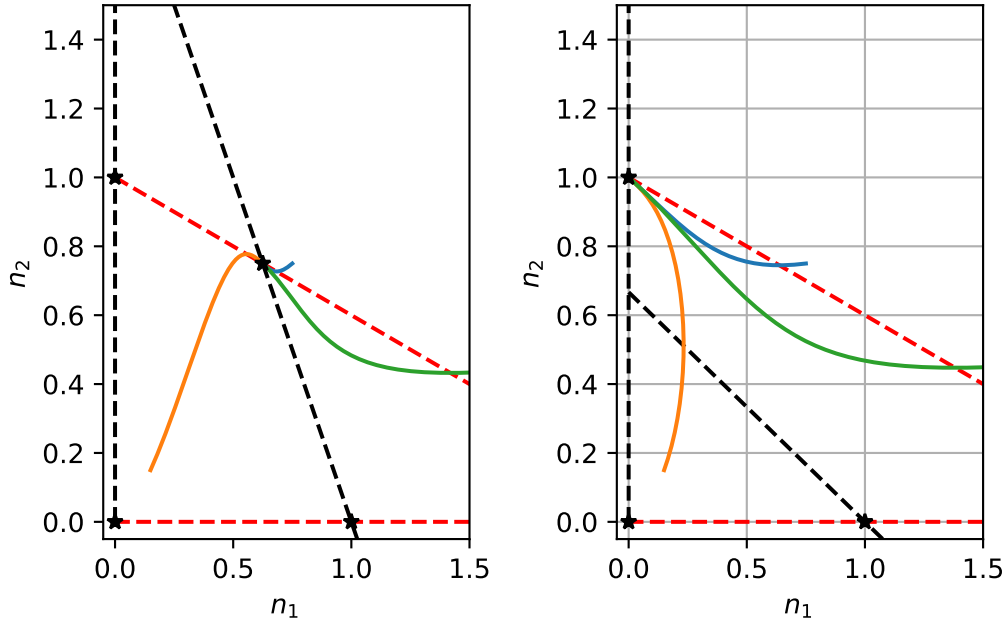


Figure 4.5: Numerical solution of the competition model

#### 4.3.6 Insight

The model therefore has four qualitatively different behaviours that are described as follows: Consider the case where  $a_{21} > 1$ . This represents the case of Population 1 strongly competing with Population 2.

- If  $a_{12} > 1$ , Population 2 also strongly competes with Population 1. In this case, there are four biologically relevant steady states, two of which are stable (1,0) and (0,1). The coexistence steady state is a saddle and thus unstable. The model is bistable and the initial conditions determine whether solutions end up at (1,0) or (0,1) (see Figure 4.5). The biological interpretation of this solution is that one species will always win and completely outcompete the other. Even if the two populations are equal ( $K_1 = K_2$  and  $a_{21} = a_{12} > 1$ ), one species will always win and the other will become extinct.
- If  $a_{12} < 1$ , Population 2 weakly competes with Population 1. There is no coexistence steady state and the only stable steady state is (1,0). Hence Population 1 always wins and Population 2 always becomes extinct.

Now consider the case where  $a_{21} < 1$ . This represents the case of Population 1 weakly competing with Population 2.

- If  $a_{12} > 1$ , Population 2 strongly competes with Population 1. There is no coexistence steady state and the only stable steady state is (0,1). Hence Population 2 always wins

and Population 1 always becomes extinct.

- If  $a_{12} < 1$ , Population 2 also weakly competes with Population 1. The coexistence steady state is stable and the steady states  $(1,0)$  and  $(0,1)$  are unstable.

## 4.4 Symbiosis/Mutualism

When the interactions between two species results in mutually benefit, it is known as mutualism or symbiosis. To study this behaviour, we consider a model of the form

$$\begin{aligned}\frac{dN_1}{dt} &= r_1 N_1 \left( 1 - \frac{N_1}{K_1} + b_{12} \frac{N_2}{K_1} \right), \\ \frac{dN_2}{dt} &= r_2 N_2 \left( 1 - \frac{N_2}{K_2} + b_{21} \frac{N_1}{K_2} \right).\end{aligned}$$

Note the only difference with the competition model is that the sign of the interaction term has changed. Hence we will not work through all the details as the analysis is the same as before.

### 4.4.1 Nondimensionalisation

Using the same nondimensionalisation as the competition model, we obtain the nondimensional equations

$$\begin{aligned}\frac{dn_1}{d\tau} &= n_1(1 - n_1 + a_{12}n_2) = f(n_1, n_2), \\ \frac{dn_2}{d\tau} &= \rho n_2(1 - n_2 + a_{21}n_1) = g(n_1, n_2).\end{aligned}$$

### 4.4.2 Steady states

This model has steady-states  $(0,0)$ ,  $(1,0)$ ,  $(0,1)$  and

$$(n_1^*, n_2^*) = \left( \frac{1 + a_{12}}{1 - a_{12}a_{21}}, \frac{1 + a_{21}}{1 - a_{12}a_{21}} \right).$$

The coexistence steady state is biological relevant if  $a_{12}a_{21} < 1$

### 4.4.3 Nullclines

The  $n_1$  nullclines satisfy

$$n_1 = 0, \quad n_2 = \frac{1}{a_{12}}(n_1 - 1).$$

The  $n_2$  nullclines satisfy

$$n_2 = 0, \quad n_2 = 1 + a_{21}n_1.$$

Note that both nullclines have a positive slope.

### 4.4.4 Linear stability

Using a similar analysis to the competition model, it can be shown that the steady states  $(0,0)$ ,  $(1,0)$  and  $(0,1)$  are unstable. When  $a_{12}a_{21} < 1$  there is a stable coexistence steady state. See Figure 4.6.

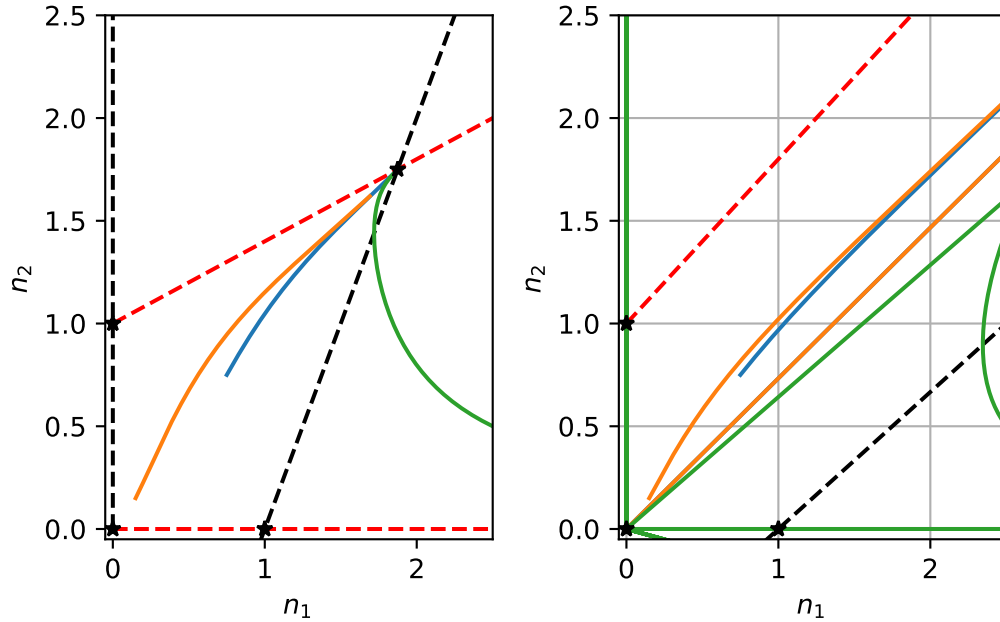


Figure 4.6: Numerical solution of the symbiosis model

### 4.4.5 Insight

The important parameter in the model is the product  $a_{12}a_{21}$ . This quantifies the total amount of cooperativity in the model. All steady states that involve the extinction of a species are unstable. In the case where if  $a_{12}a_{21} < 1$  there is a stable steady state. Note that steady states

of both variables are higher than they would be in the absence of the other species. In the  $a_{12}a_{21} > 1$  there is no coexistence steady state and both populations grow in an unbounded manner.

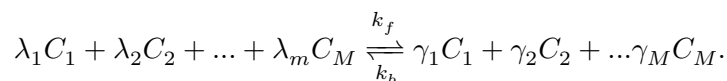
## 5 Biochemical kinetics

Biochemical kinetics concern the concentration of chemical substances in biological systems. In this section we neglect spatial variation in concentrations, effectively assuming chemical reactions occur in environments that are “well-mixed”.

### 5.1 The law of mass action

We denote the  $i^{th}$  chemical species using the notation  $C_i$ . The concentration of molecule type  $C_i$  is denoted  $[C_i]$  and represents the number of molecules of  $C_i$  per unit volume.

Suppose  $C_1, ..C_M$  undergo the reaction



the law of mass action states that the forward reaction proceeds at rate

$$k_f [C_1]^{\lambda_1} [C_2]^{\lambda_2} .. [C_M]^{\lambda_M}$$

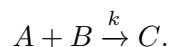
whilst the backward reaction proceeds at rate

$$k_b [C_1]^{\gamma_1} [C_2]^{\gamma_2} .. [C_M]^{\gamma_M}$$

where  $k_f$  and  $k_b$  are dimensional rate constants.

#### **i** Example

Suppose A and B react to produce C. Hence



The law of mass action states that the rate of the reaction is

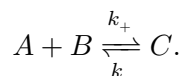
$$k[A][B].$$

Using the reaction rates, we write down ordinary differential equations that describe how concentrations of a given molecule will change in time. Hence

$$\frac{d[C]}{dt} = k[A][B].$$

### **i** Example

Consider the reversible reaction



Define dependent variables, identify reaction rates and derive ordinary differential equations that describe how concentrations evolved in time.

The dependent variables are:

$$[A](t), [B](t), [C](t)$$

Applying the law of mass action yields the reaction rates:

$$k_+[A][B] \text{ and } k_-[C]$$

The ODEs are

$$\begin{aligned}\frac{d[A]}{dt} &= -k_+[A][B] + k_-[C], \\ \frac{d[B]}{dt} &= -k_+[A][B] + k_-[C], \\ \frac{d[C]}{dt} &= k_+[A][B] - k_-[C].\end{aligned}$$

For a given set of initial conditions,

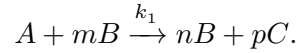
$$[A](t=0) = [A]_0, \quad [B](t=0) = [B]_0, \quad [C](t=0) = [C]_0,$$

the ODEs can be solved and hence the concentrations of the different molecules described as time evolves.

### 5.1.1 Example

The previous example have had stoichiometric constants all set to unity (i.e. all reactions involved a molecules of one species interaction with one from another). Consider the reaction

in which one molecule of species A with  $m$  of species B giving rise to  $n$  molecules of species B and  $p$  molecules of species C, i.e.



The law of mass action says that the rate of the forward reaction is

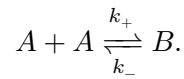
$$k_1[A][B]^m.$$

The governing ODEs are

$$\begin{aligned}\frac{d[A]}{dt} &= -k_1[A][B]^m, \\ \frac{d[B]}{dt} &= (n-m)k_1[A][B]^m, \\ \frac{d[C]}{dt} &= pk_1[A][B]^m.\end{aligned}$$

### **i** Example

Consider the reversible reaction



Define dependent variables, identify reaction rates and derive ordinary differential equations that describe how concentrations evolved in time.

The reaction rates are

$$k_+[A]^2$$

and

$$k_-[B].$$

The ODEs are

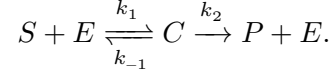
$$\begin{aligned}\frac{d[A]}{dt} &= -2k_+[A]^2 + 2k_-[B], \\ \frac{d[B]}{dt} &= k_+[A]^2 - k_-[B],\end{aligned}$$

## 5.2 Enzyme kinetics

Biochemical reactions are often regulated by enzymes (substances that convert a substrate into another substrate). Consider a chemical reaction in which substrate,  $S$ , reacts with an enzyme,



$E$ , to form a complex,  $C$ . Suppose the complex can either undergo the reverse reaction or go on to form a product,  $P$ , with the release of the enzyme, i.e.



### 5.2.1 Deriving the model equations

For notational convenience we let lower case letter denote concentrations, i.e.

$$s(t) = [S](t), \quad c(t) = [C](t), \quad e(t) = [E](t), \quad p(t) = [P](t).$$

Applying the law of mass action the above enzyme kinetic scheme can be described by the system of ODES

$$\begin{aligned} \frac{ds}{dt} &= -k_1 se + k_{-1} c, \\ \frac{de}{dt} &= -k_1 se + k_{-1} c + k_2 c, \\ \frac{dc}{dt} &= k_1 se - k_{-1} c - k_2 c, \\ \frac{dp}{dt} &= k_2 c. \end{aligned}$$

We consider initial conditions such that at  $t = 0$  the reaction has not yet started, i.e. there is no product or complex formed

$$s(0) = s_0, \quad e(0) = e_0, \quad c(0) = 0, \quad p(0) = 0,$$

where  $s_0$  and  $e_0$  represent the initial concentrations of substrate and enzyme, respectively.

### 5.2.2 Numerical solutions

In Figure 5.1 we plot numerical solutions of Equation 5.1. For this solution we have chosen initial conditions such that

$$\frac{e_0}{s_0} \ll 1.$$

Note that  $c$  and  $e$  evolve rapidly in time close to the initial data whilst those of  $s$  and  $p$  do not.

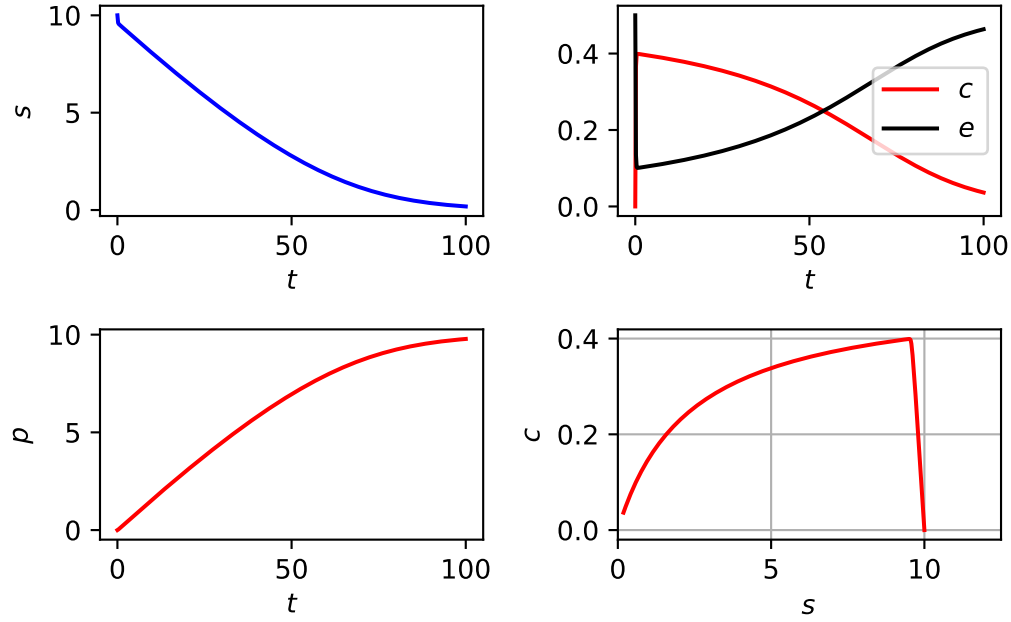


Figure 5.1: Numerical solutions of the Michaelis-Menten model

### 5.2.3 Reducing the dimensions of the model

Whilst there are four dependent variables in the problem ( $s(t)$ ,  $c(t)$ ,  $e(t)$  and  $p(t)$ ), the description of the problem can be simplified by noting that the variable  $p(t)$  does not couple back to the other variables, i.e. if we can solve the system for  $s(t)$ ,  $e(t)$  and  $c(t)$  then  $c(t)$  can be expressed as a function of time and

$$p(t) = k_2 \int c(t) dt + C_1.$$

Furthermore, there is a conserved quantity in the system. Note that

$$\frac{de}{dt} + \frac{dc}{dt} = 0,$$

which reflects the fact that enzyme exists either in free form or bound to the product. Integrating yields

$$c(t) + e(t) = C_2.$$

Evaluating at  $t = 0$ ,

$$e_0 = C_2.$$

Hence

$$c(t) + e(t) = e_0$$

and the variable  $e(t)$  can be replaced by

$$e(t) = e_0 - c(t)$$

Given

$$e(t) = e_0 - c(t),$$

substitute in

$$\begin{aligned}\frac{ds}{dt} &= -k_1 s e + k_{-1} c, \\ \frac{dc}{dt} &= k_1 s e - k_{-1} c - k_2 c,\end{aligned}$$

Hence

$$\begin{aligned}\frac{ds}{dt} &= -k_1 s (e_0 - c) + k_{-1} c, \\ \frac{dc}{dt} &= k_1 s (e_0 - c) - (k_{-1} + k_2) c,\end{aligned}$$

### 5.2.4 The quasi-steady state approximation (QSSA)

A usual approach to these equations is to assume that the timescale of complex dynamics is very fast compared to that of substrate dynamics.

To make the QSSA we assume that one of the variables (in this case  $c$ ), is in equilibrium

$$dc/dt \sim 0.$$

In which case the second equation yields

$$c = \frac{e_0 s(t)}{s(t) + K_m},$$

where

$$K_m = \frac{k_{-1} + k_2}{k_1},$$

is known as the Michaelis constant.

Now consider

$$\frac{ds}{dt} = -k_1 s(e_0 - c) + k_{-1} c,$$

By QSSA

$$-k_1 s(e_0 - c) + k_{-1} c = -k_2 c = -k_2 \frac{e_0 s(t)}{s(t) + K_m},$$

Hence

$$\frac{ds}{dt} = -k_2 \frac{e_0 s(t)}{s(t) + K_m},$$

Note this could be achieved by direct substitution for  $c(t)$ .

## 5.2.5 Enzyme kinetics: the Michaelis-Menten quasi-steady state approximation (QSSA)

### 5.2.5.1 Nondimensionalisation

Nondimensionalisation using

$$\tau = k_1 e_0 t, \quad u = \frac{s}{s_0}, \quad v = \frac{c}{e_0},$$

yields the ODEs

$$\begin{aligned} \frac{du}{d\tau} &= -u + (u + K - \lambda)v, \\ \epsilon \frac{dv}{d\tau} &= u - (u + K)v, \end{aligned}$$

where

$$\lambda = \frac{k_2}{k_1 s_0}, \quad K = \frac{k_{-1} + k_2}{k_1 s_0}, \quad \epsilon = \frac{e_0}{s_0}.$$

### 5.2.5.2 Asymptotic expansions: the outer solution

We consider the (often realised) case where the amount of enzyme in the system is small compared with the amount of substrate, i.e.

$$\epsilon \ll 1.$$

The presence of the small parameter  $\epsilon$  allows us to use perturbation theory to calculate approximate solutions to Equation 5.1. However, the problem is singular owing to the  $\epsilon dv/dt$  term.

We propose an outer solution of the form

$$\begin{aligned} u(\tau; \epsilon) &= u_0(\tau) + \epsilon u_1(\tau) + \epsilon^2 u_2(\tau) + \dots = \sum_{n=0}^{\infty} u_n(\tau) \epsilon^n, \\ v(\tau; \epsilon) &= v_0(\tau) + \epsilon v_1(\tau) + \epsilon^2 v_2(\tau) + \dots = \sum_{n=0}^{\infty} v_n(\tau) \epsilon^n. \end{aligned}$$

Substituting the expansions in Equation 5.1 yields

$$\begin{aligned} \frac{du_0}{d\tau} + \epsilon \frac{du_1}{d\tau} + \epsilon^2 \frac{du_2}{d\tau} + \dots &= -(u_0(\tau) + \epsilon u_1(\tau) + \epsilon^2 u_2(\tau) + \dots) + \\ &+ (u_0(\tau) + \epsilon u_1(\tau) + \epsilon^2 u_2(\tau) + \dots + K - \lambda)(v_0(\tau) + \epsilon v_1(\tau) + \epsilon^2 v_2(\tau) + \dots). \end{aligned}$$

Gathering terms as coefficients of the different powers of  $\epsilon$  yields

$$\frac{du_0}{d\tau} + \epsilon \frac{du_1}{d\tau} + \dots = -u_0(\tau) + (u_0(\tau) + K - \lambda)v_0 + \epsilon(-u_1(\tau) + u_1v_0 + v_1u_0 + v_1(K - \lambda)) + O(\epsilon^2),$$

where terms of order  $\epsilon^2$  have been neglected.

Considering the  $v$  equation yields

$$\begin{aligned} \epsilon \left( \frac{dv_0}{d\tau} + \epsilon \frac{dv_1}{d\tau} + \epsilon^2 \frac{dv_2}{d\tau} + \dots \right) &= u_0(\tau) + \epsilon u_1(\tau) + \epsilon^2 u_2(\tau) + \dots \\ &- (u_0(\tau) + \epsilon u_1(\tau) + \epsilon^2 u_2(\tau) + \dots + K)(v_0(\tau) + \epsilon v_1(\tau) + \epsilon^2 v_2(\tau) + \dots). \end{aligned}$$

Gathering terms as coefficients of the different powers of  $\epsilon$  yields

$$\begin{aligned} \epsilon \frac{dv_0}{d\tau} + \epsilon^2 \frac{dv_1}{d\tau} + \dots &= u_0(\tau) - u_0(\tau)v_0(\tau) - v_0K \\ &+ \epsilon(u_1(\tau) - u_0v_1 - u_1v_0 - Kv_1) + O(\epsilon^2). \end{aligned}$$

At  $O(1)$

$$\begin{aligned}\frac{du_0}{d\tau} &= -u_0 + (u_0 + K - \lambda)v_0 \\ 0 &= u_0 - u_0v_0 - v_0K.\end{aligned}$$

Solving the algebraic equation yields

$$v_0 = \frac{u_0}{u_0 + K},$$

and substituting in the  $u_0$  equation yields

$$\frac{du_0}{d\tau} = -u_0 + (u_0 + K - \lambda)\frac{u_0}{u_0 + K} = -\frac{\lambda u_0}{u_0 + K}.$$

Integrating yields

$$u_0 + K \ln u_0 = -\lambda\tau + A,$$

where  $A$  is an integration constant.

When posing a solution as an expansion, a necessary question to ask is when the expansion is a valid? On the outer scale we have shown that

$$v_0 = \frac{u_0}{u_0 + K}.$$

Note that the expression

$$v_0 = \frac{u_0}{u_0 + K}$$

does not satisfy the initial condition  $v(0) = 0$ . As  $u_0(0) = 1$ , we find that at  $\tau = 0$

$$v_0 = \frac{1}{1 + K} \neq 0.$$

Hence the proposed expansion is not valid at least near  $\tau = 0$ .

### 5.2.5.3 Asymptotic expansions: the inner solution

This issue can be rectified by proposing a different scaling for time and recalculating a series solution in the new coordinate system.

We proceed by making the change of variable

$$\sigma = \frac{\tau}{\epsilon}.$$

Note that  $\sigma = 1$  corresponds to  $\tau = \epsilon$ . Hence the proposed rescaling of time will give rise to what is called the inner solution to the problem (close to  $t = 0$ ).

To distinguish inner and outer solutions we relabel dependent variables such that

$$\begin{aligned} u(\tau; \epsilon) &= U(\sigma; \epsilon), \\ v(\tau; \epsilon) &= V(\sigma; \epsilon). \end{aligned}$$

Note that

$$\frac{d}{d\tau} = \frac{1}{\epsilon} \frac{d}{d\sigma}.$$

Hence upon changing variables

$$\begin{aligned} \frac{1}{\epsilon} \frac{dU}{d\sigma} &= -\epsilon U + \epsilon(U + K - \lambda)V, \\ \frac{1}{\epsilon} \epsilon \frac{dV}{d\sigma} &= U - (U + K)V. \end{aligned}$$

Upon tidying

$$\begin{aligned} \frac{dU}{d\sigma} &= -\epsilon U + \epsilon(U + K - \lambda)V, \\ \frac{dV}{d\sigma} &= U - (U + K)V. \end{aligned}$$

We seek series solutions to Equation 5.1 of the form

$$\begin{aligned} U(\sigma; \epsilon) &= U_0(\sigma) + \epsilon U_1(\sigma) + \epsilon^2 U_2(\sigma) + \dots = \sum_{n=0}^{\infty} U_n(\sigma) \epsilon^n, \\ V(\sigma; \epsilon) &= V_0(\sigma) + \epsilon V_1(\sigma) + \epsilon^2 V_2(\sigma) + \dots = \sum_{n=0}^{\infty} V_n(\sigma) \epsilon^n. \end{aligned}$$

Substitution yields

$$\begin{aligned}\frac{dU_0}{d\sigma} + \epsilon \frac{dU_1}{d\sigma} + \dots &= \epsilon (U_0 + \epsilon U_1 + \dots + (U_0 + \epsilon U_1 + \dots + K - \lambda)(V_0 + \epsilon V_1 + \dots)) \\ \frac{dV_0}{d\sigma} + \epsilon \frac{dV_1}{d\sigma} + \dots &= (U_0 + \epsilon U_1 + \dots) - (U_0 + \epsilon U_1 + \dots + K)(V_0 + \epsilon V_1 + \dots).\end{aligned}$$

Hence at leading order

$$\begin{aligned}\frac{dU_0}{d\sigma} &= 0 \\ \frac{dV_0}{d\sigma} &= U_0 - (U_0 + K)V_0.\end{aligned}$$

Given the initial condition  $U(0) = 1$  we obtain that the inner solution is

$$U(\sigma) = 1.$$

Substituting in the second equation gives

$$\frac{dV_0}{d\sigma} = 1 - (1 + K)V_0.$$

Given  $V_0(0) = 0$  we obtain

$$V_0 = \frac{1 - e^{-(1+K)\sigma}}{1 + K}.$$

#### 5.2.5.4 Asymptotic expansions: matching the inner and outer solutions

Inner and outer solutions are matched by taking limits

$$\lim_{\tau \rightarrow 0} (u_0(\tau), v_0(\tau)) = \lim_{\sigma \rightarrow \infty} (U_0(\sigma), V_0(\sigma)).$$

Note that

$$\lim_{\sigma \rightarrow \infty} V_0(\sigma) = \lim_{\sigma \rightarrow \infty} \frac{1 - e^{-(1+K)\sigma}}{1 + K} = \frac{1}{1 + K},$$

and

$$\lim_{\tau \rightarrow 0} (v_0(\tau)) = \lim_{\tau \rightarrow 0} \frac{u_0}{u_0 + K} = \frac{1}{1 + K}.$$



Hence the  $v$  variables are already matching in the appropriate limit.

Similarly

$$\lim_{\sigma \rightarrow \infty} U_0(\sigma) = 1,$$

hence we require that

$$\lim_{\tau \rightarrow 0} u_0(\tau) = 1.$$

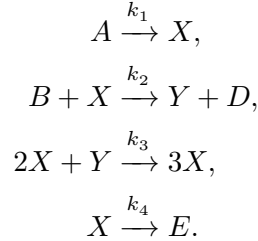
Hence for

$$(u_0 + K \ln u_0 = -\lambda\tau + A)$$

to hold as  $\tau \rightarrow 0$  implies  $A = 1$ .

### 5.3 The Brusselator

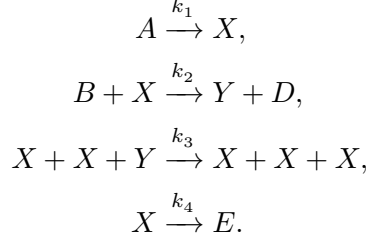
The Brusselator is an abstract model that can be used to demonstrate oscillations in (bio)-chemical systems. Consider a chemical reaction where five chemical species, A, B, D, X and Y, react according to the scheme



Assuming that the concentration of A and B ( $[A]$  and  $[B]$ , respectively) are in vast excess (i.e. the amount that A and B get depleted by the reactions is negligible compared with the total amount of A and B present), their concentration are treated as constants. Furthermore, as D and E are products but not reactants (they only appear on the right-hand side of reactions) we do not concern ourselves with their dynamics.

### 5.3.1 Deriving model equations

To make the steps leading to ODEs obvious, it is useful to rewrite the reaction scheme in the form



Applying the law of mass action yields that the four reactions occur at rates:

- $k_1[A]$ ,
- $k_2[B][X]$ ,
- $k_3[X]^2[Y]$ ,
- $k_4[X]$ .

The total time derivative of  $[X]$  is obtained by visiting each X in the reaction scheme once and adding the corresponding reaction rate to the right-hand side of the ODE, i.e.

$$\begin{aligned} \frac{d[X]}{dt} &= \overbrace{k_1[A]}^{R1-lhs} - \overbrace{k_2[B][X]}^{R2-lhs} - \overbrace{k_3[X]^2[Y]}^{R3-lhs} - \overbrace{k_3[X]^2[Y]}^{R3-lhs} + \overbrace{k_3[X]^2[Y]}^{R3-rhs} + \overbrace{k_3[X]^2[Y]}^{R3-rhs} + \overbrace{k_3[X]^2[Y]}^{R3-rhs} - \overbrace{k_4[X]}^{R4-lhs}, \\ &= k_1[A] - k_2[B][X] + k_3[X]^2[Y] - k_4[X]. \end{aligned}$$

Similarly for species Y we obtain that the total rate of change in time is

$$\frac{d[Y]}{dt} = k_2[B][X] - k_3[X]^2[Y].$$

### 5.3.2 Nondimensionalisation

Upon making the change of variables

$$x = \sqrt{\frac{k_3}{k_4}}[X], \quad y = \sqrt{\frac{k_3}{k_4}}[Y], \quad \text{and} \quad \tau = k_4 t,$$

yields

$$\begin{aligned} \frac{dx}{d\tau} &= a - bx + x^2 y - x = f(x, y), \\ \frac{dy}{d\tau} &= bx - x^2 y = g(x, y), \end{aligned} \tag{5.1}$$

where

$$a = [A] \frac{k_1}{k_4} \sqrt{\frac{k_3}{k_4}} \quad \text{and} \quad b = [B] \frac{k_2}{k_4}.$$

### 5.3.3 Steady states analysis

Seeking steady states  $(x^*, y^*)$  of equations Equation 5.1 such that

$$f(x^*, y^*) = g(x^*, y^*) = 0$$

yields

$$(x^*, y^*) = (a, \frac{b}{a}).$$

### 5.3.4 Linear stability analysis

The Jacobian matrix is

$$A = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} -b + 2xy - 1 & x^2 \\ b - 2xy & -x^2 \end{pmatrix}.$$

Evaluating the Jacobian at  $(a, b/a)$  yields

$$A = \begin{pmatrix} b - 1 & a^2 \\ -b & -a^2 \end{pmatrix}.$$

The determinant of the Jacobian is

$$\det A = -(b - 1)a^2 + a^2b = a^2 > 0,$$

Hence  $(a, b/a)$  is not a saddle. The trace of the Jacobian is

$$\text{tr} A = b - 1 - a^2.$$

Hence if  $b > 1 + a^2$  the trace is positive and, referring to Figure 4.1, the steady state is linearly unstable. Otherwise, the steady state is linearly stable.

In Figure 5.2 we numerically solve Equation 5.1 in the case of oscillatory solutions. Note that when the steady state is unstable, the numerical solution indicates that the system has limit cycle solutions. A valid question to ask is whether one can prove that this is the case.

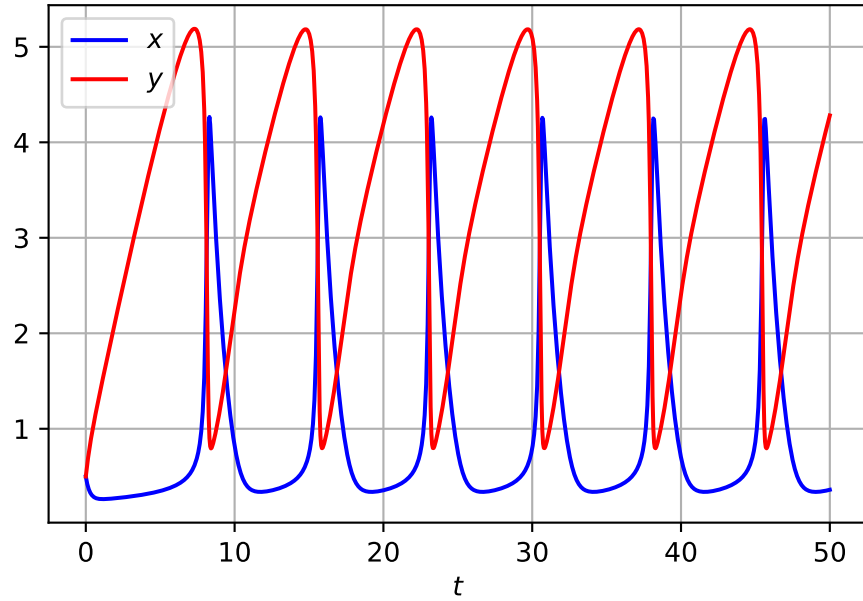


Figure 5.2: Numerical solutions of the Brusselator model

### 5.3.5 Applying the Poincaré-Bendixson theorem

Recall that the Poincaré-Bendixson theorem can be used to prove the existence of limit cycle solutions in a case where a bounding set encloses a single unstable steady state. Hence given the case of the Brusselator with  $b > a^2 + 1$ , the identification of a bounding set will allow application of the Poincaré-Bendixson theorem and hence prove the existence of limit cycle solutions.

We begin by defining the nullclines of the system. The  $x$  nullcline is given by

$$y = \frac{1+b}{x} - \frac{a}{x^2}.$$

The  $y$  nullclines are given by

$$y = \frac{b}{x} \quad \text{and} \quad x = 0.$$

In the positive quadrant, this nullcline has a single root at  $a/(1+b)$ , tends to 0 as  $x \rightarrow \infty$  and has a maximum at  $x = 2a/(1+b)$ .

In Figure 5.3 we plot the oscillatory solution together with the nullclines. Note that the nullclines separate the phase plane into regions of constant sign. Note that the  $x$  nullcline

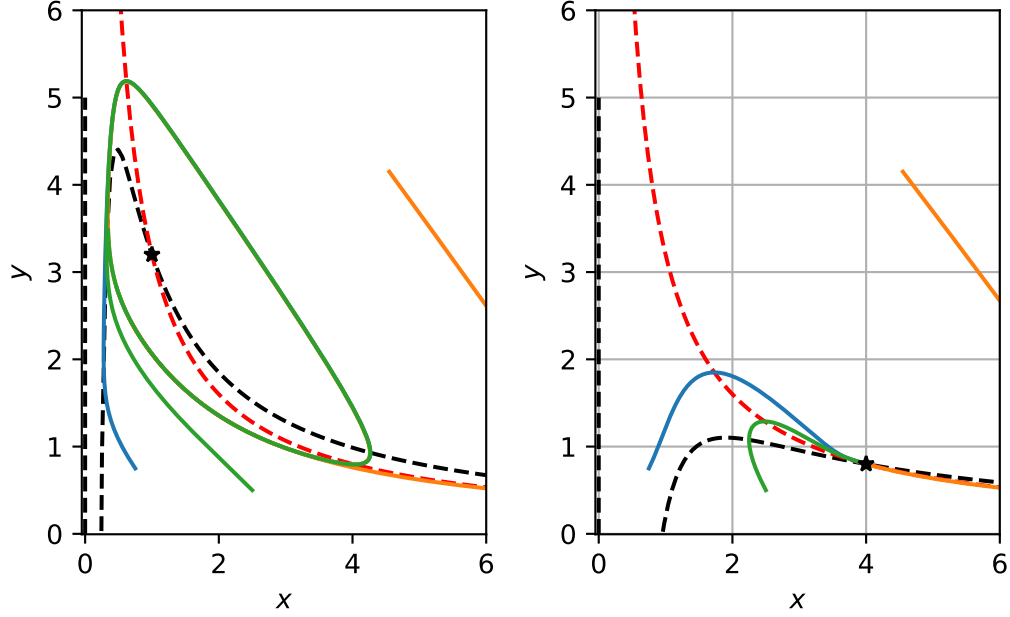


Figure 5.3: Numerical solution of the Brusselator.

separates the phase plane into regions where  $dx/dt > 0$  and  $dx/dt < 0$ . Similarly, the  $y$  nullcline separates the phase plane into regions where  $dy/dt > 0$  and  $dy/dt < 0$ .

Note that signs for  $dx/dt$  can be determined by considering behaviour of the function  $f$  as one moves away from the nullcline where, by definition,  $f = 0$ . Consider some point that sits on the  $x$  nullcline. While keeping the  $x$  value fixed, increase  $y$  so the point moves vertically in the phase plane. This implies that  $f$  has increased because the only term to change is  $x^2y$ . Hence  $f$  increases upon increasing  $y$  and therefore  $dx/dt$  is positive for all points above the  $x$  nullcline. Alternatively, note that in the Jacobian matrix we have calculated that  $\partial f/\partial y$  is positive.

#### 5.3.5.1 A confined set

To define a confined set, we construct the closed loop ABCDEA in phase space (see Figure 5.4) where the points A, B, C, D and E are defined as follows. Let the points A and B be two points on the  $x$  axis with coordinates

$$(\delta, 0)$$

and

$$\left(\frac{a}{1+b}, 0\right).$$

respectively. We choose  $\delta > a$ . The outward normal to the line segment AB is  $\mathbf{n} = [0, -1]$ . To show that the trajectories point inwards along AB, we compute

$$\mathbf{n} \cdot \left[ \frac{dx}{dt}, \frac{dy}{dt} \right] = -\frac{dy}{dt}.$$

As the line segment AB sits below the  $y$  nullcline,  $dy/dt > 0$  in this region. Hence

$$\mathbf{n} \cdot \left[ \frac{dx}{dt}, \frac{dy}{dt} \right] = -\frac{dy}{dt} < 0.$$

Hence trajectories point inwards on the line segment AB.

Let  $C$  represent the point

$$\left( \frac{a}{1+b}, b \frac{1+b}{a} \right).$$

The normal vector to the line segment BC is  $\mathbf{n} = [-1, 0]$ . As BC lies to the left of the  $x$  nullcline, in a region where  $dx/dt > 0$ , then

$$\mathbf{n} \cdot \left[ \frac{dx}{dt}, \frac{dy}{dt} \right] = -\frac{dx}{dt} < 0.$$

Hence trajectories point inwards on the line segment BC.

Consider the line segment CD where D has coordinates

$$\left( k, b \frac{1+b}{a} \right),$$

with  $k > a$

The normal to this line segment is  $[0, 1]$ . As CD lies in a region of the phase plane where  $dy/dt < 0$ ,

$$\mathbf{n} \cdot \left[ \frac{dx}{dt}, \frac{dy}{dt} \right] = \frac{dy}{dt} < 0.$$

Hence trajectories point inwards on the line segment CD.

Let E be a point that sits on the  $x$  nullcline at some position

$$\left( \delta, \frac{\delta(1+b)-a}{\delta^2} \right),$$

such that DE is a straight line with outwardly pointing normal vector  $[1, 1]$ . Along DE

$$\mathbf{n} \cdot \left[ \frac{dx}{dt}, \frac{dy}{dt} \right] = a - bx + x^2y - x + bx - x^2y = a - x < 0$$

for  $x > a$ . As  $k > a$ ,  $x > a$  for all points on DE, hence trajectories point inwards on DE.

Finally, consider the line segment EA which has normal vector  $\mathbf{n} = [1, 0]$ . As  $dx/dt < 0$  along EA

$$\mathbf{n} \cdot \left[ \frac{dx}{dt}, \frac{dy}{dt} \right] = \frac{dx}{dt} < 0.$$

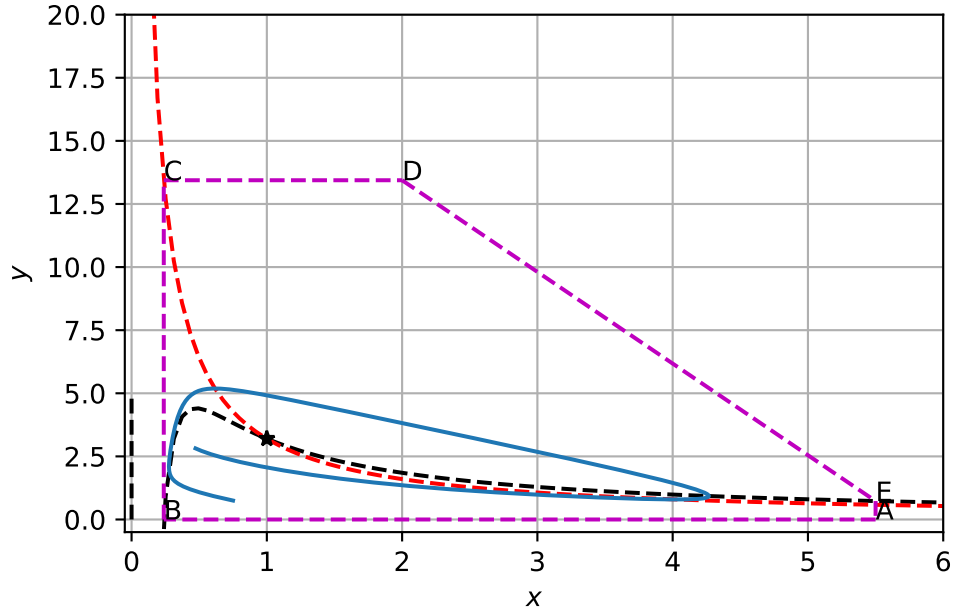


Figure 5.4: Numerical solution of the Brusselator.

Hence at all points on the closed loop ABCDEA trajectories point inwards (see Figure 5.4). Hence ABDCEA is a confined set and the Poincaré-Bendixson theorem states that the system exhibits limit cycle solutions when the steady state is unstable. This is precisely the behaviour that we see numerically in Figure Figure 5.3.

## **Part III**

# **Appendices**



# 6 Python

## 6.1 Symbolic calculations

Symbolic calculations have been performed using the Python library [SymPy](#).

This library comes with [tutorials](#).

You are encouraged to familiarise yourself with the syntax by working through some of the tutorial examples provided at the links above.

Many of the calculations that we do throughout the course involve solving systems of algebraic equations

## 6.2 Numerical solution of difference equations

Difference equations have been solved using a for loop. Routines have been written to solve either single or coupled system of difference equations.

## 6.3 Numerical integration of ODEs

Throughout the notes systems of ODEs have been integrated using the [Scipy](#) function [odeint](#).

## 6.4 Plotting

Line graphs are plotted using the Python library [Matplotlib](#).

# 7 Curve sketching

## 7.1

In the module we will be required to graph of nonlinear function.

Consider a nonlinear function

$$f(x; p), \quad x \in \mathfrak{R}.$$

where  $p$  is a constant.

For example,

$$f(x) = rx(1 - x), \quad x \in \mathfrak{R}.$$

Our goal is usually to sketch  $f$  in qualitatively distinct cases. For example: the graph of  $f$  when  $r = 1$  is *qualitatively* equivalent to the graph of  $f$  when  $r = 2$ . However, when  $r = -1$  it is qualitatively distinct.

The challenge is to firstly identify qualitatively distinct cases. Then sketch  $f$  for each of these cases.

To identify qualitatively distinct cases we test the properties of  $f$  as follows:

1. Identify the *biologically relevant* roots of  $f$ . We define  $x^*$  such that

$$f(x^*; p) = 0.$$

Biologically relevant solutions require  $x^* \geq 0$ .

Does the number of biologically relevant solutions depend on the parameter  $p$ ?

2. Identify the turning points of  $f$ .

Differentiate  $f$  with respect to  $x$ . Identify turning points,  $x_c$ , such that

$$\left. \frac{df}{dx} \right|_{x=x_c} = 0.$$

Are there any turning points? Are they biologically relevant? Does the number of biologically relevant solutions depend on the parameter  $p$ ?

3. Identify limiting behaviour as  $x \rightarrow \infty$ . Compute the limit

$$\lim_{x \rightarrow \infty} f(x; p) = 0.$$

- 4.

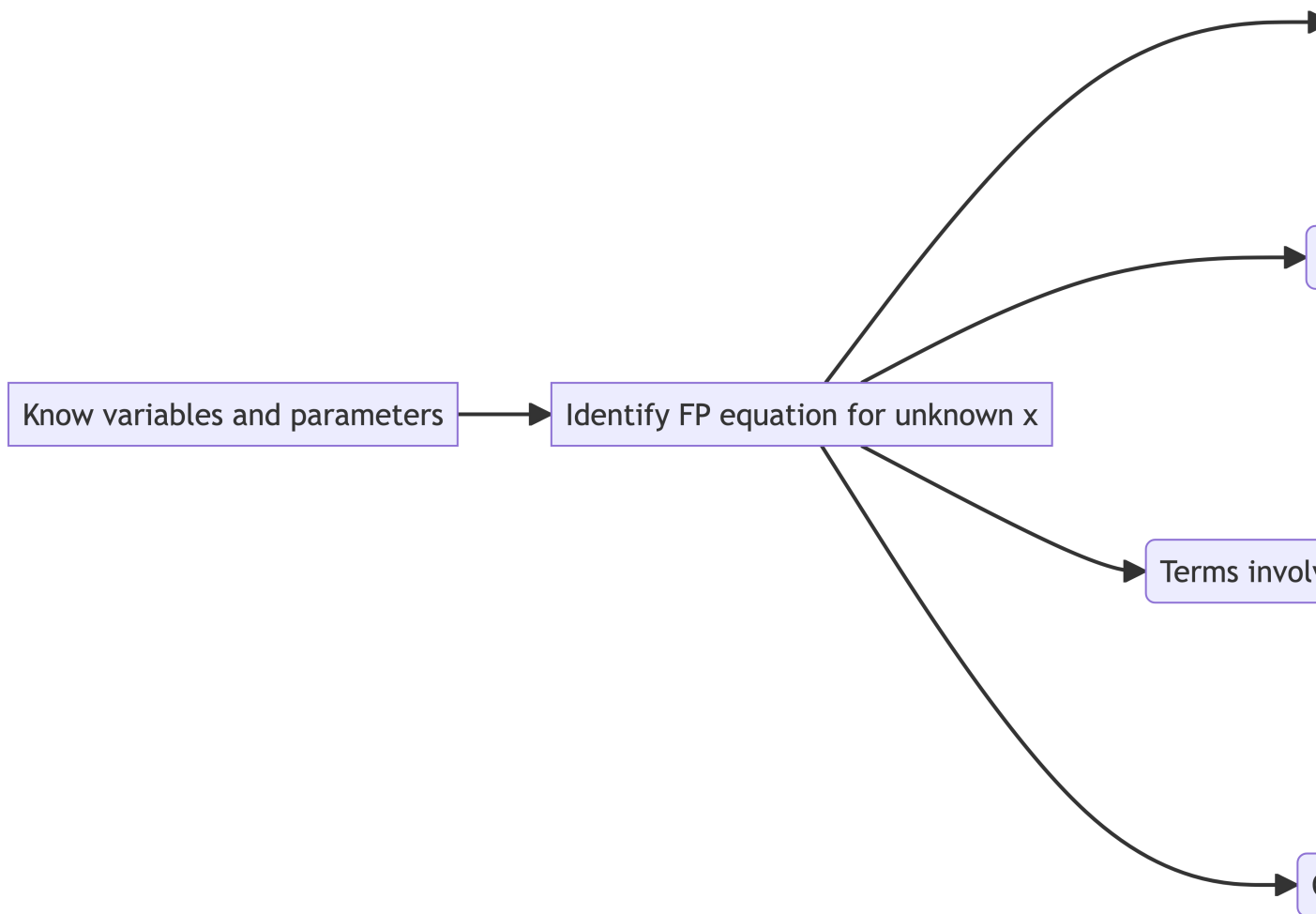
## 8 Quiz 1

### 8.1 Strategy for solving the fixed point problem

Suppose we are solving the fixed point equation

$$H(N^*; p) = N^*.$$

Here  $N^*$  is the fixed point value and  $p$  represents model parameters. In the flow chart below we represent  $N^*$  by  $x$  for notational convenience.



**i** Spotting the quadratic with a zero root

Suppose the fixed point were a quadratic equation of the form

$$aN^{*2} + bN^* + c$$

where

$$a, b, c \in \mathfrak{R}^+.$$

The solution is

$$N^* = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

However, in the case you are trying to solve  $c = 0$ .  
In this case, factorise instead, i.e.

$$aN^{*2} + bN^* = 0.$$

Hence

$$N^*(aN^* + b) = 0.$$

Hence one root is

$$N^* = 0.$$

The other is

$$N^* = -b/a.$$

In Quiz 1 the fixed point problem takes the form

$$N^* = \frac{rN^*}{1 + N^*} - hN^*. \quad (8.1)$$

We want to solve Equation 8.1 for the unknown  $N^*$ .  $r$  and  $h$  are the model parameters (these are fixed constants).

To isolate the  $N^*$  we need to deal with the denominator.

**i** Note

There is at least one term involving the variable in a denominator.

Hence multiply every term in Equation 8.1 by the denominator  $1 + N^*$ .

Therefore

$$N^*(1 + N^*) = rN^* - hN^*(1 + N^*).$$

Now gather coefficients of powers of  $N^*$ .

Hence

$$N^{*2}(1 + h) + N^*(1 + h - r) = 0.$$

**i** Note

This is of the form

$$ax^2 + bx = 0.$$

Factorise out an  $N^*$ . Hence

$$N^*((1 + h)N^* + 1 + h - r) = 0.$$

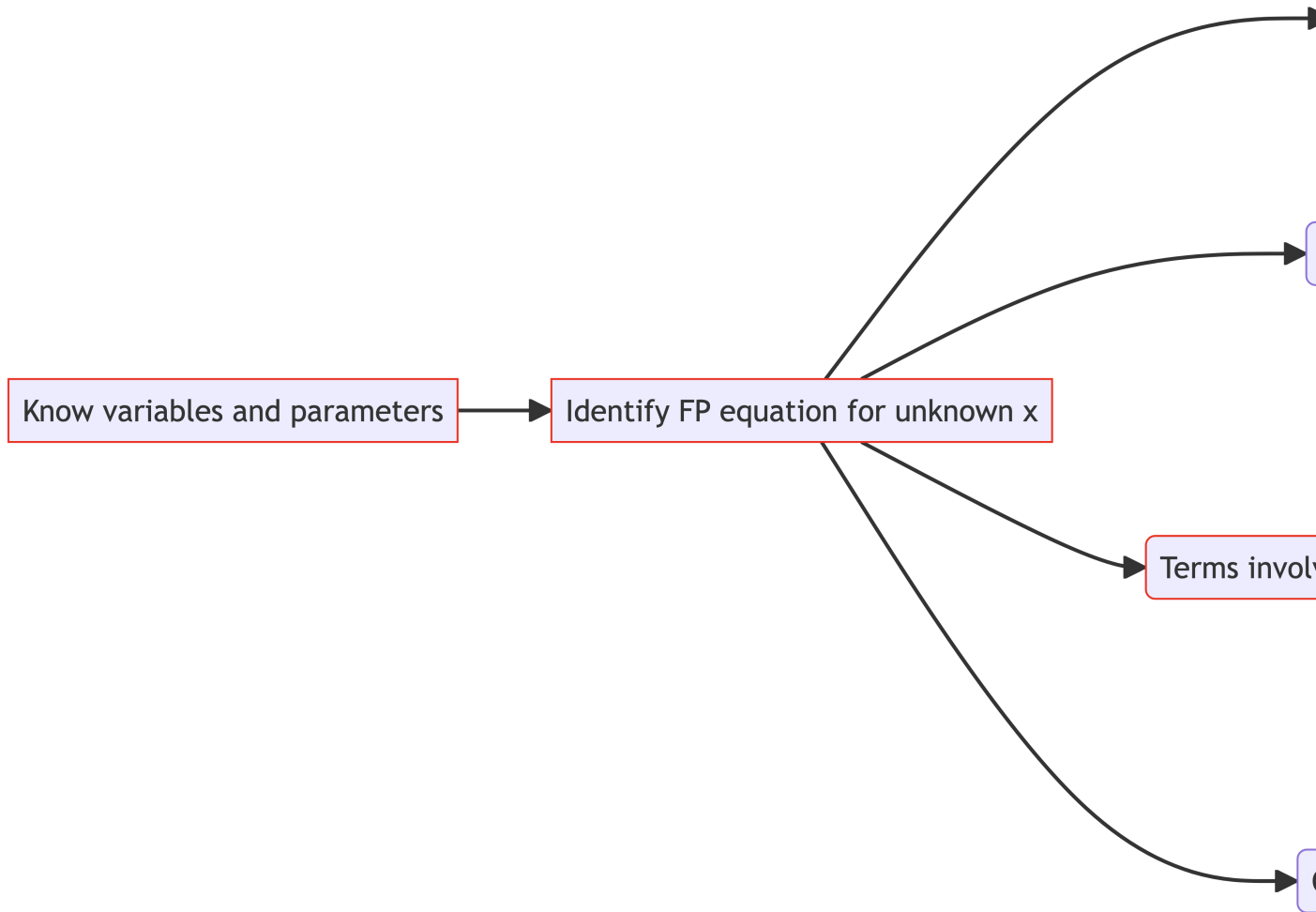


Figure 8.1: The strategy for this problem is highlighted.

Hence one solution is

$$N^* = 0.$$

The other is

$$N^* = \frac{r}{1+h} - 1.$$

## 8.2 Derivatives

$$N_{t+1} = H(N_t) = \frac{rN_t}{1+N_t} - hN_t.$$

To differentiate see below.

**i** Note

Suppose

$$f(x) = \frac{u(x)}{v(x)}.$$

Write as

$$f(x) = u(x)v(x)^{-1}.$$

Apply the product rule

$$\frac{df}{dx} = \frac{du}{dx}v(x)^{-1} + u(x) \left( -v(x)^{-2} \frac{dv}{dx} \right).$$

Hence

$$H'(N_t) = r(1+N_t)^{-1} + rN_t(-(1+N_t)^{-2}) - h.$$

Tidying

$$H'(N_t) = \frac{r}{(1+N_t)^2} - h.$$

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