

Set 1

Parker Shankin-Clarke

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1. Evaluate the fixed points and assess their stability.

$$\dot{x} = \mu \cdot x - M \cdot x^2$$

Theorem 10.3.1 Stability Criteria for $x' = g(x)$ states :

Let x_1 be a fixed point of the autonomous differential equation $x' = g(x)$, where g is differentiable at x_1 .

- (a) if $g'(x_1) < 0$, then x_1 is an asymptotically stable critical point.
- (b) if $g'(x_1) > 0$, the x_1 is an unstable critical point.

First,

$$g(x) = \mu \cdot x - M \cdot x^2$$

$$x = 0$$

$$x = \frac{\mu}{M}$$

Therefore

$$g(0) = \mu \text{ and } g\left(\frac{\mu}{M}\right) = -\mu$$

By Theorem 10.3.1 $g\left(\frac{\mu}{M}\right)$ is a asymptotically stable critical point and $g(0)$ is an unstable critical point.

2.

$$\dot{x} = x(\mu_x + x \cdot M_{xx} + y \cdot M_{xy})$$

$$\dot{y} = y(\mu_y + y \cdot M_{yy} + x \cdot M_{xy})$$

a.)

In order to find critical points find the solutions to the following system of equations:

$$\begin{cases} 0 = \mu_x \cdot x + x^2 \cdot M_{xx} + y \cdot x \cdot M_{xy} \\ 0 = \mu_y \cdot y + y^2 \cdot M_{yy} + y \cdot x \cdot M_{xy} \end{cases}$$

Matlab code used to solve system

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syms a b c d e x y p
eqn1 = a .* x + b .* x.^(2) + p .* y .* x == 0;
eqn2 = d .* y + e .* y.^(2) + c .* y .* x == 0;

sol = solve([eqn1, eqn2], [x, y]);
xSol = sol.x
ySol = sol.y
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output :

$$X.sol = \begin{bmatrix} 0 \\ -\frac{(\mu_x \cdot M_{yy} - M_{xy} \cdot \mu_y)}{-M_{xy} \cdot M_{yx} + M_{xx} \cdot M_{xy}} \\ -\frac{\mu_x}{M_{xx}} \\ 0 \end{bmatrix}$$

$$Y.sol = \begin{bmatrix} 0 \\ -\frac{(\mu_x \cdot M_{xy} - M_{xx} \cdot \mu_y)}{-(M_{xy})^2 + M_{xx} \cdot M_{xy}} \\ 0 \\ -\frac{\mu_y}{M_{yy}} \end{bmatrix}$$

Parts b and c.)

Definition: the trace of a matrix (τ) is the sum of the diagonal.

Definition : The determinant of a matrix (Δ)

Assume that $\tau^2 - 4\Delta > 0$

Therefore in general for the three non-trivial cases that are computed below there are three criteria that determine stability :

- 1.) $\tau < 0$ and $\Delta > 0$ then you have a stable solution
- 2.) $\tau > 0$ and $\Delta > 0$ then you have a stable solution
- 3.) $\Delta < 0$ then you have a saddle point

The general matrix is:

$$\begin{bmatrix} \mu_x + 2xM_{xx} + yM_{xy} & xM_{xy} \\ yM_{xy} & \mu_y + 2yM_{yy} + xM_{xy} \end{bmatrix}$$

bc_1 : for the trivial case(0,0) :

$$\begin{bmatrix} \mu_x & 0 \\ 0 & \mu_y \end{bmatrix}$$

Since the matrix is in rref form the eigenvalues are the diagonal entries of the matrix. This leads to three possibilities :

- 1.) If both of the diagonal entries are positive then the trivial solution is unstable node.
- 2.) If both of the diagonal entries are negative then the trivial solution is stable node.
- 3.) If one of the diagonal entries is negative and one of the diagonal entries is positive then you have a saddle point.

bc_2 : for the non-trivial case($-\frac{\mu_x}{M_{xx}}, 0$) :

$$\begin{bmatrix} \mu_x - 2\mu_x & M_{xy}x \\ 0 & \mu_y - \frac{M_{xy}}{M_{xx}}\mu_x \end{bmatrix}$$

bc_3 : for the non-trivial case($0, -\frac{\mu_y}{M_{yy}}$) :

$$\begin{bmatrix} \mu_x - \mu_y \frac{M_{xy}}{M_{yy}} & 0 \\ -\mu_y \frac{M_{xy}}{M_{xy}} & \mu_y - 2\mu_y \end{bmatrix}$$

bc_4 : for the non-trivial case ($-\frac{(\mu_x \cdot M_{yy} - M_{xy} \cdot \mu_y)}{-M_{xy} \cdot M_{yx} + M_{xx} \cdot M_{xy}}, -\frac{(\mu_x \cdot M_{xy} - M_{xx} \cdot \mu_y)}{-M_{xy} \cdot M_{yx} + M_{xx} \cdot M_{xy}}$)

$$\left[\begin{array}{c} \frac{2\mu_x M_{yy} M_{xx} + 2\mu_x (M_{xy})(M_{yx}) + \mu_x M_{xy} M_{xx} - M_{yy} \mu_y M_{xy} - 2\mu_y M_{xy} M_{xx}}{M_{xy} + M_{xx}} \\ \frac{2(\mu_x M_{yy} - \mu_y M_{xy})}{M_{xy} + M_{xx}} \end{array} \quad \frac{(\mu_x M_{yy} - \mu_y M_{xy})}{M_{xy} + M_{xx}} \quad \frac{3\mu_x M_{yy} M_{xy} - 2(M_{yy})^2 + \mu_y M_{xy} M_{xx}}{M_{xy}(M_{xy} + M_{xx})} \right]$$

3.)

a.) Ensure that $\dot{x}_i = x_i (\mu_i + \sum_{j=1}^N (M_{ij} \cdot x_j))$ reduce to equation (2) for $N = 2$

In order to accomplish this expand to $N = 2$:

$$\alpha(x) = \begin{cases} \dot{x}_1 = x_1(\mu_1 + M_{11}x_1 + M_{12}x_2) \\ \dot{x}_2 = x_2(\mu_2 + M_{21}x_1 + M_{22}x_2) \end{cases}$$

3b.) Take the GLV equation

$$\begin{cases} 0 = x_1(\mu_{x_1} + x_1 M_{11} + \dots + M_{1n} x_n) \\ \cdot \\ \cdot \\ \cdot \\ 0 = x_n(\mu_{x_n} + x_1 M_{n1} + \dots + M_{nn} x_n) \end{cases}$$

rewrite as:

$$\begin{bmatrix} M_{11} & M_{12} & M_{13} & \dots & M_{1n} \\ M_{21} & M_{22} & M_{23} & \dots & M_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ M_{n1} & M_{n2} & M_{n3} & \dots & M_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} = \begin{bmatrix} \mu_x \\ \mu_y \\ \dots \\ \mu_n \end{bmatrix}$$

Use the following algorithm:

for N GLV equations with variables $(x_1, x_2, x_3, \dots, x_n)$ one can find the first solution $(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n)$ by solving the above N linear system :

$$\left[\begin{array}{ccccc|c} M_{11} & M_{12} & M_{13} & \dots & M_{1n} & \mu_1 \\ M_{21} & M_{22} & M_{23} & \dots & M_{2n} & \mu_2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ M_{n1} & M_{n2} & M_{n3} & \dots & M_{nn} & \mu_n \end{array} \right]$$

The second fixed point can be found as the trivial point :

$$(0_1, 0_2, \dots, 0_n)$$

The remaining 2^{N-2} solutions can be found as follows :

Compute every permutation where greater than or equal to one variable equals zero and solve the corresponding resultant system of equations.

The sum of three classes of solutions will equal 2^N solutions.

4.) solutions to parts a and b are on github.