Differentiate your Objective

VLC = $\iiint$ Vision Learning Control $dV\,dL\,dC$

Differentiable Programming

How does pre-university calculus relate to AI and the future of computer programming?

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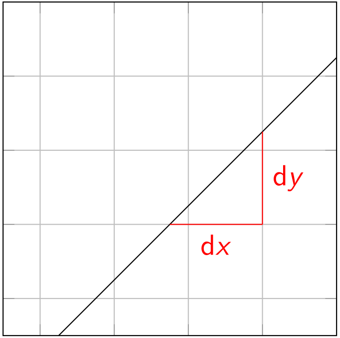
Differentiation

Recap: what is the derivative of a function of one variable?

The derivative in 1D

- Recall that the gradient of a straight line is $\frac{dy}{dx}$.



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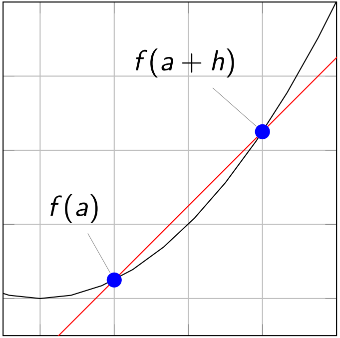

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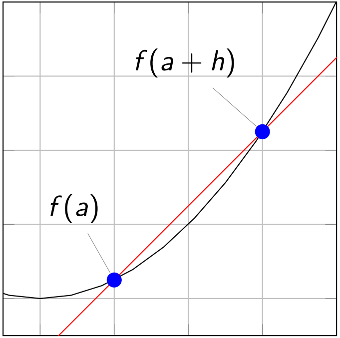

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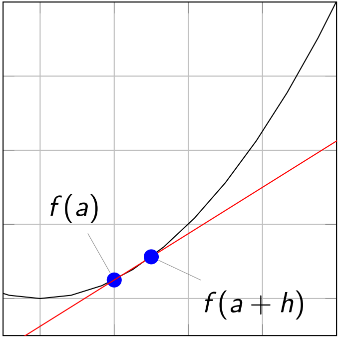

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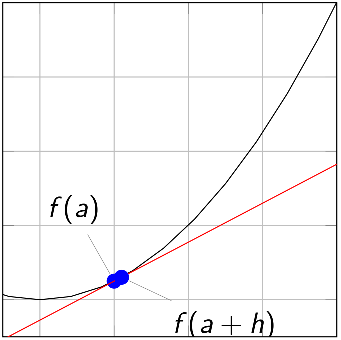
  - As h becomes smaller, the approximated derivative becomes more accurate.
  - If we take the limit as $h \to 0$, then we have an exact expression for the derivative: $\frac{df}{da} = f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$.



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The derivative of $y = x^2$ from first principles

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Intuition: What does the gradient dy/dx tell us

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- The 'rate of change' of y with respect to x.
- By how much does y change if I make a small change to the x.

- At what angle should a javelin be thrown to maximise the distance travelled?

Why should we care?

Solving a simple problem with differentiation

- At what angle should a javelin be thrown to maximise the distance travelled?
- Assume initial velocity $u = 28\,\mathrm{m\,s^{-1}}$ and $g = 9.8\,\mathrm{m\,s^{-2}}$
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- Kinematics equations:

$$x = ut\cos(\theta) = 28t\cos(\theta)$$
$$y = ut\sin(\theta) - 0.5gt^2 = 28t\sin(\theta) - 4.9t^2$$

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$$x = 28t\cos(\theta)$$
$$y = 28t\sin(\theta) - 4.9t^2$$

- Javelin hits ground when $y = 0$ and we only care about $t > 0$:

$$0 = 28t\sin(\theta) - 4.9t^2$$
$$\implies t = \frac{28}{4.9}\sin(\theta)$$

- Substituting into the horizontal component:

$$x = 28\frac{28}{4.9}\sin(\theta)\cos(\theta) = 80\sin(2\theta)$$

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Irrespective of the initial velocity maximum distance is acheived at 45°.

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  - **formulate** a *loss* or *cost* function.
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- Problems:
  - The loss must be differentiable (or rather you must be able to compute or estimate its gradient somehow).
  - The loss function could be arbitrarily complex... you might not be able to analytically compute the solution (or the gradient).
  - Some loss functions might have many minima; you might have to settle for finding a sub-optimal one (or a saddle-point).

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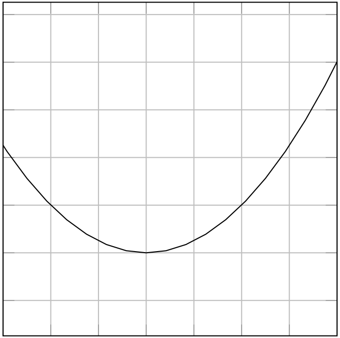
A simple algorithm for minimising a function

Gradient Descent

- How can you numerically estimate the value of the parameter θ that minimises a loss function, $\mathcal{L}(\theta)$?



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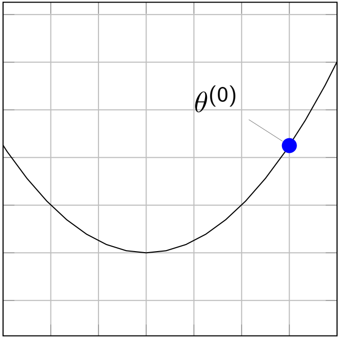

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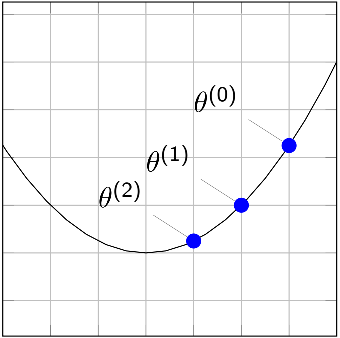
Gradient Descent:

$\theta^{(i+1)} = \theta^{(i)} - \lambda \frac{\mathrm{d}\mathcal{L}}{\mathrm{d}\theta}$ where λ is the *learning rate*



Javelin throwing again, but with Python code

Derivatives of more general functions

- Almost all complex functions can be broken into simpler parts (often with very simple derivatives).
- You can add (or subtract) sub-functions, multiply (or divide) sub-functions and make functions of functions.
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- You can add (or subtract) sub-functions, multiply (or divide) sub-functions and make functions of functions.
  - The sum rule, product rule and chain rule tell you how to differentiate these.
- If you break down functions into their constituent parts computing the derivative becomes very easy
- Example: the sin function can be written in terms of exponentials (Euler's formula) and the derivative of an exponential e^x is just e^x...

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    - In many real applications it can be *millions* of parameters.
- Partial derivatives $\frac{\partial f}{\partial x_i}$ let us compute the gradient of the i-th parameter by holding the other parameters constant.

Back to programming

Programming is really just function composition and control statements

- At the end of the day computer programs are just compositions of really simple functions that computer processors can compute: arithmetic operations (add, multiply, divide, ...), logical operations (and, or, not, comparisons...), operations that move data, etc.

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- The chain rule tells us how to compute gradients of composite functions.

So, in principle we can find the optimal "parameters" of a computer program designed to solve a specific task by following the gradients to optimise it.

Differentiating Branches

Code - *if-else* statement

```
if a > 0.5:
    b = 0
else:
    b = 2 * a
```

Math

$$b(a) = \begin{cases} 0 & \text{if } a > 0.5 \\ 2a & \text{if } a \leq 0.5 \end{cases}$$

$$\frac{\partial b}{\partial a} = \begin{cases} 0 & \text{if } a > 0.5 \\ 2 & \text{if } a \leq 0.5 \end{cases}$$

Differentiating Loops

Code - *for loop* statement

```
b = 1
for i in range(3):
    b = b + b * a
```

Math

$$b_0 = 1$$
$$b_1 = b_0 + b_0 a = 1 + a$$
$$b_2 = b_1 + b_1 a = 1 + 2a + a^2$$
$$b_3 = b_2 + b_2 a = 1 + 3a + 3a^2 + a^3$$
$$\frac{\partial b}{\partial a} = 3 + 6a + 3a^2$$

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- not every operation or function has *useful* gradients
  - discontinuities, large areas of zero-gradient, ...
- Computer science researchers are actively developing mathematical 'tricks' to circumvent many of these problems.
  - *Relaxations* of functions that behave almost the same, but have well defined gradients.
  - *Reparameterisations* of functions involving randomness.
  - *Approximations* of useable gradients for functions that have ill-posed gradients.

What kinds of functional building blocks are common?

- Today, the most common operations with parameters are:
  - *Vector addition*: the input vector to a function is added to a vector of weights.
  - *Vector-Matrix multiplication*: the input vector to the function is multiplied with a matrix of weights.
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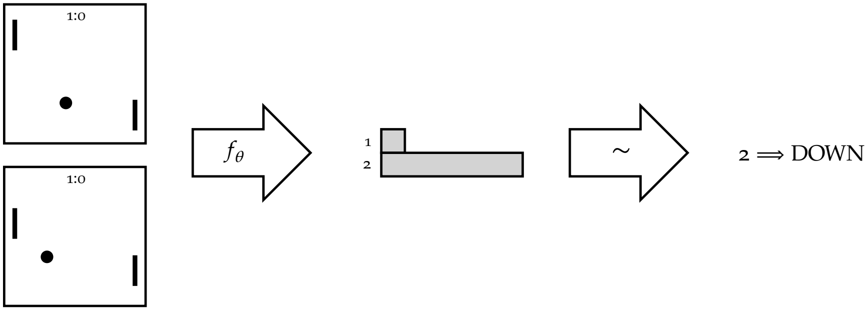
- The above operations are *linear*, so they are often combined with element-wise nonlinearities; e.g.:
  - $\max(0, x)$ aka ReLU.
  - $\tanh(x)$.
  - $\frac{1}{1+e^{-x}}$ aka *sigmoid* or the *logistic* function.

Real Examples of Differentiable Programming

Playing Games

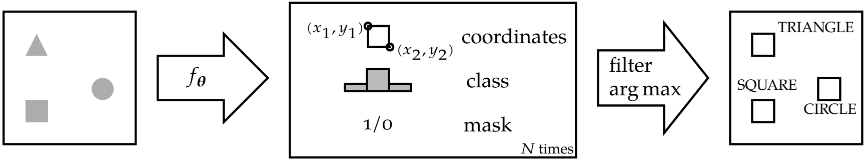
- You can use differentiable programming to write (and train) 'agents' that can play games.
- It can be hard to get a gradient from a single game involving many moves, but there is a clever trick which allows good estimates of gradients to be created over the average of *many* games.
- This is broadly the area of what is called *reinforcement learning*.



Demo: AlphaStar

Object detection

- Consider a function that takes an image as input and produces an array of *bounding boxes* and corresponding *labels*.
- With enough *training data* we can learn the parameters required to detect objects in images.



Object detection

Demo

Drawing

- We could envisage a differentiable function that takes in a set of line coordinates and turns them into an image...
- With such a function we can optimise the line coordinates so they e.g. match a photograph, thus automatically creating a *sketch*.

Where is this all going?

Software 2.0

There is a revolution happening and you're going to be part of it!

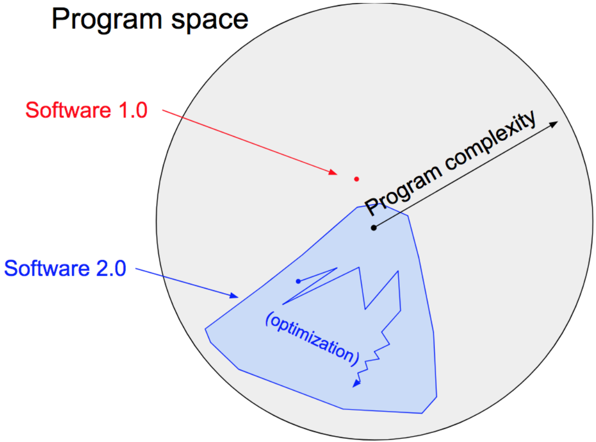


Image credit: Andrei Karpathy

https://karpathy.medium.com/software-2-0-a64152b37c35

Any Questions?