

MA32009 Lecture slides

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Lecture 1

Malthusian model - derivation and qualitative analysis

A general model

$$\frac{dN}{dt} = f(N)N = H(N),$$

Numerical solution

Dimensions and nondimensionalisation

As an example, consider the linear ODE

$$\frac{dN}{dt} = rN.$$

Introducing dimensionless variables

Steady-state analysis

Lecture 10

- ▶ Recap
- ▶ Techniques for single first order ODE (ctd)
- ▶ Example model 1: Logistic growth
- ▶ Example model 2: Spruce budworm

Linear stability analysis

Linear stability analysis (ctd)

Graphical solution

Bifurcation diagrams

Example model 1: the logistic growth equation

$$\frac{dN}{dt} = rN(t) \left(1 - \frac{N(t)}{K} \right).$$

Numerical solution

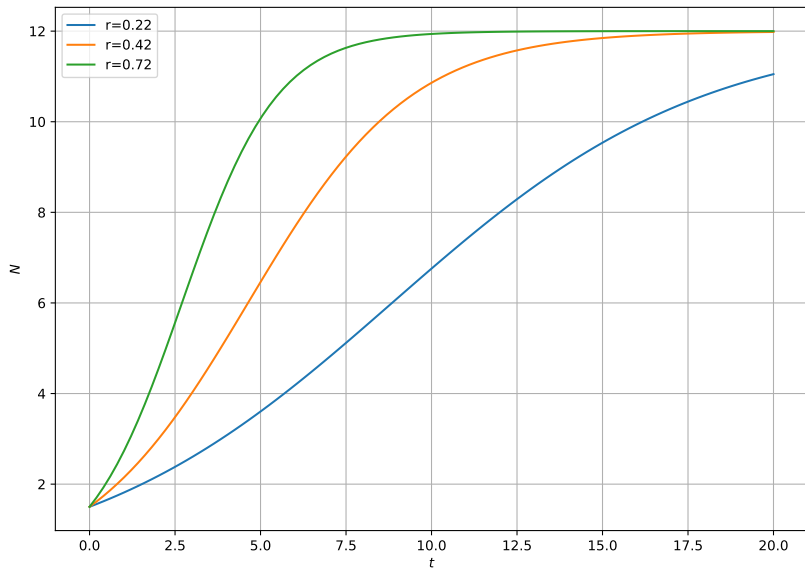


Figure 1: Numerical solution of the logistic growth model

Steady states and linear stability

Graphical analysis

An exact solution of the logistic growth equation

Example model 2: the spruce budworm model

$$\frac{dN}{dt} = r_B N \left(1 - \frac{N}{K_B} \right) - \frac{BN^2}{A^2 + N^2}, \quad (1)$$

Nondimensionalisation

$$\frac{dn}{d\tau} = rn \left(1 - \frac{n}{q} \right) - \frac{n^2}{1 + n^2} = H(n),$$

Plotting the RHS

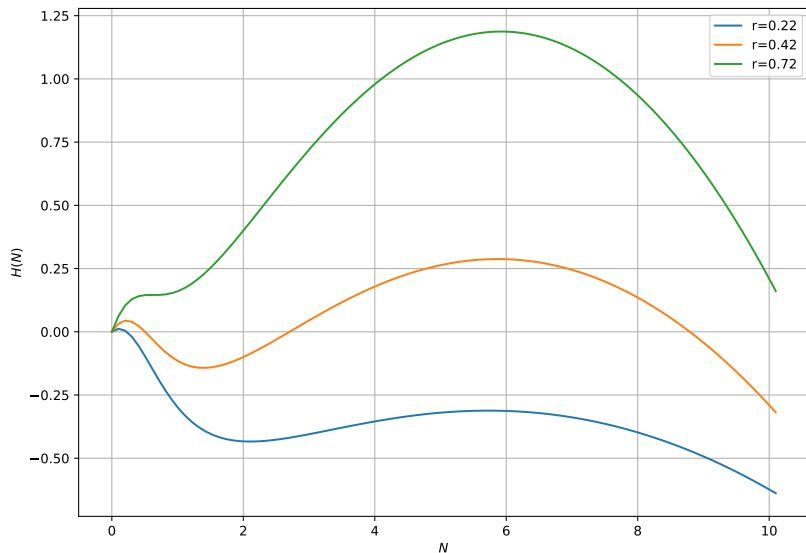
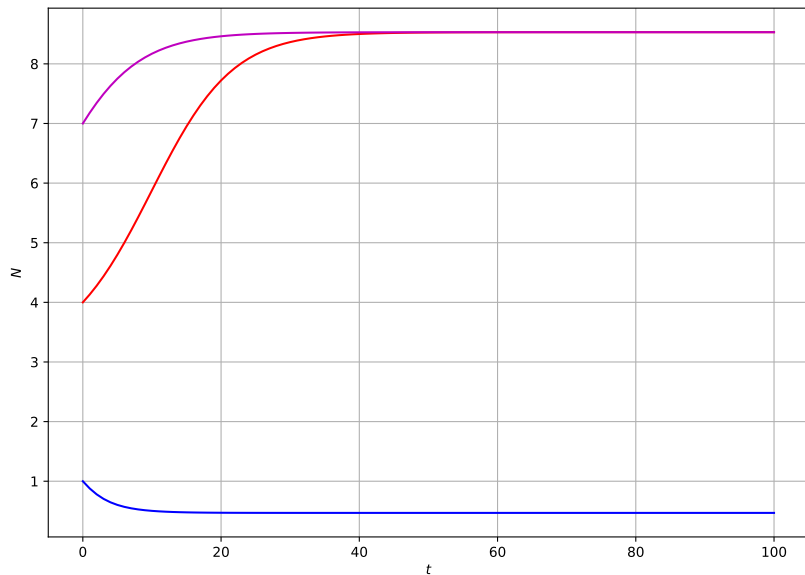


Figure 2: RHS of spr. budworm model

Numerical solution



Steady state analysis

$$rn^*(1 - \frac{n^*}{q}) - \frac{n^{*2}}{1 + n^{*2}} = 0.$$

Linear stability analysis

Lecture 12

i Recap - Spruce budworm model

$$\frac{dn}{d\tau} = rn \left(1 - \frac{n}{q}\right) - \frac{n^2}{1 + n^2} = H(n),$$

Steady states: $n^* = 0$ or

$$rn^* \left(1 - \frac{n^*}{q}\right) - \frac{n^{*2}}{1 + n^{*2}} = 0.$$

- ▶ r small - one stable steady state
- ▶ r large - one stable steady state (outbreak)
- ▶ r intermediate - bistability (two stable steady states and one unstable)

Today: bifurcation analysis, hysteresis, harvesting

Tangent bifurcations in rq space

Plotting stability regions in the rq plane

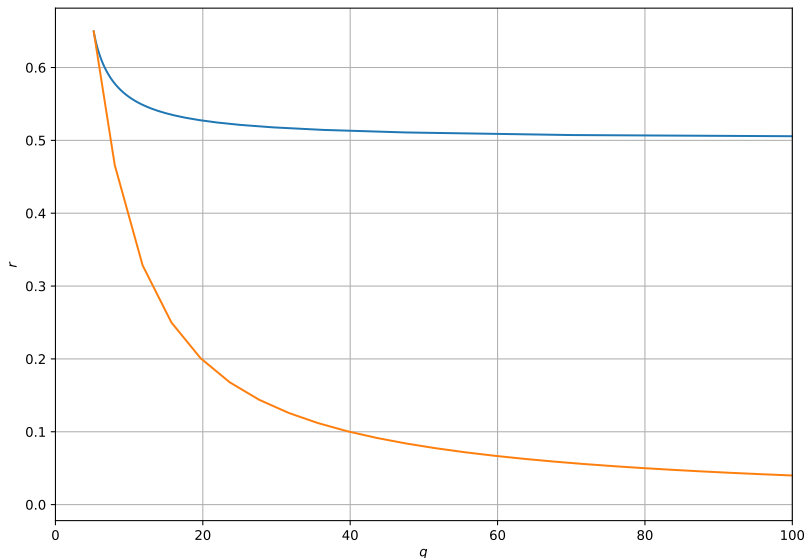


Figure 3: Bifurcations in the rq plane

Hysteresis - irreversible transitions in solution behaviour

Hysteresis

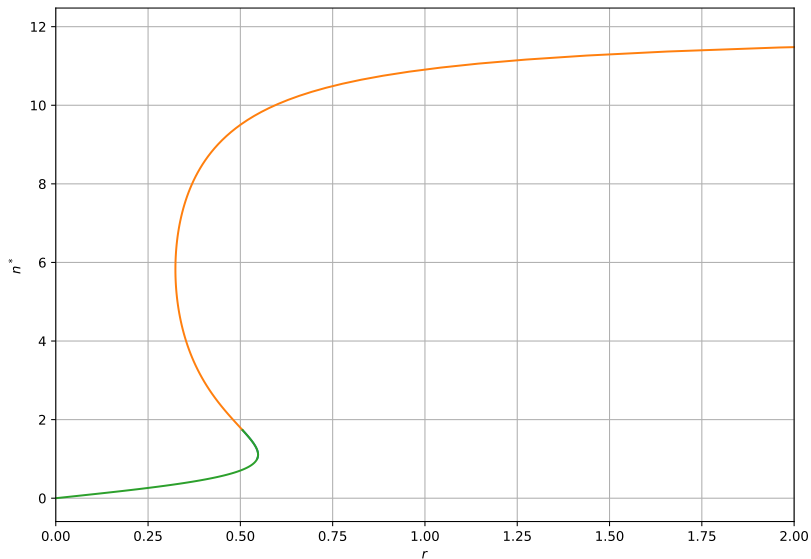


Figure 4: Bifurcations in the r q plane

Harvesting

- ▶ use models to simulate how much resource can be extracted?
- ▶ approach: take model without harvesting and add in harvesting terms

$$\frac{dN}{dt} = rN \left(1 - \frac{N}{K} \right) - EN.$$

where E is the harvesting rate.

Question: what value of E maximises the long term yield?

Delay differential equation models

$$\frac{dN}{dt} = H(N(t), N(t - T)),$$

A linear delay differential equation model

$$\frac{dN}{dt} = -N(t - T),$$

Linear stability analysis (ctd.)

$$\frac{dN}{dt} = -N(t - T),$$

Two dependent variable ODE models

$$\frac{du}{dt} = f(u, v),$$

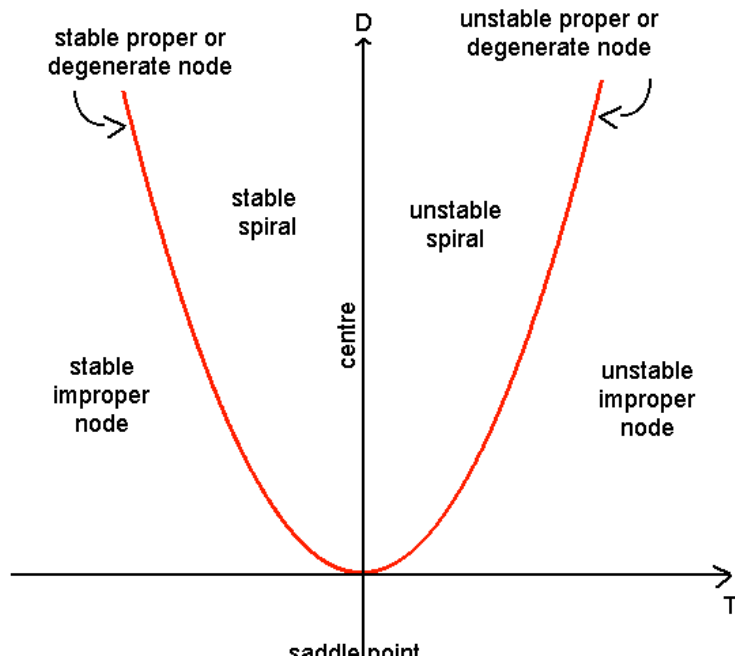
$$\frac{dv}{dt} = g(u, v).$$

Steady states

Linear stability analysis

Linear stability analysis (ctd.)

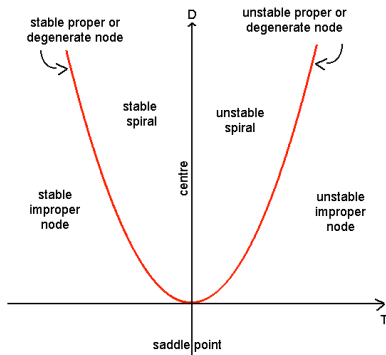
The trace determinant plane



Lecture 14

Recap

$$\begin{aligned}\frac{du}{dt} &= f(u, v), \\ \frac{dv}{dt} &= g(u, v).\end{aligned}$$



$$\lambda = \frac{\text{tr} A \pm \sqrt{\text{tr} A^2 - 4 \det A}}{2}.$$

Nullclines

Periodic solutions (Poincaré-Bendixson theorem)

- ▶ System of two ODEs
- ▶ Confined set containing unstable node or spiral
- ▶ as $t \rightarrow \infty$, the trajectory will tend towards a limit cycle.

No periodic solutions - (Dulac criterion)

- ▶ D simply connected region in the plane
- ▶ $B(x, y)$, continuously differentiable on D , with

$$\frac{\partial}{\partial u}(Bf) + \frac{\partial}{\partial v}(Bg)$$

not identically zero and does not change sign in D .

Lotka Volterra

$$\frac{dN}{dt} = aN - bNP,$$

$$\frac{dP}{dt} = cNP - dP,$$

Nondimensionalisation

$$\frac{dn}{d\tau} = n(1 - p) = f(n, p),$$

$$\frac{dp}{d\tau} = \alpha p(n - 1) = g(n, p),$$

Lecture 15 - recap

Lotka-Volterra model - predator prey interaction

n - prey p - predator

$$\begin{aligned}\frac{dn}{d\tau} &= n(1 - p) = f(n, p), \\ \frac{dp}{d\tau} &= \alpha p(n - 1) = g(n, p),\end{aligned}$$

Strategy:

- ▶ numerical solution
- ▶ steady states
- ▶ nullclines
- ▶ linear stability

Numerical solutions

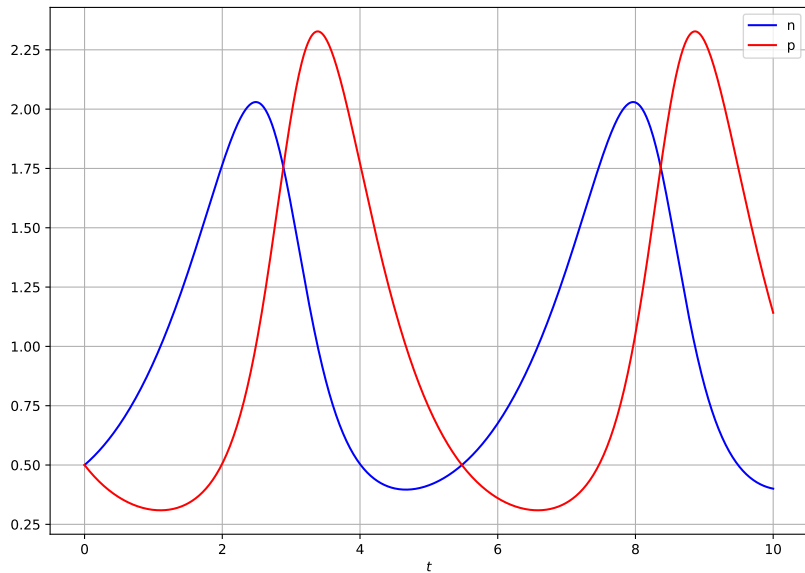


Figure 6

Steady states

Nullclines

The phase plane

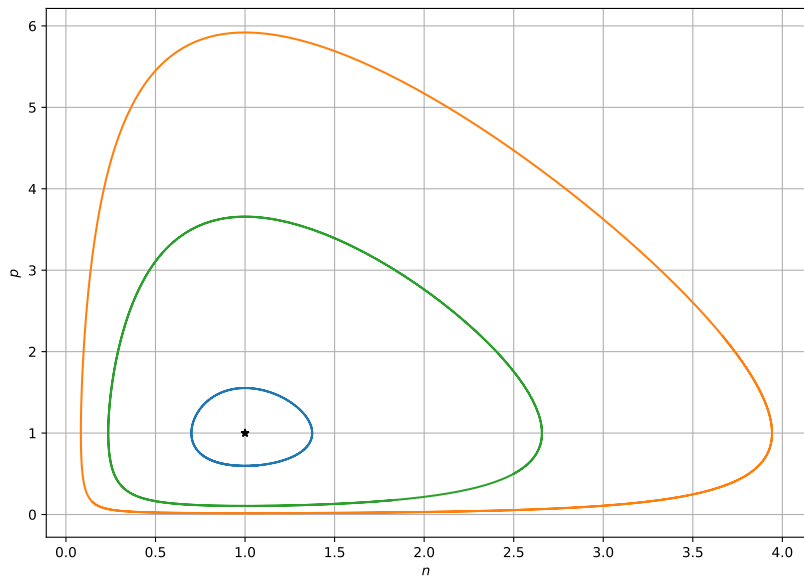


Figure 7

Linear stability

Integration

...

Lecture 16 Competition

i Recap: Lotka-Volterra model

- ▶ predator prey interaction
- ▶ n - prey
- ▶ p - predator

$$\begin{aligned}\frac{dN}{dt} &= aN - bNP, \\ \frac{dP}{dt} &= cNP - dP,\end{aligned}$$

i Aim - introduce and analyse a model of competition

Competition model

$$\begin{aligned}\frac{dN_1}{dt} &= r_1 N_1 \left(1 - \frac{N_1}{K_1} - b_{12} \frac{N_2}{K_1} \right), \\ \frac{dN_2}{dt} &= r_2 N_2 \left(1 - \frac{N_2}{K_2} - b_{21} \frac{N_1}{K_2} \right),\end{aligned}$$

- ▶ Justify why this is a model for competition
- ▶ Define model parameters
- ▶ Explain the meaning of each of the terms in the model

Nondimensionalisation

$$\frac{dn_1}{d\tau} = n_1 (1 - n_1 - a_{12}n_2) = f(n_1, n_2),$$
$$\frac{dn_2}{d\tau} = \rho n_2 (1 - n_2 - a_{21}n_1) = g(n_1, n_2),$$

► Define ρ , a_{12} and a_{21}

Nondimensionalisation (ctd.)

Steady states

Steady states (ctd)

Nullclines

Lecture 17 Competition

$$\begin{aligned}\frac{dn_1}{d\tau} &= n_1 (1 - n_1 - a_{12}n_2) = f(n_1, n_2), \\ \frac{dn_2}{d\tau} &= \rho n_2 (1 - n_2 - a_{21}n_1) = g(n_1, n_2),\end{aligned}$$

Steady states:

$$(0, 0), (1, 0), (0, 1)$$

and

$$\left(\frac{1 - a_{12}}{1 - a_{12}a_{21}}, \frac{1 - a_{21}}{1 - a_{12}a_{21}} \right).$$

Linear stability analysis

$$A_{(n_1^*, n_2^*)} = \begin{pmatrix} 1 - 2n_1 - a_{12}n_2 & -a_{12}n_1 \\ -\rho a_{21}n_2 & \rho(1 - 2n_2 - a_{21}n_1) \end{pmatrix}_{(n_1^*, n_2^*)}.$$

Linear stability analysis (ctd)

Phase portrait (one for each qualitatively distinct case)

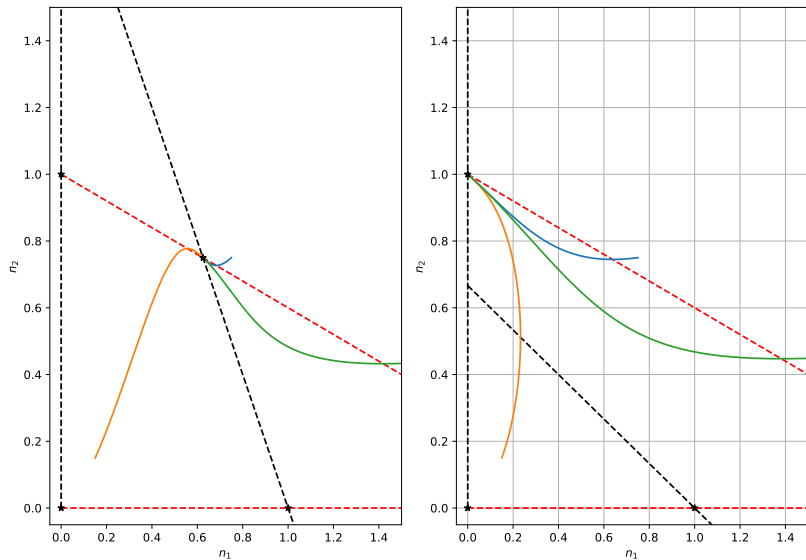


Figure 8

Insight

Mutualism/symbiosis

$$\begin{aligned}\frac{dn_1}{d\tau} &= n_1(1 - n_1 + a_{12}n_2) = f(n_1, n_2), \\ \frac{dn_2}{d\tau} &= \rho n_2(1 - n_2 + a_{21}n_1) = g(n_1, n_2).\end{aligned}$$

Steady states

This model has steady-states $(0, 0)$, $(1, 0)$, $(0, 1)$ and

$$(n_1^*, n_2^*) = \left(\frac{1 + a_{12}}{1 - a_{12}a_{21}}, \frac{1 + a_{21}}{1 - a_{12}a_{21}} \right).$$

Coexistence steady state

Summary

- ▶ Predator-prey, competition, mutualism
- ▶ techniques to analyse systems of nonlinear ODEs

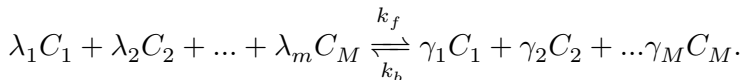
Lecture 18 Biochemical kinetics

- ▶ Cells make proteins via gene transcription and translation
- ▶ Proteins can interact
- ▶ Molecular biology is the study of the molecules that underpin biological phenomena
- ▶ For example: the cell cycle regulated by changing concentrations of cyclin/CDKs

i Our question

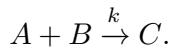
How do we mathematically describe networks of interacting chemical?

LOMA - reaction rate proportional to product of concentration of reactants

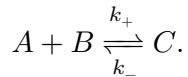


Conservation equation

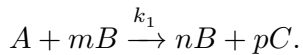
A forwards reaction



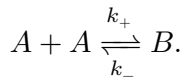
A reversible reaction



General stoichiometric constants



A reversible dimerisation



Enzyme kinetics

