Set 2

Parker Shankin-Clarke

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1. Compute the eigenvalues of the Jacobian of the following GLV equation.

$$\dot{x_a} = x_a(\mu_a + x_a \cdot M_{aa} + x_b \cdot M_{ab})$$
$$\dot{x_b} = x_b(\mu_b + x_a \cdot M_{ba} + x_b \cdot M_{bb})$$

The Jacobian of the system :

$$\begin{bmatrix} \mu_a + 2x_a M_{aa} + x_b M_{ab} & x_a M_{ab} \\ x_b M_{ba} & \mu_b + 2x_b M_{bb} + x_a M_{ba} \end{bmatrix}$$

The fixed points for the system of equations are :

1.) (0,0)

2.) (-$\frac{\mu_a}{M_{aa}}$,0)

3.) (0,$-\frac{\mu_b}{M_{bb}}$)

4.) $(-\frac{\mu_a M_{bb} - \mu_b M_{ab}}{M_{aa}M_{bb} - M_{ab}M_{ba}}, \frac{\mu_a M_{ba} - \mu_b M_{aa}}{M_{aa}M_{bb} - M_{ab}M_{ba}})$

Now plug in each fixed point to the Jacobian of the system :
1.)

$$\begin{bmatrix} \mu_a & 0 \\ 0 & \mu_b \end{bmatrix}$$

The eigenvalues for this matrix are μ_a and μ_b

2.)

$$\begin{bmatrix} -\mu_a & -\frac{M_{ab}}{M_{aa}}\mu_a \\ 0 & \mu_b - \frac{M_{ba}}{M_{aa}}\mu_a \end{bmatrix}$$

The eigenvalues are $-\mu_a$ and $\mu_b - \frac{M_{ba}}{M_{aa}}\mu_a$
3.)

$$\begin{bmatrix} \mu_a - \mu_b\frac{M_{ab}}{M_{bb}} & 0 \\ -\mu_y\frac{M_{ba}}{M_{bb}} & -\mu_b \end{bmatrix}$$

The matrix is in lower triangular form so the eigenvalues are $\mu_a - \mu_b\frac{M_{ab}}{M_{bb}}$ and $-\mu_b$

4.)

$$\begin{bmatrix} \frac{-2\mu_a(M_{aa})^2-(mu_a)(M_{bb})(M_{aa})+2(\mu_a)(M_{ab})(M_{ab})-3\mu_b M_{aa}M_{ab}}{-(M_{ba}M_{ab}-M_{aa}M_{bb})} & \frac{M_{ab}(\mu_a M_{aa}-\mu_b M_{ab})}{M_{ba}M_{ab}-M_{aa}M_{bb}} \\ \frac{M_{ba}(\mu_b M_{aa}-\mu_a m_{ab})}{M_{ba}M_{ab}-M_{aa}(M_{bb}} & \frac{-\mu_a M_{aa}M_{ba}+2\mu_a M_{bb}M_{ba}-\mu_b m_{aa}M_{bb}-2\mu_b M_{ba}M_{ab}}{-(M_{ba}M_{ab}-M_{aa}M_{bb})} \end{bmatrix}$$

In order to solve for the eigenvalues use det(A-λI) = 0:

Use substitution for values in matrix above :

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

So the eigenvalues are :

$\lambda = \frac{(-a-d)\pm\sqrt[2]{(a-d)^2-4(ad-bc)}}{2a}$

(a) if g'(x_1) $<$ 0, then x_1 is an asymptotically stable critical point.
(b) if g'(x_1) $>$ 0, the x_1 is an unstable critical point.

First,

$g(x) = \mu \cdot x - M \cdot x^2$
$x = 0$
$x = \frac{\mu}{M}$
Therefore

$g(0) = \mu$ and $g(\frac{\mu}{M}) = -\mu$

By Theorem 10.3.1 $g(\frac{\mu}{M})$ is a asymptotically stable critical point and g(0) is an unstable critical point.

For the results of the indivdual cases reference the code.

2.Non-dimensionalize the following:

$$\dot{x_a} = x_a(\mu_a + x \cdot M_{aa} + y \cdot M_{ab})$$
$$\dot{x_b} = x_b(\mu_b + y \cdot M_{bb} + x \cdot M_{ba})$$

$$\tilde{x_a} = \frac{x_a}{x_{as}}, \tilde{t} = \frac{t}{t_s}$$

$$-\frac{d(x_{as}\tilde{x_a})}{d(t_s\tilde{t})} + (x_{as}\tilde{x_a})\mu_a - (x_{as}\tilde{x_a})^2 M_{aa} + (x_{as}\tilde{x_a})(x_{bs}\tilde{x_b})M_{ab} = 0$$

$$-\frac{x_{as}}{t_s}\frac{d(\tilde{x_a})}{d(\tilde{t})} + (x_{as}\tilde{x_a})\mu_a - (x_{as}\tilde{x_a})^2 M_{aa} + (x_{as}\tilde{x_a})(x_{bs}\tilde{x_b})M_{ab} = 0$$

Now divide by the $\frac{x_{as}}{t_s}$ term

Now you can pursue a partial or complete de-dimentionalization.

Complete :

$$\begin{cases} \mu_a = \frac{1}{t_s} \\ M_{aa} = \frac{1}{x_{sa}t_s} \\ M_{ab} = \frac{1}{t_s x_{sb}} \end{cases}$$

Partial :

$$\begin{cases} \mu_a = t_s \cdot \mu_a \\ M_{aa} = \frac{1}{x_{sa}t_s} \\ M_{ab} = x_{sb} \cdot M_{ab} \end{cases}$$

Complete

$$\dot{x_a} = x_a(1 + x + y)$$

$$\dot{x_b} = x_b(1 + y + x)$$

Partial

$$\dot{x_a} = x_a(\mu_a + x + y \cdot M_{ab})$$

$$\dot{x_b} = x_b(\mu_b + y + x \cdot M_{ba})$$

3.) A function is called a Lypunov function if

(i) V(x) and the partial derivatives of V(x) are continuous

(ii) V(x) is p.d. in R

(iii) $\dot{V}$(x) is n.s.d

if the fixed point $\bar{x}$ is a.s. then it must meet these conditions :
(i) $V(\bar{x}) = 0$ and V(x) > 0
(ii) $\dot{V}$(x) < 0 in U-$\bar{x}$
The $\bar{x}$ is stable in the neighborhood U-$\bar{x}$. Take the equations

$$\alpha(x) = \begin{cases} \dot{x_1} = x_1(b_1 - x_1 - \alpha x_2) \\ \\ \dot{x_2} = x_2(b_2 - \beta x_1 - x_2) \end{cases}$$

$V(x) = \frac{\beta}{2}{x_1}^2 + \frac{\alpha}{2}{x_2}^2 - \beta b_1 x_1 - \alpha b_2 x_2 + \alpha\beta x_1 x_2$
for the critical points $V(0,0), V(-b_1,0), V(0,-b_2)$ equal zero. Clearly V(X) is p.d. assuming negative constants.
$\dot{V}$(x) is n.s.d for all values to the right of the structure $\frac{\beta}{2}{x_1}^2 + \frac{\alpha}{2}{x_2}^2 - \beta b_1 x_1 - \alpha b_2 x_2 + \alpha\beta x_1 x_2 = 0$