

Lecture slides

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Lecture 1

- ▶ Introduction to nonlinear difference equations
- ▶ The Malthusian model
- ▶ The Ricker model

Why difference equations?

A general model

Consider the first order difference equation

$$N_{t+1} = N_t f(N_t) = H(N_t), \quad (1)$$

where $f(N_t)$ is a function that defines the per capita growth rate.
The function $H(N_t)$ describes the total (net) growth rate.

The Malthusian model

The population size at time $t + 1$ is

$$N_{t+1} = N_t + bN_t - dN_t = rN_t,$$

Exercise: solve the Malthusian model and classify qualitative behaviours

Nonlinear models

- ▶ Beverton-Holt

$$N_{t+1} = \frac{rN_t}{1 + \frac{N_t}{K}},$$

- ▶ Hassell model

$$N_{t+1} = \frac{rN_t}{(1 + \frac{N_t}{K})^b},$$

- ▶ Ricker model

$$N_{t+1} = N_t e^{r(1 - \frac{N_t}{K})}.$$

Numerical simulation of the Ricker model

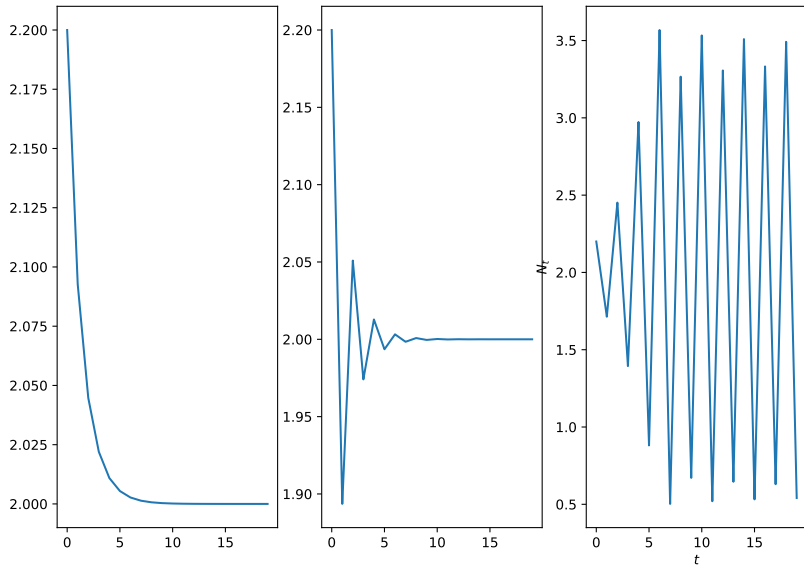


Figure 1: A plot of numerical solutions of the Ricker model. (a) $r=0.5$.

Summary

- ▶ Motivated use of difference equation models
- ▶ Introduced general model for one population
- ▶ Solved the Malthusian model
- ▶ Introduced nonlinear models

Lecture 2 - General techniques for solving nonlinear difference equations

$$N_{t+1} = N_t f(N_t) = H(N_t), \quad (2)$$

- ▶ Computational solutions
- ▶ Fixed points
- ▶ Linear stability of fixed points
- ▶ Cobweb diagrams
- ▶ Bifurcation diagrams
- ▶ Identify how model solutions depend on model parameters

Fixed points

Suppose the solution at the next iteration is equal to that at a given iteration, i.e. there exists some N^* such that

$$N^* = N_{t+1} = N_t$$



Fixed point definition

$$N^* = H(N^*), N^* \geq 0$$

Biological relevance: non-negative solutions

Linear stability analysis - how do small perturbations about N^* behave?

Linear stability analysis (ctd)

💡 Linear stability is determined by the derivative of H evaluated at N^*

$$|H'(N^*)| < 1 \implies \text{linear stability of } N^*.$$

Exercise

Identify the fixed points of the Malthusian model

$$N_{t+1} = rN_t$$

and identify their linear stability.

Cobweb diagrams

Definition

A cobweb diagram is a technique for computing graphical solutions of a difference equation.

Use previous analyses to identify different qualitative cases (one cobweb diagram for each fixed point).

For each case:

- ▶ Sketch a graph of H to evaluate iterative solutions
- ▶ Compute an iterative solution

Example

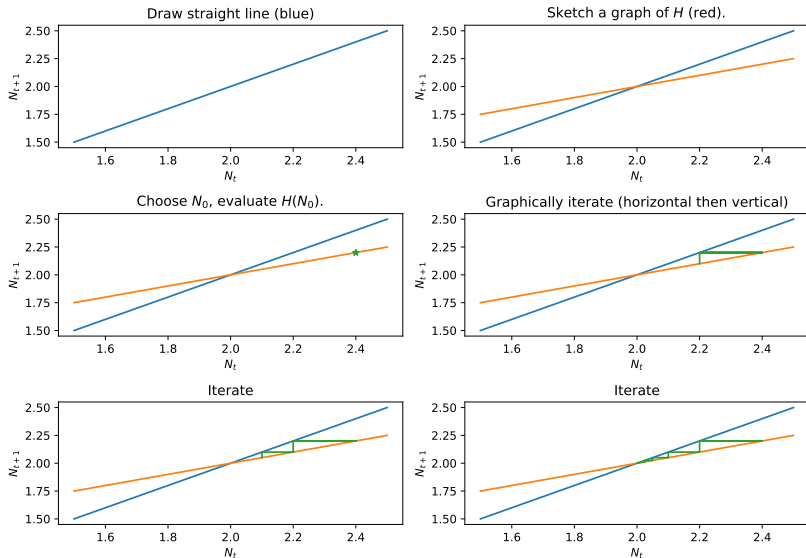


Figure 2: Generating a cobweb plot.

Bifurcation diagrams - Plot fixed points against a parameter and annotate their stability

Exercise

Draw cobweb diagrams for the Malthusian model.

$$N_{t+1} = rN_t$$

and identify their linear stability.

Lecture 3 - Preparation for tutorial 3

- Curve sketching nonlinear functions in qualitatively distinct cases

Example:

Sketch a graph of

$$f(x) = xe^{-r(1-\frac{x}{K})}, \quad r, K \in \mathfrak{R}^+, \quad x \in \mathfrak{R}, x \geq 0$$

Approach

Identify properties of H to distinguish qualitatively distinct cases

Roots

Turning points

Limit as $x \rightarrow \infty$

Limiting behaviour as $x \rightarrow 0$

Tutorial sheet 1

Lecture 4

Consider the model

$$N_{t+1} = \frac{\gamma N_t}{1 + N_t^2}, \quad \gamma \in \mathfrak{R}^+.$$

Fixed points

Linear stability

Cobweb diagrams

Bifurcations

Symbolic computations

The FPs are:

$[\{N: 0\}, \{N: -\sqrt{\gamma - 1}\}, \{N: \sqrt{\gamma - 1}\}]$

The derivative of H is:

$\gamma(1 - N^2)/(N^2 + 1)^2$

The derivative evaluated at FP 1 is:

γ

The derivative evaluated at FP 2 is:

$(2 - \gamma)/\gamma$

A model with harvesting

$$N_{t+1} = \frac{\gamma N_t}{1 + N_t^2} - hN_t, \quad (3)$$

Numerical simulation

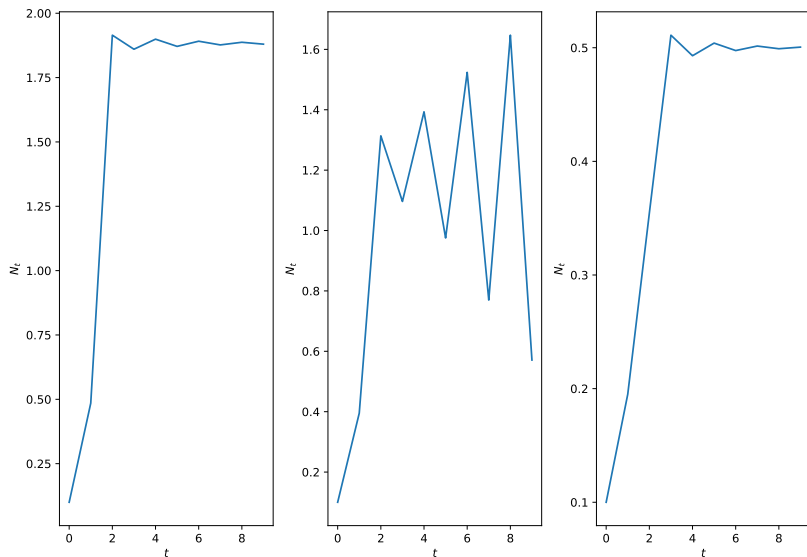


Figure 3: Time series solution for different values of h .

FPs

Linear stability