$\frac{\partial x}{\partial t} = a$

$\int \partial x \int \partial t$ Integrate

$\{x(t) = at + C$ ~~const~~ constant

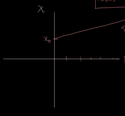
Figure out C given initial conditions:

$x(t=0) = x_0$

$x(t=0) = a(0) + C$

$C = x(t=0) + x_0$

$\boxed{x(t) = at + x_0}$



↑ Slope determined by a

$x(t) = at + x_0$ ⟸ Analytical Solution

$x(t) = \ldots$ From $\frac{\partial x}{\partial t}$

Numerical Solution

Graphical solution

$\frac{\partial x}{\partial t} = ax$ Rate of change proportional to its current value, a is the constant of proportionality

Over 1 sec: $2 \to 6$ (0.4)

next 1 sec: $6 \to 18$ (0.8)

↳ Rate of change depends on number of bacteria

$\frac{\partial x}{\partial t} = ax$

$\int \frac{\partial x}{x} \int \partial t$

$\ln x + C = at$

$e^{\ln x} = e^{C} \cdot e^{at}$

$x = \frac{1}{e^C} e^{at}$

$x = Ce^{at}$

$x_0 = Ce^{a \cdot 0}$

$C = x_0$

$\boxed{x(t) = x_0 e^{at}}$

$a = \frac{1}{\tau}$ often, τ is time constant that tells us rate of rise or fall

Exponential decline

$a = -5$

Exponential growth

$a = 5$

Simulation

Sim

$\frac{\partial x}{\partial t} = ax \Rightarrow x(t) = x_0 e^{at}$ Analytical solution

Numerical solution:

How do we get x over time if we can't write out $x(t) = \ldots$?

Or when its too annoying to...

→ Approximate x by stepping through time and estimating x at each step

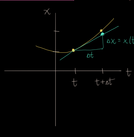
* Euler method

Trapezium rule method

Runge-Kutta methods

Euler method

$\frac{\partial x}{\partial t} = F(x,t)$



slope $= \frac{\text{rise}}{\text{run}}$

$= \frac{\Delta x}{\Delta t} = F(x,t)$

$\Delta x = F(x,t)\Delta t$

$\boxed{x(t+\Delta t) = x(t) + F(x,t)\Delta t}$

step size

$\frac{\partial x}{\partial t} = F(x,t) = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t}$

Switching to discrete notation:

$x_{n=5} = x$ after 5 steps $= x(5\Delta t)$

$t_{n=5} = t$ after 5 steps $= 5\Delta t$

Start with $n=0$, t_0, x_0 ← often 0

For n from 0 to N: ← number of steps

$x_{n+1} = x_n + F(x_n, t_n)\Delta t$

$t_{n+1} = t_n + \Delta t$

Pseudocode

Use a computer

Get x over N steps

Simulation

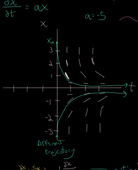
Graphical Solution

- the best for understanding the system!

Direction Fields: draw tangent line at each point

$\frac{\partial x}{\partial t} = ax$

$a = -5$



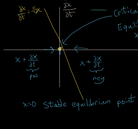
$a = 5$

$\frac{\partial x}{\partial t} = ax$

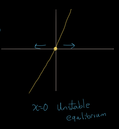
critical point

Equilibrium point

x where $\frac{\partial x}{\partial t} = 0$



x=0 stable equilibrium point



x=0 Unstable equilibrium

Simulations

$\frac{\partial x}{\partial t} = ax$ First order linear differential equation

No higher derivatives

$\frac{\partial^2 x}{\partial t^2}, \frac{\partial^3 x}{\partial t^3}$

No x^2, $\sin(x)$, etc

Analytical solution when $p(t)$ can be explicitly integrated.

General form

$\frac{\partial x}{\partial t} + p(t)x = g(t)$

Definition of chain ⟹

$x(t) = \frac{\int m g(t) \partial t + C}{m}$

where $m = \int p(t) \partial t$

$\frac{\partial x}{\partial t} - ax = 0$

$p(t) = -a$

$g(t) = 0$

$m = \int -a \partial t = \int \partial t$

$x(t) = \frac{\int 0 \partial t + C}{e^{-at}} = \frac{C}{e^{-at}} = Ce^{at}$

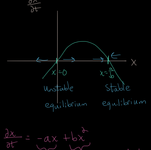
A more complex diff eq:

Bacteria

Constant of proportionality might decrease w/ # of bacteria

$\frac{\partial x}{\partial t} = (a - bx)x$

$= a - bx^2$

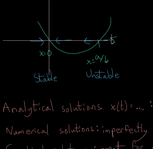


Simulation

$\frac{\partial x}{\partial t} = -ax + bx^2$

Dying

Sexual reproduction



Simulation

Analytical solutions $x(t) = \ldots$: nice but not always possible

Numerical solutions: imperfectly approximate $x(t)$, computation

Graphical solutions: great for understanding

Caveats:

→ Lots of tricks for analytical solutions of different types of diff eq

→ Focused on first order diff eq, higher orders exist