Determining the Earth's magnetic field and e/m_e using Helmholtz Coils

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This experiment examined the effect of a uniform magnetic field generated by a pair of Helmholtz coils on a beam of electrons that was accelerated through a known potential, and subsequently provided a derivation for the charge to mass ratio of the electron. As the voltage through the coils is increased, the coils' current induces larger magnetic fields. These magnetic fields change the direction of the accelerated electrons expelled by the cathode. Because magnetic fields do no work, the electrons' speed remains unchanged. In the experiment, the voltage was varied in intervals of 10V from 165V to 294V. For each set of measurements, the voltage was fixed and the current was allowed to vary; the radii were measured and so $\frac{e}{m_e}$ and the Earth's magnetic field B_e were derived from the slope and intercept respectively. The two values were found to be $\frac{e}{m_e} = 1.69 \times 10^{11} \frac{C}{kg}$ and $\mathrm{B}_e = 5.59 \times 10^{-5}$T respectively. The corresponding average uncertainties were found as well: $\Delta(e/m) = \pm 6.07 \times 10^{9} \frac{C}{kg}$ and $\Delta(\mathrm{B}_e) = \pm 5.59 \times 10^{-9}$T. It was found that the empirical $\frac{e}{m_e}$ was within $2\Delta(e/m)$ of the accepted value of the charge to mass ratio $\frac{e}{m_e} = 1.759 \times 10^{11} \frac{C}{kg}$ and that B_e was consistent with the value in Toronto 5.6×10^{-5} T [4].

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I. INTRODUCTION

In 1897, J. J. Thomson was the first person to measure the ratio between the charge and mass of the electron. He accomplished this by accelerating the electron through a known potential V and measuring the radius of the circular motion with constant speed in a uniform, measurable magnetic field. The purpose of this experiment was to determine the charge to mass ratio of an electron. This is one of the essential measurements an experimenter can make on the fundamental fermion. As an addendum, one also tried to find the empirical value of Earth's magnetic field in Toronto.

The force F on a charged particle moving in a magnetic field F is given by

$$\vec{F} = q\vec{v} \times \vec{B} \quad \text{The magnitude is just } F = evB \tag{1}$$

The charged particle in this case is just the electron, as we want the mass-to-charge ratio of the electron, with the elementary charge and mass m_e. The direction of the force is perpendicular to both $\vec{v}$ and $\vec{B}$, this results in a circular path of radius r. So this gives the centripetal acceleration v^2/r. Plugging this into [Eq. 1],

$$\begin{aligned} m_e v^2/r &= evB \\ \implies \frac{e}{m_e} &= \frac{v}{rB} \end{aligned} \tag{2}$$

Because one has circular motion, conservation of energy holds. Equating the electric potential energy $U_E = qV$, where V is the potential difference, with the classical kinetic energy,

$$T = U_E \implies \frac{1}{2} m_e v^2 = eV \implies v = \sqrt{\frac{2eV}{m_e}}$$

Substituting the velocity into [Eq. 2], one receives

$$\begin{aligned} \frac{e}{m_e} &= \sqrt{\frac{2eV}{m_e}}(rB)^{-1}, \text{ Squaring both sides} \\ \frac{e^2}{m_e^2} &= \frac{2eV}{m_e r^2 B^2} \\ \frac{e}{m_e} &= \frac{2V}{r^2 B^2} \end{aligned} \tag{3}$$

We get the charge-to-mass ratio of an electron [Eq. 3]. The Helmholtz coil configuration in the experiment has characteristics $\partial B/\partial z = \partial^2 B/\partial z^2 = 0$ at the centre of the two coils ($\rho = 0$). An illustration of the Helmholtz configuration is shown in [Fig. 1]. The axial magnetic field produced by the Helmholtz Coils at centre of the two coils has the value

$$\mathrm{B}_h = \left(\frac{8\sqrt{5}}{25}\right)\frac{\mu_0 n I}{R} \tag{4}$$

where $\mu_0 = 4\pi \cdot 10^{-7}$ T $\cdot$ m $\cdot$ A^{-1}, I is the current in amperes, R is the coil radius in metres, n is the number of turns in each coil, and B is the magnetic field at $\rho = 0$ in teslas. The direction of this field is perpendicular to the plane of the coils. The non-uniformity of the field at $\rho = 0$ could be said to be measured by $\partial^2 B/\partial z^2$. Therefore, setting this second-order differential to zero would minimise the non-uniformity of the field.

The separation of the coils in z, let us call this parameter h, must be equal to R if the condition imposed by the Helmholtz coils $\partial^2 B/\partial z^2 = 0$ is to remain true. This is what defines a Helmholtz pair. The proof starts with the formula for the axial magnetic field due to a single wire loop derived from Biot-Savart law.

$$B(z) = \frac{\mu_0 I R^2}{2(R^2 + z^2)^{3/2}}$$

Then, for an n-turn coil, one substitutes I $\mapsto$ nI.

$$B(z) = \frac{\mu_0 n I R^2}{2(R^2 + z^2)^{3/2}} \tag{5}$$

Now, I make a Taylor expansion for B(z) about the point z = h/2.

$$B(z) = B(h/2) + \frac{\partial B(h/2)}{\partial z}\left(z - \frac{h}{2}\right) + \frac{\partial^2 B(h/2)}{\partial z^2}\left(z - \frac{h}{2}\right)^2 + \mathcal{O}\left(z - \frac{h}{2}\right)^3 \tag{6}$$

I am interested only in the first three terms in the Taylor series.

$$B(h/2) = \frac{8\mu_0 n I R^2}{(4R^2 + h^2)^{3/2}}\ , \quad \frac{\partial B(h/2)}{\partial z} = \frac{-24\mu_0 n I R^2 z}{(4R^2 + h^2)^{5/2}}\ , \quad \frac{\partial^2 B(h/2)}{\partial z^2} = \frac{96\mu_0 n I R^2 (R^2 - h^2)}{(4R^2 + h^2)^{7/2}}$$

Now setting ∂^2B(h/2)/∂z^2 to zero,

$$\frac{96\mu_0 \mathrm{nIR}^2(\mathrm{R}^2 - \mathrm{h}^2)}{(4\mathrm{R}^2 + \mathrm{h}^2)^{7/2}} = 0$$

The denominator is positive definite and $96\mu_0\mathrm{nIR}^2$ is a non-zero constant. The only way the second-order differential is equal to zero is if $\mathrm{R}^2 - \mathrm{h}^2 = 0$, or in other words R=h. Therefore, the separation needs to equal the radii of the coils in order to minimise the non-uniformity of the magnetic field as expected. In this experiment, the Helmholtz coils were used to cancel out the Earth's magnetic field.

II. METHODS AND MATERIALS

The main apparatus of the experiment consisted of both a Leybold Heraeus electron gun and a pair of Leybold Heraeus Helmholtz coils. A glass bulb was used to contain the electron's trajectory. For the electron's path to be visible, the glass bulb was filled with H_2 at a low pressure. The Helmholtz coils had $n = 130$ turns. The radius and separation of the coils were both $R = 0.1605m$. The setup of the Helmholtz coils is illustrated below. In addition, an Agilent E3640A DC power supply ($< \pm 0.15\% + 5$ mA) and a Heathkit regulated high-voltage power supply were used. A Fluke 8000A digital multimeter $\pm$(0.1% of reading +1 digit) was used to measured the potential difference. The apparatus was set up as in the PHYC11 experiment 1 instruction manual [1]. A special illuminated scale with uncertainty $\pm$0.0005m or $\pm$0.5mm was used to measure the circular radii of the electron paths. A mirror was used to read off the scale measurements and to reduce parallax, and subsequently systematic error. OriginLab, instead of

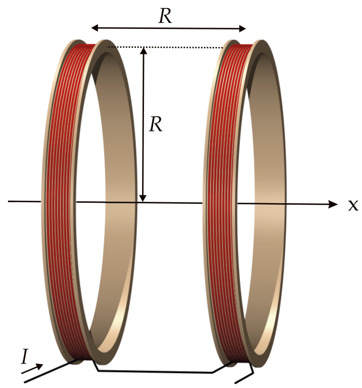

FIG. 1: The configuration of the Helmholtz coils. “Helmholtz coils” by Ansgar Hellwig.

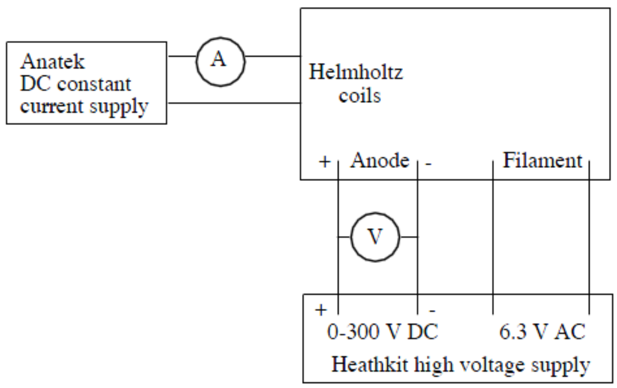

FIG. 2: The circuit of the apparatus used in the experiment [1].

Labfit, was used to do a weighted least squares fit to the data. OriginLab was preferred over Labfit because of both personal preference and the absence of computers with OSX. OriginLab was able to produce York linear fits with error, this is synonymous with the weighted fits in Labfit. MATLAB was used for error propagation.

III. EXPERIMENT PROCEDURE

The first thing done was to connect the apparatus according to the aforementioned [Fig. 2]. The glass bulb was oriented within the Helmholtz coil such that the electrons were accelerated in a direction parallel to the plane of the Helmholtz coils. The filament voltage was turned on first. After around 30 seconds, the anode voltage was turned on. Three preliminary readings of the radius of the electron beam were taken and B, and subsequently e/m were calculated as a test run. The electron beam diameter was matched exactly to the spacing on the illuminated scale instead of visual interpolation for accuracy. These preliminary readings acted as a form of calibration to make sure the values made sense before doing the measurements in earnest.

The voltage started off at 165V as I wanted a large “neighbourhood” of values that could be varied. Then, the current was varied for fields corresponding to different radii. After around three or four measurements at the fixed voltage was taken at 165V, the steps were repeated for a fixed voltage of 174V. This whole procedure was reiterated in intervals of 10V so that the range of fixed voltage varied from 165V to 294V. Overall, there were 14 sets of data that could be used for finding the charge to mass ratio. After the measurements, the anode voltage was turned off first; and finally, the filament voltage was turned off. Note that the instructions in the manual were followed first. The instructions in the manual were to match the electron beam radii to exact centimetre values and then varying the voltage and current. The values achieved were found to be less accurate than the values achieved using the visual interpolation method. This was elaborated in the discussions section.

Subsequently, calculations and data analysis followed. The charge to mass equation[Eq. 3] was linearised, and then B_h was plotted against $1/r$ with $1/r$ as the independent variable. Since the errors in $1/r$ vary as r varies, a weighted least squares fit was used in OriginLab. The slope and intercept were used to find e/m_e and B_e respectively for all 14 fixed voltages. Then, all 14 values of e/m_e and B_e were averaged so the error was reduced. Then, both of the empirical results were compared to the accepted values of e/m_e and B_e in Toronto. The uncertainty was then computed using a MATLAB program shown in appendix b.

IV. RESULTS

The three preliminary reads yielded $B_1 = 9.926 \times 10^{-4}$T, $B_2 = 1.2916 \times 10^{-4}$T, and $B_3 = 8.106 \times 10^{-4}$T. The three charge to mass ratios calculated from the preliminary measurements were $(e/m)_1 = 1.664 \times 10^{11}$C/kg, $(e/m)_2 = 6.298 \times 10^{12}$C/kg, and $(e/m)_3 = 1.5531 \times 10^{11} C/$kg respectively. These measurements served a calibration role and does not influence the final data.
With the values of V, r, and B = $(B_h + B_e)$, I calculated the values of e/m explicitly without using the slope method. The average of the calculated e/m turned out to be 1.65×10^{11} C/kg. Now the e/m and B_e are derived from the graphs' statistical data.

When [Eq. 3] is linearised, one can find the charge to mass ratio from the slope and the magnetic field of the earth B_e from the intercept. The total magnetic field is simply the superposition of the magnetic fields from the HelmHoltz coils and the Earth so B $\mapsto B_h + B_e$, with B_h known in [Eq. 4].

$$\begin{aligned} \frac{e}{m_e} &= \frac{2V}{r^2 B^2} \\ \frac{e}{m_e} &= \frac{2V}{r^2 (B_h + B_e)^2} \\ \sqrt{\frac{e}{m_e}} &= \frac{\sqrt{2V}}{r(B_h + B_e)} \\ B_h + B_e &= \sqrt{2V}\left(\frac{e}{m_e}\right)^{-1/2}\left(\frac{1}{r}\right) \\ \implies B_h &= \sqrt{2V}\left(\frac{e}{m_e}\right)^{-1/2}\left(\frac{1}{r}\right) - B_e \end{aligned} \tag{7}$$

Now when B_h is plotted against $1/r$, e/m is given by

$$\begin{aligned}
\text{Slope } &= \sqrt{2V}\left(\frac{e}{m_e}\right)^{-1/2} \\
\frac{\text{Slope}}{\sqrt{2V}} &= \left(\frac{e}{m_e}\right)^{-1/2} \\
\frac{e}{m_e} &= \frac{2V}{(\text{ Slope })^2}
\end{aligned} \tag{8}$$

And the magnetic field of the Earth is given by the negative of the intercept $B_e = -1 \times$ Intercept . The rest of the raw data is in Appendix a, while a nice summary of e/m_e and B_e is show below.

Voltage (V)	e/m (C/kg)	B_e (T)
165	152433856283	2.37E-05
174	136850406031	0.0001
184	158904139876	2.76E-05
191	163396767123	2.99E-05
200	141188484653	7.34E-05
211	158626856969	1.44E-05
228	143864693280	7.76E-05
232	133839360572	9.16E-05
244	136405397411	0.0001
254	202968549909	0.00011
266	143739001843	7.88E-05
276	164798808753	8.03E-06
287	153012408319	3.74E-05
294	162868663815	1.02E-05

TABLE I: Table of calculated e/m and B_e from the slope and intercept of the measurements

From the table of values measured in [Table. III], one can see that as the voltage through the coils is increased, the coils' current induces larger magnetic fields. The graphs of B_h vs $1/r$ for V=191 and V=254 are plotted in [Fig. 3].

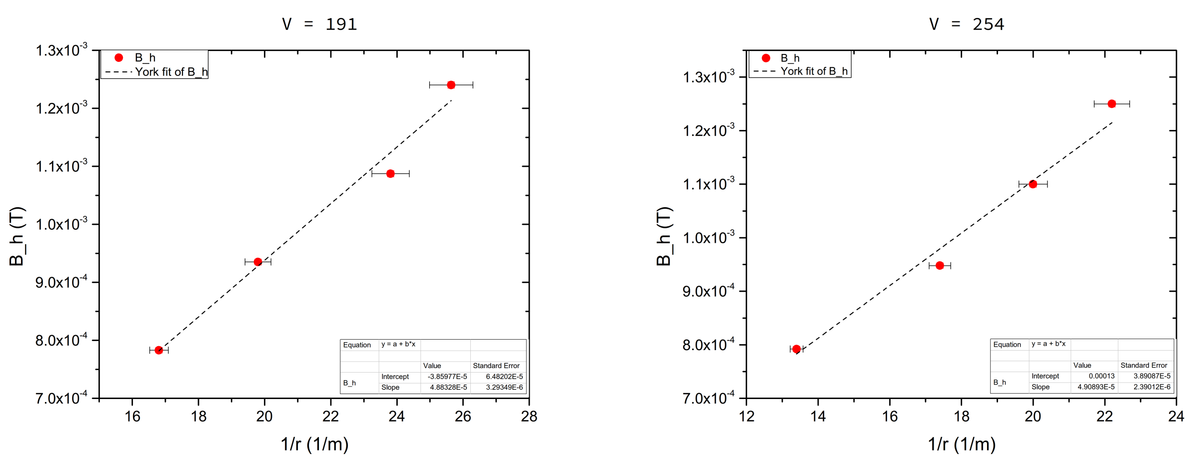

FIG. 3: Two of the B_h vs $1/r$ graphs produced for V = 191 and V = 254.

After the e/m from all 14 measurements for each fixed voltage was found, the averages of e/m and B_e were computed to be $\frac{e}{m} = 1.537 \times 10^{11} \pm \Delta(e/m)$C/kg and $B_e = 5.59 \times 10^{-5} \pm \Delta(B_e)$T. However in my final results, I decided to drop a majority of my measurements. I only kept V = (184, 191, 211, 254, 276, 294). With only these 6 values of voltage, the average of e/m was computed to be $1.69 \times 10^{11} \pm \Delta(e/m)$ C/kg. All the values were kept for B_e. The uncertainties $\Delta(e/m)$ and $\Delta(B_e)$ are calculated in the next section.

V. DISCUSSION

To find the uncertainty in e/m, one simply uses the equation below [Eq. 9].

$$\begin{aligned} \sigma^2_{e/m} &= \left(\frac{\partial f}{\partial V}\sigma_V\right)^2 + \left(\frac{\partial f}{\partial r}\sigma_r\right)^2 + \left(\frac{\partial f}{\partial I}\sigma_I\right)^2 \\ &= \left(\frac{2}{Iar^2}\sigma_V\right)^2 + \left(\frac{4V}{Iar^3}\sigma_r\right)^2 + \left(\frac{2V}{I^2ar}\sigma_I\right)^2, \text{ Where } a = B_h/I \end{aligned} \tag{9}$$

Since r has an absolute error of ± 0.001, it was found that $1/r$ had the same percentage error so that the absolute errors for $1/r$ were calculated and displayed in [Table. III]. Using the essential uncertainty rules,

$$\begin{aligned} \frac{0.001}{r} &= \frac{\Delta(1/r)}{1/r} \\ \Delta(1/r) &= (1/r) \times \frac{0.001}{r} \end{aligned} \tag{10}$$

V has a percent error of $\pm$(0.1% of reading +1 digit) and I has a percent error of $\pm 0.15\% + 5$ mA. In the calculation of B_h, only the I is varying. Everything else in the formula [Eq. 4] is the same for the set of apparatus; thus they are constant.

$$\Delta B_h = \left|\frac{8\sqrt{5}}{25}\frac{\mu_0 n}{R}\right| \times \Delta I \tag{11}$$

There are 56 measurements in total, so calculating the uncertainties by hand using [Eq. 9] would be too tedious. Using a MATLAB program, listed in Appendix b, and taking the average, one receives an average uncertainty of $\Delta(e/m) = \pm 6.07 \times 10^9$C/kg.

```
>> PropError(em, [V r B], [227.32, 0.053, 0.001], [0.227, 0.001, 0.0000052])
ans =

         [1x1 sym]       '+/-'      [6.0658e+09]
    'Percent Error'      '+/-'         [1x1 sym]
```

With the experimental value and the modern accepted value of e/m, the percentage error can also be found.

$$\begin{aligned} \% \text{ error } &= \left|\frac{\text{empirical value - theoretical value}}{\text{theoretical value}}\right| \\ &= \left|\frac{1.69 \times 10^{11} - 1.76 \times 10^{11}}{1.76 \times 10^{11}}\right| \\ &= \pm 4.14\% \end{aligned}$$

The percentage error of this e/m is quite high, compared to the $\pm 1\%$ stated in the lab manual; however, the final value for the charge to mass ratio 1.69×10^{11} is just inside $2\Delta(e/m)$, with the $\Delta(e/m)$ found above.

$$\Delta(B_e) = \sqrt{(\Delta\xi)^2 + (\Delta B_h)^2}, \text{ where } \Delta\xi = \sqrt{2}\xi\sqrt{\left[\frac{\Delta(e/m)^{-1/2}}{(e/m)^{-1/2}}\right]^2 + \left(\frac{\Delta\sqrt{V}}{\sqrt{V}}\right)^2 + \left(\frac{\Delta 1/r}{1/r}\right)^2}$$

The average uncertainty of B_e turns out to be $\Delta(B_e) = \pm 5.59 \times 10^{-9}$T . The total magnetic field in Toronto is about 5.6×10^{-5} T [4]. Therefore, my value $B_e = 5.59 \times 10^{-5}$T is consistent with the accepted value for Toronto with an uncertainty of $\pm 5.59 \times 10^{-9}$T. Unlike my error in e/m_e, the percentage error in the magnetic field of Toronto is only $\pm 0.18\%$. This is an acceptable result as the percent error ir $< \pm 1\%$.

A major source of error is the illuminated scale and parallax. At low currents and high voltages, there were many occurrences in which the electron beam's diameter was much larger than the scale; this required an "extension" of the ruler. The scale was also flawed such that the spacings between the 7th and 8th centimetre had no millimetre markings. As a result, some visual interpolation was used. Another systematic error is that the experimental velocity of the electrons do not match up with the theoretical because of the nonuniform current and also since the electrons have to collide with the hydrogen gas inside the bulb to make the electron path visible. It should also be noted that the magnetic fields were not ideal in the experiment, which may have caused some of the electrons to travel at different velocities, contrary to what was assumed in [Eq. 2]. These are the random errors in the experiment.

One strength of the experiment was the mirror used to aid with measuring the circular electron path radius. The mirror reduced the parallax, and therefore significantly reduced the systematic error. Another strength of this experiment would be that the HelmHoltz coils were connected in series instead of parallel. If the coils were connected in parallel, the two currents would run against each other and therefore cancelling each other out. Additionally, the current through the two coils is maximised when put in series, not parallel. A related weakness of the experiment was the circular glass bulb that the experimenter had to look through to make measurements. At many times, the measurements were hard to make even in a dark room.

As an addendum, if one wanted to find the e/m_e ratio using quantum mechanical laws, the Zeeman effect is manifested. Using the Zeeman effect, the charge to mass ratio m is given by:

$$\frac{e}{m_e} = \frac{4\pi c}{B(m_{j,f}g_{J,f} - m_{j,i}g_{J,i})}\frac{\delta D}{D\Delta D} \tag{12}$$

where D is the mirror separation and δD is the change in mirror separation.

VI. CONCLUSION

The empirical value of e/m_e identified from the slope, after dropping some measurements, was found to be $\frac{e}{m_e} = 1.69 \times 10^{11}$ with an average uncertainty of $\Delta(e/m) = \pm 6.07 \times 10^9$C/kg. The charge to mass ratio was also calculated directly using the values of V, r, and B and was found to be 1.65×10^{11} C/kg which is close to the empirical result 1.69×10^{11} C/kg.

The result for the Earth's field was also found to be $B_e = 5.59 \times 10^{-5}$ T with an average uncertainty of $\pm 5.59 \times 10^{-9}$ T, which is consistent with the accepted value in Toronto [4]. This value of magnetic field is not only the Earth's magnetic field. This experiment was done in a laboratory made of metal so the walls and ceiling would have affected the magnetic field.

It is noted that the uncertainty of B_h depended only on I, and as such was the only quantity considered in the error propagation. R, both the separation and radius of the coils, is assumed to be perfectly error-less.

The effect of a uniform magnetic field generated by a pair of Helmholtz coils on a beam of electrons that was accelerated through a known potential was also discovered. As the voltage through the coils is increased, the coils' current induces larger magnetic fields. These magnetic fields change the direction of the accelerated electrons expelled by the cathode. Because magnetic fields do no work, the electrons' speed remains unchanged. The radii, different voltage and current pairings produced, generated a basis in which one could compute the charge to mass ratio of an electron.

Overall, the experiment was successful. I deviated from the lab manual instructions as I did not first match the electron beam's radii to perfect centimetre measurements and then vary the current for a fixed voltage. Instead, I did the measurements on the radii after varying and fixing voltage to an arbitrary value. This resulted in a only a percentage error of $\pm 4.14\%$ instead of a percentage error of $\pm 12.6\%$ when I followed the lab manuals instructions. Both of these values are greater than the $\pm 1\%$ expected, but the final results still lie within $2\Delta(e/m)$. This is even better than the reasonable result of $3\Delta(e/m)$ from the lab manual! If not for having 56 measurements, the error may have been even worse. As for finding the empirical value for the Earth's magnetic field in Toronto, my value was perfectly consistent with the modern accepted value.

VII. REFERENCES

[1] PHYC11 Experiment 1
[2] PHYC11 Lecture 0
[3] PHYC11 Lecture 1
[4] YorkU PHYS2211 Lab 6

VIII. APPENDIX A: RAW DATA

Voltage (V)	Current (A)	Radius (m)
165	1.717	0.0365
165	1.508	0.0415
165	1.298	0.048
174	1.71	0.038
174	1.504	0.042
174	1.292	0.048
174	1.08	0.056
174	0.866	0.0695
184	1.735	0.038
184	1.519	0.042
184	1.308	0.049
184	1.093	0.0575
184	0.877	0.073
191	1.703	0.039
191	1.493	0.042
191	1.284	0.0505
191	1.075	0.0595
200	1.717	0.04025
200	1.503	0.04575
200	1.293	0.052
200	1.082	0.062
211	1.711	0.042
211	1.497	0.04725
211	1.289	0.061
211	1.078	0.063
228	1.701	0.043
228	1.49	0.0485
228	1.281	0.055
228	1.072	0.066
232	1.732	0.04275
232	1.52	0.051
232	1.308	0.05575
232	1.093	0.066
244	1.724	0.044
244	1.513	0.05
244	1.302	0.056
244	1.088	0.067
254	1.718	0.045
254	1.512	0.05
254	1.301	0.0575
254	1.087	0.0745
266	1.723	0.04575
266	1.512	0.0515
266	1.302	0.059
266	1.088	0.07
276	1.723	0.0475
276	1.51	0.052
276	1.3	0.061
276	1.087	0.0745
287	1.714	0.0475
287	1.5	0.0545
287	1.291	0.0625
294	1.708	0.048
294	1.504	0.054
294	1.295	0.0635
294	1.083	0.075

TABLE II: The raw values of radii whilst varying V and I.

Voltage	1/r (1/m)	$\Delta(1/r)$	B_h	$\Delta(B_h)$
165	2.74E+01	7.51E-01	1.25E-03	5.52E-06
165	2.41E+01	5.81E-01	1.10E-03	5.29E-06
165	2.08E+01	4.34E-01	9.45E-04	5.06E-06
174	2.63E+01	6.93E-01	1.25E-03	5.51E-06
174	2.38E+01	5.67E-01	1.10E-03	5.28E-06
174	2.08E+01	4.34E-01	9.41E-04	5.05E-06
174	1.79E+01	3.19E-01	7.87E-04	4.82E-06
174	1.44E+01	2.07E-01	6.31E-04	4.59E-06
184	2.63E+01	6.93E-01	1.26E-03	5.54E-06
184	2.38E+01	5.67E-01	1.11E-03	5.30E-06
184	2.04E+01	4.16E-01	9.53E-04	5.07E-06
184	1.74E+01	3.02E-01	7.96E-04	4.84E-06
184	1.37E+01	1.88E-01	6.39E-04	4.60E-06
191	2.56E+01	6.57E-01	1.24E-03	5.50E-06
191	2.38E+01	5.67E-01	1.09E-03	5.27E-06
191	1.98E+01	3.92E-01	9.35E-04	5.04E-06
191	1.68E+01	2.82E-01	7.83E-04	4.82E-06
200	2.48E+01	6.17E-01	1.25E-03	5.52E-06
200	2.19E+01	4.78E-01	1.09E-03	5.28E-06
200	1.92E+01	3.70E-01	9.42E-04	5.05E-06
200	1.61E+01	2.60E-01	7.88E-04	4.82E-06
211	2.38E+01	5.67E-01	1.25E-03	5.51E-06
211	2.12E+01	4.48E-01	1.09E-03	5.28E-06
211	1.64E+01	2.69E-01	9.39E-04	5.05E-06
211	1.59E+01	2.52E-01	7.85E-04	4.82E-06
228	2.33E+01	5.41E-01	1.24E-03	5.50E-06
228	2.06E+01	4.25E-01	1.09E-03	5.27E-06
228	1.82E+01	3.31E-01	9.33E-04	5.04E-06
228	1.52E+01	2.30E-01	7.81E-04	4.81E-06

Voltage	1/r (1/m)	$\delta(1/r)$	B_h	$\delta(B_h)$
232	2.34E+01	5.47E-01	1.26E-03	5.53E-06
232	1.96E+01	3.84E-01	1.11E-03	5.30E-06
232	1.79E+01	3.22E-01	9.53E-04	5.07E-06
232	1.52E+01	2.30E-01	7.96E-04	4.84E-06
244	2.27E+01	5.17E-01	1.26E-03	5.52E-06
244	2.00E+01	4.00E-01	1.10E-03	5.29E-06
244	1.79E+01	3.19E-01	9.48E-04	5.06E-06
244	1.49E+01	2.23E-01	7.92E-04	4.83E-06
254	2.22E+01	4.94E-01	1.25E-03	5.52E-06
254	2.00E+01	4.00E-01	1.10E-03	5.29E-06
254	1.74E+01	3.02E-01	9.48E-04	5.06E-06
254	1.34E+01	1.80E-01	7.92E-04	4.83E-06
266	2.19E+01	4.78E-01	1.25E-03	5.52E-06
266	1.94E+01	3.77E-01	1.10E-03	5.29E-06
266	1.69E+01	2.87E-01	9.48E-04	5.06E-06
266	1.43E+01	2.04E-01	7.92E-04	4.83E-06
276	2.11E+01	4.43E-01	1.25E-03	5.52E-06
276	1.92E+01	3.70E-01	1.10E-03	5.29E-06
276	1.64E+01	2.69E-01	9.47E-04	5.06E-06
276	1.34E+01	1.80E-01	7.92E-04	4.83E-06
287	2.11E+01	4.43E-01	1.25E-03	5.51E-06
287	1.83E+01	3.37E-01	1.09E-03	5.28E-06
287	1.60E+01	2.56E-01	9.40E-04	5.05E-06
294	2.08E+01	4.34E-01	1.24E-03	5.51E-06
294	1.85E+01	3.43E-01	1.10E-03	5.28E-06
294	1.57E+01	2.48E-01	9.43E-04	5.06E-06
294	1.33E+01	1.78E-01	7.89E-04	4.82E-06

TABLE III: The values for plotting B_h against $1/r$ with errors.

IX. APPENDIX B: ERROR PROPAGATION

```
%(c) Brad Ridder 2007
function sigma = PropError(f,varlist,vals,errs)
n = numel(varlist);
sig = vpa(ones(1,n));
for i = 1:n
    sig(i) = diff(f,varlist(i),1);
end
error1 =sqrt((sum((subs(sig,varlist,vals).^2).*(errs.^2))));
error = double(error1);
sigma = [{subs(f,varlist,vals)} {'+/-'} {error};
         {'Percent Error'} {'+/-'} {abs(100*(error)/subs(f,varlist,vals))}];
```