Appearance of chaos in reversible systems

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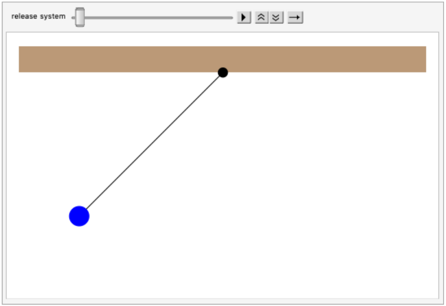
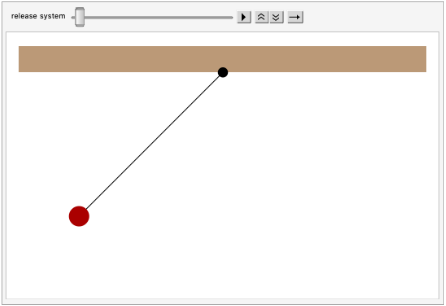
“The origin of reversible systems”

Time-reversal symmetry stands for invariance of the equations under the transformation $t \to -t$



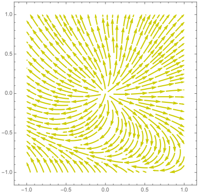
Established symmetry



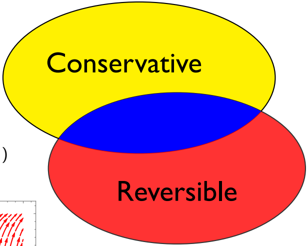
Breaking symmetry

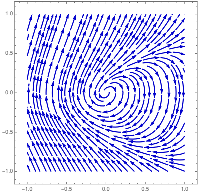
Reversible vs conservative systems

Reversible dynamical systems have an involution in phase space which reverses the direction of time.

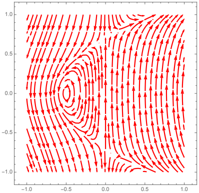


$$x' = x - y^2(1-y),\ \ y' = y + 0.3x'^2(1-x')$$





$$x' = y,\ \ y' = -x + 0.6y + 2y^2 \text{ (Hénon)}$$



$$x' = xy,\ \ y' = 2x - 2y^2 + 1$$

Some literature

Definition - Reversibility

A dynamical system $X' = V(X)$ is *reversible* if there is $\ell\colon \mathbb{R}^2 \to \mathbb{R}^2$ s.t. $\ell^2 = Id$, $D\ell(p)V(p) = -V(\ell(p))\ \forall p \in \mathbb{R}^2$.

Remark

Systems with reversing symmetries are in between conservative and dissipative systems.

Classical references:

- J.R.L. Devaney. *Trans. Amer. Math. Soc.* (1976).
- J.S.W. Lamb and J.A.G. Roberts. *Phys. D* (1998).
- M.B. Sevryuk. *Lecture Notes in Mathematics* (1986).
- M. A. Teixeira. *Phys. D* (1997).

2D reversible systems and equilibria

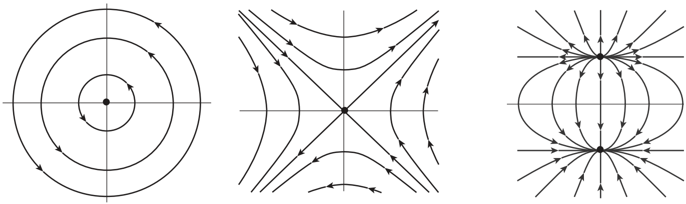
Let $\ell(x,y) = (x,-y)$, so that we consider the planar system

$$(\mathcal{S}) \qquad \begin{cases} x' = yf(x,y^2) \\ y' = g(x,y^2) \end{cases}$$

with f and g smooth.

- Line of symmetry for $(\mathcal{S})$: $\{(x,0) \colon x \in \mathbb{R}\}$.
- Categories of possible equilibria for $(\mathcal{S})$:

centers saddles attractors/repellers



Normal forms

Assume that $(\mathcal{S})$ has a symmetric equilibrium O at the $(0,0)$.

Theorem (Codimension 0)

The normal forms around O of a structurally stable reversible vector field X are:

- $X(x,y) = (y,x)$,
- $X(x,y) = (y,-x)$.

Theorem (Codimension 1)

The normal forms of one-parameter families of structurally stable reversible vector fields X_λ near O are:

(i) *saddle type:* $X_\lambda(x,y) = (xy, x - y^2 + \lambda)$,

(ii) *cusp type:* $X_\lambda(x,y) = (y, x^2 + \lambda)$,

(iii) *nodal type:*
$X_\lambda(x,y) = (xy, x + 2y^2 + \lambda)$ or $X_\lambda(x,y) = (-xy, x - 2y^2 + \lambda)$,

(iv) *focal type:* $X_\lambda(x,y) = (xy + y^3, -x + y^2 + \lambda)$.

M. A. Teixeira, *Phys. D* (1997).

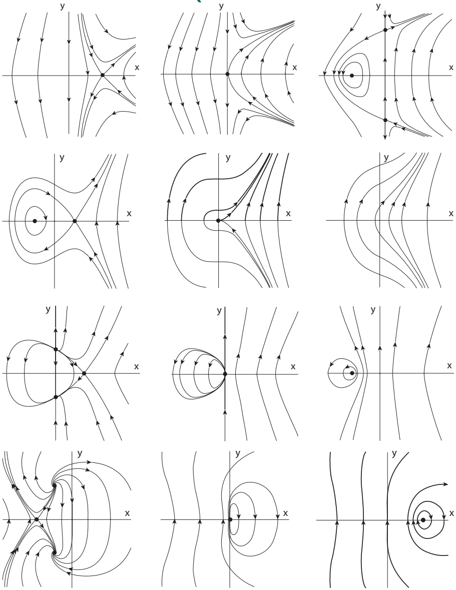
Classification of X_λ near O (Codimension 1)

(i) saddle type

(ii) cusp type

(iii) nodal type

(iv) focal type



$\lambda < 0$ $\lambda = 0$ $\lambda > 0$

Problem setting

- Consider a normal form for a one-parameter family of codimension 1 reversible vector field $X_\lambda(x, y)$.
- Suppose that the dynamical system $X' = X(x, y)$ switches in a T-periodic manner between

$$(\Diamond) \qquad \begin{cases} X' = X_{\lambda_1}(x, y), & t \in [0, \tau_1) \\ X' = X_{\lambda_2}(x, y), & t \in [\tau_1, \tau_1 + \tau_2) \end{cases}$$

  with $\lambda_1 \neq \lambda_2$ and $\tau_1 + \tau_2 = T$.
- Establish whether or not there are open sets of the parameters $\lambda_1,\ \lambda_2$ and open intervals for $\tau_1,\ \tau_2$ where there exist infinitely many T-periodic solutions as well as chaotic-like dynamics for $X' = X(x, y)$.

Symbolic dynamics

Let $h \colon \operatorname{dom} h \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ and $\emptyset \neq \mathcal{D} \subseteq \operatorname{dom} h$.

Definition - Chaos

h induces chaotic dynamics on $m \geq 2$ symbols on $\mathcal{D}$ if there exist m nonempty pairwise disjoint compact sets $\mathcal{K}_0, \dots, \mathcal{K}_{m-1} \subseteq \mathcal{D}$ such that for each two-sided sequence $(s_i)_{i\in\mathbb{Z}} \in \Sigma_m$ there exists a corresponding sequence $(w_i)_{i\in\mathbb{Z}} \in \mathcal{D}^{\mathbb{Z}}$ such that

$$w_i \in \mathcal{K}_{s_i} \text{ and } w_{i+1} = h(w_i) \text{ for all } i \in \mathbb{Z}, \tag{1}$$

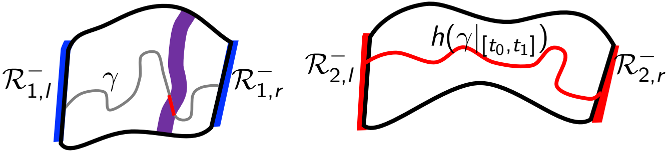
and, whenever $(s_i)_{i\in\mathbb{Z}} \in \Sigma_m$ is a k-periodic sequence for some $k \geq 1$ there exists a k-periodic sequence $(w_i)_{i\in\mathbb{Z}} \in \mathcal{D}^{\mathbb{Z}}$ satisfying (1).

A. Margheri, C. Rebelo and F. Zanolin *J. Differential Equations* (2010).

A. Medio, M. Pireddu and F. Zanolin *Int. J. Bifur. Chaos* (2009).

Stretching Along the Paths

Let $h : \mathrm{dom}\, h \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be continuous. Let $\widehat{\mathcal{R}}_1 := (\mathcal{R}_1, \mathcal{R}_1^-)$, $\widehat{\mathcal{R}}_2 := (\mathcal{R}_2, \mathcal{R}_2^-)$ two topological oriented rectangles.



Definition - SAP property

h stretches $\widehat{\mathcal{R}}_1$ to $\widehat{\mathcal{R}}_2$ along the paths, $(\mathcal{K}, h) : \widehat{\mathcal{R}}_1 \Leftrightarrow\!\!\!\rightarrow \widehat{\mathcal{R}}_2$, if there is $\mathcal{K}$ compact subset of $\mathcal{R}_1 \cap \mathrm{dom}\, h$ and for each path $\gamma : [0, 1] \to \mathcal{R}_1$ such that $\gamma(0) \in \mathcal{R}_{1,l}^-$, $\gamma(1) \in \mathcal{R}_{1,r}^-$ (or vice-versa), there is $[t_0, t_1] \subseteq [0, 1]$

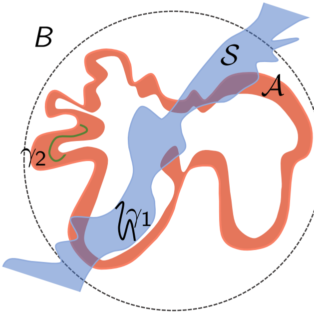
- $\gamma(t) \in \mathcal{K}$ for all $t \in [t_0, t_1]$,
- $h(\gamma(t)) \in \mathcal{R}_2$ for all $t \in [t_0, t_1]$,
- $h(\gamma(t_0))$ and $h(\gamma(t_1))$ belong to different components of $\mathcal{R}_2^-$.

A. Medio, M. Pireddu and F. Zanolin *Int.. J. Bifur. Chaos* (2009).

Rays and bridges

Let $\mathcal{S}$ be a topological strip.
Let $\mathcal{A}$ be a topological annulus with $\partial\mathcal{A} = \partial^i\mathcal{A} \cup \partial^e\mathcal{A}$.

- *A bridge in* $\mathcal{S}$ is a simple continuous curve $\gamma_1 : [a, b] \to \mathcal{S}$ such that $\gamma_1(a)$ and $\gamma_1(b)$ belongs to different components of $\partial\mathcal{S}$.
- *A ray in* $\mathcal{A}$ is a simple continuous curve $\gamma_2 : [a, b] \to \mathcal{A}$ such that $\gamma_2(a) \in \partial^i\mathcal{A}$ and $\gamma_2(b) \in \partial^e\mathcal{A}$ or viceversa.



Linkage condition & streatching

Let Γ a Jordan curve. $out(\Gamma) =$ region outside Γ, $in(\Gamma) =$ region inside Γ.
Given $\mathcal{A}$ a topological annulus and $\mathcal{S}$ a topological strip, assume $\partial^i\mathcal{A} \subset in(\partial^e\mathcal{A})$.

Linkage condition

$\mathcal{A}$ is linked with $\mathcal{S}$ if there exist a bridge γ_1 in $\mathcal{S}$, a ray γ_2 in $\mathcal{A}$, and a topological ball B containing $\mathcal{A}$ such that:

- $\gamma_1 \subset in(\partial^i\mathcal{A})$;
- $\gamma_2 \cap \mathcal{S} = \emptyset$;
- $(\mathcal{S} \setminus \gamma_1) \cap \partial B$ consists of exactly two disjoint bridges.

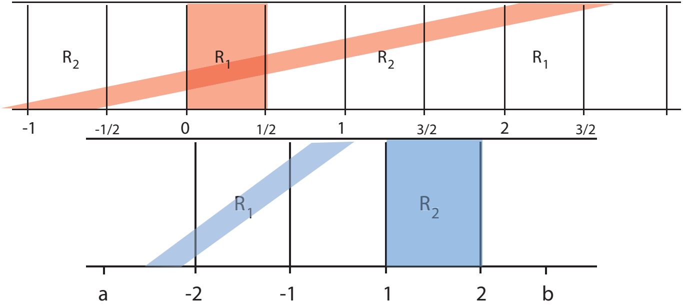
Theorem

Let $\mathcal{R}_1, \mathcal{R}_2$ be disjoint oriented topological rectangles through the linkage of $\mathcal{A}$ with $\mathcal{S}$. Let $\phi_\mathcal{A} : \mathcal{A} \to \mathcal{A}$ and $\phi_\mathcal{S} : \mathcal{S} \to \mathcal{S}$ be continuous maps that satisfy the boundary invariance conditions, and the twist conditions. Then, $\phi_\mathcal{A} \circ \phi_\mathcal{S} : \widehat{\mathcal{R}}_j \Leftrightarrow\!\!\!\!\longrightarrow^{m-1} \widehat{\mathcal{R}}_j$ and $\phi_\mathcal{S} \circ \phi_\mathcal{A} : \widehat{\mathcal{R}}_{j+1} \Leftrightarrow\!\!\!\!\longrightarrow^{m-1} \widehat{\mathcal{R}}_{j+1}$ for some $j \pmod 2$ with $m = |j_{-1} - j_1| + 1$.

D. Papini, G. Villari and F. Zanolin *Differential Integral Equations* (2019).

I. Labouriau and E.S., J. Singularities (2020).

Linkage condition & streatching



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D. Papini, G. Villari and F. Zanolin *Differential Integral Equations* (2019).

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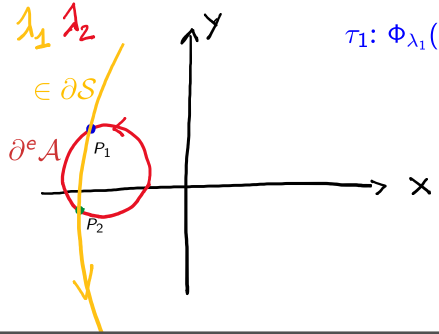
Application to reversible systems

The full dynamics of $(\lozenge)$ can be broken into

$$\begin{cases} x' = yf(x, y^2) \\ y' = g(x, y^2) + \lambda_1 \end{cases} \qquad \begin{cases} x' = yf(x, y^2) \\ y' = g(x, y^2) + \lambda_2 \end{cases}$$

The Poincaré map Φ associated with $(\lozenge)$ may be decomposed as $\Phi = \Phi_{\lambda_2} \circ \Phi_{\lambda_1}$, Φ_{λ_1} and $\Phi_{\lambda_2}(x_0, y_0)$ are the Poincaré maps associated with the two subsystems.



τ_1: $\Phi_{\lambda_1}(P_1) = P_2$

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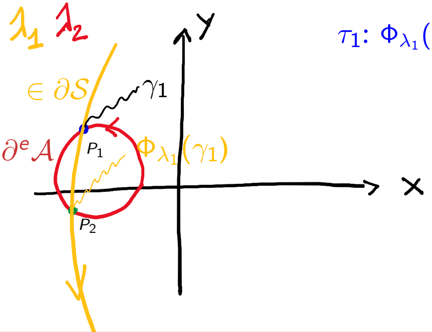

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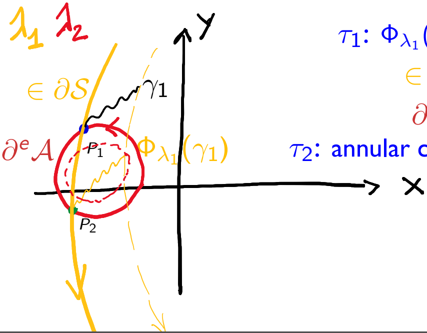


τ_1: $\Phi_{\lambda_1}(P_1) = P_2$

$\in \partial\mathcal{S}$

$\partial^i\mathcal{A}$

τ_2: annular cross condition

The saddle case

Consider the periodically forced reversible system

$$(\star) \qquad \begin{cases} x' = xy \\ y' = x - y^2 + p(t) \end{cases}$$

where

$$p(t) := \begin{cases} \lambda_1 & \text{for } t \in [0, \tau_1) \\ \lambda_2 & \text{for } t \in [\tau_1, \tau_1 + \tau_2) \end{cases}$$

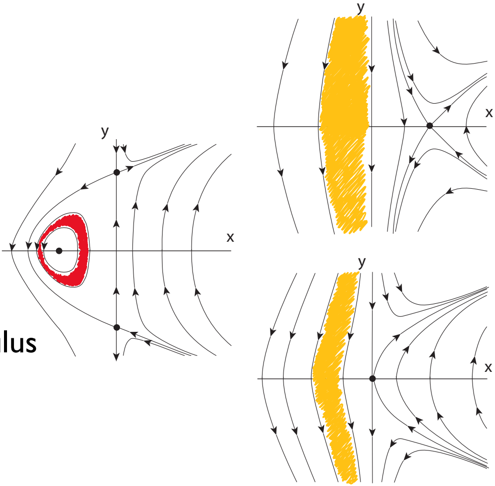
Theorem

Let Φ be the Poincaré map associated with system $(\star)$. Then for each $\lambda_1 > 0$ and each λ_2 with $\lambda_1 > \lambda_2$ and for an open set of values of τ_1 and τ_2 the map Φ induces chaotic dynamics on m symbols, for some $m \geq 2$.

I. Labouriau and E.S., J. Singularities (2020).

Phase-plane analysis

$\lambda_2 < 0$: Strip



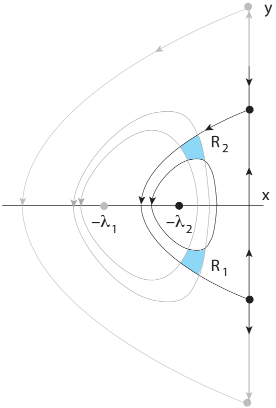
$\lambda_1 > 0$: Annulus

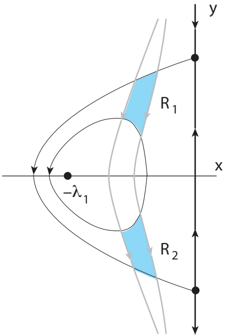
Step I: Linkage.

$\lambda_2 = 0$: Strip

Phase-plane analysis

$0 < \lambda_2 < \lambda_1$ $\qquad$ $\lambda_2 \leq 0 < \lambda_1$

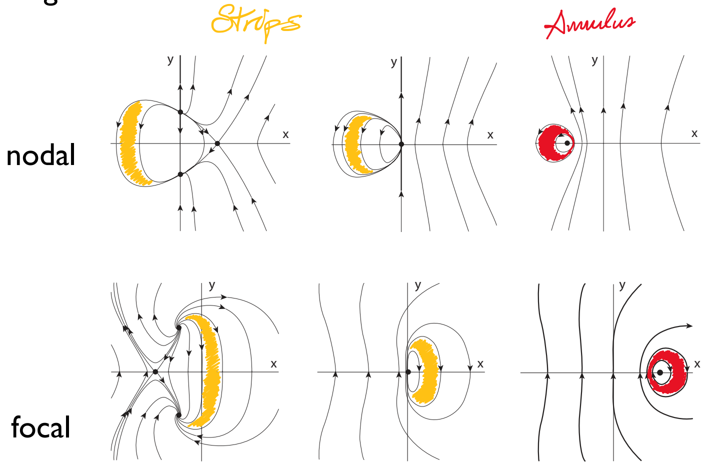




Step II: Stretching (with τ_1, τ_2 large enough).

Other results and open problems

The same approach applies to the *cusp case*! (Configuration already discussed for Ambrosetti-Prodi problems).
The *nodal case* and *focal case* have not been treated yet, but they have good chance of work well!



Grazie dell'attenzione!