Week 2 Notes (5/29/18)

During our meeting:

- Any questions about chapter 1 of Wiggins?
- Review worksheet and answers
- Review/walkthrough code and figures
- Discuss nondimensionalization of gLV equations
- Discuss Lyapunov functions for dynamical systems

For next week:

- Explicitly compute the eigenvalues of the Jacobian for the 2D gLV equations,

$$\begin{aligned} \frac{\mathrm{d}x_a}{\mathrm{d}t} &= x_a(\mu_a - M_{aa}x_a - M_{ab}x_b) \\ \frac{\mathrm{d}x_b}{\mathrm{d}t} &= x_b(\mu_b - M_{ba}x_a - M_{bb}x_b), \end{aligned} \tag{1}$$

  for the singly-existing steady states $(x^*,\ 0)$ and $(0,\ y^*)$, where you should compute x^* and y^* by hand (feel free to use symbolic computing software to check your answers). Compute the stability of these states for the following two parameter combinations: ($\mu_a = \mu_b = 1$, $M_{aa} = M_{bb} = 1$, and $M_{ab} = M_{ba} = .5$), and ($\mu_a = \mu_b = 1$, $M_{aa} = M_{bb} = 1$, and $M_{ab} = M_{ba} = 1.5$). Compare these results with numerical simulations. In your simulations, use `plt.subplot` in order to make a 2x2 grid of subfigures. Make the top row correspond to simulations that start at an initial condition of (.1, .9) and the bottom row correspond to an initial condition of (.5, .5), and make the left and right columns correspond to the simulations that use the first and second parameter sets. Make the title of each subfigure indicate which IC and which parameter set is used. Do the numerical results agree with your analytic results?

- Read chapter 2 of Wiggins
- Show that 2D gLV equations can be nondimensionalized so that

$$\begin{aligned} \frac{\mathrm{d}x_a}{\mathrm{d}t} &= x_a(\mu_a - x_a - M_{ab}x_b) \\ \frac{\mathrm{d}x_b}{\mathrm{d}t} &= x_b(\mu_b - M_{ba}x_a - x_b), \end{aligned} \tag{2}$$

  and that you may further nondimensionalize time by setting $\mu_a \to 1$.

- Show that the Lyapunov function given by Tang, Yuan and Ma in Phys. Rev. E 2013 satisfies the Lyapunov conditions given in chapter 2 of Wiggins. In our notation, the Lyapunov function they provide is

$$\begin{aligned} V(x_a, x_b) = {} & M_{ba}x_a^2/2 + M_{ab}x_b^2/2 \\ & - M_{ba}\mu_a x_a - M_{ab}\mu_b x_b + M_{ab}M_{ba}x_a x_b \end{aligned} \tag{3}$$

  Some hints: 1) remember that $\hat{x}_a$ and $\hat{x}_b$ are "directions" (like $\hat{x}$ and $\hat{y}$) in a 2-dimensional space, 2) remember that $\dot{V} = \nabla V \cdot \dot{\mathbf{x}}$, where $\dot{\mathbf{x}}$ is the vector form of the dynamical system, and 3) assume $M_{ab} > 0$ and $M_{ba} > 0$ (this condition must be satisfied in order for there to be two stable steady states). Note that this equation satifies the Lyapunov conditions except for the fact that $V(\bar{x}) = 0$ for two different $\bar{x}$, corresponding to the two stable steady states. For this reason, this is called a *split Lyapunov function.*

- Generalize your code for solving the 2-dimensional gLV equations so that it can solve N-dimensional gLV equations. Hint: consider what the commands `np.dot(np.diag(mu), Y)` and `np.dot(np.diag(np.dot(M, Y)), Y)` do, and compare them to the N-dimensional gLV equations. Test your code out using the parameters from Stein et al., Plos Comp Biol 2013— I have provided code that imports these parameters and turns them into numpy arrays for you. For usage, look at `example_import_data.py` and for the implementation itself look at `barebones_CDI.py`. Compare the parameter values from the python code with the Stein paper and ensure they agree. This code also imports experimental initial conditions from the Stein paper (e.g. `ic4` in the code). Don't plot your output, but try starting from different initial conditions (0-8 are allowed; 4 is given as an example in `example_import_data.py`) and compare your obtained steady states (`y[:, -1]`) with Table B in `S1_Appendix_revision.pdf`