$\frac{dx}{dt} = f(x)$ single diff eq

Ex single neuron actively over fire

2 reasons: $x_1, x_2$

$$\frac{dx_1}{dt} = F(x_1, x_2)$$
$$\frac{dx_2}{dt} = G(x_1, x_2)$$

First order system of differential equations

Continuous dynamical system of $x$ moving object

Linear

$$\frac{dx_1}{dt} = ax_1 + bx_2 + c_1(t)$$
$$\frac{dx_2}{dt} = dx_1 + ex_2 + c_2(t)$$

① Homogeneous $c_1/c_2 = 0$

*Easy to understand all

③ Nonhomogeneous otherwise

② Nonlinear otherwise

② Compare to discrete

## Homogeneous linear first order system of diff eq

*Seems a lot but approach will generalize

$$\frac{dx_1}{dt} = 5x_1 + 2x_2$$
$$\frac{dx_2}{dt} = 3x_1 + x_2$$

Will cover analytical & graphical solutions

Numerical: can approximate with Euler method, in time update $\vec{x}$

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 5 & 2 \\ 3 & 1 \end{bmatrix} \vec{x}$$

$$\boxed{\frac{d\vec{x}}{dt} = A\vec{x}}$$ General matrix form

Phase plane

Time not explicitly represented

Phase portrait

$\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\frac{d\vec{x}}{dt} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$



$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\frac{d\vec{x}}{dt} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

$$\frac{d\vec{x}}{dt} = A\vec{x}$$

Remember $\frac{dx}{dt} = ax \Rightarrow x = Ce^{at}$

Let's try $\vec{x} = \vec{v}e^{\lambda t}$ vector now

$$\frac{d\vec{x}}{dt} = \lambda\vec{v}e^{\lambda t} = A\vec{x}e^{\lambda t}$$
$$\lambda\vec{v} = A\vec{v}$$

Looks familiar?

$A\vec{v} = \lambda\vec{v}$

If $\vec{v}$ is eigenvector and $\lambda$ eigenvalue $\vec{x} = \vec{v}e^{\lambda t}$ will be a solution

$\vec{x} = \vec{v}_1 e^{\lambda_1 t}$ for $i = 1, 2, \ldots n$

## A with real distinct eigenvalues

If $\vec{x}_1, \vec{x}_2$ solutions of $\frac{d\vec{x}}{dt} = A\vec{x}$, $\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2$ also solution

Proof: $\frac{d}{dt}[c_1\vec{x}_1 + c_2\vec{x}_2] = c_1\frac{d\vec{x}_1}{dt} + c_2\frac{d\vec{x}_2}{dt}$

$= c_1 A\vec{x}_1 + c_2 A\vec{x}_2$

$= A[c_1\vec{x}_1 + c_2\vec{x}_2]$

$\frac{d\vec{x}}{dt} = A\vec{x}$

If $\vec{x}_1, \ldots \vec{x}_n$ are solutions for non matrix $A$:

*linearly ind $\Rightarrow$ fundamental set of solutions

General solution: $\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 + \ldots c_n\vec{x}_n$

$\lambda_1$s distinct $\rightarrow v_1$s linearly independent

$$\frac{d\vec{x}}{dt} = A\vec{x}$$

General solution of $\lambda_1$s distinct

$$\boxed{\vec{x}(t) = c_1\vec{v}_1 e^{\lambda_1 t} + c_2\vec{v}_2 e^{\lambda_2 t} + \ldots c_n\vec{v}_n e^{\lambda_n t}}$$

Find $c_i$s by initial conditions for particular solution

### 2D example

$$\frac{d\vec{x}}{dt} = A\vec{x} \text{ where } A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$$

$\lambda_1 = -2$ $\vec{v}_1 = \begin{bmatrix} -0.7 \\ 0.7 \end{bmatrix}$ $\lambda_2 = 5$ $\vec{v}_2 = \begin{bmatrix} 0.6 \\ 0.8 \end{bmatrix}$

$$\vec{x}(t) = c_1\begin{bmatrix} -0.7 \\ 0.7 \end{bmatrix}e^{-2t} + c_2\begin{bmatrix} 0.6 \\ 0.8 \end{bmatrix}e^{5t}$$

$$\frac{d\vec{x}}{dt} = A\vec{x} = 0 \Rightarrow \vec{x} = 0 \text{ is equilibrium point}$$

$c_1 = 0$, $\vec{x}(t) = c_2\begin{bmatrix} 0.6 \\ 0.8 \end{bmatrix}e^{5t}$ $t \to \infty$ $e^{5t} \to 0$

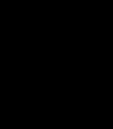
$c_2 = 0$, $\vec{x}(t) = c_1\begin{bmatrix} -0.7 \\ 0.7 \end{bmatrix}e^{-2t}$ $t \to \infty$ $e^{-2t} \to 0$

Very negative $t$: $e^{-2t} \to \infty$ $e^{5t}$ dominates

$\vec{x} = c_1\begin{bmatrix} -0.7 \\ 0.7 \end{bmatrix}e^{5t}$ $|\text{neg}| < |\text{pos}|$

Very positive $t$: $e^{5t}$ dominates

$\vec{x} = c_2\begin{bmatrix} 0.6 \\ 0.8 \end{bmatrix}e^{5t}$




Always like this if $\lambda_1$ and $\lambda_2$ are real and distinct with one negative and one positive

$\vec{x} = 0$ Equilibrium point

Saddle point unstable

## Complex eigenvalues

$\lambda_{1,2} = \pm a + ib$

$A\vec{v}$s not complex just $\lambda_1$s

if $A$ has $\lambda = a + ib$, $\vec{v} = \vec{v}^R + i\vec{v}^I$:

$$\vec{x}_1 = e^{at}(\vec{v}^R\cos bt - \vec{v}^I\sin bt)$$
$$\vec{x}_2 = e^{at}(\vec{v}^R\sin bt + \vec{v}^I\cos bt)$$

$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2$$

## Repeated eigenvalues

$\lambda_1 = \lambda_2$

Analytical solution in similar Planar system exists

Remember discrete dynamical systems are different!

First order diff eq

Linear

Homogeneous

Different categories of $\lambda_1$s

Nonhomogeneous

Nonlinear

Understanding based on homogeneous systems!

## Nonhomogeneous linear systems

$$\frac{d\vec{x}}{dt} = A\vec{x} + \vec{c}(t)$$

Let's say $\vec{c}(t) = \vec{c}$

*Analytical solutions can exist but messy

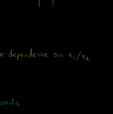
Critical points not at origin

Tractable system

$\vec{z} = \vec{x} - \vec{c}$

$\frac{d\vec{z}}{dt} = A\vec{z}$

Same matrix new dynamics




1) Find critical point $\vec{x}_c$

2) Figure out dynamics of equivalent homogeneous system

3) Translate the dynamics of this system back around the critical point

## Nonlinear systems

$$\frac{dx_1}{dt} = F(x_1, x_2)$$
$$\frac{dx_2}{dt} = G(x_1, x_2)$$

F/G nonlinear dependence on $x_1/x_2$

Can have 0, 1, 2, 3, ... critical points

$$\left.\begin{aligned} F(x_1, x_2) &= 0 \\ G(x_1, x_2) &= 0 \end{aligned}\right\} \text{Find critical points}$$

More chaotic phase portraits

Approach: Approximate nearby trajectories by using linear terms of F/G around each critical point

*Cannot realize up to a linear sys near each critical point

Linearization of F/G: 1st 3 terms of Taylor series expansion of each function around critical point $(a, b)$

$$\frac{dx_1}{dt} = F(a,b) + F_{x_1}(a,b)(x_1 - a) + F_{x_2}(a,b)(x_2 - b)$$

$$\frac{dx_2}{dt} = G(a,b) + G_{x_1}(a,b)(x_1 - a) + G_{x_2}(a,b)(x_2 - b)$$

$$\frac{dx_1}{dt} = F_{x_1}(a,b)x_1 + F_{x_2}(a,b)x_2$$

$$\frac{dx_2}{dt} = G_{x_1}(a,b)x_1 + G_{x_2}(a,b)x_2$$

$$\frac{d\vec{x}}{dt} = A\vec{x} \text{ where } A = \begin{bmatrix} F_{x_1}(a,b) & F_{x_2}(a,b) \\ G_{x_1}(a,b) & F_{x_2}(a,b) \end{bmatrix}$$

$$J = \begin{bmatrix} F_{x_1} & F_{x_2} \\ G_{x_1} & G_{x_2} \end{bmatrix}$$

Matrix of partial derivatives

$\rightarrow$ Jacobian matrix