Statement

1. Para cada uma das matrizes abaixo encontre todos os autovalores e autovetores associados:

(a) $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. (b) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. (d) $\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. (d) $\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

Solution

(1) (a) $\overbrace{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}}^{A}$, $p_n(\lambda) = \det(A - \lambda I) = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 1 \\ 1 & 1 & 0 \end{vmatrix} = 0 \Rightarrow -(1-\lambda+1-\lambda) = 0 \Rightarrow 2\lambda - 2 = 0$$

$\Rightarrow \boxed{\lambda_1 = 1 \text{ e } \lambda_2 = 1}$ Achando o autovalor associado:

$$Av = \lambda v \Rightarrow Av = v \Rightarrow (A - I)v = 0$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow v^3 = 0, \quad v^1 = -v^2 \Rightarrow \boxed{v = k\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \ k \in \mathbb{C}}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -5 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} -5 \\ 5 \\ 0 \end{bmatrix} \Big\} \rightarrow \text{só um teste}$$

(b) $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $p(\lambda) = \det\left(\begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{bmatrix}\right) =$

$= (1-\lambda)(-\lambda)^2 - (1-\lambda) = 0 \Rightarrow \lambda^3 - \lambda^2 + \lambda - 1 = 0$

$p(1) = 1 - 1 + 1 - 1 = 0 \Rightarrow \lambda_1 = 1$

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$$\begin{array}{l|l} \lambda^3 - \lambda^2 + \lambda - 1 & (\lambda - 1) \\ \hline -\lambda^3 + \lambda^2 & \lambda^2 + 1 \\ \hline \lambda - 1 & \\ -\lambda + 1 & \\ \hline 0 & \end{array}$$

$\lambda^3 - \lambda^2 + \lambda - 1 =$

$\Rightarrow \; = (\lambda^2 + 1)(\lambda - 1)$

$\lambda^2 + 1 = 0 \Rightarrow \lambda^2 = -1 \Rightarrow \lambda^2 = i$, $\lambda^3 = -i$

Achando os autovetores:

$\underline{\lambda_1 = 1}$

$$\begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix} = 0$$

$$\begin{cases} -v^2 + v^3 = 0 \\ v^2 - v^3 = 0 \end{cases} \Rightarrow v^2 = v^3 \; = v = \begin{bmatrix} m \\ n \\ n \end{bmatrix}, \; m, n \in \mathbb{C}.$$

$\underline{\lambda_2 = i}$

$$\begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{bmatrix} v = 0 \Rightarrow \begin{bmatrix} 1-i & 0 & 0 \\ 0 & -i & 1 \\ 0 & 1 & -i \end{bmatrix} v = 0$$

$$\begin{cases} (1-i)v^1 = 0 \Rightarrow v^1 = 0 \\ -iv^2 + v^3 = 0 \Rightarrow iv^2 = v^3 \\ v^2 - iv^3 = 0 \Rightarrow v^2 = iv^3 \end{cases} \Bigg] \Rightarrow v = \begin{bmatrix} 0 \\ m \\ mi \end{bmatrix}, \; m \in \mathbb{C}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ m \\ mi \end{bmatrix} = \begin{bmatrix} 0 \\ mi \\ m \end{bmatrix} \cdot i \; \Big\} \text{ Teste}$$

$\underline{\lambda_3 = -i}$

$$\begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{bmatrix} v = 0 \Rightarrow \begin{bmatrix} 1+i & 0 & 0 \\ 0 & i & 1 \\ 0 & 1 & i \end{bmatrix} v = 0$$

$$\begin{cases} v^1 = 0 \\ iv^2 + v^3 = 0 \\ v^2 + iv^3 = 0 \end{cases} \Big\} \Rightarrow v^2 = -iv^3 \Rightarrow v = \begin{bmatrix} 0 \\ -im \\ m \end{bmatrix}, \; m \in \mathbb{C}$$

(c) $A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $p_A(\lambda) = \det(A - I\lambda) = \det\left(\begin{bmatrix} -\lambda & 1 & 1 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix} \right)$

$$p(\lambda) = -\lambda^3 = 0 \Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0$$

Achando os autovetores:

$\underline{\lambda = 0}$

$$\begin{bmatrix} -\lambda & 1 & 1 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix} v = 0 \Rightarrow \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} v = 0$$

$$\left\{ v^2 = -v^3 \right. \Rightarrow v = \begin{bmatrix} m \\ n \\ -n \end{bmatrix}, \quad m, n \in \mathbb{C}$$

Reference

Link: MS512_2024S1 Lista de Autopares
Exercise: 1