Set 1

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1. Evaluate the fixed points and assess their stability.

$$\dot{x} = \mu \cdot x - M \cdot x^2$$

Theorem 10.3.1 Stability Criteria for x' = g(x) states :

Let x_1 be a fixed point of the autonomous differential equation x'=g(x), where g is differentiable at x_1.

(a) if g'(x_1) < 0, then x_1 is an asymptotically stable critical point.
(b) if g'(x_1) > 0, the x_1 is an unstable critical point.

First,

$g(x) = \mu \cdot x - M \cdot x^2$
$x = 0$
$x = \frac{\mu}{M}$
Therefore

$g(0) = \mu$ and $g(\frac{\mu}{M}) = -\mu$

By Theorem 10.3.1 $g(\frac{\mu}{M})$ is a asymptotically stable critical point and g(0) is an unstable critical point.

2.

$$\dot{x} = x(\mu_x + x \cdot M_{xx} + y \cdot M_{xy})$$
$$\dot{y} = y(\mu_y + y \cdot M_{yy} + x \cdot M_{xy})$$

a.)

In order to find critical points find the solutions to the following system of equations:

$$\begin{cases} 0 = \mu_x \cdot x + x^2 \cdot M_{xx} + y \cdot x \cdot M_{xy} \\ 0 = \mu_y \cdot y + y^2 \cdot M_{yy} + y \cdot x \cdot M_{xy} \end{cases}$$

Matlab code used to solve system

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syms a b c d e x y p
eqn1 = a .* x + b .* x.^(2) + p .* y .* x == 0;
eqn2 = d .* y + e .* y.^(2) + c .* y .* x == 0;

sol = solve([eqn1, eqn2], [x, y]);
xSol = sol.x
ySol = sol.y
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output :

$$\text{X.sol=} \begin{bmatrix} 0 \\ -\frac{(\mu_x \cdot M_{yy} - M_{xy} \cdot \mu_y)}{-M_{xy} \cdot M_{yx} + M_{xx} \cdot M_{xy}} \\ -\frac{\mu_x}{M_{xx}} \\ 0 \end{bmatrix}$$

$$\text{Y.sol=} \begin{bmatrix} 0 \\ -\frac{(\mu_x \cdot M_{xy} - M_{xx} \cdot \mu_y)}{-(M_{xy})^2 + M_{xx} \cdot M_{xy}} \\ 0 \\ -\frac{\mu_y}{M_{yy}} \end{bmatrix}$$

Parts b and c.)

Definition: the trace of a matrix (τ) is the sum of the diagonal.
Definition : The determinant of a matrix (Δ)

Assume that $\tau^2 - 4\Delta > 0$

Therefore in general for the three non-trivial cases that are computed below there are three criteria that determine stability :

1.) $\tau < 0$ and $\Delta > 0$ then you have a stable solution
2.) $\tau > 0$ and $\Delta > 0$ then you have a stable solution
3.) $\Delta < 0$ then you have a saddle point

The general matrix is:

$$\begin{bmatrix} \mu_x + 2xM_{xx} + yM_{xy} & xM_{xy} \\ yM_{xy} & \mu_y + 2yM_{yy} + xM_{xy} \end{bmatrix}$$

bc_1 : for the trivial case(0,0) :

$$\begin{bmatrix} \mu_x & 0 \\ 0 & \mu_y \end{bmatrix}$$

Since the matrix is in rref form the eigenvalues are the diagonal entries of the matrix. This leads to three possibilities :

1.) If both of the diagonal entries are positive then the trivial solution is unstable node.

2.)If both of the diagonal entries are negative then the trivial solution is stable node.

3.)If one of the diagonal entries is negative and one of the diagonal entries is positive then you have a saddle point.

bc_2 : for the non-trivial case($-\frac{\mu_x}{M_{xx}}$,0) :

$$\begin{bmatrix} \mu_x - 2\mu_x & M_{xy}x \\ 0 & \mu_y - \frac{M_{xy}}{M_{xx}}\mu_x \end{bmatrix}$$

bc_3 : for the non-trivial case(0,$-\frac{\mu_y}{M_{yy}}$) :

$$\begin{bmatrix} \mu_x - \mu_y \frac{M_{xy}}{M_{yy}} & 0 \\ -\mu_y \frac{M_{xy}}{M_{xy}} & \mu_y - 2\mu_y \end{bmatrix}$$

bc_4 : for the non-trivial case $(-\frac{(\mu_x \cdot M_{yy} - M_{xy} \cdot \mu_y)}{-M_{xy} \cdot M_{yx} + M_{xx} \cdot M_{xy}}, -\frac{(\mu_x \cdot M_{xy} - M_{xx} \cdot \mu_y)}{-M_{xy} \cdot M_{yx} + M_{xx} \cdot M_{xy}})$

$$\begin{bmatrix} \frac{2\mu_x M_{yy} M_{xx} + 2\mu_x (M_{xy})(M_{yx}) + \mu_x M_{xy} M_{xx} - M_{yy}\mu_y M_{xy} - 2\mu_y M_{xy} M_{xx}}{M_{xy} + M_{xx}} & \frac{(\mu_x M_{yy} - \mu_y M_{xy})}{M_{xy} + M_{xx}} \\ \frac{2(\mu_x M_{yy} - \mu_y M_{xy})}{M_{xy} + M_{xx}} & \frac{3\mu_x M_{yy} M_{xy} - 2(M_{yy})^2 + \mu_y M_{xy} M_{xx}}{M_{xy}(M_{xy} + M_{xx})} \end{bmatrix}$$

3.)
a.) Ensure that $\dot{x_i} = x_i$ ($\mu_i + \sum_{j=1}^{N}(M_{ij} \cdot x_j)$) reduce to equation (2) for N = 2

In order to accomplish this expand to N = 2 :

$$\alpha(x) = \begin{cases} \dot{x_1} = x_1(\mu_1 + M_{11}x_1 + M_{12}x_2) \\ \\ \dot{x_2} = x_2(\mu_2 + M_{21}x_1 + M_{22}x_2) \end{cases}$$

3b.)Take the GLV equation

$$\begin{cases} 0 = x_1(\mu_{x_1} + x_1 M_{11} + ... + M_{1n}) \\ . \\ . \\ . \\ 0 = x_n(\mu_{x_n} + x_1 M_{n1} + ... + M_{nn}) \end{cases}$$

rewrite as:

$$\begin{bmatrix} M_{11} & M_{12} & M_{13} & \dots & M_{1n} \\ M_{21} & M_{22} & M_{23} & \dots & M_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ M_{n1} & M_{n2} & M_{n3} & \dots & M_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1.Sol \\ x_2.Sol \\ \dots \\ x_n.Sol \end{bmatrix}$$

Use the following algorithm:

for N GLV equations with variables $(x_1, x_2, x_3, ..., x_n)$ one can find the first solution $(\bar{x_1}, \bar{x_2}, ..., \bar{x_n})$ by solving the above N linear system :

$$\left[\begin{array}{ccccc|c} M_{11} & M_{12} & M_{13} & \dots & M_{1n} & x_1.Sol \\ M_{21} & M_{22} & M_{23} & \dots & M_{2n} & x_2.Sol \\ \dots & \dots & \dots & \dots & \dots & \dots \\ M_{n1} & M_{n2} & M_{n3} & \dots & M_{nn} & x_n.Sol \end{array}\right]$$

The second fixed point can be found as the trivial point :
$(0_1, 0_2, ..., 0_n)$

The remaining 2^{N-2} solutions can be found as follows :
Compute every permutation where greater than or equal to one variable equals zero and solve the corresponding resultant system of equations.

The sum of three classes of solutions will equal 2^N solutions.

4.) solutions to parts a and b are on github.