Notes concerning "On the Linear Theory of the Land and Sea Breeze" (Rotunno 1983)

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1 Paper Misprints

1. Equation (37) should be

$$-\beta \mathrm{Re}\left\{\mathcal{F}^{-1}\left[\frac{1}{k}e^{-i\,\mathrm{sgn}(k)k\zeta}\int_0^\zeta \mathcal{F}\left(\frac{\partial\tilde{Q}}{\partial\xi}\right)\sin k\zeta' d\zeta' + \frac{1}{k}\sin k\zeta\int_\zeta^\infty \mathcal{F}\left(\frac{\partial\tilde{Q}}{\partial\xi}\right)e^{-i\,\mathrm{sgn}(k)k\zeta'}d\zeta'\right]\right\}.$$

2. $\tilde{Q}$ should have time dependance $e^{-i\left(\tau-\frac{\pi}{2}\right)} = ie^{-i\tau}$ not simply $\sin\tau$.

3. The expression $\tilde{b} = bh^{-1}\omega^{-3}$ in equation (13) should actually be $\tilde{b} = bh^{-1}\omega^{-2}$ to make the units come out right - assuming b has units m s^{-1}.

2 Tropical Case

We are attempting to derive the equation

$$\tilde{\psi}(\xi,\zeta,\tau) = -\beta\tilde{A}\int_0^\infty \frac{\cos k\xi e^{-\xi_0 k}}{1+k^2}\left(\sin(k\zeta+\tau) - e^{-\zeta}\sin\tau\right)dk. \tag{1}$$

Note

$$\tilde{Q} = \beta\tilde{A}\left(\frac{\pi}{2} + \tan^{-1}\frac{\xi}{\xi_0}\right)e^{-\zeta}ie^{-i\tau} \tag{2}$$

$$\Rightarrow \frac{\partial\tilde{Q}}{\partial\xi} = \frac{1}{\xi^2+\xi_0^2}\beta\tilde{A}\xi_0 e^{-\zeta}ie^{-i\tau} \tag{3}$$

$$\Rightarrow \mathcal{F}\left(\frac{\partial\tilde{Q}}{\partial\xi}\right) = \mathcal{F}\left[\frac{1}{\xi^2+\xi_0^2}\right]\tilde{A}\xi_0 e^{-\zeta}ie^{-i\tau} \tag{4}$$

$$\Rightarrow \mathcal{F}\left(\frac{\partial\tilde{Q}}{\partial\xi}\right) = \frac{\pi}{\xi_0}e^{-\xi_0|k|}\tilde{A}\xi_0 e^{-\zeta}e^{-i\tau} = \pi e^{-\xi_0|k|}\tilde{A}e^{-\zeta}ie^{-i\tau} \tag{5}$$

using the non-unitary, angular frequency form of the Fourier transform. This Fourier transform can be derived by considering the Fourier transform of $\frac{\pi}{\xi_0}e^{-\xi_0|\xi|}$, applying the inverse Fourier transform, and changing signs.

Now, starting from the corrected form of equation 37 we have,

$$-\beta\mathcal{F}^{-1}\left[\frac{1}{k}e^{-i\,\mathrm{sgn}(k)k\zeta}\int_0^{\zeta}\mathcal{F}\left(\frac{\partial\tilde{Q}}{\partial\xi}\right)\sin k\zeta' d\zeta' + \frac{1}{k}\sin k\zeta\int_{\zeta}^{\infty}\mathcal{F}\left(\frac{\partial\tilde{Q}}{\partial\xi}\right)e^{-i\,\mathrm{sgn}(k)k\zeta'}d\zeta'\right] \tag{6}$$

$$= -\frac{\beta}{2}\tilde{A}ie^{-i\tau}\int_{-\infty}^{0}e^{ik\xi}\left[\frac{1}{k}e^{ik\zeta}\int_0^{\zeta}e^{\xi_0 k}e^{-\zeta}\sin k\zeta' d\zeta' + \frac{1}{k}\sin k\zeta\int_{\zeta}^{\infty}e^{\xi_0 k}e^{-\zeta}e^{ik\zeta'}d\zeta'\right]dk \tag{7}$$

$$-\frac{\beta}{2}\tilde{A}ie^{-i\tau}\int_0^{\infty}e^{ik\xi}\left[\frac{1}{k}e^{-ik\zeta}\int_0^{\zeta}e^{-\xi_0 k}e^{-\zeta}\sin k\zeta' d\zeta' + \frac{1}{k}\sin k\zeta\int_{\zeta}^{\infty}e^{-\xi_0 k}e^{-\zeta}e^{-ik\zeta'}d\zeta'\right]dk \tag{8}$$

$$= -\frac{\beta}{2}\tilde{A}ie^{-i\tau}\int_0^{\infty}e^{-ik\xi}\left[\frac{1}{k}e^{-ik\zeta}\int_0^{\zeta}e^{-\xi_0 k}e^{-\zeta}\sin k\zeta' d\zeta' + \frac{1}{k}\sin k\zeta\int_{\zeta}^{\infty}e^{-\xi_0 k}e^{-\zeta}e^{-ik\zeta'}d\zeta'\right]dk \tag{9}$$

$$-\frac{\beta}{2}\tilde{A}ie^{-i\tau}\int_0^{\infty}e^{ik\xi}\left[\frac{1}{k}e^{-ik\zeta}\int_0^{\zeta}e^{-\xi_0 k}e^{-\zeta}\sin k\zeta' d\zeta' + \frac{1}{k}\sin k\zeta\int_{\zeta}^{\infty}e^{-\xi_0 k}e^{-\zeta}e^{-ik\zeta'}d\zeta'\right]dk \tag{10}$$

$$= -\beta ie^{-i\tau}\tilde{A}\int_0^{\infty}\cos(k\xi)\,e^{-\xi_0 k}\left[\frac{1}{k}e^{-ik\zeta'}\int_0^{\zeta}e^{-\zeta'}\sin k\zeta' d\zeta' + \frac{1}{k}\sin k\zeta\int_{\zeta}^{\infty}e^{-\zeta'}e^{-ik\zeta'}d\zeta'\right]dk \tag{11}$$

$$= -\beta\int_0^{\infty}\cos(k\xi)\left[e^{-ik\zeta'}\int_0^{\zeta}\left(\frac{\partial\tilde{Q}}{\partial\xi}\right)\sin k\zeta' d\zeta' + \sin k\zeta\int_{\zeta}^{\infty}\left(\frac{\partial\tilde{Q}}{\partial\xi}\right)e^{-ik\zeta'}d\zeta'\right]\frac{dk}{k}. \tag{12}$$

Note (12) matches what's in Rotunno's notes. From (12) the derivation is easy.

My attempt at deriving equation (1) - equation (38) in Rotunno's paper - is very messy. It didn't quite work originally as I was using the incorrect version of (37). Now having the correct version, let's see if it works. We have from (6),

$$-\beta\mathcal{F}^{-1}\left[\frac{1}{k}e^{-i\,\mathrm{sgn}(k)k\zeta}\int_0^{\zeta}\left(\pi e^{-\xi_0|k|}\tilde{A}e^{-\zeta}ie^{-i\tau}\right)\sin k\zeta' d\zeta'\right. \tag{13}$$

$$\left.+\frac{1}{k}\sin k\zeta\int_{\zeta}^{\infty}\left(\pi e^{-\xi_0|k|}\tilde{A}e^{-\zeta}ie^{-i\tau}\right)e^{-i\,\mathrm{sgn}(k)k\zeta'}d\zeta'\right] \tag{14}$$

$$= -\beta\tilde{A}\pi ie^{-i\tau}\mathcal{F}^{-1}\left[\frac{1}{k}e^{-i\,\mathrm{sgn}(k)k\zeta}e^{-\xi_0|k|}\int_0^{\zeta}e^{-\zeta'}\sin k\zeta' d\zeta'\right. \tag{15}$$

$$\left.+\frac{1}{k}\sin k\zeta e^{-\xi_0|k|}\int_{\zeta}^{\infty}e^{-\zeta'}e^{-i\,\mathrm{sgn}(k)k\zeta'}d\zeta'\right] \tag{16}$$

$$= -\beta\tilde{A}\pi ie^{-i\tau}\mathcal{F}^{-1}\left\{\frac{1}{k}e^{-i\,\mathrm{sgn}(k)k\zeta}e^{-\xi_0|k|}\left[\frac{-e^{-\zeta'}}{k^2+1}\left(k\cos k\zeta' + \sin k\zeta'\right)\right]_0^{\zeta}\right\} \tag{17}$$

$$-\beta\tilde{A}\pi ie^{-i\tau}\mathcal{F}^{-1}\left\{\frac{1}{k}\sin k\zeta e^{-\xi_0|k|}\left[\frac{1}{-i\,\mathrm{sgn}(k)k-1}e^{(-i\,\mathrm{sgn}(k)k-1)\zeta'}\right]_{\zeta}^{\infty}\right\}, \tag{18}$$

$$= -\beta\tilde{A}\pi ie^{-i\tau}\mathcal{F}^{-1}\left\{\frac{1}{k}e^{-i\,\mathrm{sgn}(k)k\zeta}e^{-\xi_0|k|}\left(\frac{-e^{-\zeta}}{k^2+1}\left(k\cos k\zeta + \sin k\zeta\right) + \frac{k}{k^2+1}\right)\right\} \tag{19}$$

$$-\beta\tilde{A}\pi ie^{-i\tau}\mathcal{F}^{-1}\left\{\frac{1}{k}\sin k\zeta e^{-\xi_0|k|}\left[\frac{1}{-i\,\mathrm{sgn}(k)k-1}e^{(-i\,\mathrm{sgn}(k)k-1)\zeta'}\right]_{\zeta}^{\infty}\right\}, \tag{20}$$

where the first integral (17) can be calculated by performing integration by parts. Consider the first term of the sum, i.e. (19). We have

$$
\begin{aligned}
&-\beta\tilde{A}\pi i e^{-i\tau}\mathcal{F}^{-1}\left\{\frac{1}{k}e^{-i\,\mathrm{sgn}(k)k\zeta}e^{-\xi_0|k|}\left(\frac{-e^{-\zeta}}{k^2+1}\left(k\cos k\zeta+\sin k\zeta\right)+\frac{k}{k^2+1}\right)\right\} && (21)\\
&=-\beta\tilde{A}ie^{-i\tau}\frac{1}{2}\int_{-\infty}^{\infty}\frac{1}{k}e^{ik\xi}e^{-i\,\mathrm{sgn}(k)k\zeta}e^{-\xi_0|k|}\left(\frac{-e^{-\zeta}}{k^2+1}\left(k\cos k\zeta+\sin k\zeta\right)+\frac{k}{k^2+1}\right)dk && (22)\\
&=-\beta\tilde{A}ie^{-i\tau}\frac{1}{2}\int_{-\infty}^{0}\frac{1}{k}e^{ik\xi}e^{ik\zeta}e^{\xi_0 k}\left(\frac{-e^{-\zeta}}{k^2+1}\left(k\cos k\zeta+\sin k\zeta\right)+\frac{k}{k^2+1}\right)dk && (23)\\
&-\beta\tilde{A}ie^{-i\tau}\frac{1}{2}\int_{0}^{\infty}\frac{1}{k}e^{ik\xi}e^{-ik\zeta}e^{-\xi_0 k}\left(\frac{-e^{-\zeta}}{k^2+1}\left(k\cos k\zeta+\sin k\zeta\right)+\frac{k}{k^2+1}\right)dk && (24)\\
&=-\beta\tilde{A}ie^{-i\tau}\frac{1}{2}\int_{0}^{\infty}\frac{1}{-k}e^{-ik\xi}e^{-ik\zeta}e^{-\xi_0 k}\left(\frac{-e^{-\zeta}}{k^2+1}\left(-k\cos k\zeta-\sin k\zeta\right)-\frac{k}{k^2+1}\right)dk && (25)\\
&-\beta\tilde{A}ie^{-i\tau}\frac{1}{2}\int_{0}^{\infty}\frac{1}{k}e^{ik\xi}e^{-ik\zeta}e^{-\xi_0 k}\left(\frac{-e^{-\zeta}}{k^2+1}\left(k\cos k\zeta+\sin k\zeta\right)+\frac{k}{k^2+1}\right)dk && (26)\\
&=-\beta\tilde{A}ie^{-i\tau}\frac{1}{2}\int_{0}^{\infty}\frac{1}{k}\left(e^{ik\xi}+e^{-ik\xi}\right)e^{-ik\zeta}e^{-\xi_0 k}\left(\frac{-e^{-\zeta}}{k^2+1}\left(k\cos k\zeta+\sin k\zeta\right)+\frac{k}{k^2+1}\right)dk && (27)\\
&=-\beta\tilde{A}\left(i\cos\tau+\sin\tau\right)\int_{0}^{\infty}\frac{1}{k}\cos k\xi e^{-ik\zeta}e^{-\xi_0 k}\frac{1}{k^2+1}\left(-e^{-\zeta}\left(k\cos k\zeta+\sin k\zeta\right)+k\right)dk && (28)\\
&=-\beta\tilde{A}\left(i\cos\tau+\sin\tau\right)\int_{0}^{\infty}\frac{1}{k}\cos k\xi\left(\cos k\zeta-i\sin k\zeta\right) && (29)\\
&\times e^{-\xi_0 k}\frac{1}{k^2+1}\left(-e^{-\zeta}\left(k\cos k\zeta+\sin k\zeta\right)+k\right)dk && (30)\\
&=-\beta\tilde{A}\int_{0}^{\infty}\frac{1}{k}\sin k\xi\left(i\cos\tau+\sin\tau\right)\left(\cos k\zeta-i\sin k\zeta\right) && (31)\\
&\times e^{-\xi_0 k}\frac{1}{k^2+1}\left(-e^{-\zeta}\left(k\cos k\zeta+\sin k\zeta\right)+k\right)dk && (32)\\
&=-\beta\tilde{A}\int_{0}^{\infty}\frac{1}{k}\sin k\xi\left(i\cos\tau\cos k\zeta+\cos\tau\sin k\zeta+\sin\tau\cos k\zeta-i\sin\tau\sin k\zeta\right) && (33)\\
&\times e^{-\xi_0 k}\frac{1}{k^2+1}\left(-e^{-\zeta}\left(k\cos k\zeta+\sin k\zeta\right)+k\right)dk && (34)
\end{aligned}
$$

The real part of this is then

$$
\begin{aligned}
&-\beta\tilde{A}\int_{0}^{\infty}\frac{1}{k}\sin k\xi\left(\cos\tau\cos k\zeta+\sin\tau\sin k\zeta\right)e^{-\xi_0 k}\frac{1}{k^2+1}\left(-e^{-\zeta}\left(k\cos k\zeta+\sin k\zeta\right)+k\right)dk && (35)\\
&=-\beta\tilde{A}\int_{0}^{\infty}\frac{1}{k}\sin k\xi\sin\left(\tau+k\zeta\right)e^{-\xi_0 k}\frac{1}{k^2+1}\left(-e^{-\zeta}\left(k\cos k\zeta+\sin k\zeta\right)+k\right)dk && (36)\\
&=-\beta\tilde{A}\int_{0}^{\infty}\frac{1}{k}\sin k\xi\frac{1}{k^2+1}e^{-\xi_0 k} && (37)\\
&\times\left(-\sin\left(\tau+k\zeta\right)e^{-\zeta}k\cos k\zeta-\sin\left(\tau+k\zeta\right)e^{-\zeta}\sin k\zeta+k\sin\left(\tau+k\zeta\right)\right)dk. && (38)
\end{aligned}
$$

Consider now the second term. We have

$$-\beta\tilde{A}\pi i e^{-i\tau}\mathcal{F}^{-1}\left\{\frac{1}{k}\sin k\zeta e^{-\xi_0|k|}\left[\frac{1}{-i\,\mathrm{sgn}(k)k-1}e^{(-i\,\mathrm{sgn}(k)k-1)\zeta'}\right]_\zeta^\infty\right\} \tag{39}$$

$$= -\beta\tilde{A}ie^{-i\tau}\frac{1}{2}\int_{-\infty}^0\frac{1}{k}e^{ik\xi}\left(\sin k\zeta e^{\xi_0 k}\left[\frac{1}{ik-1}e^{(ik-1)\zeta'}\right]_\zeta^\infty\right)dk \tag{40}$$

$$-\beta\tilde{A}ie^{-i\tau}\frac{1}{2}\int_0^\infty\frac{1}{k}e^{ik\xi}\left(\sin k\zeta e^{-\xi_0 k}\left[\frac{1}{-ik-1}e^{(-ik-1)\zeta'}\right]_\zeta^\infty\right)dk \tag{41}$$

$$= -\beta\tilde{A}ie^{-i\tau}\frac{1}{2}\int_{-\infty}^0\frac{1}{k}e^{ik\xi}\left(\sin k\zeta e^{\xi_0 k}\frac{1}{1-ik}e^{(ik-1)\zeta}\right)dk \tag{42}$$

$$-\beta\tilde{A}ie^{-i\tau}\frac{1}{2}\int_0^\infty\frac{1}{k}e^{ik\xi}\left(\sin k\zeta e^{-\xi_0 k}\frac{1}{1+ik}e^{(-ik-1)\zeta}\right)dk \tag{43}$$

$$= -\beta\tilde{A}ie^{-i\tau}\frac{1}{2}\int_0^\infty\frac{1}{-k}e^{-ik\xi}\left(-\sin k\zeta e^{-\xi_0 k}\frac{1}{1+ik}e^{(-ik-1)\zeta}\right)dk \tag{44}$$

$$-\beta\tilde{A}ie^{-i\tau}\frac{1}{2}\int_0^\infty\frac{1}{k}e^{ik\xi}\left(\sin k\zeta e^{-\xi_0 k}\frac{1}{1+ik}e^{(-ik-1)\zeta}\right)dk \tag{45}$$

$$= -\beta\tilde{A}ie^{-i\tau}\int_0^\infty\frac{1}{k}\cos k\xi e^{-\xi_0 k}e^{-ik\zeta}\left(\sin k\zeta\frac{1-ik}{1+k^2}e^{-\zeta}\right)dk \tag{46}$$

$$= -\beta\tilde{A}ie^{-i\tau}\int_0^\infty\frac{1}{k}\cos k\xi e^{-\xi_0 k}\left(\cos k\zeta - i\sin k\zeta\right)(1-ik)\left(\sin k\zeta\frac{1}{1+k^2}e^{-\zeta}\right)dk \tag{47}$$

$$= -\beta\tilde{A}\int_0^\infty\frac{1}{k}\cos k\xi e^{-\xi_0 k}\left(\sin k\zeta\frac{1}{1+k^2}e^{-\zeta}\right) \tag{48}$$

$$\times\left(i\cos\tau+\sin\tau\right)\left(\cos k\zeta - ik\cos k\zeta - i\sin k\zeta - k\sin k\zeta\right)dk. \tag{49}$$

The real part of this is then

$$-\beta\tilde{A}\int_0^\infty\frac{1}{k}\sin k\xi e^{-\xi_0 k}\left(\sin k\zeta\frac{1}{1+k^2}e^{-\zeta}\right) \tag{50}$$

$$\times\left(k\cos\tau\cos k\zeta+\cos\tau\sin k\zeta+\sin\tau\cos k\zeta-k\sin\tau\sin k\zeta\right)dk \tag{51}$$

$$= -\beta\tilde{A}\int_0^\infty\frac{1}{k}\sin k\xi e^{-\xi_0 k}\left(\sin k\zeta\frac{1}{1+k^2}e^{-\zeta}\right) \tag{52}$$

$$\times\left(k\cos\left(\tau+k\zeta\right)+\sin\left(\tau+k\zeta\right)\right)dk \tag{53}$$

$$= -\beta\tilde{A}\int_0^\infty\frac{1}{k}\sin k\xi e^{-\xi_0 k}\frac{1}{1+k^2} \tag{54}$$

$$\times\left(\sin k\zeta e^{-\zeta}k\cos\left(\tau+k\zeta\right)+\sin k\zeta e^{-\zeta}\sin\left(\tau+k\zeta\right)\right)dk \tag{55}$$

Summing (38) and (55) gives

$$\tilde{\psi}(\xi,\zeta,\tau) = -\beta\tilde{A}\int_0^\infty\frac{\cos k\xi e^{-\xi_0 k}}{1+k^2}\left(\sin(k\zeta+\tau)-e^{-\zeta}\sin\tau\right)dk \tag{56}$$

as required!

3 Calculating Potential Temperature

From equation (4) Rotunno (1983) we have

$$\frac{\partial b}{\partial t}+N^2 w = Q.$$

The non-dimensional form of this is

$$\frac{\partial \tilde{b}}{\partial \tau} h\omega^3 + N^2 \tilde{w} h\omega = \tilde{Q} h\omega^3$$
$$= \frac{\partial \tilde{b}}{\partial \tau} + \left(\frac{N}{\omega}\right)^2 \tilde{w} = \tilde{Q}.$$

Note that $\left(\frac{N}{\omega}\right)^2$ is essentially the Berger number with $H = L$. Thus can solve for $\tilde{b}$ using

$$\frac{\tilde{b}_{k+1} - \tilde{b}_k}{\Delta \tau} = \tilde{Q}_k - \left(\frac{N}{\omega}\right)^2 \tilde{w}_k$$
$$= \tilde{b}_{k+1} - \tilde{b}_k = \Delta \tau \left(\tilde{Q}_k - \left(\frac{N}{\omega}\right)^2 \tilde{w}_k\right).$$

This produces a linear system of τ_N linearly independent equations in τ_N unknowns. However, in this form the system is singular - so substitute the equation for $\tilde{b_1}$ for $\tilde{b_1} + \tilde{b}_{\frac{\tau_n}{2}} = 0$ to impose symmetry on the bouyancy. Note can use $\tilde{b_1} + \tilde{b}_{\lfloor \frac{\tau_n}{2} \rfloor}$! This works - we can solve for bouyancy even without initial conditions! From bouyancy can extract potential temperature!

4 Choosing $\tilde{A}$

Note that from equation (4) we have at the surface

$$\frac{\partial b}{\partial t} = \frac{g}{\theta_0} \frac{\partial \theta'}{\partial t} = Q,$$

as $w = 0$ at the surface. Thus letting θ_M and θ_m denote the land surface temperature maxima and minima respectively, and noting that $\left(\frac{\pi}{2} + \tan^{-1} \frac{x}{x_0}\right)$ maps onto $(0, \pi)$, we have

$$\frac{g}{\theta_0} \frac{\theta_M - \theta_m}{12 \cdot 60 \cdot 60} = \left(\sin \frac{\pi}{2} - \sin \frac{3\pi}{2}\right) A\pi = 2A\pi,$$

with $A\pi$ the supremum of H at the surface, and it taking 12 hours to go from maximum to minimum temperature. Multiplying both sides by $h^{-1}\omega^{-3}$ gives

$$\tilde{A} = \frac{g}{2\pi\theta_0} \frac{\theta_M - \theta_m}{12 \cdot 60 \cdot 60} h^{-1} \omega^{-3}.$$

References

Rotunno, R. (1983), 'On the linear theory of the land and sea breeze', *Journal of the Atmospheric Sciences* **40**(8), 1999–2009.