Statement

6. Sejam $A \in \mathbb{R}^{m\times m}$, $B \in \mathbb{R}^{m\times n}$, $C \in \mathbb{R}^{n\times n}$ e defina $T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$. Prove que $\Lambda(T) = \Lambda(A) \cup \Lambda(C)$.

Solution

→ $A \in \mathbb{R}^{m\times m}$, $B \in \mathbb{R}^{m\times n}$ e $C \in \mathbb{R}^{n\times n}$, $T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$

Seja (λ, v) um autopar de T.

$v = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $u_1 \in \mathbb{R}^{m\times m}$ $u_2 \in \mathbb{R}^{n\times n}$. Assim,

$\underline{\lambda \in \Lambda(T) \Rightarrow \lambda \in \Lambda(A) \cup \Lambda(C)}$

$$Tv = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} A \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} B \\ C \end{bmatrix} u_2 = \begin{bmatrix} Au_1 + Bu_2 \\ Cu_2 \end{bmatrix} \overset{\text{por hipótese}}{=} \begin{bmatrix} \lambda u_1 \\ \lambda u_2 \end{bmatrix}$$

$\Rightarrow Cu_2 = \lambda u_2 \Rightarrow \lambda \in \Lambda(A) \cup \Lambda(C)$

$\underline{\lambda \in \Lambda(A) \cup \Lambda(C) \Rightarrow \lambda \in \Lambda(T)}$

Seja (λ, u_1) um autopar de A.

Queremos provar que $\exists\, v \in \mathbb{R}^{m+n}$ t.q. $Tv = \lambda v$.

Tome $v = \begin{bmatrix} u_1 \\ 0 \end{bmatrix}$. Assim,

$$Tv = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \begin{bmatrix} u_1 \\ 0 \end{bmatrix} = \begin{bmatrix} A \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} B \\ C \end{bmatrix} 0 = \begin{bmatrix} Au_1 \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda u_1 \\ 0 \end{bmatrix} = \lambda v$$

Agora, seja (λ, u_2) um autopar de C t.q. $\lambda \notin \Lambda(A)$ (repare que $[\lambda \in \Lambda(A)$ e $\lambda \in \Lambda(C)]$ já foi provado acima)

Queremos provar que $\exists\, v$ t.q. $Tv = \lambda v$.

$$\begin{bmatrix} A \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} B \\ C \end{bmatrix} u_2 = \begin{bmatrix} Au_1 + Bu_2 \\ Cu_2 \end{bmatrix} = \begin{bmatrix} \lambda u_1 \\ \lambda u_2 \end{bmatrix}$$ → RASCUNHO

$Au_1 + Bu_2 = \lambda u_1 \Rightarrow (A - \lambda I)u_1 = -Bu_2$

Sabemos que $(A - \lambda I)^{-1}$ existe, pois $\lambda \notin \Lambda(A)$. (se λ torna a matriz $(A - \lambda I)$ singular, λ é autovalor de A por definição.)

Logo, $u_1 = -(A - \lambda I)^{-1} Bu_2$.

Tome $v = \begin{bmatrix} -(A - \lambda I)^{-1} Bu_2 \\ u_2 \end{bmatrix}$. Assim, $Tv = \begin{bmatrix} A \\ 0 \end{bmatrix} -(A - \lambda I)^{-1} Bu_2 + \begin{bmatrix} B \\ C \end{bmatrix} u_2 =$

$$= \begin{bmatrix} -A(A - \lambda I)^{-1} Bu_2 + Bu_2 \\ Cu_2 \end{bmatrix} = \lambda \begin{bmatrix} (A - \lambda I)^{-1} Bu_2 \\ u_2 \end{bmatrix}$$

eu não consegui fazer esse passo

$\square$.

Reference

Link: MS512_2024S1 Lista de Autopares

Exercise: 6