Lecture slides

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Lecture 1

A general model

Consider the first order difference equation

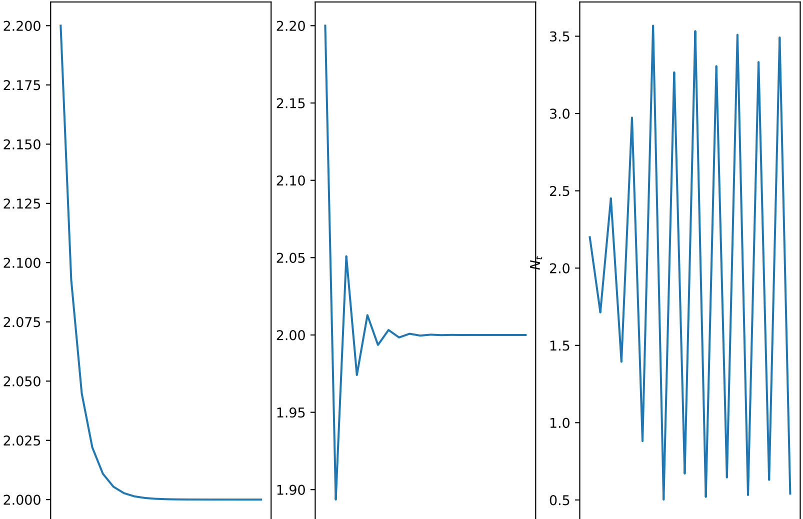
$$N_{t+1} = N_t f(N_t) = H(N_t), \tag{1}$$

where $f(N_t)$ is a function that defines the per capita growth rate. The function $H(N_t)$ describes the total (net) growth rate.

The Malthusian model

The Ricker model

$$N_{t+1} = N_t e^{r(1-\frac{N_t}{K})}.$$



Steady state calculation

Compute the steady states of the system of ODEs

$$\frac{du}{dt} = 1 - u,$$
$$\frac{dv}{dt} = 1 - uv - v.$$

Suppose (u^*, v^*) is a steady state. Hence

$$0 = 1 - u^*$$

and

$$0 = 1 - u^* v^* - v^*$$

The steady state is $(1, 1/2)$.

Linear stability calculation

Deduce, by considering the form for the eigenvalues that, for example, the conditions $\det(A) > 0$, $\mathrm{tr}(A) > 0$ with $\mathrm{tr}(A)^2 < 4\det(A)$ imply that the steady state is an unstable spiral.

The eigenvalues are given by

$$\lambda = \frac{\mathrm{tr}A \pm \sqrt{\mathrm{tr}A^2 - 4\det A}}{2}$$

Jacobian calculation

Compute the Jacobian matrix for the system of ODEs

$$\frac{du}{dt} = 1 - u,$$
$$\frac{dv}{dt} = 1 - uv - v.$$

Evaluate the Jacobian matrix at the steady state and hence determine its linear stability.

The Jacobian is given by

$$A = \left(\begin{array}{cc} -1 & 0 \\ -v & -u-1 \end{array} \right).$$

At (1,1/2)

$$A = \left(\begin{array}{cc} -1 & 0 \\ -\frac{1}{2} & -2 \end{array} \right).$$

In this case

$$tr(A) = -3$$

and

$$\det(A) = 2$$

Nullclines calculation

Sketch the nullclines of the system of ODEs

$$\frac{du}{dt} = 1 - u,$$
$$\frac{dv}{dt} = 1 - uv - v.$$

The u nullcline is $u = 1$. The v nullcline is $v = 1/(1 + u)$.

Biochemical kinetics I

Suppose A and B react to produce C. Hence

$$A + B \xrightarrow{k} C.$$

The law of mass action states that the rate of the reaction is

$$k[A][B].$$

Using the reaction rates, we write down ordinary differential equations that describe how concentrations of a given molecule will change in time. Hence

$$\frac{d[C]}{dt} = k[A][B].$$

Biochemical kinetics

Consider the reversible reaction

$$A + B \underset{k_-}{\overset{k_+}{\rightleftharpoons}} C.$$

Define dependent variables, identify reaction rates and derive ordinary differential equations that describe how concentrations evolved in time.

The dependent variables are:

$$[A](t), \quad [B](t), \quad [C](t)$$

.

Applying the law of mass action yields the reaction rates:

$$k_+[A][B] \quad \text{and} \quad k_-[C]$$

.

Ctd.

The ODEs are

$$\begin{aligned}
\frac{d[A]}{dt} &= -k_+[A][B] + k_-[C], \\
\frac{d[B]}{dt} &= -k_+[A][B] + k_-[C], \\
\frac{d[C]}{dt} &= k_+[A][B] - k_-[C].
\end{aligned}$$

For a given set of initial conditions,

$$[A](t=0) = [A]_0, \quad [B](t=0) = [B]_0, \quad [C](t=0) = [C]_0,$$

the ODEs can be solved and hence the concentrations of the different molecules described as time evolves.