Statement

1. Para cada uma das matrizes abaixo encontre todos os autovalores e autovetores associados:

(a) $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. (b) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. (d) $\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. (d) $\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

Solution

(1) (a) $\overbrace{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}}^{A}$, $p_n(\lambda) = \det(A - \lambda I) = 0$

$$\begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)^2(-\lambda) - ((1-\lambda) + (1-\lambda)) = 0$$

$$\Rightarrow (-\lambda)(1 - 2\lambda + \lambda^2) + 2\lambda - 2 = 0 \Rightarrow -\lambda + 2\lambda^2 - \lambda^3 + 2\lambda - 2 = 0$$

$$\Rightarrow -\lambda^3 + 2\lambda^2 + \lambda - 2 = 0$$

Rascunho

$$\begin{array}{l|l} -\lambda^3 + 2\lambda^2 + \lambda - 2 & \lambda - 1 \\ \hline -(-\lambda^3 + \lambda^2) & -\lambda^2 + \lambda + 2 \\ \hline 0 + \lambda^2 + \lambda - 2 & \\ -(\lambda^2 - \lambda) & \\ \hline 0 + 2\lambda - 2 & \\ -(2\lambda - 2) & \\ \hline 0 & \end{array}$$

Chutando $\lambda = 1 \Rightarrow -1 + 2 + 1 - 2 = 0$

$$p(\lambda) = (\lambda - 1)(-\lambda^2 + \lambda + 2) = 0$$

$$-\lambda^2 + \lambda + 2 = 0 \Rightarrow \lambda = \frac{-1 \pm \sqrt{1 - 4 \cdot (-1) 2}}{-2} =$$

$$= \frac{-1 \pm 3}{-2} \begin{cases} 2 \\ -1 \end{cases}$$

Logo, $\underline{\lambda_1 = 1,\ \lambda_2 = -1,\ \lambda_3 = 2}$

Autovetor de $\lambda_1 = 1$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0 \Rightarrow \begin{cases} v_3 = 0 \\ v_2 = -v_1 \end{cases} \Rightarrow v = \begin{bmatrix} t \\ -t \\ 0 \end{bmatrix}, \ t \in \mathbb{R}$$

Autovetor $\lambda_2 = -1$

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0 \Rightarrow \begin{cases} 2v_1 + v_3 = 0 \Rightarrow v_3 = -2v_1 \\ 2v_2 + v_3 = 0 \Rightarrow v_3 = -2v_2 \Rightarrow v = \begin{bmatrix} t \\ t \\ -2t \end{bmatrix}, \ t \in \mathbb{R} \end{cases}$$

Autovalor $\lambda_3 = 2$

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{cases} -v_1 + v_3 = 0 \Rightarrow v_1 = v_3 \\ -v_2 + v_3 = 0 \Rightarrow v_3 = v_2 \end{cases} \Rightarrow v = \begin{bmatrix} t \\ t \\ t \end{bmatrix}, \ t \in \mathbb{R}$$

(b) $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $p(A) = \det\left(\begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{bmatrix}\right) =$

$= (1-\lambda)(-\lambda)^2 - (1-\lambda) = 0 \Rightarrow \lambda^3 - \lambda^2 + \lambda - 1 = 0$

$p(1) = 1 - 1 + 1 - 1 = 0 \Rightarrow \lambda_1 = 1$

↳ chute

$$\begin{array}{l|l} \lambda^3 - \lambda^2 + \lambda - 1 & (\lambda - 1) \\ \hline -\lambda^3 + \lambda^2 & \lambda^2 + 1 \\ \hline \lambda - 1 & \\ -\lambda + 1 & \\ \hline 0 & \end{array}$$

$\lambda^3 - \lambda^2 + \lambda - 1 =$

$\Rightarrow \quad = (\lambda^2 + 1)(\lambda - 1)$

$\lambda^2 + 1 = 0 \Rightarrow \lambda^2 = -1 \Rightarrow \lambda^2 = i, \quad \lambda^3 = -i$

Achando os autovetores:

$\underline{\lambda_1 = 1}$

$$\begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix} = 0$$

$$\begin{cases} -v^2 + v^3 = 0 \\ v^2 - v^3 = 0 \end{cases} \Rightarrow v^2 = v^3 \quad = v = \begin{bmatrix} m \\ n \\ n \end{bmatrix}, \quad m, n \in \mathbb{C}.$$

$\underline{\lambda_2 = i}$

$$\begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{bmatrix} v = 0 \Rightarrow \begin{bmatrix} 1-i & 0 & 0 \\ 0 & -i & 1 \\ 0 & 1 & -i \end{bmatrix} v = 0$$

$$\begin{cases} (1-i)v^1 = 0 \Rightarrow v^1 = 0 \\ -iv^2 + v^3 = 0 \Rightarrow iv^2 = v^3 \\ v^2 - iv^3 = 0 \Rightarrow v^2 = iv^3 \end{cases} \Rightarrow v = \begin{bmatrix} 0 \\ m \\ mi \end{bmatrix},\ m \in \mathbb{C}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ m \\ mi \end{bmatrix} = \begin{bmatrix} 0 \\ mi \\ m \end{bmatrix} \cdot i \quad \Big\} \text{ Teste}$$

$\underline{\lambda_3 = -i}$

$$\begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{bmatrix} v = 0 \Rightarrow \begin{bmatrix} 1+i & 0 & 0 \\ 0 & i & 1 \\ 0 & 1 & i \end{bmatrix} v = 0$$

$$\begin{cases} v^1 = 0 \\ iv^2 + v^3 = 0 \\ v^2 + iv^3 = 0 \end{cases} \Rightarrow v^2 = -iv^3 \Rightarrow v = \begin{bmatrix} 0 \\ -im \\ m \end{bmatrix},\ m \in \mathbb{C}$$

(c) $A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $p_A(\lambda) = \det(A - I\lambda) = \det\left(\begin{bmatrix} -\lambda & 1 & 1 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix}\right)$

$p(\lambda) = -\lambda^3 = 0 \Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0$

Achando os autovetores:

$\underline{\lambda = 0}$

$\begin{bmatrix} -\lambda & 1 & 1 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix} v = 0 \Rightarrow \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} v = 0$

$\{ v^2 = -v^3 \Rightarrow v = \begin{bmatrix} m \\ n \\ -n \end{bmatrix}$, $m, n \in \mathbb{C}$

Reference

Link: MS512_2024S1 Lista de Autopares
Exercise: 1