Approximate Functions

Going Deep

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Overview

- No free lunch and universal approximation
- Why go deep?
- Problems of going deep
- Some fixes:
  - Improving gradient flow with skip connections
  - Regularising with Dropout

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  - **No machine learning algorithm is universally better than any other!**
  - Fortunately, in the real world, data is generated by a small subset of generating distributions...

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$\implies$ simple neural networks can represent a wide variety of interesting functions when given appropriate parameters.

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    - The optimiser might not find a good solution[2].
    - The training algorithm might just choose the wrong solution as a result of overfitting.
    - *There is no known universal procedure for examining a set of examples and choosing a function that will generalise to points out of the training set.*

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Then Why Go Deep?

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- **Empirically, deeper networks just seem to generalise better!**

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  - Overfitting

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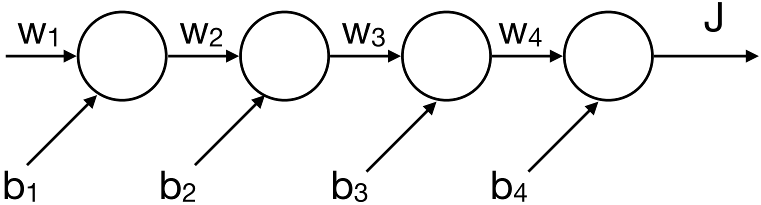
- In principle, optimisers that rescale the gradients of each weight should be able to deal with this issue (as long as numeric precision doesn't become problematic).

Issues with Going Deep



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- Is this the full story though? Skip connections also break symmetries, which could be much more important...

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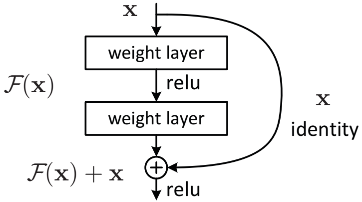

Figure 2. Residual learning: a building block.

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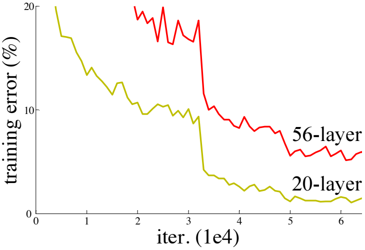

Figure 1. Training error (left) and test error (right) on CIFAR-10 with 20-layer and 56-layer "plain" networks. The deeper network has higher training error, and thus test error. Similar phenomena on ImageNet is presented in Fig. 4.

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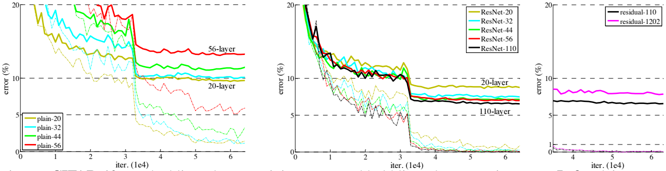

Figure 6. Training on **CIFAR-10**. Dashed lines denote training error, and bold lines denote testing error. **Left**: plain networks. The error of plain-110 is higher than 60% and not displayed. **Middle**: ResNets. **Right**: ResNets with 110 and 1202 layers.

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  - Regularise by smoothing the optimisation landscape (e.g. Batch Normalisation)

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- Motivation: the best way to regularise a fixed size model is to average predictions over all possible parameter settings, weighting each setting by the posterior probability given the training data.
  - Clearly this isn't actually tractable - dropout is an approximation of this idea.
  - The idea of averaging predictions to resolve the bias-variance dilemma is called ensembling.

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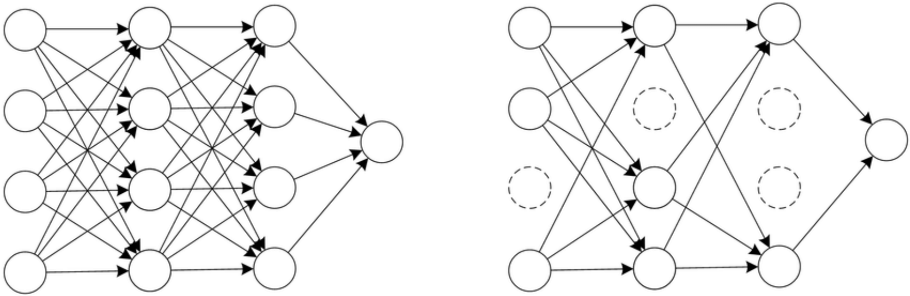

(a) Standard Neural Network

(b) Network after Dropout

Image from: `https://www.researchgate.net/figure/Dropout-neural-network-model-a-is-a-standard-neural-network-b-is-the-same-network_fig3_309206911`

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- **Inverse Dropout** scales the activations with their probability to maintain the overall magnitude of the response when dropout is disabled at evaluation/test time.

How is Inverted Dropout implemented?

- We define a random binary mask $\boldsymbol{m}^{(l)}$ which is used to remove neurons and is generated by sampling a Bernoulli distribution with $P(x = 1) = p$, and note, $m^{(l)}$ changes for each iteration of the backpropagation algorithm.

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- The gradient (during training) is simply the hadamard product of the incoming gradient with $\boldsymbol{m}/p$.

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- By breaking co-adaptation, each unit will ultimately find more general features.
- By ensembling (averaging) multiple networks, each relying on different (but overlapping) features we get a more effective machine.