

# Recap of Basic Neural Networks

(and some Deep Network Fundamentals)

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## Classical Types of Learning

- Supervised Learning - learn to predict an output when given an input vector
- Unsupervised Learning - discover a good internal representation of the input
- Reinforcement Learning - learn to select an action to maximize the expectation of future rewards (payoff)
- Semi-supervised Learning - learn with few labelled examples and many unlabelled ones

## Other Types of Learning

- Self-supervised Learning - learn with targets induced by a prior on the unlabelled training data
- Active Learning - learn by seeking guidance from human or oracle when needed (iterative semi-supervised learning)
- Continual Learning - learn new tasks/classes sequentially (iterative supervised/unsupervised learning)
- Online learning - learning one example at a time sequentially (iterative supervised learning)

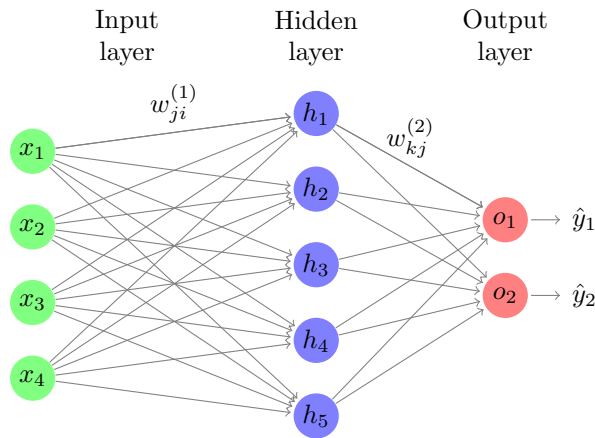
## Two Types of Supervised Learning

- Regression: The machine is asked predict  $k$  numerical values given some input. The machine is a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^k$ .
- Classification: The machine is asked to specify which of  $k$  categories some input belongs to.
  - Multiclass classification - target is one of the  $k$  classes
  - Multilabel classification - target is some number of the  $k$  classes
  - In both cases, the machine is a function  $f : \mathbb{R}^n \rightarrow \{1, \dots, k\}$ .
- It is most common for both types of algorithms to actually learn  $\hat{f} : \mathbb{R}^n \rightarrow \mathbb{R}^k$ .
- Note that there are lots of exceptions in the form the inputs (and outputs) can take though! We'll see lots of variations in the coming weeks.

## How Supervised Learning Typically Works

- Start by choosing a model-class:  $\hat{y} = f(\mathbf{x}; \mathbf{W})$  where the model-class  $f$  is a way of using some numerical parameters,  $\mathbf{W}$ , to map each input vector  $\mathbf{x}$  to a predicted output  $\hat{y}$ .
- *Learning* means adjusting the parameters to reduce the discrepancy between the true target output  $y$  on each training case and the output  $\hat{y}$ , predicted by the model.

Let's look at a Multilayer Perceptron (without biases)...



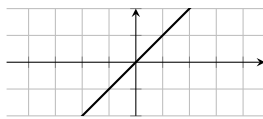
Without loss of generality, we can write the above as:

$$\hat{\mathbf{y}} = g(f(\mathbf{x}; \mathbf{W}^{(1)}); \mathbf{W}^{(2)}) = g(\mathbf{W}^{(2)} f(\mathbf{W}^{(1)} \mathbf{x}))$$

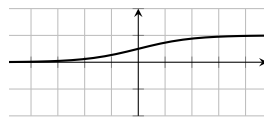
where  $f$  and  $g$  are activation functions.

### Common Activation Functions

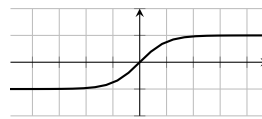
- Identity
- Sigmoid (aka Logistic)
- Hyperbolic Tangent (tanh)
- Rectified Linear Unit (ReLU) (aka Threshold Linear)



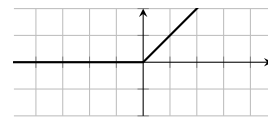
Identity



Sigmoid



tanh



ReLU

### Final layer activations

$$\hat{\mathbf{y}} = g(\mathbf{W}^{(2)} f(\mathbf{W}^{(1)} \mathbf{x}))$$

- What form should the final layer function  $g$  take?
- It depends on the task (and on the chosen loss function)...
  - For regression it is typically linear (e.g. identity), but you might choose others if you say wanted to clamp the range of the network.
  - For binary classification (MLP has a single output), one would choose Sigmoid
  - For multilabel classification, typically one would choose Sigmoid
  - For multiclass classification, typically you would use the Softmax function

### Softmax

The softmax is an activation function used at the output layer of a neural network that forces the outputs to sum to 1 so that they can represent a probability distribution across a discrete mutually exclusive alternatives.

$$\text{softmax}(\mathbf{z})_i = \frac{e^{z_i}}{\sum_{j=1}^K e^{z_j}} \quad \forall i = 1, 2, \dots, K$$

- Note that unlike the other activation functions you've seen, softmax makes reference to all the elements in the output.
- The output of a softmax layer is a set of positive numbers which sum up to 1 and can be thought of as a probability distribution.

- Note:

$$\begin{aligned}\frac{\partial \text{softmax}(\mathbf{z})_i}{\partial z_i} &= \text{softmax}(z_i)(1 - \text{softmax}(z_i)) \\ \frac{\partial \text{softmax}(\mathbf{z})_i}{\partial z_j} &= \text{softmax}(z_i)(1(i=j) - \text{softmax}(z_j)) \\ &= \text{softmax}(z_i)(\delta_{ij} - \text{softmax}(z_j))\end{aligned}$$

## Ok, so let's talk loss functions

- The choice of loss function depends on:
  1. the task (e.g. classification/regression/something else)
  2. the activation function of the last layer
    - For numerical reasons (see Log-Sum-Exp in a few slides)

many times the activation is computed directly within the loss rather than being part of the model
- Some classification losses require *raw outputs* (e.g. a linear layer) of the network as their input
  - These are often called *unnormalised log probabilities* or *logits*
  - An example would be hinge-loss used to create a Support Vector Machine that maximises the margin — e.g.:  $\ell_{\text{hinge}}(\hat{y}, y) = \max(0, 1 - y \cdot \hat{y})$  with a true label,  $y \in \{-1, 1\}$  for binary classification.
- There are many different loss functions we might encounter (MSE, Cross-Entropy, KL-Divergence, huber, L1 (MAE), CTC, Triplet, ...) for different tasks.

## The Loss Function (measure of discrepancy)

Recall from Foundations of Machine Learning:

- Mean Squared Error (MSE) loss for a single data point (here assumed to be a vector, but equally applicable to a scalar)
- is given by
- $$\ell_{MSE}(\hat{\mathbf{y}}, \mathbf{y}) = \sum_i (\hat{y}_i - y_i)^2 = (\hat{\mathbf{y}} - \mathbf{y})^\top (\hat{\mathbf{y}} - \mathbf{y})$$
- We often multiply this by a constant factor of  $\frac{1}{2}$  — can anyone guess/remember why?
  - $\ell_{MSE}(\hat{\mathbf{y}}, \mathbf{y})$  is the predominant choice for regression problems with linear activation in the last layer
  - For a classification problem with Softmax or Sigmoidal (or really anything non-linear) activations MSE can cause slow learning, especially if the predictions are very far off the targets
    - Gradients of  $\ell_{MSE}$  are proportional to the difference in target and predicted value, multiplied by the gradient of the activation function<sup>1</sup>
    - **The Cross-Entropy loss function is generally a better choice in this case**

## Binary Cross-Entropy

For the binary classification case:

$$\ell_{BCE}(\hat{y}, y) = -y \log(\hat{y}) - (1 - y) \log(1 - \hat{y})$$

- The cross-entropy loss function is non-negative,  $\ell_{BCE} > 0$
- $\ell_{BCE} \approx 0$  when the prediction and targets are equal (i.e.  $y = 0$  and  $\hat{y} = 0$  or when  $y = 1$  and  $\hat{y} = 1$ )
- With Sigmoidal final layer,  $\frac{\partial \ell_{BCE}}{\partial \mathbf{W}_i^{(2)}}$  is proportional to just the error in the output ( $\hat{y} - y$ ) and therefore, the larger the error, the faster the network will learn!
- Note that the BCE is the negative log likelihood of the Bernoulli Distribution

<sup>1</sup><http://neuralnetworksanddeeplearning.com/chap3.html>

## Binary Cross-Entropy — Intuition

- The cross-entropy can be thought of as a **measure of surprise**.
- Given some input  $x_i$ , we can think of  $\hat{y}_i$  as the estimated probability that  $x_i$  belongs to class 1, and  $1 - \hat{y}_i$  is the estimated probability that it belongs to class 0.
- Note the extreme case of infinite cross-entropy, if your model believes that a class has 0 probability of occurrence, and yet the class appears in the data, the ‘surprise’ of your model will be infinitely great.

## Binary Cross-Entropy for multiple labels

In the case of multi-label classification with a network with multiple sigmoidal outputs you just sum the BCE over the outputs:

$$\ell_{BCE} = - \sum_{k=1}^K [y_k \log(\hat{y}_k) + (1 - y_k) \log(1 - \hat{y}_k)]$$

where  $K$  is the number of classes of the classification problem,  $\hat{y} \in \mathbb{R}^K$ .

## Numerical Stability: The Log-Sum-Exp trick

$$\ell_{BCE}(\hat{y}, y) = -y \log(\hat{y}) - (1 - y) \log(1 - \hat{y})$$

- Consider what might happen early in training when the model might confidently predict a positive example as negative
  - $\hat{y} = \sigma(z) \approx 0 \implies z \ll 0$
  - if  $\hat{y}$  is small enough, it will become 0 due to limited precision of floating-point representations
  - but then  $\log(\hat{y}) = -\infty$ , and everything will break!
- To tackle this problem implementations usually combine the sigmoid computation and BCE into a single loss function that you would apply to a network with linear outputs (e.g. `BCEWithLogitsLoss`).
- Internally, a trick called ‘log-sum-exp’ is used to *shift* the centre of an exponential sum so that only numerical underflow can potentially happen, rather than overflow<sup>2</sup>
  - Ultimately this means you’ll always get a numerically reasonable result (and will avoid NaNs and Infs originating from this point).

## Multiclass classification with Softmax Outputs

- Softmax can be thought of making the  $K$  outputs of the network mimic a probability distribution.
- The target label  $y$  could also be represented as a distribution with a single 1 and zeros everywhere else.
  - e.g. they are “one-hot encoded”.
- In such a case, the obvious loss function is the *negative log likelihood* of the Categorical distribution (aka Multinoulli, Generalised Bernoulli, Multinomial with one sample)<sup>3</sup>:  $\ell_{NLL} = - \sum_{k=1}^K y_k \log \hat{y}_k$ 
  - Note that in practice as  $y_k$  is zero for all but one class you don’t actually do this summation, and if  $y$  is an integer class index you can write  $\ell_{NLL} = -\log \hat{y}_y$ .
- Analogously to what we saw for BCE,

Log-Sum-Exp can be used for better numerical stability. PyTorch combines LogSoftmax with NLL in one loss and calls this “Categorical Cross-Entropy” (so you would use this with a **linear output layer**)

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<sup>2</sup><https://www.xarg.org/2016/06/the-log-sum-exp-trick-in-machine-learning/>

<sup>3</sup>Note: Keras calls this function ‘Categorical Cross-Entropy’; you would need to have a Softmax output layer to use this

### Reminder: Gradient Descent

- Define total loss as  $\mathcal{L} = \sum_{(\mathbf{x}, y) \in \mathcal{D}} \ell(g(\mathbf{x}, \boldsymbol{\theta}), y)$  for some loss function  $\ell$ , dataset  $\mathcal{D}$  and model  $g$  with learnable parameters  $\boldsymbol{\theta}$ .
- Define how many passes over the data to make (each one known as an Epoch)
- Define a learning rate  $\lambda$

Gradient Descent updates the parameters  $\boldsymbol{\theta}$  by moving them in the direction of the negative gradient with respect to the **total loss**  $\mathcal{L}$  by the learning rate  $\lambda$  multiplied by the gradient:

```
for each Epoch:
     $\boldsymbol{\theta} \leftarrow \boldsymbol{\theta} - \lambda \nabla_{\boldsymbol{\theta}} \mathcal{L}$ 
```

### Reminder: Stochastic Gradient Descent

- Define loss function  $\ell$ , dataset  $\mathcal{D}$  and model  $g$  with learnable parameters  $\boldsymbol{\theta}$ .
- Define how many passes over the data to make (each one known as an Epoch)
- Define a learning rate  $\lambda$

Stochastic Gradient Descent updates the parameters  $\boldsymbol{\theta}$  by moving them in the direction of the negative gradient with respect to the loss of a **single item**  $\ell$  by the learning rate  $\lambda$  multiplied by the gradient:

```
for each Epoch:
    for each  $(\mathbf{x}, y) \in \mathcal{D}$ :
         $\boldsymbol{\theta} \leftarrow \boldsymbol{\theta} - \lambda \nabla_{\boldsymbol{\theta}} \ell$ 
```

### A Quick Introduction to Tensors

Broadly speaking a tensor is defined as a linear mapping between sets of algebraic objects<sup>4</sup>.

A tensor  $T$  can be thought of as a generalization of scalars, vectors and matrices to a single algebraic object.

We can just think of this as a multidimensional array<sup>5</sup>.

- A  $0D$  tensor is a scalar
- A  $1D$  tensor is a vector
- A  $2D$  tensor is a matrix
- A  $3D$  tensor can be thought of as a vector of identically sized matrices
- A  $4D$  tensor can be thought of as a matrix of identically sized matrices or a sequence of  $3D$  tensors
- ...

### Aside: Tensor Decompositions

- Just in the same way a matrix can be decomposed into a product of matrices (EVD, SVD, QR, LU, Cholesky, ...), there are tensor decompositions:
  - PARAFAC / Canonical polyadic / HO-SVD / Tucker
  - These have found their way into some deep learning models as a form of structural regularisation or weight reduction

### Operations on Tensors in PyTorch

- PyTorch lets you do all the standard matrix operations on  $2D$  tensors
  - including important things you might not yet have seen like the **hadamard product** of two  $N \times M$  matrices:  $\mathbf{A} \odot \mathbf{B}$ )
- You can do element-wise add/divide/subtract/multiply to  $ND$ -tensors

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<sup>4</sup>This statement is always entirely true

<sup>5</sup>This statement will upset mathematicians and physicists because its not always true for them (but it is for us!).

- and even apply scalar functions element-wise (log, sin, exp, ...)
- you can slice, reshape, and *even index a single element* (**generally don't do that!**)
- PyTorch often lets you *broadcast* operations (just like in numpy)
  - if a PyTorch operation supports broadcast, then its Tensor arguments can be automatically expanded to be of equal sizes (without making copies of the data).<sup>6</sup>

## Tensors, batches and vectorisation

- The reality of training a model is that we neither use gradient descent or stochastic gradient descent; we do something in-between called mini-batch SGD.
- This works on batches of data (e.g. small subsets of the training set)
- These batches are assembled into a tensor
- Broadcasting is used to apply operations/functions to all the samples in the batch tensor *in parallel* to compute a loss vector
- the loss vector is summed/averaged using a *vectorised* method (e.g. `.sum()`)

## Tensor implementation

It's important to understand something about how tensors are implemented in software and particularly how memory copies can be avoided...

## We are siamese...

An important and clever trick:

## Homework

PyTorch Tensor 101: <https://colab.research.google.com/gist/jonhare/d98813b2224dddabb234d2031510878e1/notebook.ipynb>

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<sup>6</sup>Important - read and understand this after the lab: <https://pytorch.org/docs/stable/notes/broadcasting.html>