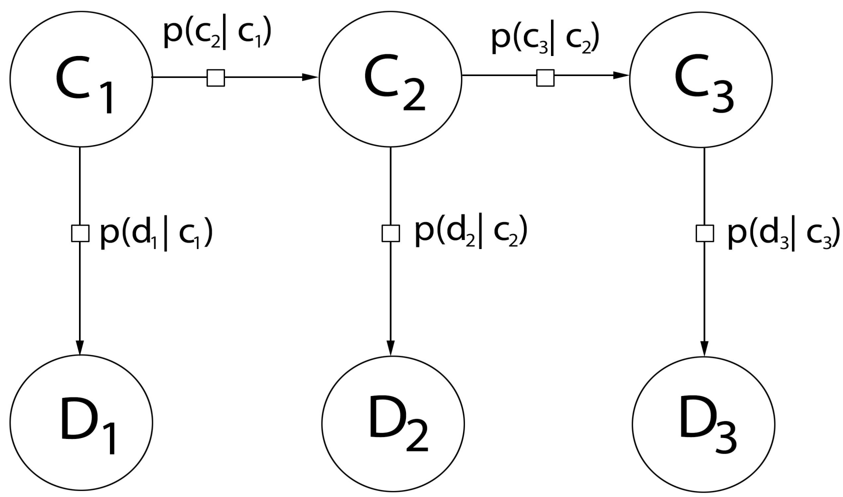
CS221 Fall 2015 Homework [car]

Problem 1: Warmup

We were encouraged to draw out the (factor) graph, here it is:



1a.

To calculate the posterior distribution $P(C_2=1 \mid D_2=0)$ let's consider:

$$\begin{aligned} P(C_2=1) &= P(C_1=0, C_2=1) + P(C_1=1, C_2=1) \\ &= P(C_1=0)\ P(C_2=1 \mid C_1=0) + P(C_1=1)\ P(C_2=1 \mid C_1=1) \\ &= 0.5\ \epsilon + 0.5\ (1-\epsilon) = 0.5 \end{aligned}$$

Therefore $P(C_2=0) = 1-P(C_2=1) = 0.5$

Using Bayes' rule:

$$P(C_2=1 \mid D_2=0) = \frac{P(C_2=1, D_2=0)}{P(D_2=0)}$$

The numerator $P(C_2=1, D_2=0) = P(C_2=1)\ P(D_2=0 \mid C_2=1)$
$= 0.5\ \eta$

The denominator $P(D_2=0) = P(C_2=0, D_2=0) + P(C_2=1, D_2=0)$
$= P(C_2=0)\ P(D_2=0 \mid C_2=0) + 0.5\eta$
$= 0.5\ (1-\eta) + 0.5\eta = 0.5$

Therefore $P(C_2=1 \mid D_2=0) = \dfrac{0.5\eta}{0.5} = \eta$

1b.

To calculate the posterior distribution $P(C_2=1 \mid D_2=0, D_3=1)$ let's consider:

$$\begin{aligned} P(C_3=1) &= P(C_3=1, C_2=0) + P(C_3=1, C_2=1) \\ &= P(C_2=0)\ P(C_3=1 \mid C_2=0) + P(C_2=1)\ P(C_3=1 \mid C_2=1) \\ &= 0.5\ \epsilon + 0.5\ (1-\epsilon) = 0.5 \end{aligned}$$

Therefore $P(C_3=0) = 1-P(C_3=1) = 0.5$

Using Bayes rule: $P(C_2=1 \mid D_2=0, D_3=1) = \dfrac{P(C_2=1, D_2=0, D_3=1)}{P(D_2=0, D_3=1)}$

Numerator $P(C_2=1, D_2=0, D_3=1) = \sum_{u,v\in\{0,1\}} P(C_1=u, C_3=v, C_2=1, D_2=0, D_3=1) =$

$$P(C_1=0, C_2=1, D_2=0, C_3=0, D_3=1) + P(C_1=1, C_2=1, D_2=0, C_3=0, D_3=1) +$$
$$P(C_1=0, C_2=1, D_2=0, C_3=1, D_3=1) + P(C_1=1, C_2=1, D_2=0, C_3=1, D_3=1) =$$

$$0.5 . \epsilon . \eta . \epsilon . \eta + 0.5 . \epsilon . \eta . (1-\epsilon) . (1-\eta) + 0.5 . (1-\epsilon) . \eta . \epsilon . \eta + 0.5 . (1-\epsilon) . \eta . (1-\epsilon) . (1-\eta)$$

Which reduces the numerator equal to $0.5\eta(1-\epsilon-\eta+2\epsilon\eta)$

Now the denominator $P(D_2=0, D_3=1) = P(C_2=0, D_2=0, D_3=1) + P(C_2=1, D_2=0, D_3=1) =$
$P(C_2=0, D_2=0, D_3=1) + 0.5\eta(1-\epsilon-\eta+2\epsilon\eta)$

Where we must calculate

$$P(C_2=0, D_2=0, D_3=1) = \sum_{u,v\in\{0,1\}} P(C_1=u, C_3=v, C_2=0, D_2=0, D_3=1) =$$
$$P(C_1=0, C_2=0, D_2=0, C_3=0, D_3=1) + P(C_1=1, C_2=0, D_2=0, C_3=0, D_3=1) +$$
$$P(C_1=0, C_2=0, D_2=0, C_3=1, D_3=1) + P(C_1=1, C_2=0, D_2=0, C_3=1, D_3=1) =$$

$$0.5 . (1-\epsilon) . (1-\eta) . (1-\epsilon) . \eta + 0.5 . (1-\epsilon) . (1-\eta) . \epsilon . (1-\eta) +$$
$$0.5 . \epsilon . (1-\eta) . (1-\epsilon) . \eta + 0.5 . \epsilon . (1-\eta) . \epsilon . (1-\eta) = 0.5(1-\eta)(\epsilon+\eta-2\epsilon\eta)$$

Which makes the denominator equal to $0.5(1-\eta)(\epsilon+\eta-2\epsilon\eta) + 0.5\eta(1-\epsilon-\eta+2\epsilon\eta) =$
$2\eta^2\epsilon+\eta+0.5\epsilon -\eta^2-2\eta\epsilon$

Which results in the answer $P(C_2=1 \mid D_2=0, D_3=1) = \dfrac{0.5\eta(1-\epsilon-\eta+2\epsilon\eta)}{2\eta^2\epsilon+\eta+0.5\epsilon -\eta^2-2\eta\epsilon}$

(note: graph drawn in previous page is still the same DAG)

1c.
Supposing $\epsilon = 0.1$ and $\eta = 0.2$

i.

$$P(C_2 = 1 \mid D_2 = 0) = \eta = \mathbf{0.2}$$

$$P(C_2 = 1 \mid D_2 = 0, D_3 = 1) = \frac{0.5\eta(1 - \epsilon - \eta + 2\epsilon\eta)}{2\eta^2\epsilon + \eta + 0.5\epsilon - \eta^2 - 2\eta\epsilon} =$$

$$\frac{((.5*.2)*(1.0-.1-.2+(2.0*.1*.2)))}{((2.0*(.2*.2)*.1)+.2+(.5*.1)-(.2*.2)-(2*.2*.1))} \approx \mathbf{0.4157}$$

ii.
Adding the second sensor reading $D_3 = 1$ increases the probability that $C_2 = 1$
This is because $D_3 = 1$ supplies more evidence for $C_3 = 1$ which supports $C_2 = 1$ since D_3 depends on C_3 and C_3 depends on C_2

iii.

In order to satisfy $P(C_2 = 1 \mid D_2 = 0) = P(C_2 = 1 \mid D_2 = 0, D_3 = 1)$ let's set epsilon as follows:

Supposing $\eta = 0.2$

$$P(C_2 = 1 \mid D_2 = 0, D_3 = 1) = \frac{0.1(0.8 - 0.6\epsilon)}{0.18\epsilon + 0.16} = .2$$

Therefore $\epsilon = 0.5$ as requested.

The intuition is that with $\epsilon = 0.5$ the car will move half the time. With this higher value of epsilon, there is a higher probability that the car will move positions C_2 to C_3 such that $C_2 \neq C_3$ which explains $D_3 = 1$ even when there is a low probability that $C_2 = 1$ such as is the case in this example.

Problem 5: Which car is it?

a.

An expression for the conditional distribution $P(C_{11}, C_{12} \mid E_1 = e_1)$ as a function of the probability density function of a Gaussian $p_N(\nu; \mu, \sigma^2)$ and the prior probability $p(c_{11})$ and $p(c_{12})$ over car locations can be calculated as follows:

$$P(C_{11}=c_{11}, C_{12}=c_{12} \mid E_1=e_1) = \frac{P(C_{11}=c_{11}, C_{12}=c_{12}, E_1=e_1)}{P(E_1=e_1)} =$$

$$\frac{0.5\,P(C_{11}=c_{11}, C_{12}=c_{12}, D_{11}=e_{11}, D_{12}=e_{12}) + 0.5\,P(C_{11}=c_{11}, C_{12}=c_{12}, D_{11}=e_{12}, D_{12}=e_{11})}{P(E_1=e_1)} =$$

$$\frac{\begin{array}{l} p(c_{11})p(c_{12})\;\; p_N(e_{11};\ \|a_1-c_{11}\|, \sigma^2)\;\; p_N(e_{12};\ \|a_1-c_{12}\|, \sigma^2) \\ +\;\; p(c_{11})p(c_{12})\;\; p_N(e_{12};\ \|a_1-c_{11}\|, \sigma^2)\;\; p_N(e_{11};\ \|a_1-c_{12}\|, \sigma^2) \end{array}}{2\mathrm{P}(E_1=e_1)}$$

Therefore $P(C_{11}=c_{11}, C_{12}=c_{12} \mid E_1=e_1) \propto$ *(is proportional to)*

$$\begin{array}{l} p(c_{11})p(c_{12})\;\; p_N(e_{11};\ \|a_1-c_{11}\|, \sigma^2)\;\; p_N(e_{12};\ \|a_1-c_{12}\|, \sigma^2) \\ +\;\; p(c_{11})p(c_{12})\;\; p_N(e_{12};\ \|a_1-c_{11}\|, \sigma^2)\;\; p_N(e_{11};\ \|a_1-c_{12}\|, \sigma^2) \end{array}$$

b. $P(C_{11}=c_{11}, \ldots, C_{1K}=c_{1K} \mid E_1=e_1) = \dfrac{P(C_{11}=c_{11}, \ldots, C_{1K}=c_{1K}, E_1=e_1)}{P(E_1=e_1)}$

$$\propto \sum_{\pi \in \prod} P(C_{11}=c_{11}, \ldots, C_{1K}=c_{1K}, D_{11}=\pi(e_{11}), \ldots, D_{1K}=\pi(e_{1K}))$$

Here π is a permutation over the set $\{e_{11}, \ldots, e_{1K}\}$ and $\prod$ is the set of all $K!$ permutations. If the assignment $(C_{11}=c_{11}, \ldots, C_{1K}=c_{1K})$ maximizes $P(C_{11}=c_{11}, \ldots, C_{1K}=c_{1K} \mid E_1=e_1)$ then consider a permutation σ over the set $\{c_{11}, \ldots, c_{1K}\}$

Thus $P(C_{11}=\sigma(c_{11}), \ldots, C_{1K}=\sigma(c_{1K}) \mid E_1=e_1) \propto$

$\sum_{\pi \in \prod} P(C_{11}=\sigma(c_{11}), \ldots, C_{1K}=\sigma(c_{1K}), D_{11}=\pi(e_{11}), \ldots, D_{1K}=\pi(e_{1K}))$ which is equal to

$\sum_{\pi \in \prod} P(C_{11}=c_{11}, \ldots, C_{1K}=c_{1K}, D_{11}=\sigma^{-1}(\pi(e_{11})), \ldots, D_{1K}=\sigma^{-1}(\pi(e_{1K})))$ which is equal to

$\sum_{\pi \in \prod} P(C_{11}=c_{11}, \ldots, C_{1K}=c_{1K}, D_{11}=\pi(e_{11}), \ldots, D_{1K}=\pi(e_{1K}))$ because the permutation of a permutation yields another permutation and importantly $p(c_{1i})$ is the same for all *i*

Therefore for any permutation σ over the set $\{c_{11}, \ldots, c_{1K}\}$ the probability $P(C_{11}=\sigma(c_{11}), \ldots, C_{1K}=\sigma(c_{1K}) \mid E_1=e_1)$ is also the same maximum value. Now since there are *K!* such σ the number of maximizing assignments is at least *K!*