FROM LAGRANGE TO FEYNMAN

A different approach

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PHYC54H3

LAGRANGE

$$\mathcal{L} = T - V$$

- In PHYC54's section on variational calculus, we calculated the action $\mathcal{S} = \int_{t_1}^{t_2} \mathcal{L}[x, \dot{x}; t] \, \mathrm{d}t$
- This $\mathcal{S}$ is fundamental in Feynman's path integral

SCHRÖDINGER

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- Ψ is the position-space wave-unction

SOLUTION METHODS

- Recall that we solve Schrödinger's equation by finding the energy eigenvalues
- We dealt in the quantity called the wave-function
- In the path integral approach, we deal in the quantity known as the kernel

FEYNMAN PATH INTEGRAL

AN ANALOGY: BLAST TO THE PAST!

Figure: Two slits

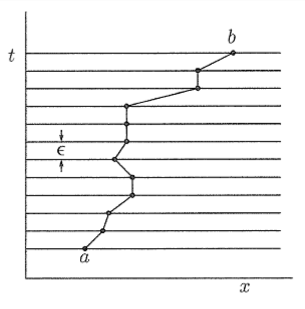
Figure: Infinite slits

One can think of this approach as an infinite slit experiment.

- The double slit experiment was to demonstrate that light and matter could behave like classical particles AND waves.
- The event of an electron going from a source to a destination has one amplitude while passing through one of the two slits and has another amplitude for passing through the other slit
- The total amplitude of this event is then the superposition of the two amplitudes from the two respective slits.

CONTRIBUTING PATHS

- In quantum mechanics (unlike classical mechanics), there is not just a single path (of least action) contributing to the amplitude.
- In addition to the classical path, there are paths that fluctuate around it.
- This is known as quantum fluctuation $x(t) = x_c(t) + \delta x(t)$
- This differentiates classical and quantum particles, a classical particle cannot escape a potential barrier, but a quantum particle can tunnel through



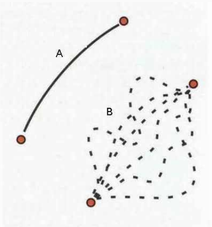


Figure: Case A includes only the classical trajectory while case B includes all possible paths.

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- $$\phi[x(t)] = \text{ constant } e^{i\,\mathcal{S}[x(t)]/\hbar} \tag{1}$$
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- $\mathcal{D}$ is the weight of all the possible paths

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- $$K(b, a) = Fe^{i\,\mathcal{S}_{cl}/\hbar} \tag{3}$$

- The amplitude of a path with the mapping a $\to$ c $\to$ b is the sum of the sum of the amplitudes from a $\to$ c and the one from c $\to$ b.
- The action for multiple events is

$$\mathcal{S}[b, a] = \mathcal{S}[b, c] + \mathcal{S}[c, a]$$

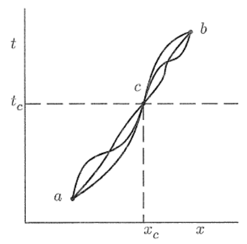

Figure: A path with multiple events.

CONT'D

- $K(b,a) = \int_a^b \mathcal{D}\,x(t) e^{i(\mathcal{S}[b,c]+\mathcal{S}[c,a])/\hbar}$
- $K(b,a) = \int_{-\infty}^{\infty} \int_c^b \mathcal{D}\,x(t) K(c,a) e^{i\,\mathcal{S}[b,c]/\hbar} \mathrm{d}x_c$
- $K(b,a) = \int_{-\infty}^{\infty} K(b,c) K(c,a) \mathrm{d}x_c$

FAMILIAR EXAMPLES

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- $$K_0(b, a) = \left(\frac{m}{2\pi i\hbar(t_b - t_a)}\right)^{1/2} \exp\left[\frac{im(x_b - x_a)^2}{2\hbar(t_b - t_a)}\right] \tag{6}$$

- Fix time, vary distance

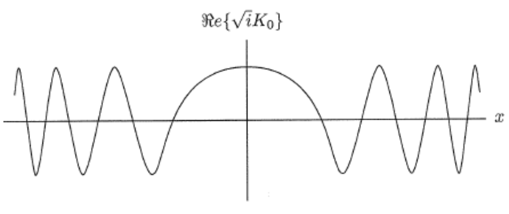
- 

Figure: The plot of the real part of $\sqrt{i}$ times our kernel with varying distance.

- $$2\pi = \frac{m(x+\lambda)^2}{2\hbar t} - \frac{mx^2}{2\hbar t} = \frac{mx\lambda}{\hbar t} + \frac{m\lambda^2}{2\hbar t} \tag{7}$$

- $$\lambda = \frac{2\pi\hbar}{mxt^{-1}} = \frac{h}{p} \tag{8}$$

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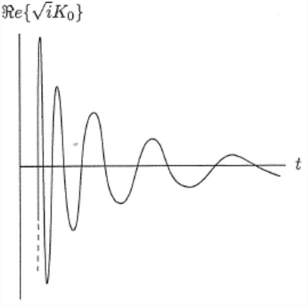

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$$2\pi = \frac{mx^2}{2\hbar t} - \frac{mx^2}{2\hbar(t+T)} = \frac{mx^2}{2\hbar t^2}\left(\frac{T}{1+T/t}\right) \tag{9}$$

$$\omega \approx \frac{m}{2\hbar}\left(\frac{x}{t}\right)^2 = E/\hbar \tag{10}$$

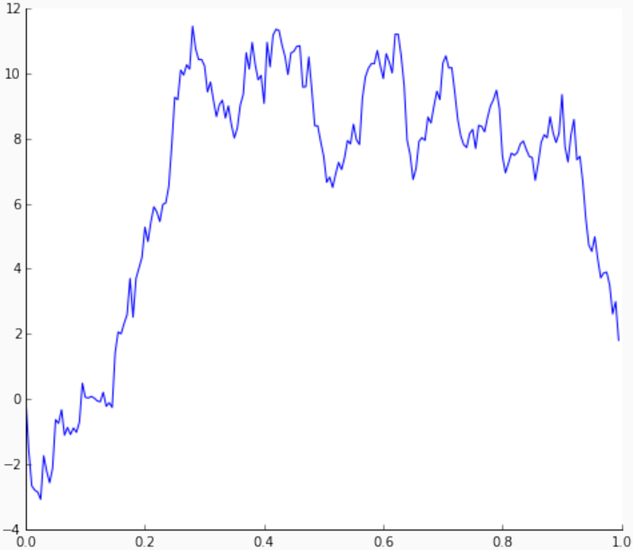
Schrödinger's equation Have to find energy eigenvalues

Path Integral Have only to find action

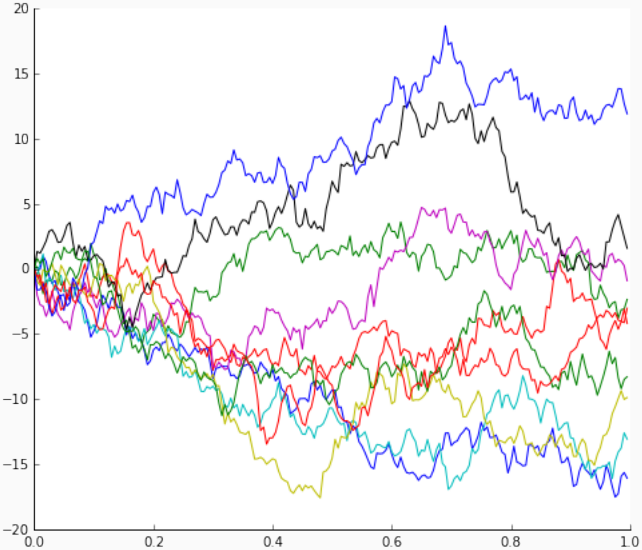
BEYOND!

- A Wiener process is a stochastic (random) time-continuous process.
- It is also known as Brownian Motion.
- It is important to note that it forms a basis to the path integral through the Feynman-Kac formula.
- Using python, we can simulate a Wiener process

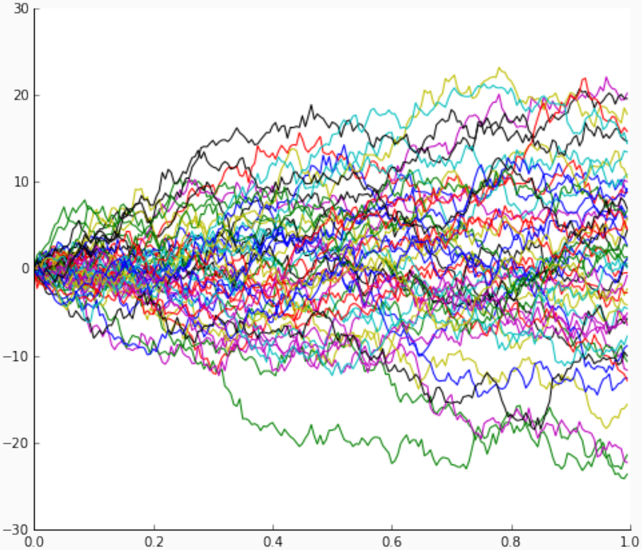
1 PATH



10 PATHS



50 PATHS



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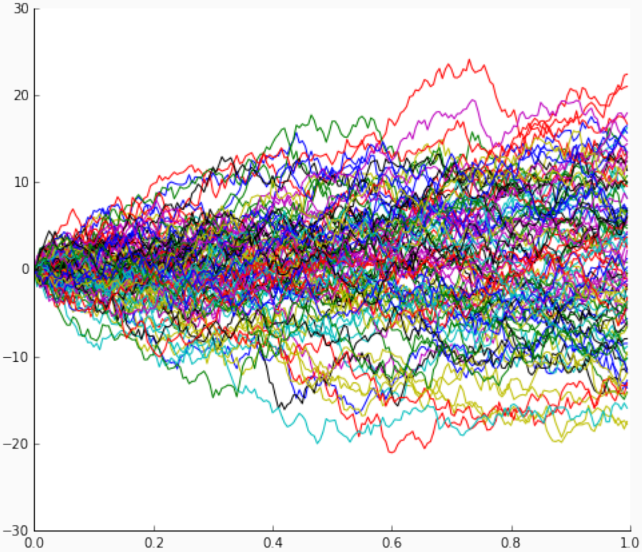

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- Done

Questions?