MA32009 Lecture slides

Philip Murray

Lecture 1

Malthusian model - derivation and qualitative analysis

A general model

$$\frac{dN}{dt} = f(N)N = H(N),$$

Numerical solution

Dimensions and nondimensionalisation

As an example, consider the linear ODE

$$\frac{dN}{dt} = rN.$$

Introducing dimensionless variables

Steady-state analysis

Lecture 10

Linear stability analysis

Linear stability analysis (ctd)

Graphical solution

Bifurcation diagrams

Example model 1: the logistic growth equation

$$\frac{dN}{dt} = rN(t)\left(1 - \frac{N(t)}{K}\right).$$
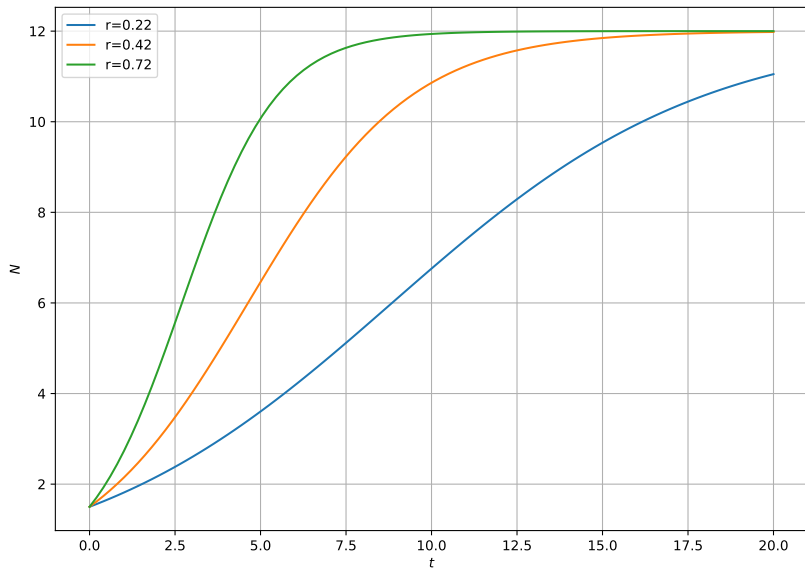
Numerical solution

Figure 1: Numerical solution of the logistic growth model

Steady states and linear stability

Graphical analysis

An exact solution of the logistic growth equation

Example model 2: the spruce budworm model

$$\frac{dN}{dt} = r_B N \left(1 - \frac{N}{K_B}\right) - \frac{BN^2}{A^2 + N^2}, \tag{1}$$

Nondimensionalisation

$$\frac{dn}{d\tau} = rn\left(1 - \frac{n}{q}\right) - \frac{n^2}{1 + n^2} = H(n),$$
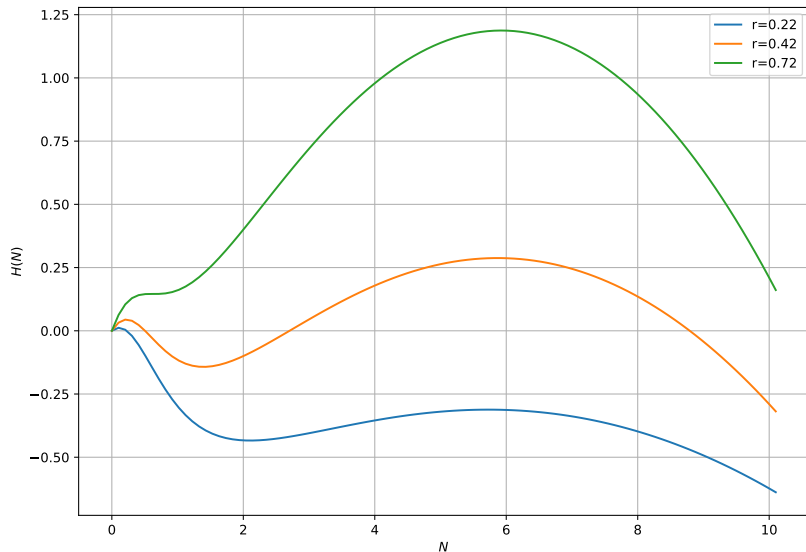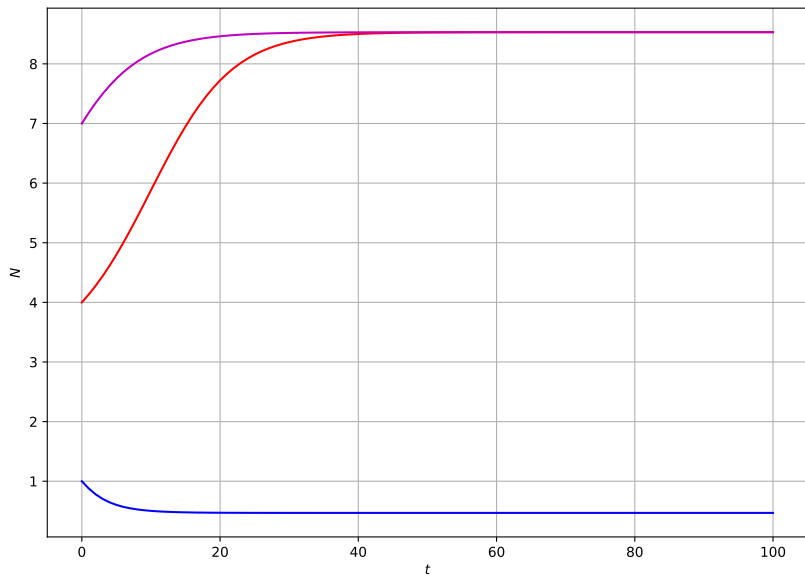
Plotting the RHS

Figure 2: RHS of spr. budworm model

Numerical solution

Steady state analysis

$$rn^*(1 - \frac{n^*}{q}) - \frac{n^{*2}}{1 + n^{*2}} = 0.$$

Linear stability analysis

Lecture 12

> **i** Recap - Spruce budworm model
>
> $$\frac{dn}{d\tau} = rn\left(1 - \frac{n}{q}\right) - \frac{n^2}{1 + n^2} = H(n),$$
>
> Steady states: $n^* = 0$ or
>
> $$rn^*(1 - \frac{n^*}{q}) - \frac{n^{*2}}{1 + n^{*2}} = 0.$$
>
> ▶ r small - one stable steady state
> ▶ r large - one stable steady state (outbreak)
> ▶ r intermediate - bistability (two stable steady states and one unstable)

Today: bifurcation analysis, hysteresis, harvesting

Tangent bifurcations in rq space
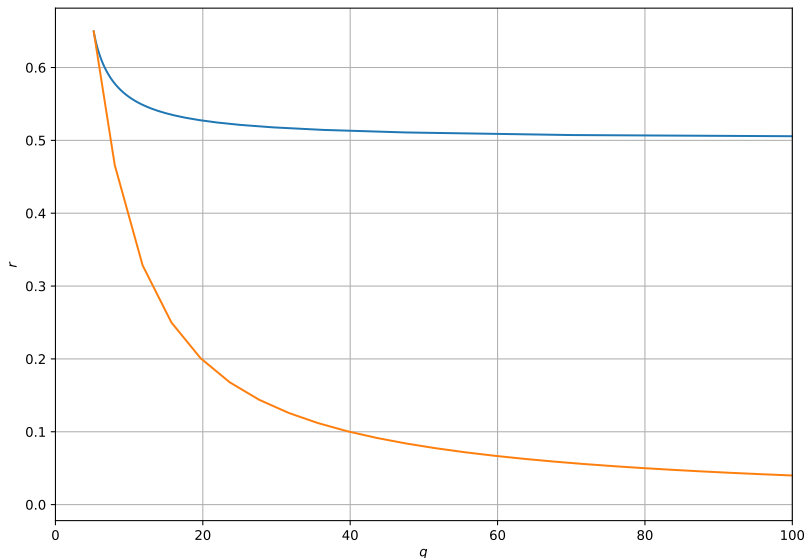
Plotting stability regions in the rq plane

Figure 3: Bifurcations in the rq plane

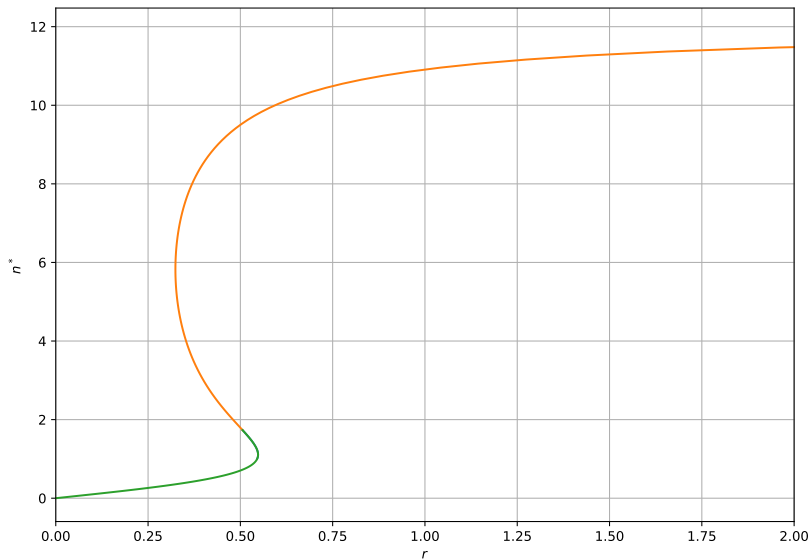# Hysteresis - irreversible transitions in solution behaviour

Hysteresis

Figure 4: Bifurcations in the rq plane

Harvesting

▶ use models to simulate how much resource can be extracted?

▶ approach: take model without harvesting and add in harvesting terms

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - EN.$$

where E is the harvesting rate.

Question: what value of E maximises the long term yield?

Delay differential equation models

$$\frac{dN}{dt} = H(N(t), N(t-T)),$$

A linear delay differential equation model

$$\frac{dN}{dt} = -N(t - T),$$

Linear stability analysis (ctd.)

$$\frac{dN}{dt} = -N(t - T),$$

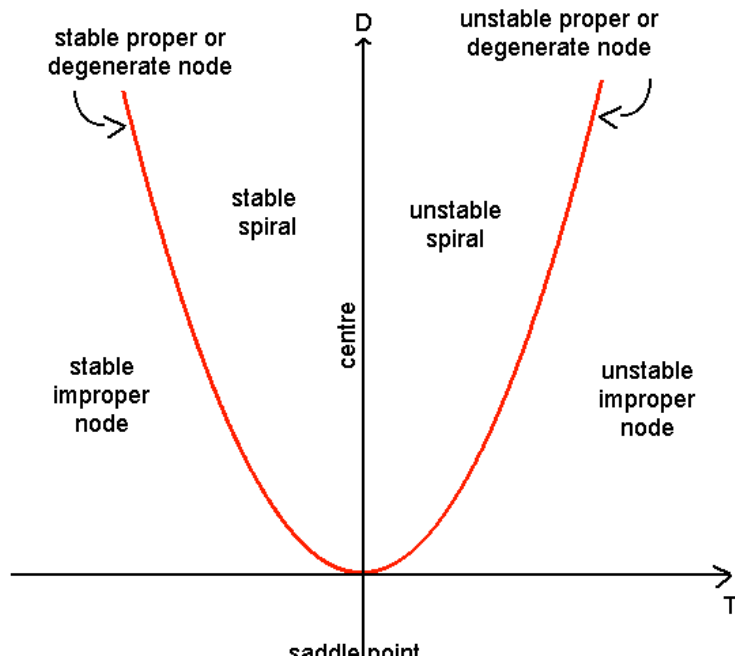# Two dependent variable ODE models

$$\frac{du}{dt} = f(u, v),$$
$$\frac{dv}{dt} = g(u, v).$$

Steady states

Linear stability analysis

Linear stability analysis (ctd.)
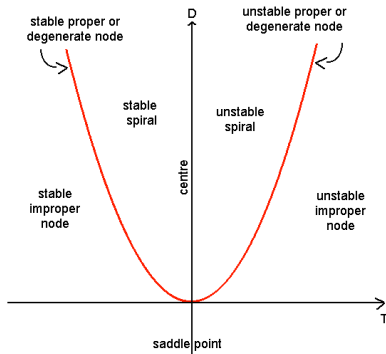
The trace determinant plane

Lecture 14

Recap

$$\frac{du}{dt} = f(u, v),$$
$$\frac{dv}{dt} = g(u, v).$$

$$\lambda = \frac{\operatorname{tr} A \pm \sqrt{\operatorname{tr} A^2 - 4 \det A}}{2}.$$

Nullclines

Periodic solutions (Poincaire-Bendixson theorem)

▶ System of two ODEs
▶ Confined set containing unstable node or spiral
▶ as $t \to \infty$, the trajectory will tend towards a limit cycle.

No periodic solutions - (Dulac criterion)

- D simply connected region in the plane
- $B(x, y)$, continuously differentiable on D, with

$$\frac{\partial}{\partial u}(Bf) + \frac{\partial}{\partial v}(Bg)$$

not identically zero and does not change sign in D.

Lotka Volterra

$$\frac{dN}{dt} = aN - bNP,$$
$$\frac{dP}{dt} = cNP - dP,$$

Nondimensionalisation

$$\frac{dn}{d\tau} = n(1 - p) = f(n, p),$$

$$\frac{dp}{d\tau} = \alpha p(n - 1) = g(n, p),$$

Lecture 15 - recap

Lotka-Volterra model - predator prey interaction

n - prey p - predator

$$\frac{dn}{d\tau} = n(1-p) = f(n,p),$$
$$\frac{dp}{d\tau} = \alpha p(n-1) = g(n,p),$$

Strategy:

▶ numerical solution
▶ steady states
▶ nullclines
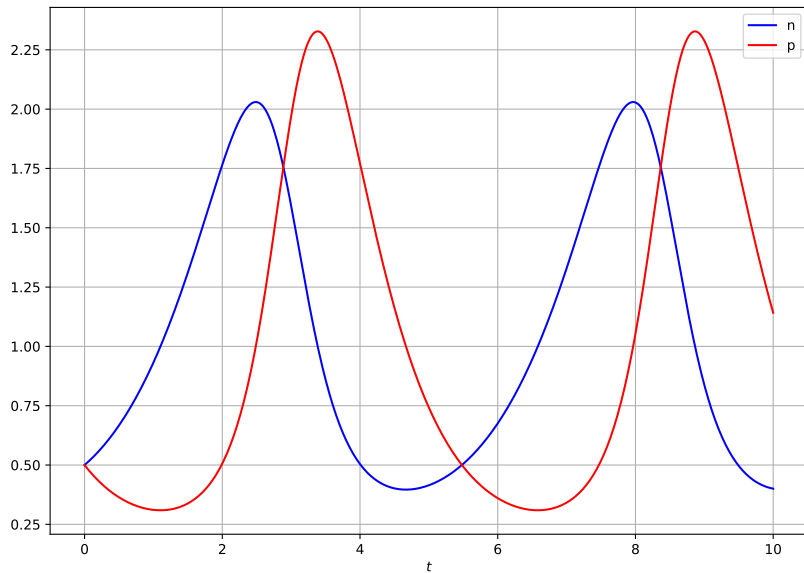▶ linear stability

Numerical solutions

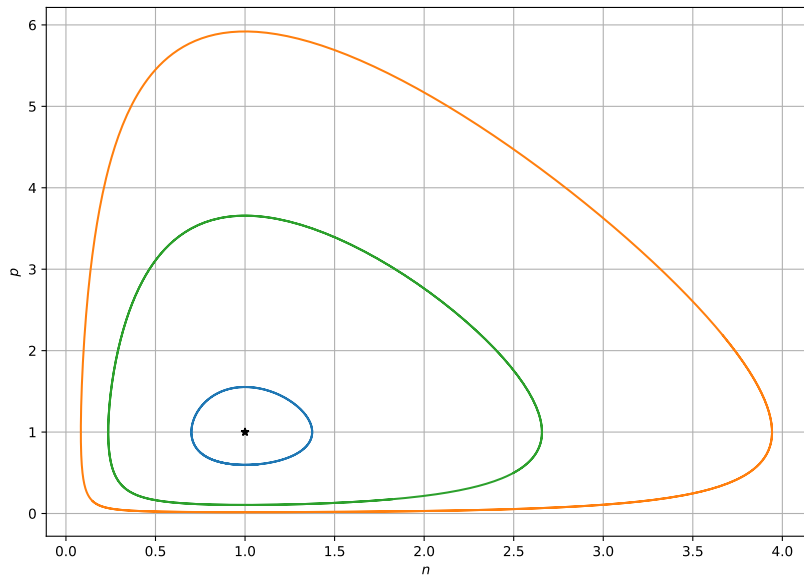
Figure 6

Steady states

Nullclines

The phase plane

Figure 7

Linear stability

Integration

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Lecture 16 Competition

> **i** Recap: Lotka-Volterra model
>
> ▶ predator prey interaction
> ▶ n - prey
> ▶ p - predator
>
> $$\frac{dN}{dt} = aN - bNP,$$
> $$\frac{dP}{dt} = cNP - dP,$$

> **i** Aim - introduce and analyse a model of competition

Competition model

$$\frac{dN_1}{dt} = r_1 N_1 \left(1 - \frac{N_1}{K_1} - b_{12}\frac{N_2}{K_1}\right),$$

$$\frac{dN_2}{dt} = r_2 N_2 \left(1 - \frac{N_2}{K_2} - b_{21}\frac{N_1}{K_2}\right),$$

▶ Justify why this a is a model for competition
▶ Define model parameters
▶ Explain the meaning of each of the terms in the model

Nondimensionalisation

$$\frac{dn_1}{d\tau} = n_1 \left(1 - n_1 - a_{12}n_2\right) = f(n_1, n_2),$$

$$\frac{dn_2}{d\tau} = \rho n_2 \left(1 - n_2 - a_{21}n_1\right) = g(n_1, n_2),$$

▶ Define ρ, a_{12} and a_{21}

Nondimensionalisation (ctd.)

Steady states

Steady states (ctd)

Nullclines

Lecture 17 Competition

$$\frac{dn_1}{d\tau} = n_1 \left(1 - n_1 - a_{12}n_2\right) = f(n_1, n_2),$$

$$\frac{dn_2}{d\tau} = \rho n_2 \left(1 - n_2 - a_{21}n_1\right) = g(n_1, n_2),$$

Steady states:

$$(0,0), \ (1,0), \ (0,1)$$

and

$$\left(\frac{1 - a_{12}}{1 - a_{12}a_{21}}, \frac{1 - a_{21}}{1 - a_{12}a_{21}}\right).$$

Linear stability analysis

$$A_{(n_1^*,n_2^*)} = \begin{pmatrix} 1 - 2n_1 - a_{12}n_2 & -a_{12}n_1 \\ -\rho a_{21}n_2 & \rho(1 - 2n_2 - a_{21}n_1) \end{pmatrix}_{(n_1^*,n_2^*)}.$$

Linear stability analysis (ctd)

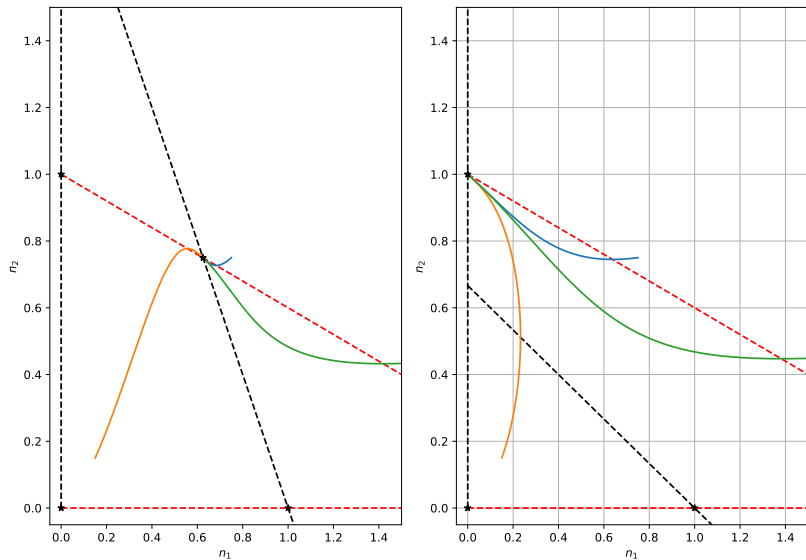# Phase portrait (one for each qualitatively distinct case)

Figure 8

Insight

Mutualism/symbiosis

$$\frac{dn_1}{d\tau} = n_1(1 - n_1 + a_{12}n_2) = f(n_1, n_2),$$
$$\frac{dn_2}{d\tau} = \rho n_2(1 - n_2 + a_{21}n_1) = g(n_1, n_2).$$

Steady states

This model has steady-states $(0,0)$, $(1,0)$, $(0,1)$ and

$$(n_1{}^*, n_2{}^*) = \left(\frac{1 + a_{12}}{1 - a_{12}a_{21}}, \frac{1 + a_{21}}{1 - a_{12}a_{21}} \right).$$

Coexistence steady state

Summary

▶ Predator-prey, competition, mutualism
▶ techniques to analyse systems of nonlinear ODEs

Biochemical kinetics

LOMA

$$\lambda_1 C_1 + \lambda_2 C_2 + ... + \lambda_m C_M \underset{k_b}{\overset{k_f}{\rightleftharpoons}} \gamma_1 C_1 + \gamma_2 C_2 + ...\gamma_M C_M.$$

Write down ODEs describing this chemical reaction

$$A + B \xrightarrow{k} C.$$