MATH 551 (Elementary Topology) Course Notes

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This document was motivated by Professor Botong Wang's MATH 551 lectures during the Fall 2020 semester at UW-Madison. The course covered the majority of Chapters 2 - 3 from James Munkres' *Topology (2nd Edition)* (link). This document should follow the structure of the chapters in the textbook but aims to not require a supporting text.

I welcome feedback on this document. Please feel free to reach out to me by email at egalles@wisc.edu to share your thoughts or concerns. For the most recent edition, please visit GitHub (link).

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- *Emmett Galles*

Chapter 2: Topological Spaces and Continuous Functions

Although it is common to use facets of topological spaces in other mathematics classes, we now look to rigorously explain and flesh out the concept. Something I recommend before continuing: set aside the definitions of open and closed sets that were most likely taught in a real analysis course to allow for a more thorough construction of a topological space.

2.1 Topological spaces

We will first define what a topology is.

Definition 2.1.1: Topology

Let a set X be given. A topology $\mathcal{T}$ on the set X is a collection of subsets of X such that

i. $\varnothing$ and X are in $\mathcal{T}$.
ii. Any union of a collection of sets from $\mathcal{T}$ is also in $\mathcal{T}$.
iii. Any finite intersection of elements from $\mathcal{T}$ is also in $\mathcal{T}$.

Recognize that $\mathcal{T}$ is a set of sets!

Please note that throughout this document, the words "subcollection" and "subset" can be used synonymously, as well as "collection" and "set". However, in most situations, collection and subcollection will indicate that both our set of interest is a set whose elements are sets. For example, (ii.) from Definition 2.1.1 can be rewritten as "The union of the elements from any subcollection of $\mathcal{T}$ is also in $\mathcal{T}$." since $\mathcal{T}$ and any subset of $\mathcal{T}$ itself is a set whose elements are sets.

Let's examine how the distinction between finite and any intersection from (iii.) is important with the following example.

Example 2.1.2: Infinite intersection not in set

Find a collection of sets C where an intersection of elements from C is not in C.
Let $C = \left\{\left[0, \frac{1}{n}\right] : n \in \mathbb{N}\right\}$ and let $P = \bigcap_{p\in C} p$. From previous coursework we know that $P = \{0\}$, but $P \notin C$ since there does not exist an $n \in \mathbb{N}$ such that $\{0\} = \left[0, \frac{1}{n}\right]$.

Once we construct a topology on a set, we can now look to define what it means for a set to be open.

Definition 2.1.3: Open set

Let a set X be given and let $\mathcal{T}$ be a topology on X. A subset U of X is an open set if U belongs to the topology $\mathcal{T}$.

In other words, $U \subset X$ is open $\iff U \in \mathcal{T}$.

Below are some topologies that are commonly used and have special names.

Remark 2.1.4: Notable topologies

For a set X, the topology $\mathcal{T} = \{\varnothing, X\}$ is called the trivial topology.

For a set X, the topology $\mathcal{T} = \{t : t \subset X\}$ is called the discrete topology.

Theorem 2.1.5: Finite complement topology

Let a set X be given and have $\mathcal{T}_f$ be the collection of all subsets U of X such that $X \setminus U$ is either finite or is all of X. We declare $\mathcal{T}_f$ to be a topology on X.

Proof: To show that $\mathcal{T}_f$ is a topology on X, we must uphold the three conditions outlined in Definition 2.1.1.

i. $\varnothing \in \mathcal{T}_f$ since $X \setminus \varnothing = X$ and $X \in \mathcal{T}_f$ since $X \setminus X = \varnothing$ is a finite set.

ii. If we let $\{U_i\}$ be a collection of sets from $\mathcal{T}_f$, we need to show that $(X \setminus \bigcup_{i \in I} U_i) \in \mathcal{T}_f$. By De Morgan's laws, we know

$$X \setminus \bigcup_{i \in I} U_i = \bigcap_{i \in I} (X - U_i)$$

is either the entire set X or a finite set, since the intersection of finite sets is finite. Therefore, $(X \setminus \bigcup_{i \in I} U_i) \in \mathcal{T}_f$.

iii. If we let $\{U_j\}$ be a finite collection of sets from $\mathcal{T}_f$, we need to show that $(X \setminus \bigcup_{j \in J}) \in \mathcal{T}_f$. We will utilize De Morgan's laws again to see that

$$X \setminus \bigcap_{j \in J} U_j = \bigcup_{j \in J} (X \setminus U_j)$$

which is either the entire set X or a finite set, since a finite union of finite sets is finite. Therefore, $(X \setminus \bigcup_{j \in J}) \in \mathcal{T}_f$.

This concludes the proof. $\square$