Lecture slides

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Lecture 1

- Introduction to nonlinear difference equations
- The Malthusian model
- The Ricker model

Why difference equations?

A general model

Consider the first order difference equation

$$N_{t+1} = N_t f(N_t) = H(N_t), \tag{1}$$

where $f(N_t)$ is a function that defines the per capita growth rate. The function $H(N_t)$ describes the total (net) growth rate.

The Malthusian model

The population size at time $t + 1$ is

$$N_{t+1} = N_t + bN_t - dN_t = rN_t,$$

Exercise: solve the Malthusian model and classify qualitative behaviours

Nonlinear models

- Beverton-Holt
$$N_{t+1} = \frac{rN_t}{1 + \frac{N_t}{K}},$$
- Hassell model
$$N_{t+1} = \frac{rN_t}{(1 + \frac{N_t}{K})^b},$$
- Ricker model
$$N_{t+1} = N_t e^{r(1-\frac{N_t}{K})}.$$

Numerical simulation of the Ricker model

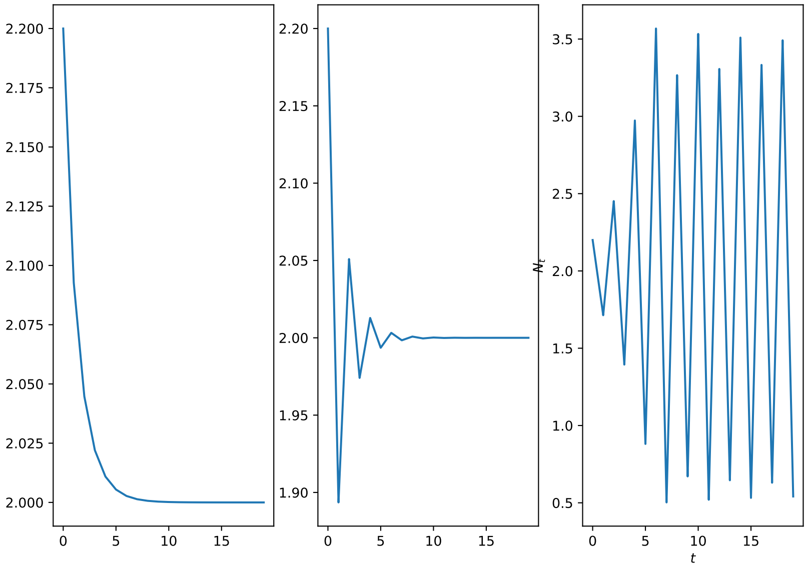

Figure 1: A plot of numerical solutions of the Ricker model. (a)r=0.5.

Summary

- Motivated use of difference equation models
- Introduced general model for one population
- Solved the Malthusian model
- Introduced nonlinear models

Lecture 2 - General techniques for solving nonlinear difference equations

$$N_{t+1} = N_t f(N_t) = H(N_t), \qquad (2)$$

- Computational solutions
- Fixed points
- Linear stability of fixed points
- Cobweb diagrams
- Bifurcation diagrams
- Identify how model solutions depend on model parameters

Fixed points

Suppose the solution at the next iteration is equal to that at a given iteration, i,e. there exists some N^* such that

$$N^* = N_{t+1} = N_t$$

Fixed point definition

$$N^* = H(N^*), N^* \geq 0$$

Biological relevance: non-negative solutions

Linear stability analysis - how do small perturbations about N^* behave?

Linear stability analysis (ctd)

Linear stability is determined by the derivative of H evaluated at N^*

$$|H'(N^*)| < 1 \implies \text{linear stability of } N^*.$$

Exercise

Identify the fixed points of the Malthusian model

$$N_{t+1} = rN_t$$

and identify their linear stability.

Cobweb diagrams

Definition

A cobweb diagram is a technique for computing graphical solutions of a difference equation.

Use previous analyses to identify different qualitative cases (one cobweb digram for each fixed point).

For each case:

- Sketch a graph of H to evaulate iterative solutions
- Compute an iterative solution

Example

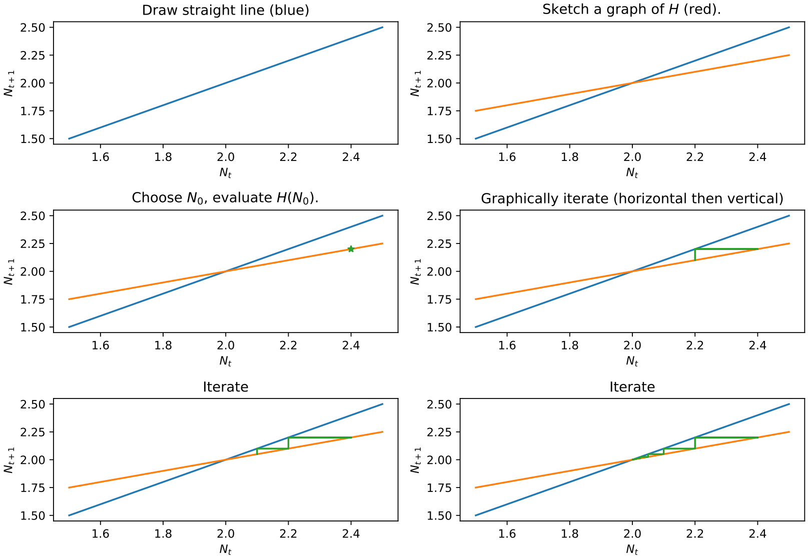

Figure 2: Generating a cobweb plot.

Bifurcation diagrams - Plot fixed points against a parameter and annotate their stability

Exercise

Draw cobweb diagrams for the Malthusian model.

$$N_{t+1} = rN_t$$

and identify their linear stability.

Lecture 3 - Preparation for tutorial 3

- Curve sketching nonlinear functions in qualitatively distinct cases

Example:

Sketch a graph of

$$f(x) = xe^{-r(1-\frac{x}{K})}, \quad r, K \in \Re^+, \quad x \in \Re, x \geq 0$$

Approach

Identify properties of H to distinguish qualitatively distinct cases

Roots

Turning points

Limit as $x \to \infty$

Limiting behaviour as $x \to 0$

Tutorial sheet 1

Lecture 4

Consider the model

$$N_{t+1} = \frac{\gamma N_t}{1 + N_t^2}, \quad \gamma \in \mathfrak{R}^+.$$

Fixed points

Linear stability

Cobweb diagrams

Bifurcations

Symbolic computations

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The FPs are:
[{N: 0}, {N: -sqrt(gamma - 1)}, {N: sqrt(gamma - 1)}]
The derivative of H is:
gamma*(1 - N**2)/(N**2 + 1)**2
The derivative evaluated at FP 1 is:
gamma
The derivative evaluated at FP 2 is:
(2 - gamma)/gamma
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Lecture 4 .. A model with harvesting

$$N_{t+1} = \frac{\gamma N_t}{1 + N_t^2} - hN_t, \quad \gamma > 0, \quad h \geq 0 \tag{3}$$

Numerical simulation

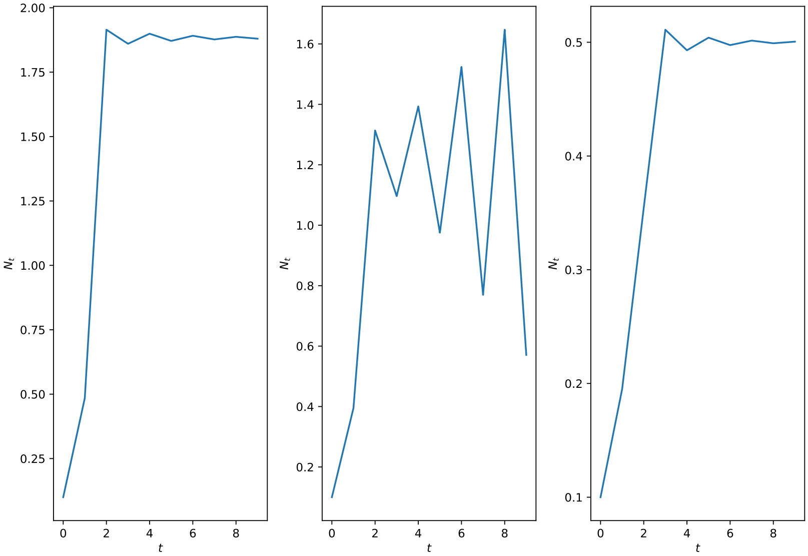


Figure 3: Time series solution for different values of h.

Computing FPs

Linear stability

Deriving expressions for linear stability boundaries in the $h\gamma$ plane

Sketch of stability boundaries

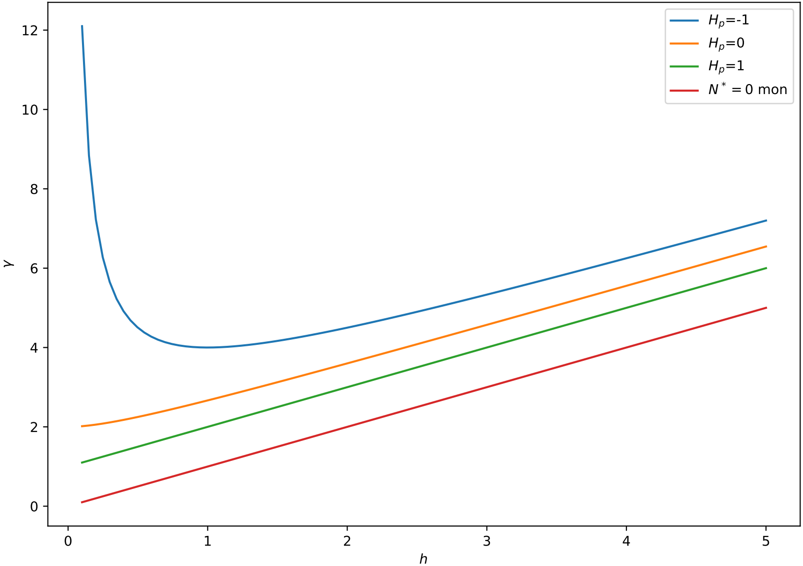
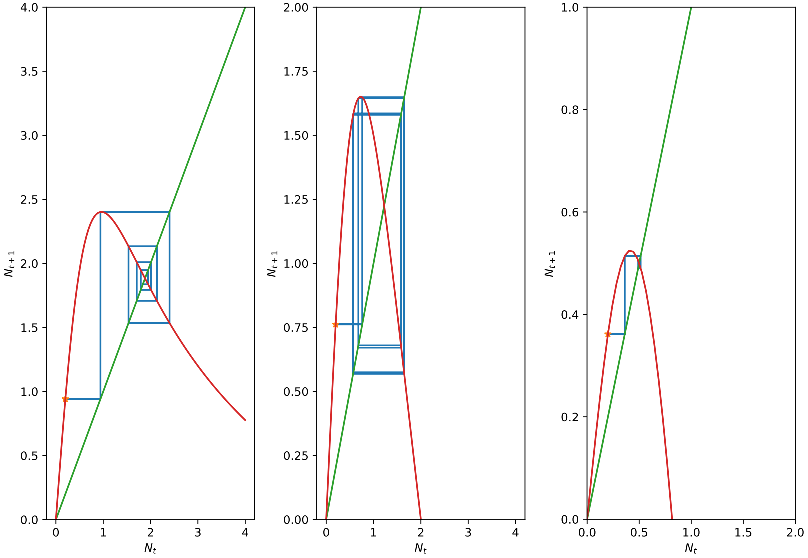


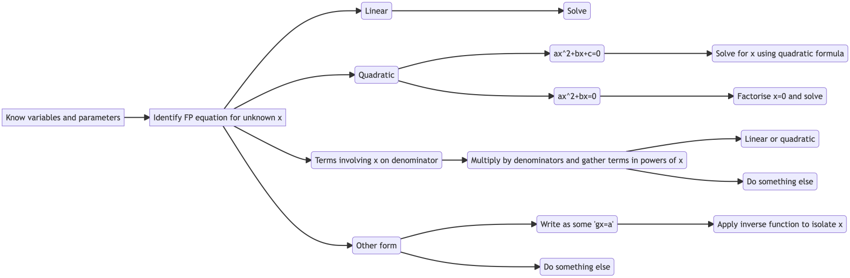
Figure 4: Stability regions for the harvesting model.

Cobweb diagrams



Lecture 5

Quiz review


Know variables and parameters
Identify FP equation for unknown x
Linear
Solve
Quadratic
ax^2+bx+c=0
Solve for x using quadratic formula
ax^2+bx=0
Factorise x=0 and solve
Terms involving x on denominator
Multiply by denominators and gather terms in powers of x
Linear or quadratic
Do something else
Other form
Write as some 'gx=a'
Apply inverse function to isolate x
Do something else

Oscillatory solutions

$$N_{t+1} = H(N_t) \tag{4}$$

A solution to Equation 4 is defined to be periodic with period T if

$$N_{t+T} = N_t \quad \forall t, N_{t+\tau} \quad \neq N_t \quad \forall t, \quad \tau < T.$$

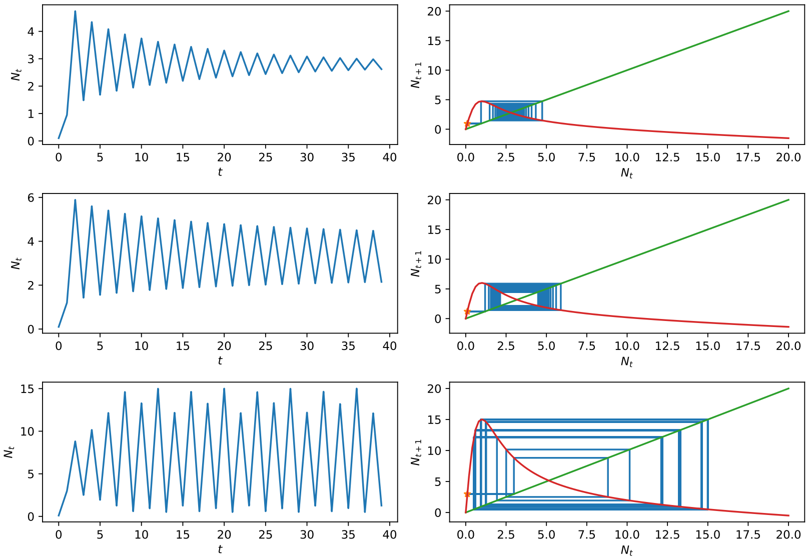


Figure 5: Transition form period 2 to period 4 solutions in the harvesting model

Exercise: identify an equation satisfied by period 2 solutions of the logistic map

$$N_{t+1} = rN_1(1 - N_t).$$

Systems of coupled difference equations

General form of model

We consider governing equations of the form

$$N_{t+1} = f(N_t, P_t),$$
$$P_{t+1} = g(N_t, P_t).$$

Fixed points

The fixed points (N^*, P^*) satisfy

$$\begin{aligned} N^* &= g(N^*, P^*) \\ P^* &= f(N^*, P^*). \end{aligned}$$

Example

Compute the fixed points of the model

$$
\begin{aligned}
N_{t+1} &= P_t \\
P_{t+1} &= N_t + P_t.
\end{aligned}
$$

Linear stability analysis

Let

$$N_t = N^* + \hat{N}_t,$$
$$P_t = P^* + \hat{P}_t.$$

Eigenvalues of the Jacobian matrix determine stability

For Linear stability - real part of both eigenvalues is less than 1 in magnitude

Example

Compute the Jacobian matrix of the model

$$
\begin{aligned}
N_{t+1} &= P_t \\
P_{t+1} &= N_t + P_t.
\end{aligned}
$$

and hence determine linear stability of the FP.

When the eigenvalues cannot be easily explicitly computed

Jury conditions

Consider the characteristic equation

$$P(\lambda) = \lambda^2 + a\lambda + b = 0,$$

where $a, b \in \mathfrak{R}$.
The Jury conditions state that $|\lambda_i| < 1 \forall \quad i$ if, and only if,

- $b < 1$,
- $1 + a + b > 0$,
- *$1 - a + b > 0$.