Conservation of the Hamiltonian in the 1-D and 2-D Harmonic Oscillators

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We shall first solve the systems of 1-D and 2-D harmonic oscillators using MATLAB's ODE45 method, and then show that as the time step h $\rightarrow$ 0, the Hamiltonian is conserved.

We consider a 1 dimensional harmonic oscillator first. With a potential shown below.

$$\mathrm{V} = \frac{1}{2}\mathrm{kq}^2 + \mathcal{O}(\mathrm{q}^4) \tag{1}$$

where k is the spring constant, q is the generalised coordinate and $\mathcal{O}(\mathrm{q}^4)$ represents the higher order terms, which we neglect.

The system of differential equations is simply

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \mathrm{F}(q) \tag{2}$$

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{p}{m} \tag{3}$$

where p is the momentum, m is the mass, and F(q) is the force. This system of equations resembles Hamilton's equations.

The analytical solution is

$$q(t) = q_0 \sin(\omega_0 t) \tag{4}$$

where ω is the natural frequency given by $\omega = \sqrt{\frac{k}{m}}$.

The Hamiltonian is simply H = T + V.

$$H = \frac{p^2}{2m} + \frac{1}{2}kq^2 \tag{5}$$

```
function dh = harmonic(t,h)
q0 = 10; % initial position
p0 = 200; % initial momentum
k = 30; % spring constant
m = 40; % mass
omega = sqrt(k/m); % define the natural frequency
tstep = 500; % timestep or "tspan"
dh = zeros(2,1);
dh(1) = -k*h(2);
dh(2) = h(1)/m;
```

We then call the function and plot the phase space [Fig. 1].

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```
[T,H] = ode45(@harmonic,[0 tstep],[p0 q0]);

plot(H(:,1),H(:,2),'r:')
xlabel('Position')
ylabel('Momentum')
grid on
grid minor
```

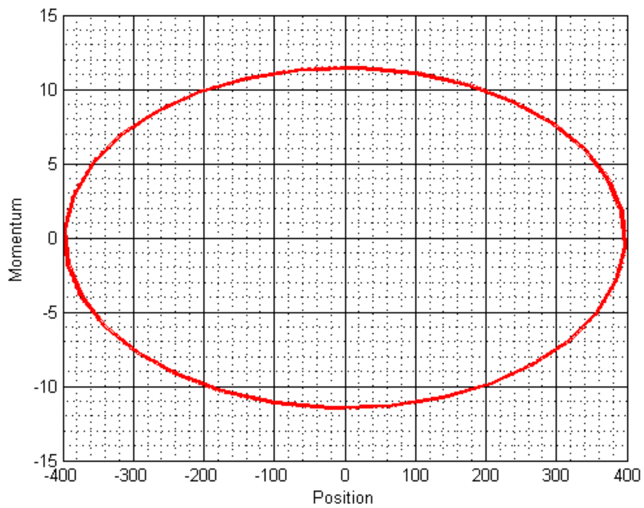

FIG. 1: The phase space of the 1-Dimensional harmonic oscillator

This phase portrait resembles an oval, which agrees with the analytical solutions.

Looking at the output of [T,H], we can compare the calculated energies at varying values of tstep. Our initial conditions $p_0 = 20$, $q_0 = 10$ give an initial energy of $E_0 = 1505J$ from [Eq. 5]. The following table shows the subsequent calculations.

TABLE I: Relationship between time-step and energy

Timestep	Momentum	Position	Energy
0	2E2	1E1	1505
500	3.3424E2	6.0774	1950.4765
1000	3.8894E2	8.6545E-1	1902.1641
5000	-2.7820E2	6.2646	1556.118

As one can see, as tstep increases, E_n goes closer to the initial value.

After that, we consider the 2 dimensional case, with a similar potential. Using x and y as the coordinates,

$$\mathrm{V} = \frac{1}{2}\mathrm{kr}^2 = \frac{1}{2}\mathrm{k}(\mathrm{x}^2 + \mathrm{y}^2) \tag{6}$$

Since the differential equations are harmonic again, the solutions are similar

```
function dy = harmonic2(t,y)
x0 = 10; % initial x-position
y0 = 10; % initial y-position
px0 = 20; % initial x-momentum
py0 = 20; % initial y-momentum
k = 30;
m = 40;
```

```
omega = sqrt(k/m)
tstep = 500;

dy = zeros(4,1);
dy(1) = y(2); % dx/dt = x'
dy(2) = -k/m*y(1); % d^2x/dt^2 = -k/m x
dy(3) = y(4); % dy/dt = y'
dy(4) = -k/m*y(3); % d^2y/dt^2 = -k/m y
```