

**Make a
forward pass
before the
backward pass**

Backpropagation: Understanding the implications of the chain rule

Jonathon Hare

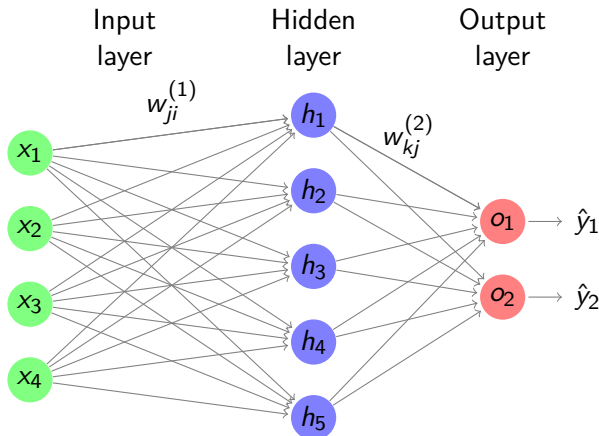
Vision, Learning and Control
University of Southampton

A lot of the ideas in this lecture come from Andrej Karpathy's blog post on backprop (<https://medium.com/@karpathy/yes-you-should-understand-backprop-e2f06eab496b>) and his CS231n Lecture Notes (<http://cs231n.github.io/optimization-2/>)



- A quick look at an MLP again
- The chain rule (again)
- Unintuitive gradient effects
- A closer look at basic stochastic gradient descent algorithms

The unbiased Multilayer Perceptron (again)...



Without loss of generality, we can write the above as:

$$\hat{\mathbf{y}} = g(f(\mathbf{x}; \mathbf{W}^{(1)}); \mathbf{W}^{(2)}) = g(\mathbf{W}^{(2)} f(\mathbf{W}^{(1)} \mathbf{x}))$$

where f and g are activation functions.

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- (But we're not that crazy!)

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$$\text{so } \nabla_{[x, y, z]} f = [z, z, q]$$

A computational graph perspective

$$f(x, y, z) = (x + y)z$$

An intuition of the chain rule

- Notice how every operation in the computational graph given its inputs can immediately compute two things:
 - ① its output value
 - ② the *local* gradient of its inputs with respect to its output value
- The chain rule tells us literally that each operation should take its local gradients and multiply them by the gradient that *flows* backwards into it

This is backpropagation

- The backprop algorithm is just the idea that you can perform the forward pass (computing and caching the local gradients as you go),
- and then perform a backward pass to compute the total gradient by applying the chain rule and re-utilising the cached local gradients

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- Backprop is just another name for 'Reverse Mode Automatic Differentiation'...

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 - **Hence you need to always pay attention to data normalisation!**

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 - What's the implication of this in a deep network with sigmoid activations?

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- These are dead ReLUs - ones that never fire for all training data
 - Sometimes you can find that you have a large fraction of these
 - if you get them from the beginning, check weight initialisation and data normalisation
 - if they're appearing during training, maybe λ is too big?

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- Consider $z = a \prod_n^\infty b$
 - $z \rightarrow 0$ if $|b| < 1$
 - $z \rightarrow \infty$ if $|b| > 1$
- Same thing happens in the backward pass of an RNN (although with matrices rather than scalars, so the reasoning applies to the largest eigenvalue)