Statement

Solution

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SVD of a matrix A is $A = U\Sigma V^T$, where U and V are orthogonal and Σ is nonnegative real diagonal.

Now, let X be orthogonal. Note that $X = U\Sigma V^T$, where $U := X$ is orthogonal, $\Sigma := \mathrm{I}$ is diagonal, and $V := \mathrm{I}$ is orthogonal. So, singular values are all equal to 1.

Or, you can use the definition by which the singular values of X are the absolute square roots of the eigenvalues of $X^T X$. In case of an orthogonal X, eigenvalues of $X^T X = \mathrm{I}$ are all equal to one, so the singular values of X are all equal to 1.

As for your second question, I don't think the statement is true. Let

$$X = \mathrm{I} = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}, \quad Y = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}.$$

If the set of the orthogonal matrix is convex, then $Z := \frac{1}{2}(X+Y)$ is also orthogonal. But,

$$Z = \frac{1}{2}(X+Y) = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

is singular, so it cannot be orthogonal. We can even check directly: $Z^T Z = Z \neq \mathrm{I}$.

You can find a topic on the convex hull of the set of orthogonal matrices here.

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Vedran Šego
11.4k ● 1 ■ 31 ▲ 55

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The SVD for a symmetric matrix $A = U\Sigma V^T$, where U and V are unitary matrices with $U = [u_1|u_2|\dots|u_n]$, $V = [v_1|v_2|\dots|v_n]$ and Σ is a diagonal matrix with non-negative diagonal entries and $v_i = \pm u_i$

For a symmetric matrix the following decompositions are equivalent to SVD. (Well, almost equivalent if you do not worry about the signs of the vectors).

1. **Eigen-value decomposition** i.e. $A = X\Lambda X^{-1}$. When A is symmetric, the eigen values are real and the eigenvectors can be chosen to be orthonormal and hence $X^T X = XX^T = I$ i.e. $X^{-1} = X^T$. The only difference is that the singular values are the magnitudes of the eigen values and hence the column of X needs to be multiplied by a negative sign if the eigen value turns out to be negative to get the singular value decomposition. Hence, $U = X$ and $\sigma_i = |\lambda_i|$.
2. **Orthogonal decomposition** i.e. $A = PDP^T$, where P is a unitary matrix and D is a diagonal matrix. This exists only when matrix A is symmetric and is the same as eigen value decomposition.
3. **Schur decomposition** i.e. $A = QSQ^T$, where Q is a unitary matrix and S is an upper triangular matrix. This can be done for any matrix. When A is symmetric, then S is a diagonal matrix and again is the same as the eigen value decomposition and orthogonal decomposition.

I do not remember the cost for each of these operations i.e. I don't remember the coefficients before the leading order n^3 term. If my memory is right, the typical algorithm for orthogonal decomposition is slightly cheaper than the other two, though I cannot guarantee.

Reference