An Investigation of Nonlinear Oscillators (Duffing's Equation)

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1. Introduction

Hooke's law states that the magnitude of the restoring force of a load bearing spring is proportional to the displacement caused by the load. Hooke's law, though useful, is not always an accurate model for a spring's behavior or restoring force. In this paper we will examine springs that are not accurately modeled by Hooke's law. To be precise, we will examine the nonlinear oscillation modeled by Duffing's equation, provided below:

$$x''+cx'+kx+lx^3=f(t) \quad (1.1)$$

Here x is a function of time, t, written implicitly, x = x(t), and will be written this way for the remainder of this paper. Where c is the damping force, and in the cases of our study, is always constant. The restoring force of Duffing's equation is represented as Rx= -kx-lx3 where x is the displacement from the equilibrium of the oscillation. For the purposes of this paper, kand l will be constants. Finally f(t) is the forcing term of our system and is an oscillatory function dependent on t or is equal to 0.

2. Methods

2.1 Materials

The following materials were used for this project:
Differential Equations with Boundary Value Problems, second edition
http://www.wolframalpha.com
Maple
Microsoft Word
Google Documents

2.2 Procedure

Section 3.a:

We used the following maple code to numerically solve and graph solutions to the differential equation:
x''+3x+x3=0, x0=0, x'0=0.:

```
with(plots);
ode := diff(x(t),t,t) + 3*x(t)+x(t)^3 = 0;
ics := x(0) = A, (D(x))(0)=0;
p := dsolve({ode, ics}, x(t), type=numeric, range=0..20);
odeplot(p);
```

Where A is equal to a constant value between 0.1 and 10. We ran this code 21 times, changing A each time, starting with A=0.1 and incrementing by 0.5, ending with A=10. For each trial we recorded the period as indicated by the graph. After collecting all of the data, we graphed the period vs the amplitude for the trials.

Similarly we ran the same experiment for the linear differential equation:
x''+3x=0 in order to contrast the results with those of the non-linear case.

Section 4.a:
We used the following maple code to graph solutions to the differential equation:
x''+x'+x+3x3=20cos(ωt)

```
with(plots);
ode := diff(x(t),t,t) + diff(x(t),t) + x(t) + 3*x(t)^3 = 20*cos(W*t);
ics := x(0) = A, (D(x))(0)=0;
p := dsolve({ode, ics}, x(t), type=numeric, range=0..20);
odeplot(p);
```

Our goal was to find a graph that indicated a steady state emergence. We began by holding A at a constant 0 and repeatedly ran the code for slowly incrementing values of W starting at W=1. After we found a W that seemed to indicate the emergence of a steady state solution we kept W constant at that value and began slowly incrementing the values of A in our code until we found a solution that demonstrated the emergence of a steady state solution.

Section 4.b:

We used the following maple code to graph solutions to the differential equation: $x''+x'+x+3x^3=20\cos\omega t$, $x_0=0$, $x'_0=0$

```
with(plots);
ode := diff(x(t),t,t) + diff(x(t),t) + x(t) + 3*x(t)^3 = 20*cos(W*t);
ics := x(0) = 0, (D(x))(0)=0;
p := dsolve({ode, ics}, x(t), type=numeric, range=0..20);
odeplot(p);
```

We ran several trials of the above code changing the values of W starting with W=2 and ending with W=6. For each trial we recorded the amplitude as indicated by the graph. After collecting all of the data we graphed the amplitude vs the driving frequency.

Section 4.c:

Same as the procedure listed for section 4.b except $x_0=6$.

Section 5.e:

Used the following maple code:

```
with(plots);
contourplot(-1/2*x(t)^2+1/4*x(t)^4+1/2*v(t)^2, x=-3..3, v=-3..3, grid=[75, 75], contours=[-.1,0,.1,1]);
```

to graph the level curves of $E(x,v)=-\frac{1}{2}x^2+\frac{1}{4}x^4+\frac{1}{2}v^2=C$ for C=-.1,0,.1,1

3. The period of an undamped, unforced oscillation

3.a) In this section we will examine the relationship between the amplitude and the period of our non-linear oscillation system modeled by Duffing’s equation, specifically when Duffing’s equation is undamped and unforced, c=0=f(t) from equation (1.1). We will also set k=3 and l=1. The resulting equation is as follows:

$$x''+3x+x^3=0 \qquad (3.1)$$

To observe the behavior of the amplitude vs the period we graphed the solution, x(t) vs t, to our above differential equation for various set parameters to our system. For each trial we recorded the amplitude and the period of the steady state solution as indicated by the resulting graph. From there we graphed the period vs the amplitude for the sum of our trials. In order to use a numerical solver to graph solutions for equation 3.1 we had to make some assumptions about our system. For starters we assumed that the system started at rest in every trial, x’(0)=0. Secondly we decided to run 21 cases for the starting location of our system, x(0)=A for A starting at .1 and incremented by .5 for each case, ending at A=10.

The following is a graph of the period vs the amplitude for each solution, x(t), to our equation 3.1 obeying the initial conditions of x(0)=A and x'(0)=0.

(Figure 1)

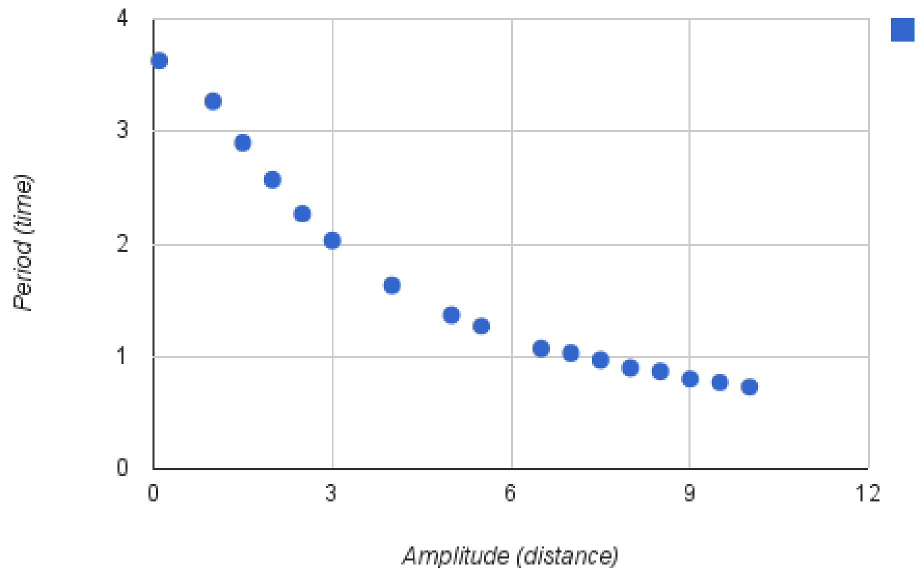


Figure 1: Period vs amplitude of nonlinear oscillation, undamped and unforced.

Data for Figure 1:

Amplitude	Period		Amplitude	Period
0.1	3.63		5	1.37
0.5	3.5		5.5	1.27
1	3.27		6	1.17
1.5	2.9		6.5	1.07
2	2.57		7	1.03
2.5	2.27		7.5	0.97
3	2.03		8	0.9
3.5	1.83		8.5	0.87
4	1.63		9	0.8
4.5	1.5		9.5	0.77
			10	0.73

3.b) This question asked us to analyze the equation:

$x''+3x+x^3=0$; $x(0)=A$; $x'(0)=0$ for values of A between 0.1 and 10, $0.1\leq A\leq 10$

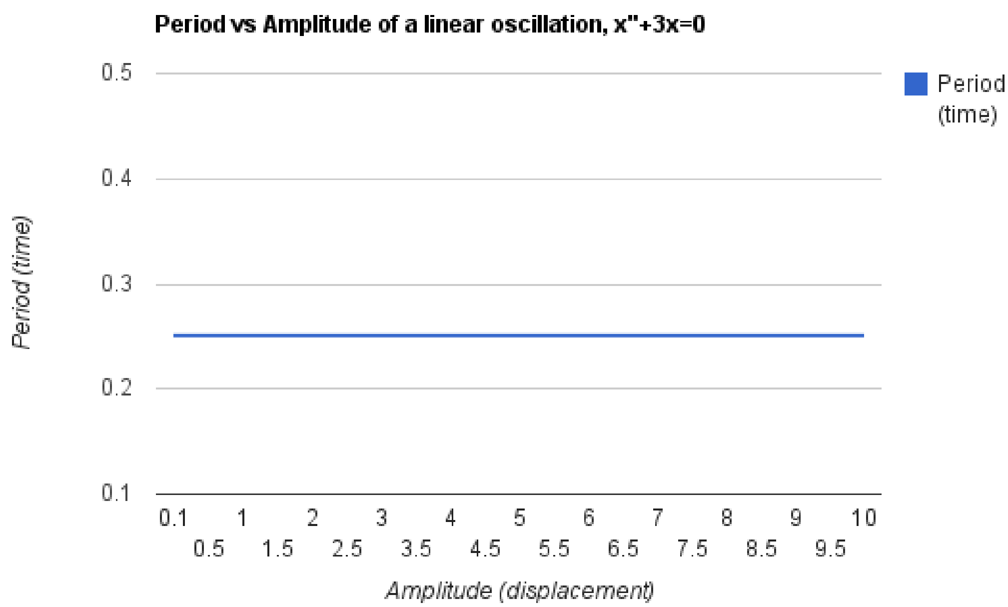
This is an undamped, unforced non-linear oscillator. As we can see from the graph, amplitude and period are inversely proportional. This is opposite of what we expect to see in linear oscillator where there is no relation between the amplitude and the period.

The following is the graph of period vs. amplitude for solutions to the linear differential equation,

x''+3x=0

In other words, l = 0 in Duffing's Equation, resulting in a linear oscillator with a spring restoring force obeying Hooke's Law. Trials were executed in a similar fashion to those of the non-linear case.

Figure 1b: Period vs Amplitude of a linear, undamped, unforced oscillator: x''+ 3x=0



As indicated by figure 1b, the amplitude and period act independently of each other for the case of linear oscillators, unlike the non-linear case.

4. Frequency Response of a Hard Oscillator

In this section we will be analyzing various properties of a non-linear oscillator modeled by Duffing's equation with a damping term and a forcing function.

4.a) In part a we will modify equation 1.1 such that c=1, k=1, l=3 and set the forcing term, f(t), to 20cos(ωt) resulting in the following equation:

x''+x'+x+3x3=20cosωt (4.1)

We will now examine graphs of numerical solutions, x(t), to equation 4.1 for various driving frequencies, ω, and initial conditions x(0) and x'(0). We are in search of a solution that results in a steady state oscillation. One such solution results when ω=4.5, x(0)=14 and x'(0)=0. The graph of this solution is shown in figure 2.

(Figure 2)

Figure 2: Graph of the solution to $x''+x'+3x3=20\cos(4.5t)$ with initial conditions $x(0)=14$ and $x'(0)=0$, demonstrating the transition of the transient solution (from $t=0$ to about $t<7$) to the steady state solution ($t\geq7$).

4.b) In this part we will look at solving equation 4.1 for various driving frequencies, ω, of our driving function, $20\cos(\omega t)$, while holding the initial conditions of our system constant, $x(0)=0$ and $x'(0)=0$. We will numerically solve equation 4.1 nine times starting with $\omega=2$ and incrementing ω by 0.5 each trial, ending with $\omega=6$. For each trial we will graph the solution and record the amplitude of the steady state solution as seen on the graph.

The following graph displays the amplitude vs the driving frequency for each solution to equation 4.1.

(Figure 3)

Figure 3: Amplitude vs driving frequency of solutions to $x''+x'+x+3x3=20\cos\omega t$ for values of ω between 2 and 6, $x(0)=0$, $x'(0)=0$

Data for Figure 3:

Frequency	Amplitude		Frequency	Amplitude
2	2.5		4	3.1
2.5	2.6		4.5	1.2
3	2.8		5	0.9
3.5	3		5.5	0.7
			6	0.56

4.c) The experiment performed in this section is identical to section 4.b, with one exception, $x(0)=6$.

(Figure 4)

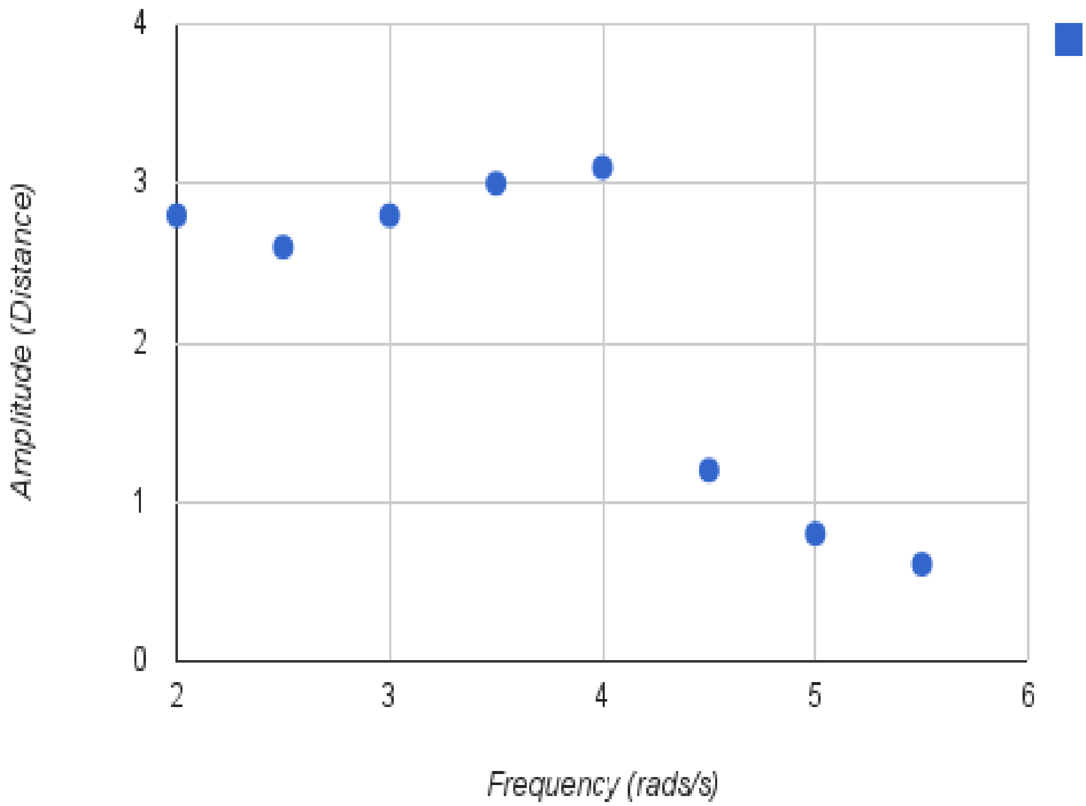


Figure 4: Amplitude vs driving frequency of solutions to x''+x'+x+3x3=20cosωt for values of ω between 2 and 6 and x(0)=6, x'(0)=0

Data for Figure 4:

Frequency	Amplitude		Frequency	Amplitude
2	2.8		4	3.1
2.5	2.6		4.5	1.2
3	2.8		5	0.8
3.5	3		5.5	0.61
			6	0.59

4.d) In this section we will analyze the Figures 3 and 4 superimposed on one another.

(Figure 5)

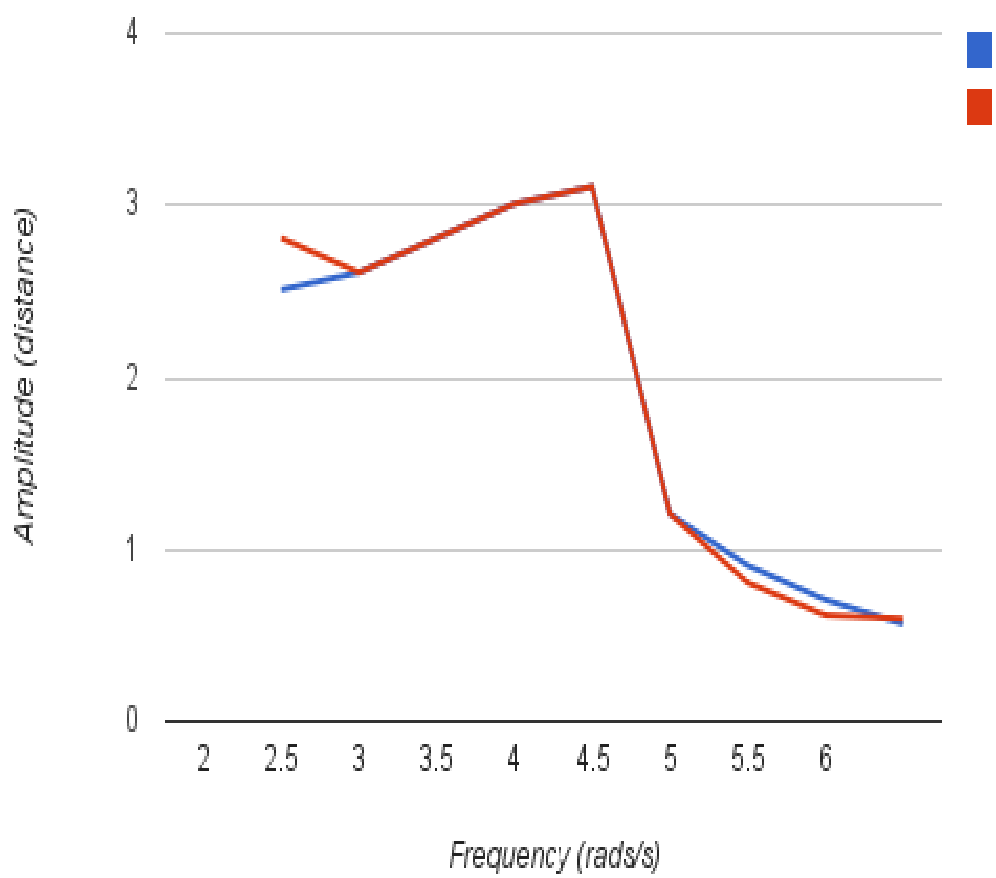


Figure 5: Figures 3 and 4 superimposed on each other, representing *Duffing's hysteresis*.

Figure 5 demonstrates what is known as Duffing's hysteresis: for oscillators modeled by Duffing's equation, as the driving frequency of the forcing term increases so does the amplitude of the system up until a certain driving frequency, at which pont the amplitude suddenly drops and decreases as the driving frequency increases from that rate forward. Secondly notice that the relationship between frequency and amplitude for a specific oscillator system remains constant even for various starting amplitudes of said system.

4.e) Summary: question 4 asked us to analyze the relationship between the driving frequency and the resulting amplitude of the system modeled by the equation:

$$x''+x'+x+3x^3=20\cos\omega t$$

We solved the system using various values of ω from 2.0 to 6.0 and discovered that the system follows what is known as Duffing's hysteresis (explained at the end of section 4.d). The relationship between frequency and amplitude remains nearly constant regardless of the starting amplitude of the system.

The above results do not hold true for linear oscillators. Once again we will modify equation 4.1 such that l in Duffing's equation is set to 0 resulting in the following linear differential equation:

$x''+x'+x=20\cos(\omega t)$ (4.2)

Next we run the same experiment as section 4.b, this time with equation 4.2 and x(0)=0, x’(0)=0. Below is a graph of the results that followed.

(Figure 6)

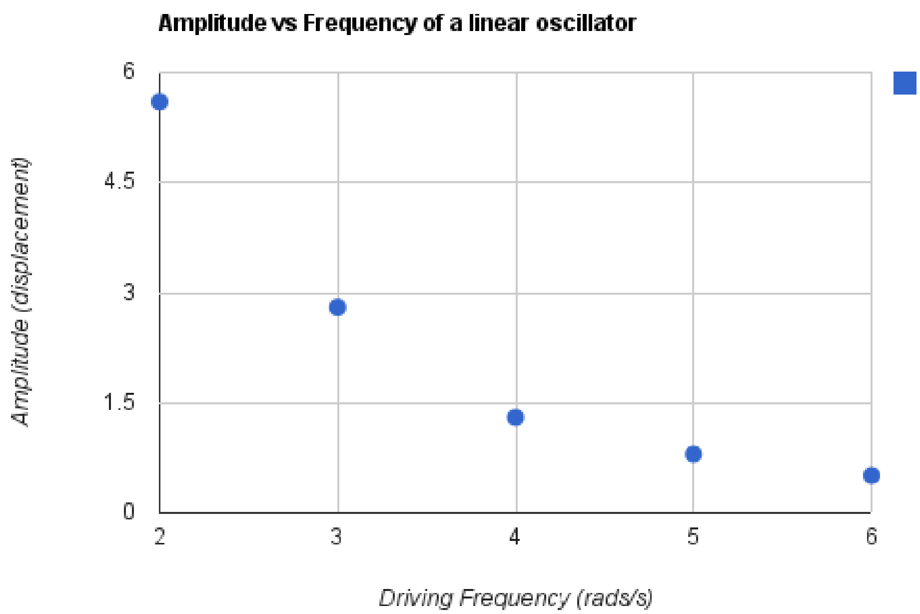


Figure 6: Amplitude vs driving frequency of solutions to the linear oscillator: $x''+x'+x=20\cos\omega t$ for values of ω between 2 and 6 and x(0)=0, x’(0)=0

Notice that the amplitude of our linear oscillator exponentially declines as the frequency linearly increases unlike that of the non-linear oscillator. Another difference is that the linear case does not appear to have any anomalies or jumps in the amplitude as the driving frequency changes, as such we had in the non-linear case.

5. Conservative systems

In this section you will work with the Duffing equation

	$x''+\alpha x'+\beta x+\gamma x3=0$	(5.1)

where α, β, γ are system parameters.

a. Introduce the variable v=x' and write equation (5.1) as a system of two first-order equations.

$x''+ \alpha x'+ \beta x+ \gamma x3=0$ (5.1)
$x''= -\alpha x'-\beta x-\gamma x3$ (5.2)

Introduce a new variable, v and let v=x' then we have a system of first order equations:

1. x'= v=Fv
2. v'= -αv- βx-γx3=G(x,v)

b. Since the independent variable t does not appear explicitly in the equation, the system is autonomous (recall Section 2.9 of the textbook). The system you wrote in (a) has the form

	x'=Fx,v, v'=G(x,v)	(5.2)

The trajectories of a two-dimensional autonomous system can sometimes be found by solving a related first-order differential equation. From Equations (5.2) we have

	dvdx=dv/dtdx/dt=G(x,v)F(x,v)	(5.3)

which is first-order in the variables x and v. Note that such a reduction is not usually possible if F and G depend also on t. Write equation (5.3) as a differential form and then determine the conditions on α, β, γ so that the differential form is exact.

dvdx=dvdtdxdt=Gx,vFv=-αv+βx+γx3v (5.3)

v dv+ αv+βx+γx3 dx=0 (5.4)

Lets introduce new function, let Px,v=v and Qx,v= αv+βx+γx3 then in order for equation 5.4 to be exact the partial derivative of P with respect to x must be equal to the partial derivative of Q with respect to v:

∂P∂x=∂Q∂v

∂P∂x=0

∂Q∂v= α

Therefore α=0 in order to have 5.4 in exact differential form, leading to:

v dv+βx+γx3 dx=0 (5.5)

c. Solutions of an exact equation can be written implicitly in the form

	Ex,v=c.	(5.6)

This equation determines the integral curves (trajectories) for equation (5.3). In other words, the trajectories lie on the level curves of E(x,v). With α=0, β=-1, and γ=1, find the function E.

5.5 is now a separable differential equation and can be solved:

-v dv= βx+γx3 dx

-12v2=12βx2+14γx4+ C, C is a constant of integration.

$E(x,v)=\frac{1}{2}\beta x^2+\frac{1}{4}\gamma x^4+\frac{1}{2}v^2=C$

For $\alpha=0$, $\beta=-1$, $\gamma=1$

$E(x,v)=-\frac{1}{2}x^2+\frac{1}{4}x^4+\frac{1}{2}v^2=C$ (5.6)

d. Find the critical points of E and classify them as either local or global minima, maxima, or saddle point.

solving $E(x,v)$ for minima, maxima and saddle points:
$\frac{\partial E}{\partial x}=-x+x^3$, $\frac{\partial E}{\partial x}=0$ for $x=-1,0,1$.

$\frac{\partial E}{\partial x^2}=-1+3x^2$ and $\frac{\partial E}{\partial x\partial v}=0$

$\frac{\partial E}{\partial v}=v$, $\frac{\partial E}{\partial v}=0$ for $v=0$.

$\frac{\partial E}{\partial v^2}=1$ and $\frac{\partial E}{\partial v\partial x}=0=\frac{\partial E}{\partial x\partial v}$

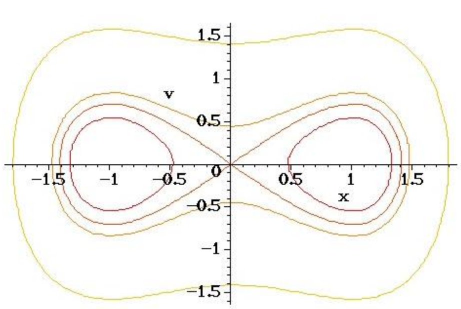
By the second partial derivative test:
$M(x,v)=\begin{vmatrix}E_{xx} & E_{xv}\\ E_{vx} & E_{vv}\end{vmatrix}=E_{xx}E_{vv}-E_{xv}^2=-1+3x^2$

for critical points a and b of $E(x,v)$:
If $M(a,b)>0$ and $E_{xx}(a,b)>0$, then $E(a,b)$ is a local minimum.
If $M(a,b)>0$ and $E_{xx}(a,b)<0$ for, then $E(a,b)$ is a local maximum.
If $M(a,b)<0$, then $E(a,b)$ is a saddle point.

$M(-1,0)=2>0$ and $E_{xx}(-1,0)=2>0$ Therefore $E(-1,0)$ is a local minimum.
$M(0,0)=-1<0$ Therefore $E(0,0)$ is a saddle point.
$M(1,0)=2>0$ and $E_{xx}(1,0)=2>0$ Therefore $E(1,0)$ is a local minimum.

e. Use Maple to plot the level curves associated with c=-.1, 0, .1, and 1. You should adapt the Maple commands given below. Note that the x,v-plane is called the phase plane.

```
> contourplot(-1/2*x(t)^2+1/4*x(t)^4+1/2*v(t)^2, x=-3..3, v=-3..3, grid=[75, 75], contours=[-.1,0,.1,1]);
```



f. Label the trajectories (level curves) on the graph of the phase plane and use arrows to indicate the direction of the trajectories.

g. [Optional – for extra credit] Carefully consider your phase plane plot and explain the physical significance of each level curve. (Recall Exercise 18 on p. 149.)

To conclude, a quantity like Ex,v which is constant along integral curves is called a conserved quantity. In particular, the function E represents the total energy of the system. To see this, imagine a particle of mass equal to one unit and consider the terms in the function E. Do you recognize the expressions for kinetic energy? Notice that the remaining terms depend only on x, giving an expression for the potential energy. The Duffing oscillator has application to many areas including signal detection and shock protection in portable electronics.