Maths behind constant function market makers

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June 2021

1 Definition

1.1 Constant function

A peer-to-pool swap denoted $(\Delta x, \Delta y)$ is where an agent (*aka* taker) transfers (or sells) $\Delta x > 0$ amount in asset X to a liquidity pool (*aka* maker), and in return receives (or buys) $\Delta y > 0$ amount in asset Y from the same pool. In the perspective of pool L, Δx amount of asset X is 'swapped in' and Δy amount of asset Y is 'swapped out'.

For a liquidity pool L with asset X and asset Y with reserve balances x and y respectively, if there exists a function Θ where the following equality holds before and after swaps in a no-fee environment, then L is a constant function market maker (CFMM).

$$L : \Theta(x, y) = const. \tag{1}$$

The reserves of pool L after a swap is $x+\Delta x$ in X and $y-\Delta y$ in Y respectively, hence the relationship between Δx and Δy is

$$\Theta(x + \Delta x, y - \Delta y) = \Theta(x, y). \tag{2}$$

We define the relative swap-in size with respect to asset X λ_x and the relative swap-out size with respect to asset Y λ_y as the ratio between the swap size and the pre-swap reserve balance of their respective asset

$$(\lambda_x, \lambda_y) \coloneqq \left(\frac{\Delta x}{x}, \frac{\Delta y}{y} \right). \tag{3}$$

We assume asset balances are positive $x > 0$ and $y > 0$ unless mentioned otherwise. We also assume no fees $f = 0$ are involved in swapping unless mentioned otherwise.

1.2 Pricing and valuing

The execution (or actual) price of Y denominated in X p_e in swap $(\Delta x, \Delta y)$ is

$$p_e = \frac{\Delta x}{\Delta y}, \tag{4}$$

whereas the virtual (or marginal) price of Y denominated in X p_v is

$$p_v = \frac{\partial \Delta x}{\partial \Delta y}\Big|_{(\Delta x, \Delta y) \to (0,0)}, \tag{5}$$

The virtual values of asset X and asset Y in pool L pre-swap, as well as the virtual values of swap-in and swap-out are denoted as ϕ_x, ϕ_y, $\phi_{\Delta x}$, and $\phi_{\Delta y}$ respectively with the following forms

$$(\phi_x, \phi_y, \phi_{\Delta x}, \phi_{\Delta y}) = (x, p_v y, \Delta x, p_v \Delta y)\,. \tag{6}$$

The total virtual value of the pool Φ denominated in X is the sum of virtual values of asset X and Y

$$\Phi = \phi_x + \phi_y, \tag{7}$$

such that we can define the swap-to-pool ratio Λ as the ratio between swap-in size and the total virtual value of the pool

$$\Lambda \coloneqq \frac{\Delta x}{\Phi}. \tag{8}$$

A constant function curve is usually visualised on a x-y chart in form of $y(x)$ with a slope M, which is usually related to the marginal price p_v

$$M(p_v) = \frac{\partial y}{\partial x}. \tag{9}$$

1.3 Slippage and impermanent loss

Slippage S in takers' perspective is the relative loss in value between post- and pre-swaps, which can be written as

$$S \coloneqq \frac{\phi_{\Delta y} - \phi_{\Delta x}}{\phi_{\Delta x}} = \frac{p_v}{p_e} - 1. \tag{10}$$

As time passes, the state of pool L has transformed from (x, y) to (x', y'). We define the ratio in virtual price from p_v to p'_v due to the imbalance of aggregated swap flows as $P > 0$ such that

$$P = \frac{p'_v}{p_v}. \tag{11}$$

Impermanent loss V in makers' perspective is the relative loss in asset value between pooling the assets and holding the assets for a given P.

$$V \coloneqq \frac{\phi_{x'} + \phi_{y'}}{\phi_x + \phi_y P} - 1, \tag{12}$$

where $\phi_{x'} = x'$ and $\phi_{y'} = y' \cdot p'_v = y' \cdot P p_v$ represent new virtual value of asset X and Y in pool L respectively.

2 Common constant functions

2.1 Constant product market maker

A liquidity pool is considered as a constant product market maker (CPMM) if the product of reserves X and Y remains constant. We use K^2 as the constant term in (13) instead of the commonly known $xy = k$ for the sake of consistency.

$$\Theta_{\text{cpmm}}(x, y) = x \cdot y = K^2 \tag{13}$$

Substituting (13) into (2), the relative swap-out size λ_y is

$$\lambda_y := \frac{\Delta y}{y} = 1 - \frac{x}{x + \Delta x} = \frac{\Delta x}{x + \Delta x} = \frac{\lambda_x}{1 + \lambda_x}, \tag{14}$$

for a given Δx. Rearranging (14), the relative swap-in size λ_x is

$$\lambda_x := \frac{\Delta x}{x} = \frac{y}{y - \Delta y} - 1 = \frac{\Delta y}{y - \Delta y} = \frac{\lambda_y}{1 - \lambda_y}, \tag{15}$$

for a given Δy. It is trivial that $\lambda_x < \lambda_y$.

Using (5), the virtual price of Y denominated in X p_v is

$$p_v = \frac{\partial \Delta x}{\partial \Delta y}\bigg|_{(\Delta x, \Delta y) \to (0,0)} = \lim_{\Delta y \to 0} \frac{xy}{(y - \Delta y)^2} = \frac{x}{y}, \tag{16}$$

such that by substituting (16) into (6), we found that the virtual value of asset X ϕ_x and Y ϕ_y in pool L are equal

$$\phi_x = x \cdot 1 \equiv y \cdot p_v = \phi_y. \tag{17}$$

Therefore, the total virtual pool value Φ is equivalent to $2\phi_x$, and the swap-to-pool ratio Λ is

$$\Lambda = \frac{\Delta x}{2\phi_x} = \frac{\lambda_x}{2}. \tag{18}$$

On the other hand, the relative swap size λ_x and λ_y can be formulated in p_e and p_v

$$(\lambda_x, \lambda_y) = \left(\frac{p_e - p_v}{p_v}, \frac{p_e - p_v}{p_e} \right). \tag{19}$$

Usually, a CPMM is visualised in a x-y chart in form of $y = \frac{K^2}{x}$ with a slope $M \in (-\infty, 0)$ that is equivalent to the negative of the inverse of the marginal price p_v

$$M = \frac{\partial y}{\partial x} = -\frac{K^2}{x^2} = -\frac{y}{x} = -\frac{1}{p_v}. \tag{20}$$

Substituting (4), (14), and (16) into (10), slippage S can be derived as

$$S = -\lambda_y = \frac{\lambda_y}{\lambda_x} - 1 = \frac{1}{1 + \lambda_x} - 1. \tag{21}$$

Substituting (18), we can express slippage as a function of swap-to-pool ratio Λ such that

$$S(\Lambda) = \frac{1}{1+2\Lambda} - 1. \tag{22}$$

It is trivial that $S < 0$ (hence 'loss'). Its first derivative with respect to Λ gives

$$\frac{\partial S}{\partial \Lambda} = -\frac{2}{(1+2\Lambda)^2} < 0, \tag{23}$$

which implies S monotonically decreases (or the magnitude of loss monotonically increases) from $S = 0$ at $(\lambda_x, \lambda_y) \to (0, 0)$ to $S = -1$ at $(\lambda_x, \lambda_y) \to (\infty, 1)$. This means that a larger swap size entails a larger slippage.

Since $\phi_x = \phi_y$ and $\phi_{x'} = \phi_{y'}$, we have

$$\phi_{x'}^2 = \phi_{x'} \cdot \phi_{y'} = x'y'Pp_v = xyPp_v = \phi_x \cdot \phi_y P = \phi_x^2 P, \tag{24}$$

the impermanent loss in (12) can be expressed in term of P

$$V(P) = \frac{2\sqrt{P}}{1+P} - 1. \tag{25}$$

Its first derivative is

$$\frac{dV}{dP} = \frac{1-P}{\sqrt{P}(1+P)^2}, \tag{26}$$

which has a root at $P^* = 1$. This is a global maximum point as confirmed by the first derivative test, which implies $V(P = 1) = 0$ and $V(P \neq 1) < 0$. In other words, pooling assets will incur a loss (negative V) unless the new marginal price is the same as the old one $p'_v = p_v$. Also, impermanent loss V is inversely symmetric with respect to P

$$V(P) = V\left(\frac{1}{P}\right). \tag{27}$$

This is also trivial as the definitions of asset X and Y are arbitrary in calculating impermanent loss.

2.2 Concentrated liquidity

From (5), a CPMM provides a price range of $p_v(x > 0, y > 0) \in (0, \infty)$. If a market maker only intends to provide liquidity in a finite price range $p_v \in [p_L, p_H]$, the constant product function can be modified with the following form

$$\Theta_{cl}(x, y) = (x + c_1) \cdot (y + c_2) = K^2, \tag{28}$$

for some constants c_1 and c_2 such that either (but not both simultaneously) asset balance can reach zero at either end of the price range.

$$x \geq 0, y \geq 0, x + y > 0 \tag{29}$$

Substituting (x, y) with $(x+c_1, y+c_2)$ in (16), we have

$$p_v = \frac{x+c_1}{y+c_2} = \frac{(x+c_1)^2}{K^2} = \frac{K^2}{(y+c_2)^2}. \tag{30}$$

The same substitution in (20) gives the same slope to marginal price relationship

$$M = -\frac{1}{p_v}, \tag{31}$$

whereas substituting in (21) suggests that the slippage S in a concentrated liquidity pool is smaller than a CPMM with the same asset balances (hence the liquidity is 'concentrated').

We define p_L and p_H as the marginal price p_v at $x = 0$ and $y = 0$ respectively. It is trivial that

$$\frac{c_1}{y_{\max}+c_2} = p_L < p_H = \frac{x_{\max}+c_1}{c_2}, \tag{32}$$

hence their subscripts. Evaluating the marginal price p_v at $x = 0$ and $y = 0$ give

$$(c_1, c_2) = \left(K\sqrt{p_L}, \frac{K}{\sqrt{p_H}}\right). \tag{33}$$

Therefore, the constant term of a concentrated liquidity pool with a price range $[p_L, p_H]$ is

$$\Theta_{cl} = (x + K\sqrt{p_L}) \cdot \left(y + \frac{K}{\sqrt{p_H}}\right) = K^2. \tag{34}$$

It converges to a CPMM constant in (13) if $(p_L, p_H) \to (0, \infty)$.

2.3 Constant mean market maker

Constant mean market maker (CMMM) is a generalisation of CPMM that allows multiple assets X_i with reserves x_i in the same liquidity pool L carrying different weights w_i. For an n-asset CMMM, the constant can be formulated as the product of asset reserves raised to the power of their weights $\prod_{i=1}^n x_i^{w_i} = K^*$ with all the positive weights $w_i \in (0, 1)$ sum to one $\sum_{i=1}^n w_i = 1$. However, a swap only involves two assets X and Y, hence it can reduced to

$$\Theta_{\text{cmmm}} = x^w y^{1-w} = K. \tag{35}$$

The weight w and the constant K here are normalised to ignore irrelevant assets in the pool with $x = x_1$, $y = x_2$, $\frac{w_2}{w_1} = \frac{1-w}{w}$, and $K = \left(\frac{K^*}{\prod_{i=3}^n x_i^{w_i}}\right)^{\frac{w}{w_1}}$. When $w = \frac{1}{2}$, it further reduces to the square root representation of (13), which is the definition of a CPMM.

Using (2) and (3), the relative swap-out size λ_y is

$$\lambda_y \coloneqq \frac{\Delta y}{y} = 1 - \left(\frac{x}{x+\Delta x}\right)^{\frac{w}{1-w}}, \tag{36}$$

for a given Δx. Rearranging, the relative swap-in size λ_x is

$$\lambda_x \coloneqq \frac{\Delta x}{x} = \left(\frac{y}{y - \Delta y}\right)^{\frac{1-w}{w}} - 1, \tag{37}$$

for a given Δy. The marginal price p_v is

$$p_v = \frac{\partial \Delta x}{\partial \Delta y}\bigg|_{(\Delta x, \Delta y) \to (0,0)} = \lim_{\Delta y \to 0} \frac{1-w}{w} \frac{x y^{\frac{1-w}{w}}}{(y - \Delta y)^{\frac{1-w}{w}+1}} = \frac{\frac{x}{w}}{\frac{y}{1-w}}. \tag{38}$$

Using (38), the swap-to-pool ratio Λ is

$$\Lambda = \frac{\Delta x}{x + p_v y} = w \cdot \xi_x. \tag{39}$$

On the x-y hyperplane of an CMMM, the slope M is

$$M = \frac{\partial y}{\partial x} = -\frac{w}{1-w} \frac{y}{x} = -\frac{1}{p_v} \tag{40}$$

Applying the definition of slippage S in (10), we get

$$S = \frac{1-w}{w} \frac{1}{\xi_x} \left(1 - \left(\frac{1}{1 + \xi_x}\right)^{\frac{w}{1-w}}\right) - 1, \tag{41}$$

and the expression of S as a function of ξ_y can be derived in a similar fashion. We can express S as a function of w and Λ such that

$$S(w, \Lambda) = \frac{1-w}{\Lambda} \left(1 - \left(\frac{w}{w + \Lambda}\right)^{\frac{w}{1-w}}\right) - 1. \tag{42}$$

If $w = \frac{1}{2}$, this converges to the same slippage expression of a CPMM.

While finding the analytical forms of optima is cumbersome, we can linearise the expression using Taylor's series around $\Lambda = 0$ such that the slippage for small Λ can be approximated as

$$\bar{S} \coloneqq S|_{\Lambda=0} + \frac{\partial S}{\partial \Lambda}\bigg|_{\Lambda=0} \cdot \Lambda + \mathcal{O}(\Lambda^2). \tag{43}$$

Ignoring second or higher order terms, the linearised slippage is

$$\bar{S}(w, \Lambda) = -\frac{\Lambda}{2w(1-w)}. \tag{44}$$

Its first derivative with respect to w is

$$\frac{\partial \bar{S}}{\partial w} = \frac{1 - 2w}{2w^2(1-w)^2} \cdot \Lambda \tag{45}$$

It is apparent that $\bar{S}$ is maximised at $w^* = \frac{1}{2}$ using the first derivative test, and slippage is symmetric around $w = \frac{1}{2}$ locally such that

$$\bar{S}(w, \Lambda \to 0) = \bar{S}(1-w, \Lambda \to 0). \tag{46}$$

In fact, this linearisation is a decent approximation for most realistic Λ that we have witnessed. The symmetry breaks and the optimal weight w^* shifts for larger Λ.

Given

$$\phi_x = \frac{w}{1-w}\phi_y, \tag{47}$$

and

$$\phi_{x'}^{w} \cdot \phi_{y'}^{1-w} = \phi_x^{w} \cdot \phi_y^{1-w} P^{1-w}, \tag{48}$$

the impermanent loss V can be written as

$$V(P) = \frac{P^{1-w}}{w+(1-w)P} - 1. \tag{49}$$

Differentiate it with respect to P gets

$$\frac{\partial V}{\partial P} = \frac{(P-1)(w-1)wP^{-w}}{(P+w-Pw)^2}, \tag{50}$$

which also has a root $P^* = 1$.

2.4 Time-decaying constant mean market maker

A time-decaying constant mean market maker (TD-CMMM) is a two-asset CMMM where the asset weights change over time.

$$\Theta_{\text{td-cmmm}}(x, y) = x^{w(t)} \cdot y^{1-w(t)} = K \tag{51}$$

It inherits most properties from CMMM. The marginal price p_v is

$$p_v(t) = \frac{x}{y}\frac{1-w(t)}{w(t)}. \tag{52}$$

For a linear weight evolution from time $t = 0$ to $t = T$, it can be represented as

$$w(t) = \begin{cases} w_0 + (w_T - w_0)\frac{t}{T} & \text{if } 0 \leq t < T \\ w_T & \text{if } t \geq T \end{cases}, \tag{53}$$

the time derivative of the marginal price $p_v(t)$ is

$$\frac{dp_v}{dt} = \frac{dp_v}{dw}\frac{dw}{dt} = -\frac{x}{yw^2}\frac{w_T - w_0}{T}. \tag{54}$$

In general, $p_v(t)$ decreases over time if $w(t)$ increases over time.

$$sgn\left(\frac{dp_v}{dt}\right) = -sgn\left(\frac{dw}{dt}\right) \tag{55}$$