MA32009 Lecture slides

Philip Murray

Lecture 1

Malthusian model - derivation and qualitative analysis

A general model

$$\frac{dN}{dt} = f(N)N = H(N),$$

Numerical solution

Dimensions and nondimensionalisation

As an example, consider the linear ODE

$$\frac{dN}{dt} = rN.$$

Introducing dimensionless variables

Steady-state analysis

Lecture 10

- Recap
- Techniques for single first order ODE (ctd)
- Example model 1: Logistic growth
- Example model 2: Spruce budworm

Linear stability analysis

Linear stability analysis (ctd)

Graphical solution

Bifurcation diagrams

Example model 1: the logistic growth equation

$$\frac{dN}{dt} = rN(t)\left(1 - \frac{N(t)}{K}\right).$$

Numerical solution

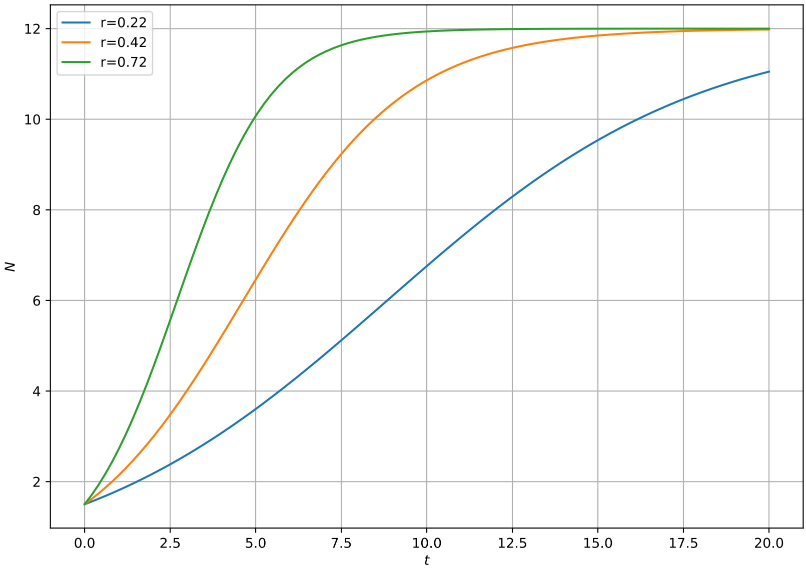


Figure 1: Numerical solution of the logistic growth model

Steady states and linear stability

Graphical analysis

An exact solution of the logistic growth equation

Example model 2: the spruce budworm model

$$\frac{dN}{dt} = r_B N \left(1 - \frac{N}{K_B}\right) - \frac{BN^2}{A^2 + N^2}, \tag{1}$$

Nondimensionalisation

$$\frac{dn}{d\tau} = rn\left(1 - \frac{n}{q}\right) - \frac{n^2}{1+n^2} = H(n),$$

Plotting the RHS

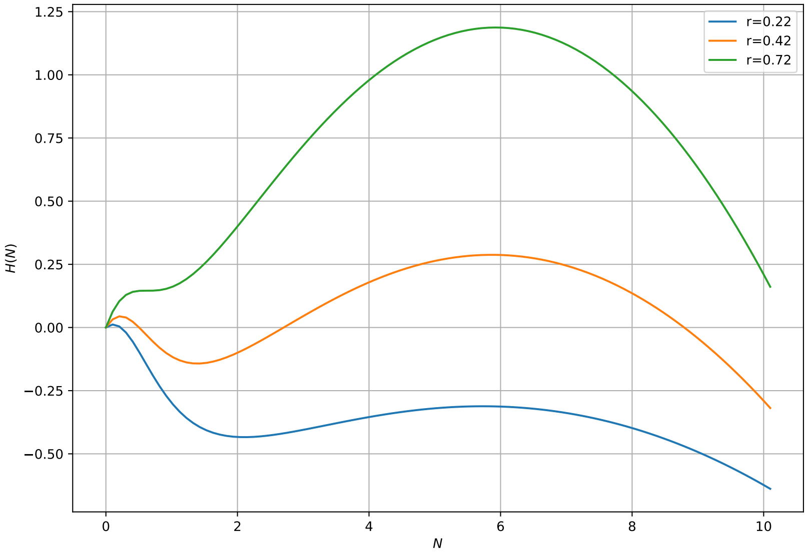
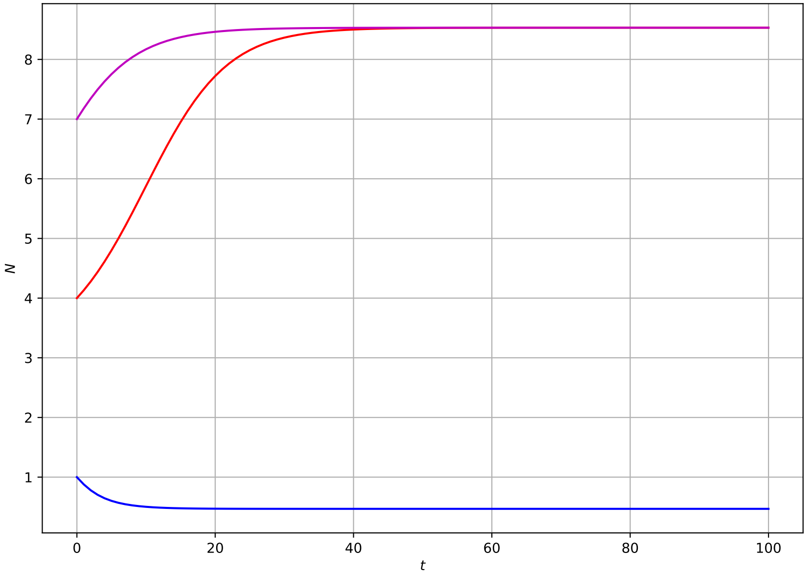


Figure 2: RHS of spr. budworm model

Numerical solution



Steady state analysis

$$rn^*(1 - \frac{n^*}{q}) - \frac{n^{*2}}{1 + n^{*2}} = 0.$$

Linear stability analysis

Lecture 12

Recap - Spruce budworm model

$$\frac{dn}{d\tau} = rn\left(1 - \frac{n}{q}\right) - \frac{n^2}{1+n^2} = H(n),$$

Steady states: $n^* = 0$ or

$$rn^*(1 - \frac{n^*}{q}) - \frac{n^{*2}}{1+n^{*2}} = 0.$$

- r small - one stable steady state
- r large - one stable steady state (outbreak)
- r intermediate - bistability (two stable steady states and one unstable)

Today: bifurcation analysis, hysteresis, harvesting

Tangent bifurcations in rq space

Plotting stability regions in the rq plane

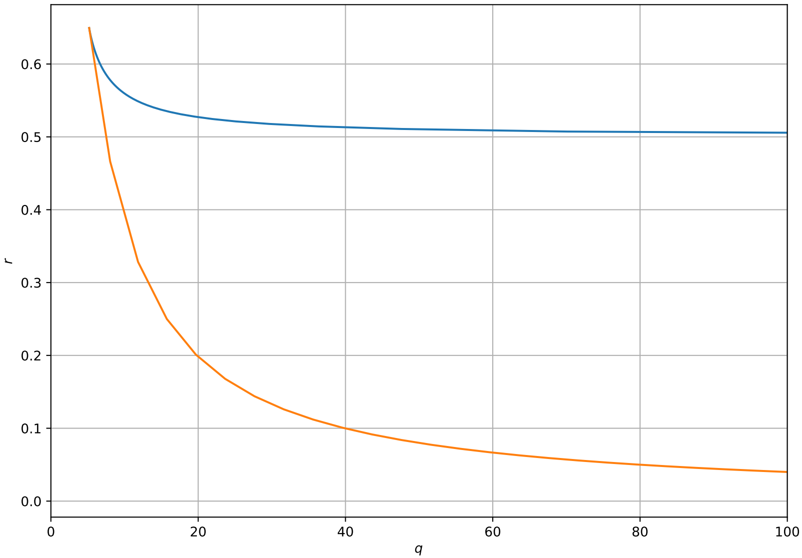


Figure 3: Bifurcations in the rq plane

Hysteresis - irreversible transitions in solution behaviour

Hysteresis

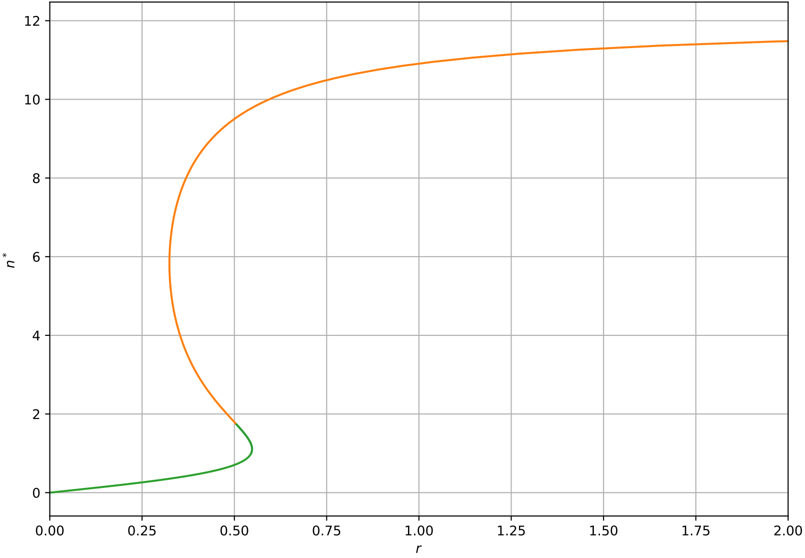


Figure 4: Bifurcations in the rq plane

Harvesting

- use models to simulate how much resource can be extracted?
- approach: take model without harvesting and add in harvesting terms

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - EN.$$

where E is the harvesting rate.

Question: what value of E maximises the long term yield?

Delay differential equation models

$$\frac{dN}{dt} = H(N(t), N(t - T)),$$

A linear delay differential equation model

$$\frac{dN}{dt} = -N(t-T),$$

Linear stability analysis (ctd.)

$$\frac{dN}{dt} = -N(t-T),$$

Two dependent variable ODE models

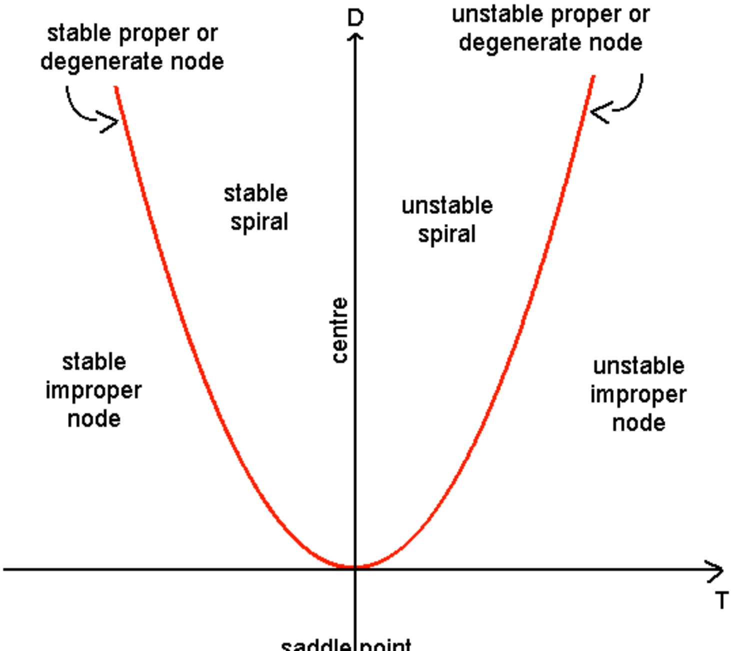
$$\frac{du}{dt} = f(u, v),$$
$$\frac{dv}{dt} = g(u, v).$$

Steady states

Linear stability analysis

Linear stability analysis (ctd.)

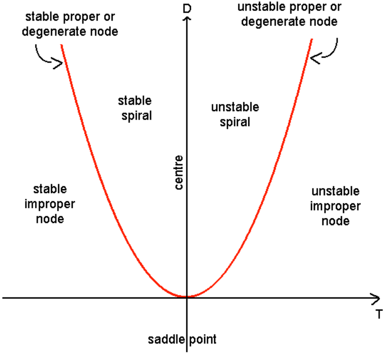
The trace determinant plane



Lecture 14

Recap

$$\frac{du}{dt} = f(u,v),$$
$$\frac{dv}{dt} = g(u,v).$$



$$\lambda = \frac{\mathrm{tr} A \pm \sqrt{\mathrm{tr} A^2 - 4 \det A}}{2}.$$

Nullclines

Periodic solutions (Poincaire-Bendixson theorem)

- System of two ODEs
- Confined set containing unstable node or spiral
- as $t \to \infty$, the trajectory will tend towards a limit cycle.

No periodic soltutions - (Dulac criterion)

- D simply connected region in the plane
- $B(x, y)$, continuously differentiable on D, with

$$\frac{\partial}{\partial u}(Bf) + \frac{\partial}{\partial v}(Bg)$$

not identically zero and does not change sign in D.

Lotka Volterra

$$
\begin{aligned}
\frac{dN}{dt} &= aN - bNP, \\
\frac{dP}{dt} &= cNP - dP,
\end{aligned}
$$

Nondimensionalisation

$$
\begin{aligned}
\frac{dn}{d\tau} &= n(1-p) = f(n,p), \\
\frac{dp}{d\tau} &= \alpha p(n-1) = g(n,p),
\end{aligned}
$$

Lecture 15 - recap

Lotka-Volterra model - predator prey interaction

n - prey p - predator

$$\frac{dn}{d\tau} = n(1-p) = f(n,p),$$
$$\frac{dp}{d\tau} = \alpha p(n-1) = g(n,p),$$

Strategy:

- numerical solution
- steady states
- nullclines
- linear stability

Numerical solutions

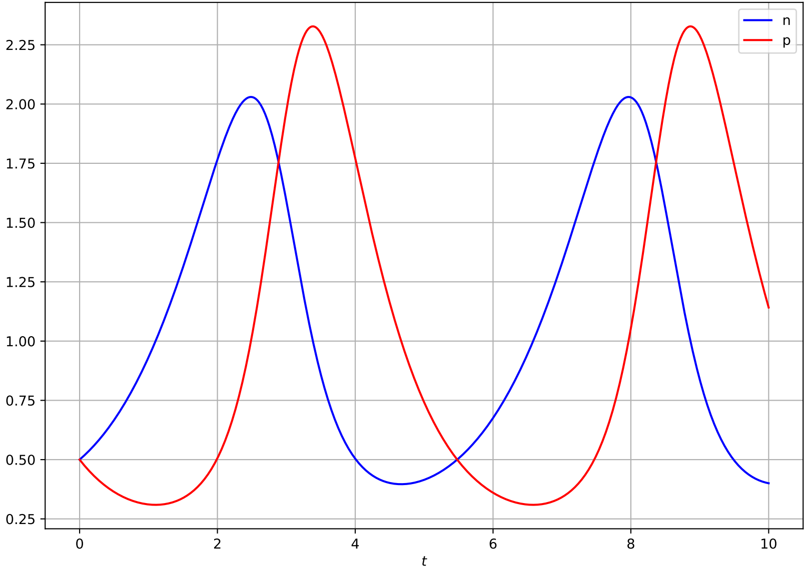


Figure 6

Steady states

Nullclines

The phase plane

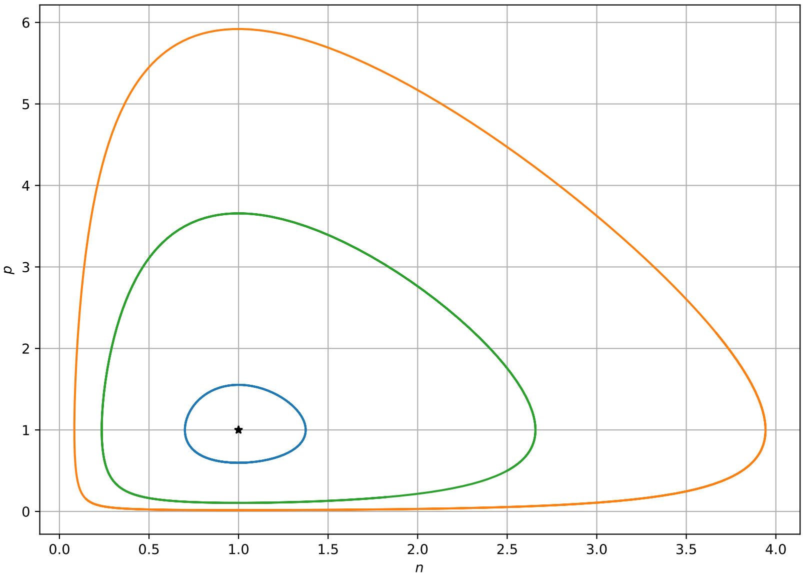


Figure 7

Linear stability

Integration

Lecture 16 Competition

Recap: Lotka-Volterra model

- predator prey interaction
- n - prey
- p - predator

$$\frac{dN}{dt} = aN - bNP,$$
$$\frac{dP}{dt} = cNP - dP,$$

Aim - introduce and analyse a model of competition

Competition model

$$\frac{dN_1}{dt} = r_1 N_1 \left(1 - \frac{N_1}{K_1} - b_{12}\frac{N_2}{K_1}\right),$$
$$\frac{dN_2}{dt} = r_2 N_2 \left(1 - \frac{N_2}{K_2} - b_{21}\frac{N_1}{K_2}\right),$$

- Justify why this a is a model for competition
- Define model parameters
- Explain the meaning of each of the terms in the model

Nondimensionalisation

$$\frac{dn_1}{d\tau} = n_1\left(1 - n_1 - a_{12}n_2\right) = f(n_1, n_2),$$
$$\frac{dn_2}{d\tau} = \rho n_2\left(1 - n_2 - a_{21}n_1\right) = g(n_1, n_2),$$

- Define ρ, a_{12} and a_{21}

Nondimensionalisation (ctd.)

Steady states

Steady states (ctd)

Nullclines

Lecture 17 Competition

$$\frac{dn_1}{d\tau} = n_1\left(1 - n_1 - a_{12}n_2\right) = f(n_1, n_2),$$
$$\frac{dn_2}{d\tau} = \rho n_2\left(1 - n_2 - a_{21}n_1\right) = g(n_1, n_2),$$

Steady states:

$$(0,0),\ (1,0),\ (0,1)$$

and

$$\left(\frac{1 - a_{12}}{1 - a_{12}a_{21}}, \frac{1 - a_{21}}{1 - a_{12}a_{21}}\right).$$

Linear stability analysis

$$A_{(n_1^*,n_2^*)} = \left(\begin{array}{rr} 1-2n_1-a_{12}n_2 & -a_{12}n_1 \\ -\rho a_{21}n_2 & \rho(1-2n_2-a_{21}n_1) \end{array} \right)_{({n_1}^*,{n_2}^*)}.$$

Linear stability analysis (ctd)

Phase portrait (one for each qualitatively distinct case)

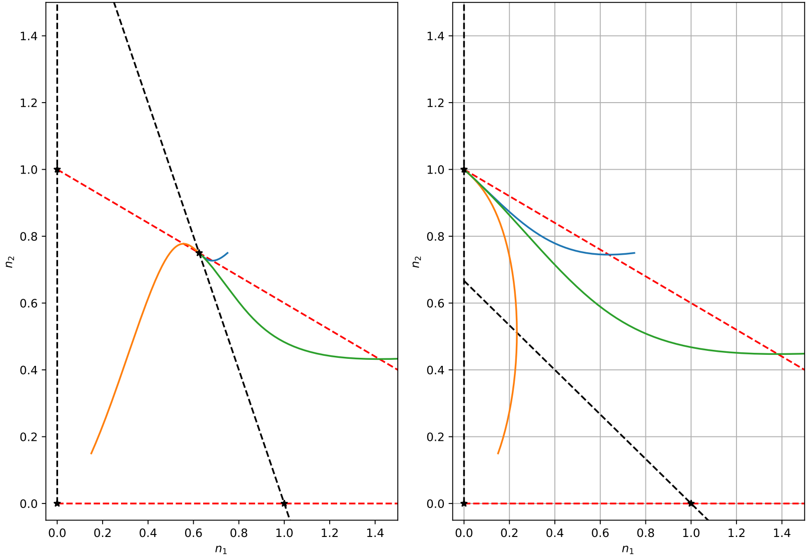


Figure 8

Insight

Mutualism/symbiosis

$$\frac{dn_1}{d\tau} = n_1(1 - n_1 + a_{12}n_2) = f(n_1, n_2),$$
$$\frac{dn_2}{d\tau} = \rho n_2(1 - n_2 + a_{21}n_1) = g(n_1, n_2).$$

Steady states

This model has steady-states $(0, 0)$, $(1, 0)$,$(0, 1)$ and

$$({n_1}^*, {n_2}^*) = \left(\frac{1 + a_{12}}{1 - a_{12}a_{21}}, \frac{1 + a_{21}}{1 - a_{12}a_{21}} \right).$$

Coexistence steady state

Summary

- Predator-prey, competition, mutualism
- techniques to analyse systems of nonlinear ODEs

Lecture 18 Biochemical kinetics

- Cells make proteins via gene transcription and translation
- Proteins can interact
- Molecular biology is the study of the molecules that underpin biological phenomena
- For example: the cell cycle regulated by changing concentrations of cyclin/CDKs

Our question

How do we mathematically describe networks of interacting chemical?

LOMA - reaction rate proportional to product of concentration of reactants

$$\lambda_1 C_1 + \lambda_2 C_2 + ... + \lambda_m C_M \underset{k_b}{\overset{k_f}{\rightleftharpoons}} \gamma_1 C_1 + \gamma_2 C_2 + ... \gamma_M C_M.$$

$\#$ Conservation equation

A forwards reaction

$$A + B \xrightarrow{k} C.$$

A reversible reaction

$$A + B \underset{k_-}{\overset{k_+}{\rightleftharpoons}} C.$$

General stochiometric constants

$$A + mB \xrightarrow{k_1} nB + pC.$$

A reversible dimerisation

$$A + A \underset{k_-}{\overset{k_+}{\rightleftharpoons}} B.$$

Enzyme kinetics

$$S + E \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} C \xrightarrow{k_2} P + E.$$

Numerical solution

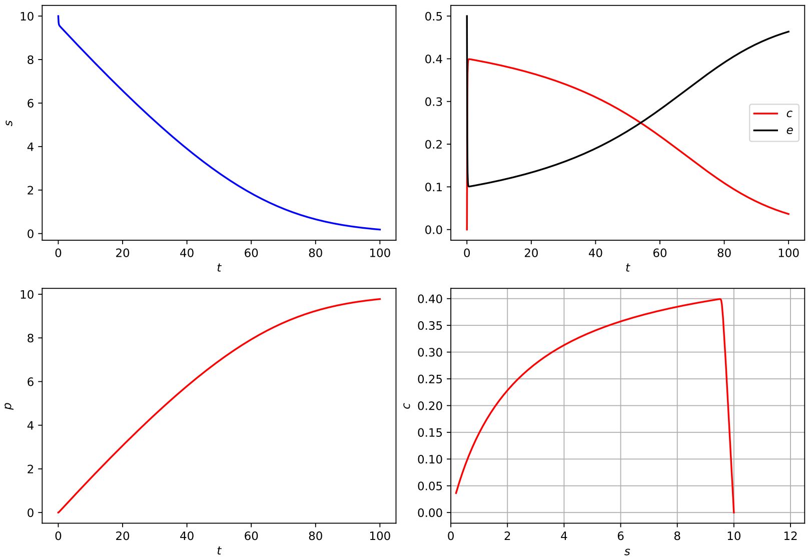


Figure 9: Numerical solutions of the Michaelis Menten model

Dimension reduction

QSSA

Nondimensionalisation

$$\frac{du}{d\tau} = -u + (u + K - \lambda)v,$$
$$\epsilon\frac{dv}{d\tau} = u - (u + K)v,$$

where

$$\lambda = \frac{k_2}{k_1 s_0}, \quad K = \frac{k_{-1} + k_2}{k_1 s_0}, \quad \epsilon = \frac{e_0}{s_0}.$$

Propose asymptotic expansion: outer solution

$$u(\tau;\epsilon) = u_0(\tau) + \epsilon u_1(\tau) + \epsilon^2 u_2(\tau) + ... = \sum_{n=0}^{\infty} u_n(\tau)\epsilon^n,$$

$$v(\tau;\epsilon) = v_0(\tau) + \epsilon v_1(\tau) + \epsilon^2 v_2(\tau) + ... = \sum_{n=0}^{\infty} v_n(\tau)\epsilon^n.$$

Substitute:

Gather terms in powers of epsilon

Leading order solution

An inner solution

Rescale time:

$$\sigma = \frac{\tau}{\epsilon}.$$

Define

$$u(\tau;\epsilon) = U(\sigma;\epsilon),$$
$$v(\tau;\epsilon) = V(\sigma;\epsilon).$$

The inner problem

$$\frac{dU}{d\sigma} = -\epsilon U + \epsilon(U + K - \lambda)V,$$
$$\frac{dV}{d\sigma} = U - (U + K)V.$$

Seek series solutions

$$U(\sigma;\epsilon) = U_0(\sigma) + \epsilon U_1(\sigma) + \epsilon^2 U_2(\sigma) + ... = \sum_{n=0}^{\infty} U_n(\sigma)\epsilon^n,$$

$$V(\sigma;\epsilon) = V_0(\sigma) + \epsilon V_1(\sigma) + \epsilon^2 V_2(\sigma) + ... = \sum_{n=0}^{\infty} V_n(\sigma)\epsilon^n.$$

Mathching inner and outer solutions

The Brusselator

$$\begin{aligned} A &\xrightarrow{k_1} X, \\ B + X &\xrightarrow{k_2} Y + D, \\ 2X + Y &\xrightarrow{k_3} 3X, \\ X &\xrightarrow{k_4} E. \end{aligned}$$

Nondimensional form

$$\begin{aligned}
\frac{dx}{d\tau} &= a - bx + x^2 y - x = f(x, y), \\
\frac{dy}{d\tau} &= bx - x^2 y = g(x, y),
\end{aligned} \tag{2}$$