MATH 551 (Elementary Topology) Course Notes

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This document was motivated by Professor Botong Wang's MATH 551 lectures during the Fall 2020 semester at UW-Madison. The course covered the majority of Chapters 2 - 3 from James Munkres' *Topology (2nd Edition)* (link). This document should follow the structure of the chapters in the textbook but aims to not require a supporting text.

I welcome feedback on this document. Please feel free to reach out to me by email at egalles@wisc.edu to share your thoughts or concerns. This document was generated on September 29, 2020; for the most recent edition, please visit GitHub (link).

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- *Emmett Galles*

Chapter 2: Topological Spaces and Continuous Functions

Although it is common to use facets of topological spaces in other mathematics classes, we now look to rigorously explain and flesh out the concept. Something I recommend before continuing: set aside the definitions of open and closed sets that were most likely taught in a real analysis course to allow for a more thorough construction of a topological space.

2.1 Topological spaces

We will first define what a topology is.

Definition 2.1.1: Topology

Let a set X be given. A topology $\mathcal{T}$ on the set X is a collection of subsets of X such that

i. $\varnothing$ and X are in $\mathcal{T}$.
ii. Any union of a collection of sets from $\mathcal{T}$ is also in $\mathcal{T}$.
iii. Any finite intersection of elements from $\mathcal{T}$ is also in $\mathcal{T}$.

Recognize that $\mathcal{T}$ is a set of sets!

If a set X has a specified topology $\mathcal{T}$, we denote a topological space by the ordered pair $(X, \mathcal{T})$. It is not uncommon to omit the mention of $\mathcal{T}$ when referring to a topological space.

Please note that throughout this document, the words "subcollection" and "subset" can be used synonymously, as well as "collection" and "set". However, in most situations, collection and subcollection will indicate that both our set of interest is a set whose elements are sets. For example, (ii.) from Definition 2.1.1 can be rewritten as "The union of the elements from any subcollection of $\mathcal{T}$ is also in $\mathcal{T}$." since $\mathcal{T}$ and any subset of $\mathcal{T}$ itself is a set whose elements are sets.

Let's examine how the distinction between finite and any intersection from (iii.) is important with the following example.

Example 2.1.2: Infinite intersection not in set

Find a collection of sets C where an intersection of elements from C is not in C.
Let $C = \left\{ \left[0, \frac{1}{n}\right] : n \in \mathbb{N} \right\}$ and let $P = \bigcap_{p \in C} p$. From previous coursework we know

that $P = \{0\}$, but $P \notin C$ since there does not exist an $n \in \mathbb{N}$ such that $\{0\} = \left[0, \frac{1}{n}\right]$.

Once we construct a topology on a set, we can now look to define what it means for a set to be open.

Definition 2.1.3: Open set

Let a set X be given and let $\mathfrak{T}$ be a topology on X. A subset U of X is an open set if U belongs to the topology $\mathfrak{T}$.

In other words, $U \subset X$ is open $\iff U \in \mathfrak{T}$.

Below are some topologies that are commonly used and have special names.

Remark 2.1.4: Notable topologies

For a set X, the topology $\mathfrak{T} = \{\varnothing, X\}$ is called the trivial topology.

For a set X, the topology $\mathfrak{T} = \{T : T \subset X\}$ is called the discrete topology.

Theorem 2.1.5: Finite complement topology

Let a set X be given and have $\mathfrak{T}_f$ be the collection of all subsets U of X such that $X \setminus U$ is either finite or is all of X. We declare $\mathfrak{T}_f$ to be a topology on X.

Proof: To show that $\mathfrak{T}_f$ is a topology on X, we must uphold the three conditions outlined in Definition 2.1.1.

i. $\varnothing \in \mathfrak{T}_f$ since $X \setminus \varnothing = X$ and $X \in \mathfrak{T}_f$ since $X \setminus X = \varnothing$ is a finite set.

ii. If we let $\{U_i\}$ be a collection of sets from $\mathfrak{T}_f$, we need to show that $(X \setminus \bigcup_{i \in I} U_i) \in \mathfrak{T}_f$. By De Morgan's laws, we know

$$X \setminus \bigcup_{i \in I} U_i = \bigcap_{i \in I} (X - U_i)$$

is either the entire set X or a finite set, since the intersection of finite sets is finite. Therefore, $(X \setminus \bigcup_{i \in I} U_i) \in \mathfrak{T}_f$.

iii. If we let $\{U_j\}$ be a finite collection of sets from $\mathfrak{T}_f$, we need to show that $(X \setminus$

$\bigcup_{j\in J}) \in \mathcal{T}_f$. We will utilize De Morgan's laws again to see that

$$X \setminus \bigcap_{j\in J} U_j = \bigcup_{j\in J} (X \setminus U_j)$$

which is either the entire set X or a finite set, since a finite union of finite sets is finite. Therefore, $(X \setminus \bigcup_{j\in J}) \in \mathcal{T}_f$.

This concludes the proof. □

We now look at a way to compare different topologies on the same set.

Definition 2.1.6: Coarser and finer

Let a set X be given and have $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on X. If $\mathcal{T} \subset \mathcal{T}'$, we say that $\mathcal{T}'$ is finer than $\mathcal{T}$ and $\mathcal{T}$ is coarser than $\mathcal{T}'$. Furthermore, we say that $\mathcal{T}$ and $\mathcal{T}'$ are comparable if the coarser and finer relationships can be established.

We can also construct the notion of strictness. If $\mathcal{T} \subset \mathcal{T}'$ and $\mathcal{T} \neq \mathcal{T}'$, we say that $\mathcal{T}'$ is strictly finer than $\mathcal{T}$ and $\mathcal{T}$ is strictly coarser than $\mathcal{T}'$.

A way to conceptualize coarser and finer is to think about the resolution of a screen. If we allow the pixels and any union of pixels to be a topology on the screen, then a coarser topology would be a screen with lower resolution.

2.2 Basis for a topology

Bases are introduced to us as the building blocks for various topologies on a set. Using a basis to describe a topology is often seen as an easier way to define a topology rather than explicitly describe every set in a topology.

Definition 2.2.1: Basis

Let a set X be given. A basis $\mathcal{B}$ on X is a collection of subsets of X (called basis elements) such that

i. For each $x \in X$, there is a basis element $(B \subset X) \in \mathcal{B}$ such that $x \in B$.

ii. If a point $x \in X$ belongs to the intersection of two basis elements B_1 and B_2, there exists a basis element B_3 such that $x \in B_3$ and $B_3 \subset (B_1 \cap B_2)$.

We now look to generate a topology from a basis.

Theorem 2.2.2: Generating a topology from a basis

Let a set X and a basis $\mathcal{B}$ be given. We can generate a topology $\mathcal{T}$ on X by the following: a set $U \subset X$ is open in $\mathcal{T}$ (meaning $U \in \mathcal{T}$) if for each $x \in U$, there exists a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$.

Proof: To show that $\mathcal{T}$ is a topology on X, we must uphold the three conditions outlined in Definition 2.1.1.

i. Note that when $U = \varnothing$, we have $U \in \mathcal{T}$ vacuously. Furthermore, when $U = X$, we can find a basis element B for each $x \in X$ where $x \in B$ and $B \subset X$. Recall from Definition 2.2.1 that each $B \in \mathcal{B}$ is a subset of X and each $x \in X$ must belong to a basis element $B \in \mathcal{B}$.

ii. Let $\{U_i\}$ be a subcollection of $\mathcal{T}$. We need to show that $U = (\bigcup_{i \in I} U_i) \in \mathcal{T}$. For each $x \in U$, there exists some $i \in I$ such that $x \in U_i$. Since $U_i \in \mathcal{T}$, there exists a $B \in \mathcal{B}$ such that $x \in B \subset U_i \subset U$, so $(\bigcup_{i \in I} U_i) \in \mathcal{T}$.

iii. Let $\{U_j\}$ be a finite subcollection of $\mathcal{T}$. We need to show that $U = (\bigcap_{j \in J} U_j) \in \mathcal{T}$. Let's first take two sets U_1 and U_2. For any $x \in U_1 \cap U_2$, let $B_1 \subset U_1$ such that $x \in B_1$ and let $B_2 \subset U_2$ such that $x \in B_2$. By Definition 2.2.1, there exists a $B_3 \subset (B_1 \cap B_2)$ such that $x \in B_3$. Since $B_3 \subset (U_1 \cap U_2)$, we declare that $(U_1 \cap U_2) \in \mathcal{T}$. We can craft the rest of the proof by induction. The case for $n = 1$ is straightforward, so let's assume that the finite intersection of n sets is true and prove it for $n+1$ terms. Observe how

$$(U_1 \cap U_2 \cap \ldots \cap U_n \cap U_{n+1}) = (U_1 \cap U_2 \cap \ldots \cap U_n) \cap U_{n+1}$$

We assumed $(U_1 \cap U_2 \cap \ldots \cap U_n) \in \mathcal{T}$, so now we are intersecting two elements of $\mathcal{T}$. However, we just proved that to be true, so $(U_1 \cap U_2 \cap \ldots \cap U_n \cap U_{n+1}) \in \mathcal{T}$, which means $(\bigcap_{j \in J} U_j) \in \mathcal{T}$.

This concludes the proof. $\square$

With the above proven, this means $B \in \mathcal{B} \implies B \in \mathcal{T}$. Furthermore, note that only one topology $\mathcal{T}$ can be generated from a basis $\mathcal{B}$.

At this point, recall that any element $T \in \mathcal{T}$ and $B \in \mathcal{B}$ are both subsets of the set X that they are defined on. In essence, Theorem 2.2.2 is saying that a set $T \subset X$ belongs to the topology $\mathcal{T}$ generated by $\mathcal{B}$ if for every $x \in T$, we can fit a basis element B within T such that $x \in B$. Another way to describe the topology generated by a basis is seen below.

Lemma 2.2.3: Interpretation of a topology from a basis

Let a set X be given and let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. We declare $\mathcal{T}$ to be the collection of all unions of sets from $\mathcal{B}$.

Proof: By Theorem 2.2.2, we know that each set $B \in \mathcal{B}$ is an element of $\mathcal{T}$. Since $\mathcal{T}$ is a topology, we know the union of $(\{B_i\} \subset \mathcal{B}) \in \mathcal{T}$. This means that $\{\text{all unions from } \mathcal{B}\} \subset \mathcal{T}$.

Let any set $T \in \mathcal{T}$ be given. For each $t \in T$, find a basis element $B_t \in \mathcal{B}$ such that $t \in B_t \subset T$. We now have $T = \bigcap_{t \in T} B_t$, which means $\mathcal{T} \subset \{\text{all unions from } \mathcal{B}\}$.

We have shown that two sets are both subsets of each other. This implies an equality between the two sets, thus the proof is concluded. $\square$

In Theorem 2.2.2 and Theorem 2.2.2, we went from basis to topology. The lemma below is a method to extract a basis from a topology.

Lemma 2.2.4: From topology to basis

Let $(X, \mathcal{T})$ be a topological space. Furthermore, let a subcollection $\mathcal{C} \subset \mathcal{T}$ be given such that for each $T \in \mathcal{T}$ and each $t \in T$, there exists a set $C \in \mathcal{C}$ such that $t \in C \subset T$. We declare $\mathcal{C}$ to be a basis for the topology $\mathcal{T}$ on X.

Proof: To start, we must show that $\mathcal{C}$ is a basis.

i. We've constructed $\mathcal{C}$ such that for all $t \in T$ where $T \in \mathcal{T}$, there exists a $(C \subset X) \in \mathcal{C}$ such that $t \in C$. This satisfies the first requirement for a basis when we let $T = X$ (which is always present by Definition 2.1.1).
ii. Let $x \in X$ and $x \in C_1 \cap C_2$ where $C_1, C_2 \in \mathcal{C}$. Since $C_1, C_2 \in \mathcal{T}$, we know that $(C_1 \cap C_2) \in \mathcal{T}$. By construction of $\mathcal{C}$, there exists a set $C_3 \in \mathcal{C}$ such that $x \in C_3 \subset (C_1 \cap C_2)$.

We now must show that the topology $\mathcal{T}'$ generated by $\mathcal{C}$ equals our topology $\mathcal{T}$.

For a set $T \in \mathcal{T}$ and $t \in T$, there exists a set $C \in \mathcal{C}$ such that $t \in C \subset T$. Therefore, by Theorem 2.2.2, we have that $T \in \mathcal{T}'$, or that $\mathcal{T} \subset \mathcal{T}'$.

Now, if we let a set $Q \in \mathcal{T}'$, we know by Lemma 2.2.3 that Q is simply a union of elements from $\mathcal{C}$. Since each set $C \in \mathcal{C}$ belongs to $\mathcal{T}$ (meaning $C \in \mathcal{T}$) and $\mathcal{T}$ is a topology, then surely $Q \in \mathcal{T}$ as well. This means $\mathcal{T}' \subset \mathcal{T}$.

We have concluded that $\mathcal{T} \subset \mathcal{T}'$ and $\mathcal{T}' \subset \mathcal{T}$, so this means $\mathcal{T} = \mathcal{T}'$. $\square$

With two topologies established from two different bases, we would like to establish which topology is finer based purely on what their bases are. We discuss a process for this in the following lemma.

Lemma 2.2.5: Comparable topologies from bases

Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively, on X. The following statements are equivalent:

i. $\mathcal{T}'$ is finer than $\mathcal{T}$.
ii. For each $x \in X$ and each basis element $B \in \mathcal{B}$ containing x, there is a basis element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.

Proof: Let's first assume that (ii.) is given. We want to show that if given a set $T \in \mathcal{T}$, we also have $T \in \mathcal{T}'$. Let $t \in T$. Since the basis $\mathcal{B}$ generates the topology $\mathcal{T}$, we know by Theorem 2.2.2 that there exists a set $B \in \mathcal{B}$ such that $x \in B \subset T$. We are told in (ii.) that there exists an element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$. This means $x \in B' \subset B \subset T$, so by Theorem 2.2.2 we have $T \in \mathcal{T}'$, implying $\mathcal{T} \subset \mathcal{T}'$.

Let's now assume that (i.) is given. Furthermore, assume that an $x \in X$ and $B \in \mathcal{B}$ with $x \in B$ is given. By Lemma 2.2.3 we know $B \in \mathcal{T}$ and by (i.) we have $\mathcal{T} \subset \mathcal{T}'$, so therefore $B \in \mathcal{T}'$. Since $\mathcal{T}'$ is generated by $\mathcal{B}'$, we know there is an element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.

The proof is concluded. $\square$

The important thing to remember from Lemma 2.2.5 is that if we have finer basis elements, then the topology generated from the basis of those finer elements will be finer than the topology generated from coarser basis elements.

Topologies on the real line

We often look at the real line, both within the scope of this course and in general situations outside of the classroom. We will define some topologies on the real line $\mathbb{R}$.

Definition 2.2.6: Standard topology on the real line

If $\mathcal{B}$ is the collection of all open intervals

$$(a, b) = \{x : a < x < b\}$$

where $a, b, x \in \mathbb{R}$, the topology generated by $\mathcal{B}$ is called the standard topology on the real line.

If we do not specify a topology when $X = \mathbb{R}$, assume that we are using this topology.

Definition 2.2.7: Lower limit topology on the real line

If $\mathcal{B}$ is the collection of all open intervals

$$[a, b) = \{x : a \leq x < b\}$$

where $a, b, x \in \mathbb{R}$, the topology generated by $\mathcal{B}$ is called the lower limit topology on the real line.

When $\mathbb{R}$ is given the lower limit topology, we denote it by $\mathbb{R}_l$.

Please note that although we do not prove that the bases in Definition 2.2.7 and Definition 2.2.6 are legitimate, we can see that the intersection of two basis elements is either another basis element or is empty.

There is a relationship between the standard and lower limit topologies on the real line; we examine this relationship below.

Lemma 2.2.8: Lower limit strictly finer than standard

The topology of $\mathbb{R}_l$ is strictly finer than the topology of $\mathbb{R}$.

Proof: Let $\mathcal{T}$ and $\mathcal{T}'$ be the topologies of $\mathbb{R}$ and $\mathbb{R}_l$, respectively. We would like to show that $\mathcal{T} \subset \mathcal{T}'$ with $\mathcal{T} \neq \mathcal{T}'$.

Given a basis element (a, b) for $\mathcal{T}$ and a point $x \in (a, b)$, the basis element $[x, b)$ for $\mathcal{T}'$ contains the point x and is contained in (a, b). Conversely, if we are given a basis element $[x, d)$ for $\mathcal{T}'$, there does not exist a basis element (a, b) of $\mathcal{T}$ that contains x and is also contained in $[x, d)$. Therefore, by Lemma 2.2.5, we conclude that $\mathcal{T} \subset \mathcal{T}'$ and $\mathcal{T}' \not\subset \mathcal{T}$, so $\mathcal{T}'$ is strictly finer than $\mathcal{T}$. $\square$

Subbasis

We know from Lemma 2.2.3 that a basis generates a topology that is equivalent to all of the possible unions from the basis. We will now examine when we have a given collection of sets and wish to make a topology from them.

Definition 2.2.9: Subbasis

Let a set X be given. A subbasis $\mathcal{S}$ for a topology on X is a collection of subsets whose union equals X.

This is all fine and well, but we would like to generate a topology on X from this subbasis. We do that in the following theorem.

Theorem 2.2.10: Generating a topology from a subbasis

Let a subbasis $\mathcal{S}$ be given. The topology $\mathcal{T}$ generated by the subbasis $\mathcal{S}$ is the collection of all unions of finite intersections from $\mathcal{S}$.

Proof: Instead of going through the criterion for a topology from Definition 2.1.1, we can instead show that the collection $\mathcal{B}$ of all finite intersections from $\mathcal{S}$ is a basis, since our topology $\mathcal{T}$ would be modeled off the situation in Lemma 2.2.3.

i. Since $\bigcup \mathcal{S} = X$, we know that for each $x \in X$, there exists a set $S \in \mathcal{S}$, and therefore $S \in \mathcal{B}$, that contains x.
ii. Let $B_1 = S_1 \cap ... \cap S_m$ and $B_2 = S'_1 \cap ... \cap S'_n$ be two elements of $\mathcal{B}$. The intersection

$$B_1 \cap B_2 = (S_1 \cap ... \cap S_m) \cap (S'_1 \cap ... \cap S'_n)$$

is another finite intersection from $\mathcal{S}$, so $(B_1 \cap B_2) \in \mathcal{S}$.

This concludes the proof. $\square$

2.3 The order topology

Before we dive into the topology aspect of this section, let's first define what an order relation is.

Definition 2.3.1: Order relation

Let a set X be given. An order relation (also called simple order relation or linear order relation) on the set X is a relation $\prec$ such that

i. If $x, y \in X$ and $x \neq y$, then either $x \prec y$ or $x \succ y$.
ii. There does not exist an element $x \in X$ such that $x \prec x$.
iii. If $x \prec y$ and $y \prec z$, then $x \prec z$.

Please note that when we use the symbol we often don't explicitly define an ordering on a set. When this is the case, assume that $a \prec b \iff a < b$, a order relation we notate by $<$. When a, b are of higher dimensions, use the lexicographic ordering with the relation $<$.

We will now define some common intervals we encounter when dealing with simply-ordered sets.

Definition 2.3.2: Intervals

Let a set X be given that has a simple order relation $\prec$ and have $a, b \in X$ such that $a \prec b$. We can define four types of intervals as follows:

i. Open interval: $(a, b) = \{x : a \prec x \prec b\}$
ii. Closed interval: $[a, b] = \{x : a \preceq x \preceq b\}$
iii. Half-open interval: $(a, b] = \{x : a \prec x \preceq b\}$ or $[a, b) = \{x : a \preceq x \prec b\}$

Note how we use the word "open" in Definition 2.3.2. This should indicate that open intervals are open sets in the order topology we put on X. This is precisely what happens in the following definition.

Definition 2.3.3: Basis for the order topology

Let a set X with a simple order relation be given, and assume further that X has more than one element. Let $\mathcal{B}$ be the collection of sets that satisfy one of the following criterion:

i. All open intervals (a, b) contained in X.
ii. All intervals of the form $[a_0, b)$ where a_0 is the smallest element (if one exists) of X.

iii. All intervals of the form $(a, b_0]$ where b_0 is the largest element (if one exists) of X.

We define $\mathcal{B}$ to be the basis for the order topology on X.

Note that if X does not have a smallest or largest element, then we do not have any intervals outlined in (ii.) or (iii.).

Although we make a strong claim that $\mathcal{B}$ from Definition 2.3.3 is a basis, we will not be constructing a proof for it.

In addition to intervals, we can also define other types of sets called rays.

Definition 2.3.4: Rays

Let a set X be given that has a simple order relation $\prec$ and have $a \in X$. We can define four types of rays as follows:

i. Open rays: $(a, +\infty) = \{x : x \succ a\}$ or $(-\infty, a) = \{x : x \prec a\}$.
ii. Closed rays: $[a, +\infty) = \{x : x \succeq a\}$ or $(-\infty, a] = \{x : x \preceq a\}$.

Once again, we would like open rays to also be open sets in the order topology from Definition 2.3.3. Note that if the set X has a largest element b_0, then we have $(a, +\infty) = (a, b_0]$. If X has no largest element, then $(a, +\infty)$ is the union of all basis elements of the form (a, x) where $x > a$. A similar process can be used for smallest elements of the set X.

We now cover an interesting conclusion that uses rays.

Theorem 2.3.5: Open rays form a subbasis for the order topology

For a given set X, its open rays form a subbasis for the order topology on X.

Proof: Recall from Definition 2.2.9 that a subbasis is a collection of subsets of X whose union is X. Furthermore, from Theorem 2.2.10 we know that the topology generated by a subbasis is the collection of all unions of finite intersections from the subbasis. We will denote the topology generated by the collection of open rays as $\mathcal{T}_r$ and the order topology by $\mathcal{T}_{or}$.

Please note that the open rays are in the order topology, so the topology they generate $\mathcal{T}_r$ is contained in $\mathcal{T}_{or}$, or $\mathcal{T}_r \subset \mathcal{T}_{or}$. This is true since the topology generated by a subbasis is constructed through finite intersections, and any finite intersection of sets within a topology is also in the topology (see Definition 2.1.1).

By taking a closer look, we can see that every basis element for the order topology (see Definition 2.3.3) is in fact a finite intersection of open rays. Any interval (a, b) is the intersection of $(-\infty, b)$ and $(a, +\infty)$, while if $[a_0, b)$ and $(a, b_0]$ exist, they are both open rays. Since a subbasis generates a topology by collecting all unions of finite intersections, we have $\mathcal{T}_{or} \subset \mathcal{T}_r$.

We have shown that $\mathcal{T}_r \subset \mathcal{T}_{or}$ and $\mathcal{T}_{or} \subset \mathcal{T}_r$, so we conclude $\mathcal{T}_r = \mathcal{T}_{or}$. The proof is concluded. $\square$

2.4 The product topology

Before we get talking about topologies, let's first define what the Cartesian product between two sets X and Y is.

Definition 2.4.1: Cartesian product

Let two sets X and Y be given. The Cartesian product $X \times Y$ is the set of all ordered pairs (x, y) where elements $x \in X$ are first and elements $y \in Y$ are second.

In set notation, we can say $X \times Y = \{(x, y) : x \in X \text{ and } y \in Y\}$

Now that we've defined the Cartesian product between two sets, let's define the product topology.

Definition 2.4.2: Product topology

Let X and Y be two topological spaces. The product topology on the set $X \times Y$ is the topology that has a basis $\mathcal{B}$ comprised of all sets of the form $U \times V$ where U is an open subset of X and V is an open subset of Y.

We shall prove that $\mathcal{B}$ from Definition 2.4.2 is in fact a basis.

Lemma 2.4.3: Basis for product topology

A basis $\mathcal{B}$ for the product topology on $X \times Y$ is comprised of all sets of the form $U \times V$ where U is an open subset of X and V is an open subset of Y.

Proof: To prove that $\mathcal{B}$ is a basis, we must satisfy the criterion from Definition 2.2.1.

i. Note that we can find any ordered pair $(x, y) \in (X \times Y)$ in the set $X \times Y$. Note that $X \times Y$ is a basis element since X is open in X and Y is open in Y.

ii. The second criteria utilizes a useful identity with Cartesian products:

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2)$$

Note that $(U_1 \cap U_2)$ and $(V_1 \cap V_2)$ are both finite intersections, so they are open in X and Y, respectively.

This concludes the proof. $\square$

The following theorem allows us to generate a basis for the product topology by using the bases for X and Y.

Theorem 2.4.4: Finding the basis for product topology

Let two topological spaces X and Y be given. If $\mathcal{B}$ is a basis for the topology of X and $\mathcal{C}$ is a basis for the topology of Y, then

$$\mathcal{D} = \{B \times C : B \in \mathcal{B} \text{ and } C \in \mathcal{C}\}$$

is a basis for the product topology of $X \times Y$.

Proof: If we are given an open set W from the product topology of $X \times Y$ and a point $(x, y) \in W$, we know from Theorem 2.2.2 that there exists a basis element $U \times V$ such that $(x, y) \in (U \times V) \subset W$.

Since $\mathcal{B}$ and $\mathcal{C}$ are bases for X and Y, respectively, we can find a basis element $B \in \mathcal{B}$ such that $x \in B \subset U$ and a basis element $C \in \mathcal{C}$ such that $y \in C \subset V$. Please note that $(B \times C) \in \mathcal{D}$. We have that $(x, y) \in (B \times C) \subset (U \times V) \subset W$. This satisfies the criterion for Lemma 2.2.4, so we conclude that $\mathcal{D}$ is a basis for the product topology of $X \times Y$. $\square$

Projections

We can view projections as functions from the set $X \times Y$ to either the set X or the set Y. Let's explicitly define what they are.

Definition 2.4.5: Projections of a Cartesian product

Let a set $X \times Y$ be given. We can define the function $\pi_1 : (X \times Y) \to X$ to be

$$\pi_1(x, y) = x$$

and the function $\pi_2 : (X \times Y) \to Y$ to be

$$\pi_2(x, y) = y$$

Note that we can also define the preimages π_1^{-1} and π_2^{-1} as follows:

$$\pi_1^{-1}(U) = U \times Y$$

and

$$\pi_2^{-1}(V) = X \times V$$

We can use these preimages to help us with the following theorem.

Theorem 2.4.6: Subbasis for the product topology

Let the product topology on $X \times Y$ be given where X and Y are two topological spaces. We declare that the collection

$$\mathcal{S} = \{\pi_1^{-1}(U) : U \text{ open in } X\} \cup \{\pi_2^{-1}(V) : V \text{ open in } Y\}$$

is a subbasis for the product topology on $X \times Y$.

Proof: Let's first denote the product topology on $X \times Y$ to be $\mathcal{T}$ and the topology generated by $\mathcal{S}$ to be $\mathcal{T}'$.

Note that every element $S \in \mathcal{S}$ belongs to $\mathcal{T}$ since U and V are open in X and Y, respectively, and X and Y are both open sets. This means that arbitrary unions of finite intersections from $\mathcal{S}$ are also in $\mathcal{T}$ (since a topology contains all finite intersections from itself and arbitrary unions from itself). Therefore, we have $\mathcal{T}' \subset \mathcal{T}$.

When we look at a basis element $U \times V$ for the product topology $\mathcal{T}$, we can view it as

$$(U \times V) = \pi_1^{-1}(U) \cap \pi_2^{-1}(V)$$

This means that $(U \times V)$ is just a finite intersection of elements from $\mathcal{S}$, so we have $(U \times V) \in \mathcal{T}'$. Note that every union of sets from $\mathcal{T}'$ is also in $\mathcal{T}'$ (by definition of a topology from Definition 2.1.1) and a topology can be generated from a basis by taking every union of sets from the basis (see Lemma 2.2.3), so if each basis element $U \times V$ for $\mathcal{T}$ is also is $\mathcal{T}'$, we have $\mathcal{T} \subset \mathcal{T}'$.

This conclude the proof. $\square$