Lecture slides

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Lecture 1

- Introduction to nonlinear difference equations
- The Malthusian model
- The Ricker model

Why difference equations?

A general model

Consider the first order difference equation

$$N_{t+1} = N_t f(N_t) = H(N_t), \tag{1}$$

where $f(N_t)$ is a function that defines the per capita growth rate. The function $H(N_t)$ describes the total (net) growth rate.

The Malthusian model

The population size at time $t + 1$ is

$$N_{t+1} = N_t + bN_t - dN_t = rN_t,$$

Exercise: solve the Malthusian model and classify qualitative behaviours

Nonlinear models

- Beverton-Holt
$$N_{t+1} = \frac{rN_t}{1 + \frac{N_t}{K}},$$
- Hassell model
$$N_{t+1} = \frac{rN_t}{(1 + \frac{N_t}{K})^b},$$
- Ricker model
$$N_{t+1} = N_t e^{r(1-\frac{N_t}{K})}.$$

Numerical simulation of the Ricker model

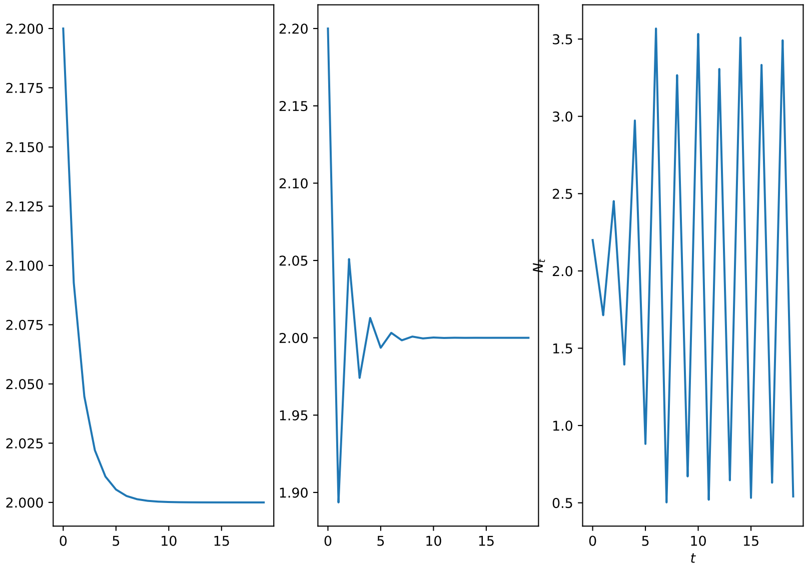

Figure 1: A plot of numerical solutions of the Ricker model. (a)r=0.5.

Summary

- Motivated use of difference equation models
- Introduced general model for one population
- Solved the Malthusian model
- Introduced nonlinear models

Lecture 2 - General techniques for solving nonlinear difference equations

$$N_{t+1} = N_t f(N_t) = H(N_t), \tag{2}$$

- Computational solutions
- Fixed points
- Linear stability of fixed points
- Cobweb diagrams
- Bifurcation diagrams
- Identify how model solutions depend on model parameters

Fixed points

Suppose the solution at the next iteration is equal to that at a given iteration, i,e. there exists some N^* such that

$$N^* = N_{t+1} = N_t$$

Fixed point definition

$$N^* = H(N^*), N^* \geq 0$$

Biological relevance: non-negative solutions

Linear stability analysis - how do small perturbations about N^* behave?

Linear stability analysis (ctd)

Linear stability is determined by the derivative of H evaluated at N^*

$$|H'(N^*)| < 1 \implies \text{linear stability of } N^*.$$

Exercise

Identify the fixed points of the Malthusian model

$$N_{t+1} = rN_t$$

and identify their linear stability.

Cobweb diagrams

Definition

A cobweb diagram is a technique for computing graphical solutions of a difference equation.

Use previous analyses to identify different qualitative cases (one cobweb digram for each fixed point).

For each case:

- Sketch a graph of H to evaulate iterative solutions
- Compute an iterative solution

Example

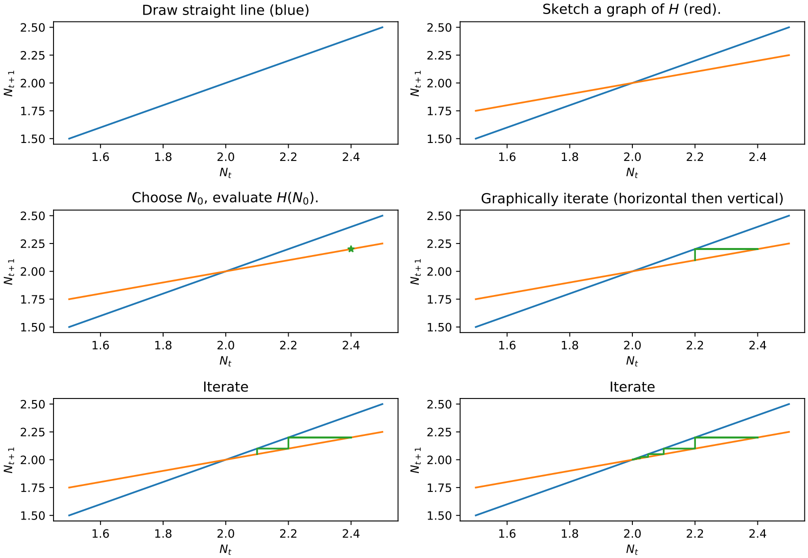

Figure 2: Generating a cobweb plot.

Bifurcation diagrams - Plot fixed points against a parameter and annotate their stability

Exercise

Draw cobweb diagrams for the Malthusian model.

$$N_{t+1} = rN_t$$

and identify their linear stability.

Lecture 3 - Preparation for tutorial 3

- Curve sketching nonlinear functions in qualitatively distinct cases

Example:

Sketch a graph of

$$f(x) = xe^{-r(1-\frac{x}{K})}, \quad r, K \in \mathfrak{R}^{+}, \quad x \in \mathfrak{R}, x \geq 0$$

Approach

Identify properties of H to distinguish qualitatively distinct cases

Roots

Turning points

Limit as $x \to \infty$

Limiting behaviour as $x \to 0$

Tutorial sheet 1

Lecture 4

Consider the model

$$N_{t+1} = \frac{\gamma N_t}{1 + N_t^2}, \quad \gamma \in \Re^+.$$

Fixed points

Linear stability

Cobweb diagrams

Bifurcations

Symbolic computations

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The FPs are:
[{N: 0}, {N: -sqrt(gamma - 1)}, {N: sqrt(gamma - 1)}]
The derivative of H is:
gamma*(1 - N**2)/(N**2 + 1)**2
The derivative evaluated at FP 1 is:
gamma
The derivative evaluated at FP 2 is:
(2 - gamma)/gamma
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Lecture 4 .. A model with harvesting

$$N_{t+1} = \frac{\gamma N_t}{1 + N_t^2} - hN_t, \quad \gamma > 0, \quad h \geq 0 \tag{3}$$

Numerical simulation

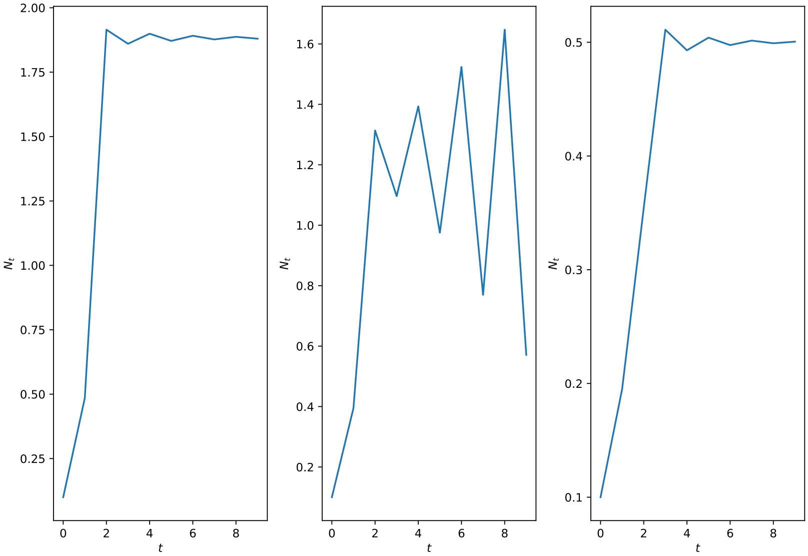


Figure 3: Time series solution for different values of h.

Computing FPs

Linear stability

Deriving expressions for linear stability boundaries in the $h\gamma$ plane

Sketch of stability boundaries

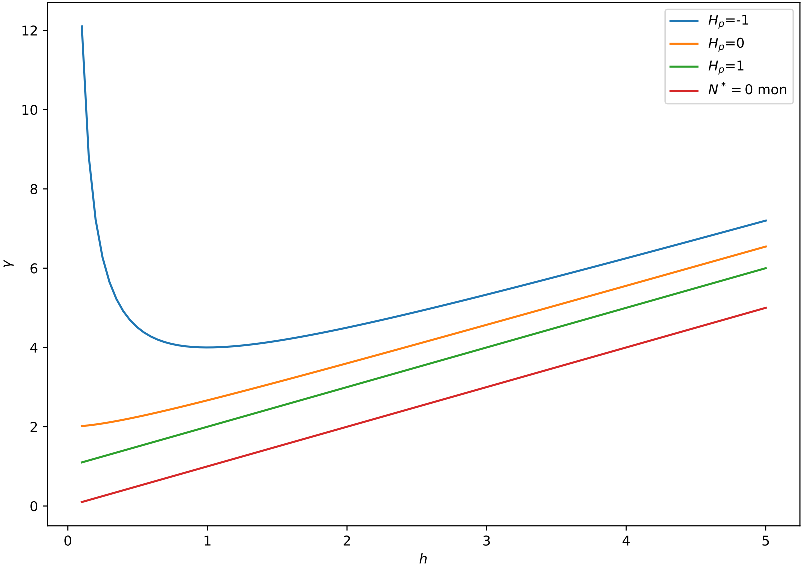
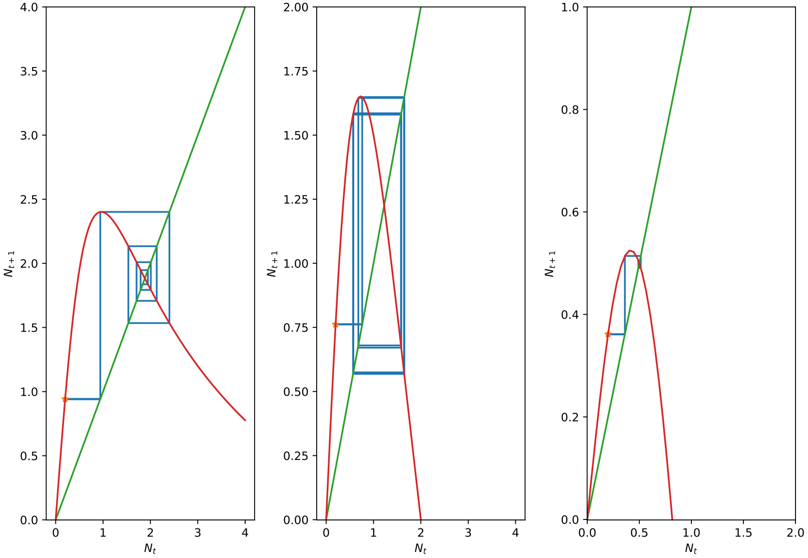


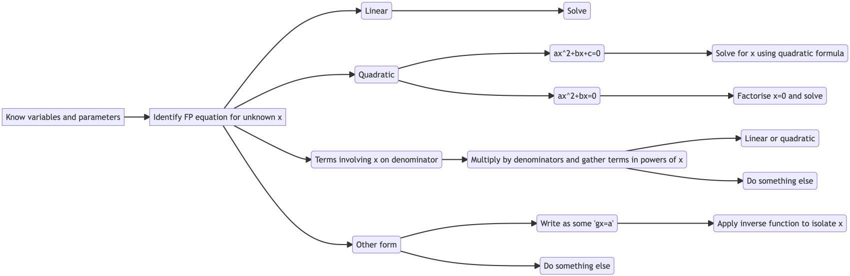
Figure 4: Stability regions for the harvesting model.

Cobweb diagrams



Lecture 5

Quiz review



Know variables and parameters
Identify FP equation for unknown x
Linear
Solve
Quadratic
ax^2+bx+c=0
Solve for x using quadratic formula
ax^2+bx=0
Factorise x=0 and solve
Terms involving x on denominator
Multiply by denominators and gather terms in powers of x
Linear or quadratic
Do something else
Other form
Write as some 'gx=a'
Apply inverse function to isolate x
Do something else

Oscillatory solutions

$$N_{t+1} = H(N_t) \tag{4}$$

A solution to Equation 4 is defined to be periodic with period T if

$$N_{t+T} = N_t \quad \forall t, N_{t+\tau} \quad \neq N_t \quad \forall t, \quad \tau < T.$$

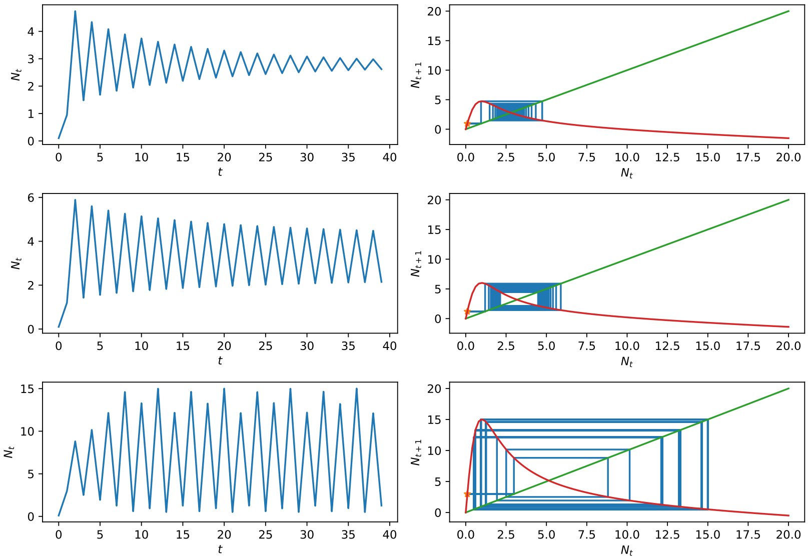


Figure 5: Transition form period 2 to period 4 solutions in the harvesting model

Exercise: identify an equation satisfied by period 2 solutions of the logistic map

$$N_{t+1} = rN_1(1 - N_t).$$

Systems of coupled difference equations

General form of model

We consider governing equations of the form

$$\begin{aligned} N_{t+1} &= f(N_t, P_t), \\ P_{t+1} &= g(N_t, P_t). \end{aligned}$$

Fixed points

The fixed points (N^*, P^*) satisfy

$$N^* = g(N^*, P^*)$$
$$P^* = f(N^*, P^*).$$

Example

Compute the fixed points of the model

$$\begin{aligned} N_{t+1} &= P_t \\ P_{t+1} &= N_t + P_t. \end{aligned}$$

Linear stability analysis

Let

$$N_t = N^* + \hat{N}_t,$$
$$P_t = P^* + \hat{P}_t.$$

Eigenvalues of the Jacobian matrix determine stability

Recap

We are studying equations of the form

$$N_{t+1} = f(N_t, P_t),$$
$$P_{t+1} = g(N_t, P_t).$$

The solution of the linearised system takes the form

$$\mathbf{w}_t = \sum_{i=1}^{2} C_i \lambda_i^t \mathbf{c}_i,$$

where λ_i $i = 1, 2$ are eigenvalues of the Jacobian matrix and $\mathbf{c}_i$ corresponding eigenvector.

Note

For linear stability - real part of both eigenvalues is less than 1 in magnitude

Example

Compute the Jacobian matrix of the model

$$\begin{aligned} N_{t+1} &= P_t \\ P_{t+1} &= N_t + 2P_t. \end{aligned}$$

and hence determine linear stability of the FP (0,0).

Trace determinant form

Suppose

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

The characteristic equation is

$$|A - \lambda \mathbf{1}| = 0$$

Hence

$$\lambda^2 - \lambda(a_{11} + a_{22}) + (a_{11}a_{22} - a_{12}a_{21}) = 0.$$

This can be written as

$$\lambda^2 - \lambda(\mathrm{tr}A) + \det A = 0.$$

The eigenvalues are

$$\lambda = \frac{-\mathrm{tr}A \pm \sqrt{\mathrm{tr}A^2 - 4\det A}}{2}$$

When the eigenvalues cannot be easily explicitly computed

Jury conditions

Consider the characteristic equation

$$P(\lambda) = \lambda^2 + a\lambda + b = 0,$$

where $a, b \in \mathfrak{R}$.
The Jury conditions state that $|\lambda_i| < 1 \;\; \forall \;\; i$ if, and only if,

- $b < 1$,
- $1 + a + b > 0$,
- $1 - a + b > 0$.

- Testing of the Jury conditions is sufficient to determine linear stability.
- This is often easier than explicitly working out the eigenvalues.

Jury conditions in trace-determinant form

Jury conditions

Consider the characteristic equation

$$P(\lambda) = \lambda^2 - \mathrm{tr}A\lambda + \det A = 0,$$

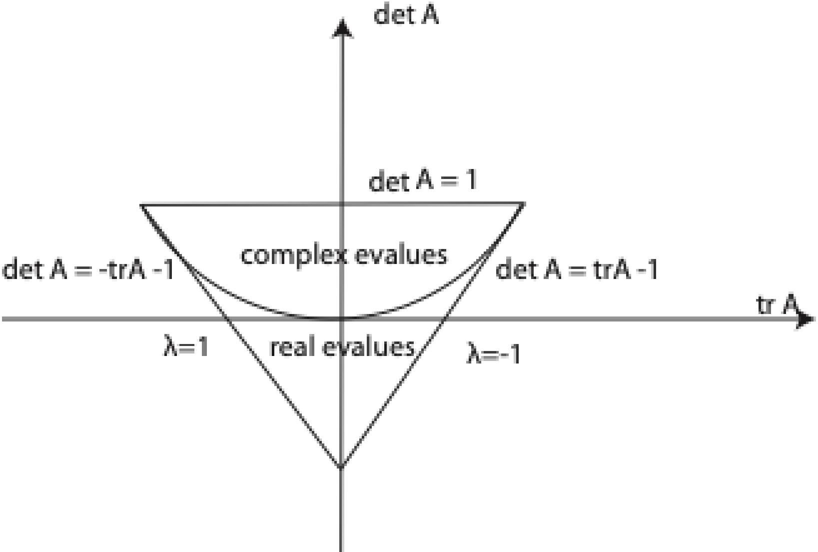
where $a, b \in \Re$.
The Jury conditions state that $|\lambda_i| < 1 \;\; \forall \;\; i$ if, and only if,

- $\det A < 1$,
- $1 - \mathrm{tr}A + \det A > 0$,
- $1 + \mathrm{tr}A + \det A > 0$.

- Compute the trace and determinant of the Jacobian
- Testing the Jury conditions to determine linear stability
- This is often easier than explicitly working out the eigenvalues.

Interpretation in the trace-determinant plane

$$P(\lambda) = \lambda^2 - \mathrm{tr}A\lambda + \det A = 0,$$



Proof of the Jury conditions

Consider the characteristic equation

$$P(\lambda) = \lambda^2 + a\lambda + b = 0,$$

The roots of $P(\lambda)$ are

$$\lambda_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}.$$

Complex roots - $a^2 - 4b < 0$

JC 1

$$b = \lambda_1 \lambda_2 = |\lambda_1|^2 = |\lambda_2|^2$$

JC 2 and 3

$$(a^2 - 4b) = (|a|^2 - 2)^2 + ...$$

Real roots - $a^2 - 4b \geq 0$

Define the largest of the roots to be

$$R = max\{|\lambda_1|, |\lambda_2|\} = \frac{|a| + \sqrt{a^2 - 4b}}{2}.$$

Consider $R = 1$.

Lecture 7 - The Nicholson-Bailey model

$$
\begin{aligned}
N_{t+1} &= R_0 N_t f(N_t, P_t), \\
P_{t+1} &= C N_t (1 - f(N_t, P_t)).
\end{aligned}
$$

Numerical solution

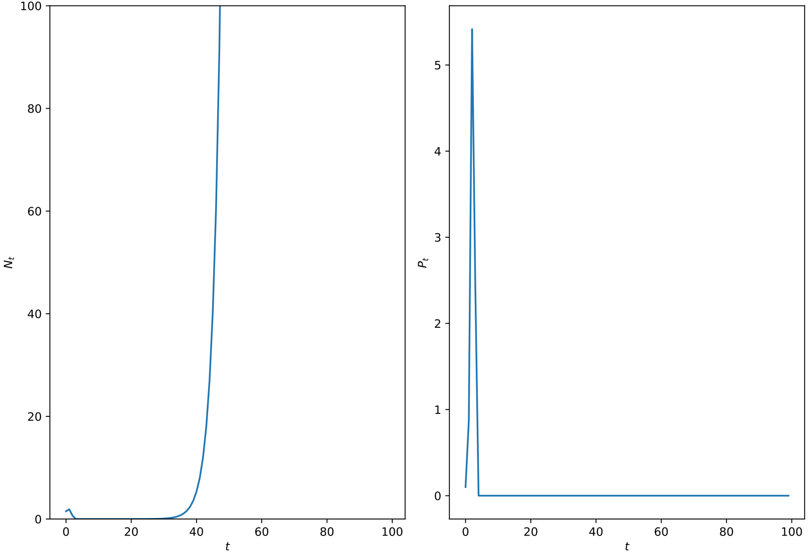


Figure 6: A plot of the Nicholson Bailey model solution.

Fixed points

The fixed points satisfy

$$\begin{aligned} N^* &= R_0 N^* e^{-aP^*}, \\ P^* &= C N^* (1 - e^{-aP^*}). \end{aligned}$$

FPs ctd

Linear stability

The Jacobian matrix is given by

$$A_{(N_t,P_t)} = \left(\begin{array}{cc} R_0 e^{-aP_t} & -R_0 a N_t e^{-aP_t} \\ c(1-e^{-aP_t}) & aCN_t e^{-aP_t} \end{array} \right).$$

At (0,0) ... 5cm

Linear stability of the nontrivial FP

The Jacobian matrix is given by

$$A_{(N_t,P_t)} = \left(\begin{array}{cc} R_0 e^{-aP_t} & -R_0 a N_t e^{-aP_t} \\ c(1-e^{-aP_t}) & aCN_t e^{-aP_t} \end{array} \right).$$

The FP is

$$\left(\frac{R_0 \ln R_0}{aC(R_0-1)}, \frac{1}{a} \ln R_0 \right).$$

5cm

Applying the Jury conditions

Symbolic computation in Python

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The FPs are:
{N: 0, P: 0}
{N: R_0*log(R_0)/(C*a*(R_0 - 1)), P: log(R_0)/a}
The Jacobian of H is:
[[R_0*exp(-P*a), -N*R_0*a*exp(-P*a)], [C - C*exp(-P*a), C*N
The Jacobian evaluated at FP 0 is:
[[R_0, 0], [0, 0]]
The Jacobian evaluated at FP 1 is:
[[1, -R_0*log(R_0)/(C*(R_0 - 1))], [C*(1 - 1/R_0), log(R_0)
```