Notes concerning "On the Linear Theory of the Land and Sea Breeze" (Rotunno 1983)

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1 Paper Misprints

1. Equation (37) should be

$$-\beta \mathrm{Re}\left\{\mathcal{F}^{-1}\left[\frac{1}{k}e^{-i\,\mathrm{sgn}(k)k\zeta}\int_0^\zeta \mathcal{F}\left(\frac{\partial \tilde{Q}}{\partial \xi}\right)\sin k\zeta' d\zeta' + \frac{1}{k}\sin k\zeta \int_\zeta^\infty \mathcal{F}\left(\frac{\partial \tilde{Q}}{\partial \xi}\right)e^{-i\,\mathrm{sgn}(k)k\zeta'}d\zeta'\right]\right\}.$$

2. $\tilde{Q}$ should have time dependance $e^{-i\left(\tau-\frac{\pi}{2}\right)} = ie^{-i\tau}$ not simply $\sin\tau$.

3. The expression $\tilde{b} = bh^{-1}\omega^{-3}$ in equation (13) should actually be $\tilde{b} = bh^{-1}\omega^{-2}$ to make the units come out right - assuming b has units m s^{-1}.

2 Tropical Case

We are attempting to derive the equation

$$\tilde{\psi}(\xi,\zeta,\tau) = -\beta\tilde{A}\int_0^\infty \frac{\cos k\xi e^{-\xi_0 k}}{1+k^2}\left(\sin(k\zeta+\tau) - e^{-\zeta}\sin\tau\right)dk. \tag{1}$$

Note

$$\tilde{Q} = \beta\tilde{A}\left(\frac{\pi}{2} + \tan^{-1}\frac{\xi}{\xi_0}\right)e^{-\zeta}ie^{-i\tau} \tag{2}$$

$$\Rightarrow \frac{\partial \tilde{Q}}{\partial \xi} = \frac{1}{\xi^2+\xi_0^2}\beta\tilde{A}\xi_0 e^{-\zeta}ie^{-i\tau} \tag{3}$$

$$\Rightarrow \mathcal{F}\left(\frac{\partial \tilde{Q}}{\partial \xi}\right) = \mathcal{F}\left[\frac{1}{\xi^2+\xi_0^2}\right]\tilde{A}\xi_0 e^{-\zeta}ie^{-i\tau} \tag{4}$$

$$\Rightarrow \mathcal{F}\left(\frac{\partial \tilde{Q}}{\partial \xi}\right) = \frac{\pi}{\xi_0}e^{-\xi_0|k|}\tilde{A}\xi_0 e^{-\zeta}e^{-i\tau} = \pi e^{-\xi_0|k|}\tilde{A}e^{-\zeta}ie^{-i\tau} \tag{5}$$

using the non-unitary, angular frequency form of the Fourier transform. This Fourier transform can be derived by considering the Fourier transform of $\frac{\pi}{\xi_0}e^{-\xi_0|\xi|}$, applying the inverse Fourier transform, and changing signs.

Now, starting from the corrected form of equation 37 we have,

$$-\beta\mathcal{F}^{-1}\left[\frac{1}{k}e^{-i\,\mathrm{sgn}(k)k\zeta}\int_0^\zeta \mathcal{F}\left(\frac{\partial\tilde{Q}}{\partial\xi}\right)\sin k\zeta' d\zeta' + \frac{1}{k}\sin k\zeta\int_\zeta^\infty \mathcal{F}\left(\frac{\partial\tilde{Q}}{\partial\xi}\right)e^{-i\,\mathrm{sgn}(k)k\zeta'}d\zeta'\right] \tag{6}$$

$$= -\frac{\beta}{2}\tilde{A}ie^{-i\tau}\int_{-\infty}^0 e^{ik\xi}\left[\frac{1}{k}e^{ik\zeta}\int_0^\zeta e^{\xi_0 k}e^{-\zeta}\sin k\zeta' d\zeta' + \frac{1}{k}\sin k\zeta\int_\zeta^\infty e^{\xi_0 k}e^{-\zeta}e^{ik\zeta'}d\zeta'\right]dk \tag{7}$$

$$-\frac{\beta}{2}\tilde{A}ie^{-i\tau}\int_0^\infty e^{ik\xi}\left[\frac{1}{k}e^{-ik\zeta}\int_0^\zeta e^{-\xi_0 k}e^{-\zeta}\sin k\zeta' d\zeta' + \frac{1}{k}\sin k\zeta\int_\zeta^\infty e^{-\xi_0 k}e^{-\zeta}e^{-ik\zeta'}d\zeta'\right]dk \tag{8}$$

$$= -\frac{\beta}{2}\tilde{A}ie^{-i\tau}\int_0^\infty e^{-ik\xi}\left[\frac{1}{k}e^{-ik\zeta}\int_0^\zeta e^{-\xi_0 k}e^{-\zeta}\sin k\zeta' d\zeta' + \frac{1}{k}\sin k\zeta\int_\zeta^\infty e^{-\xi_0 k}e^{-\zeta}e^{-ik\zeta'}d\zeta'\right]dk \tag{9}$$

$$-\frac{\beta}{2}\tilde{A}ie^{-i\tau}\int_0^\infty e^{ik\xi}\left[\frac{1}{k}e^{-ik\zeta}\int_0^\zeta e^{-\xi_0 k}e^{-\zeta}\sin k\zeta' d\zeta' + \frac{1}{k}\sin k\zeta\int_\zeta^\infty e^{-\xi_0 k}e^{-\zeta}e^{-ik\zeta'}d\zeta'\right]dk \tag{10}$$

$$= -\beta ie^{-i\tau}\tilde{A}\int_0^\infty \cos(k\xi)\,e^{-\xi_0 k}\left[\frac{1}{k}e^{-ik\zeta'}\int_0^\zeta e^{-\zeta'}\sin k\zeta' d\zeta' + \frac{1}{k}\sin k\zeta\int_\zeta^\infty e^{-\zeta'}e^{-ik\zeta'}d\zeta'\right]dk \tag{11}$$

$$= -\beta\int_0^\infty \cos(k\xi)\left[e^{-ik\zeta'}\int_0^\zeta\left(\frac{\partial\tilde{Q}}{\partial\xi}\right)\sin k\zeta' d\zeta' + \sin k\zeta\int_\zeta^\infty\left(\frac{\partial\tilde{Q}}{\partial\xi}\right)e^{-ik\zeta'}d\zeta'\right]\frac{dk}{k}. \tag{12}$$

Note (12) matches what's in Rotunno's notes. From (12) the derivation is easy.

My attempt at deriving equation (1) - equation (38) in Rotunno's paper - is very messy. It didn't quite work originally as I was using the incorrect version of (37). Now having the correct version, let's see if it works. We have from (6),

$$-\beta\mathcal{F}^{-1}\left[\frac{1}{k}e^{-i\,\mathrm{sgn}(k)k\zeta}\int_0^\zeta\left(\pi e^{-\xi_0|k|}\tilde{A}e^{-\zeta}ie^{-i\tau}\right)\sin k\zeta' d\zeta'\right. \tag{13}$$

$$\left.+\frac{1}{k}\sin k\zeta\int_\zeta^\infty\left(\pi e^{-\xi_0|k|}\tilde{A}e^{-\zeta}ie^{-i\tau}\right)e^{-i\,\mathrm{sgn}(k)k\zeta'}d\zeta'\right] \tag{14}$$

$$= -\beta\tilde{A}\pi ie^{-i\tau}\mathcal{F}^{-1}\left[\frac{1}{k}e^{-i\,\mathrm{sgn}(k)k\zeta}e^{-\xi_0|k|}\int_0^\zeta e^{-\zeta'}\sin k\zeta' d\zeta'\right. \tag{15}$$

$$\left.+\frac{1}{k}\sin k\zeta e^{-\xi_0|k|}\int_\zeta^\infty e^{-\zeta'}e^{-i\,\mathrm{sgn}(k)k\zeta'}d\zeta'\right] \tag{16}$$

$$= -\beta\tilde{A}\pi ie^{-i\tau}\mathcal{F}^{-1}\left\{\frac{1}{k}e^{-i\,\mathrm{sgn}(k)k\zeta}e^{-\xi_0|k|}\left[\frac{-e^{-\zeta'}}{k^2+1}\left(k\cos k\zeta' + \sin k\zeta'\right)\right]_0^\zeta\right\} \tag{17}$$

$$-\beta\tilde{A}\pi ie^{-i\tau}\mathcal{F}^{-1}\left\{\frac{1}{k}\sin k\zeta e^{-\xi_0|k|}\left[\frac{1}{-i\,\mathrm{sgn}(k)k-1}e^{(-i\,\mathrm{sgn}(k)k-1)\zeta'}\right]_\zeta^\infty\right\}, \tag{18}$$

$$= -\beta\tilde{A}\pi ie^{-i\tau}\mathcal{F}^{-1}\left\{\frac{1}{k}e^{-i\,\mathrm{sgn}(k)k\zeta}e^{-\xi_0|k|}\left(\frac{-e^{-\zeta}}{k^2+1}\left(k\cos k\zeta + \sin k\zeta\right) + \frac{k}{k^2+1}\right)\right\} \tag{19}$$

$$-\beta\tilde{A}\pi ie^{-i\tau}\mathcal{F}^{-1}\left\{\frac{1}{k}\sin k\zeta e^{-\xi_0|k|}\left[\frac{1}{-i\,\mathrm{sgn}(k)k-1}e^{(-i\,\mathrm{sgn}(k)k-1)\zeta'}\right]_\zeta^\infty\right\}, \tag{20}$$

where the first integral (17) can be calculated by performing integration by parts. Consider the first term of the sum, i.e. (19). We have

$$
\begin{aligned}
&-\beta\tilde{A}\pi i e^{-i\tau}\mathcal{F}^{-1}\left\{\frac{1}{k}e^{-i\,\mathrm{sgn}(k)k\zeta}e^{-\xi_0|k|}\left(\frac{-e^{-\zeta}}{k^2+1}\left(k\cos k\zeta+\sin k\zeta\right)+\frac{k}{k^2+1}\right)\right\} && (21)\\
&=-\beta\tilde{A}ie^{-i\tau}\frac{1}{2}\int_{-\infty}^{\infty}\frac{1}{k}e^{ik\xi}e^{-i\,\mathrm{sgn}(k)k\zeta}e^{-\xi_0|k|}\left(\frac{-e^{-\zeta}}{k^2+1}\left(k\cos k\zeta+\sin k\zeta\right)+\frac{k}{k^2+1}\right)dk && (22)\\
&=-\beta\tilde{A}ie^{-i\tau}\frac{1}{2}\int_{-\infty}^{0}\frac{1}{k}e^{ik\xi}e^{ik\zeta}e^{\xi_0 k}\left(\frac{-e^{-\zeta}}{k^2+1}\left(k\cos k\zeta+\sin k\zeta\right)+\frac{k}{k^2+1}\right)dk && (23)\\
&\quad-\beta\tilde{A}ie^{-i\tau}\frac{1}{2}\int_{0}^{\infty}\frac{1}{k}e^{ik\xi}e^{-ik\zeta}e^{-\xi_0 k}\left(\frac{-e^{-\zeta}}{k^2+1}\left(k\cos k\zeta+\sin k\zeta\right)+\frac{k}{k^2+1}\right)dk && (24)\\
&=-\beta\tilde{A}ie^{-i\tau}\frac{1}{2}\int_{0}^{\infty}\frac{1}{-k}e^{-ik\xi}e^{-ik\zeta}e^{-\xi_0 k}\left(\frac{-e^{-\zeta}}{k^2+1}\left(-k\cos k\zeta-\sin k\zeta\right)-\frac{k}{k^2+1}\right)dk && (25)\\
&\quad-\beta\tilde{A}ie^{-i\tau}\frac{1}{2}\int_{0}^{\infty}\frac{1}{k}e^{ik\xi}e^{-ik\zeta}e^{-\xi_0 k}\left(\frac{-e^{-\zeta}}{k^2+1}\left(k\cos k\zeta+\sin k\zeta\right)+\frac{k}{k^2+1}\right)dk && (26)\\
&=-\beta\tilde{A}ie^{-i\tau}\frac{1}{2}\int_{0}^{\infty}\frac{1}{k}\left(e^{ik\xi}+e^{-ik\xi}\right)e^{-ik\zeta}e^{-\xi_0 k}\left(\frac{-e^{-\zeta}}{k^2+1}\left(k\cos k\zeta+\sin k\zeta\right)+\frac{k}{k^2+1}\right)dk && (27)\\
&=-\beta\tilde{A}\left(i\cos\tau+\sin\tau\right)\int_{0}^{\infty}\frac{1}{k}\cos k\xi e^{-ik\zeta}e^{-\xi_0 k}\frac{1}{k^2+1}\left(-e^{-\zeta}\left(k\cos k\zeta+\sin k\zeta\right)+k\right)dk && (28)\\
&=-\beta\tilde{A}\left(i\cos\tau+\sin\tau\right)\int_{0}^{\infty}\frac{1}{k}\cos k\xi\left(\cos k\zeta-i\sin k\zeta\right) && (29)\\
&\quad\times e^{-\xi_0 k}\frac{1}{k^2+1}\left(-e^{-\zeta}\left(k\cos k\zeta+\sin k\zeta\right)+k\right)dk && (30)\\
&=-\beta\tilde{A}\int_{0}^{\infty}\frac{1}{k}\sin k\xi\left(i\cos\tau+\sin\tau\right)\left(\cos k\zeta-i\sin k\zeta\right) && (31)\\
&\quad\times e^{-\xi_0 k}\frac{1}{k^2+1}\left(-e^{-\zeta}\left(k\cos k\zeta+\sin k\zeta\right)+k\right)dk && (32)\\
&=-\beta\tilde{A}\int_{0}^{\infty}\frac{1}{k}\sin k\xi\left(i\cos\tau\cos k\zeta+\cos\tau\sin k\zeta+\sin\tau\cos k\zeta-i\sin\tau\sin k\zeta\right) && (33)\\
&\quad\times e^{-\xi_0 k}\frac{1}{k^2+1}\left(-e^{-\zeta}\left(k\cos k\zeta+\sin k\zeta\right)+k\right)dk && (34)
\end{aligned}
$$

The real part of this is then

$$
\begin{aligned}
&-\beta\tilde{A}\int_{0}^{\infty}\frac{1}{k}\sin k\xi\left(\cos\tau\cos k\zeta+\sin\tau\sin k\zeta\right)e^{-\xi_0 k}\frac{1}{k^2+1}\left(-e^{-\zeta}\left(k\cos k\zeta+\sin k\zeta\right)+k\right)dk && (35)\\
&=-\beta\tilde{A}\int_{0}^{\infty}\frac{1}{k}\sin k\xi\sin\left(\tau+k\zeta\right)e^{-\xi_0 k}\frac{1}{k^2+1}\left(-e^{-\zeta}\left(k\cos k\zeta+\sin k\zeta\right)+k\right)dk && (36)\\
&=-\beta\tilde{A}\int_{0}^{\infty}\frac{1}{k}\sin k\xi\frac{1}{k^2+1}e^{-\xi_0 k} && (37)\\
&\times\left(-\sin\left(\tau+k\zeta\right)e^{-\zeta}k\cos k\zeta-\sin\left(\tau+k\zeta\right)e^{-\zeta}\sin k\zeta+k\sin\left(\tau+k\zeta\right)\right)dk. && (38)
\end{aligned}
$$

Consider now the second term. We have

$$
\begin{aligned}
&-\beta\tilde{A}\pi i e^{-i\tau}\mathcal{F}^{-1}\left\{\frac{1}{k}\sin k\zeta e^{-\xi_0|k|}\left[\frac{1}{-i\,\mathrm{sgn}(k)k-1}e^{(-i\,\mathrm{sgn}(k)k-1)\zeta'}\right]_\zeta^\infty\right\} && (39)\\
&=-\beta\tilde{A}ie^{-i\tau}\frac{1}{2}\int_{-\infty}^{0}\frac{1}{k}e^{ik\xi}\left(\sin k\zeta e^{\xi_0 k}\left[\frac{1}{ik-1}e^{(ik-1)\zeta'}\right]_\zeta^\infty\right)dk && (40)\\
&-\beta\tilde{A}ie^{-i\tau}\frac{1}{2}\int_{0}^{\infty}\frac{1}{k}e^{ik\xi}\left(\sin k\zeta e^{-\xi_0 k}\left[\frac{1}{-ik-1}e^{(-ik-1)\zeta'}\right]_\zeta^\infty\right)dk && (41)\\
&=-\beta\tilde{A}ie^{-i\tau}\frac{1}{2}\int_{-\infty}^{0}\frac{1}{k}e^{ik\xi}\left(\sin k\zeta e^{\xi_0 k}\frac{1}{1-ik}e^{(ik-1)\zeta}\right)dk && (42)\\
&-\beta\tilde{A}ie^{-i\tau}\frac{1}{2}\int_{0}^{\infty}\frac{1}{k}e^{ik\xi}\left(\sin k\zeta e^{-\xi_0 k}\frac{1}{1+ik}e^{(-ik-1)\zeta}\right)dk && (43)\\
&=-\beta\tilde{A}ie^{-i\tau}\frac{1}{2}\int_{0}^{\infty}\frac{1}{-k}e^{-ik\xi}\left(-\sin k\zeta e^{-\xi_0 k}\frac{1}{1+ik}e^{(-ik-1)\zeta}\right)dk && (44)\\
&-\beta\tilde{A}ie^{-i\tau}\frac{1}{2}\int_{0}^{\infty}\frac{1}{k}e^{ik\xi}\left(\sin k\zeta e^{-\xi_0 k}\frac{1}{1+ik}e^{(-ik-1)\zeta}\right)dk && (45)\\
&=-\beta\tilde{A}ie^{-i\tau}\int_{0}^{\infty}\frac{1}{k}\cos k\xi e^{-\xi_0 k}e^{-ik\zeta}\left(\sin k\zeta\frac{1-ik}{1+k^2}e^{-\zeta}\right)dk && (46)\\
&=-\beta\tilde{A}ie^{-i\tau}\int_{0}^{\infty}\frac{1}{k}\cos k\xi e^{-\xi_0 k}\left(\cos k\zeta-i\sin k\zeta\right)(1-ik)\left(\sin k\zeta\frac{1}{1+k^2}e^{-\zeta}\right)dk && (47)\\
&=-\beta\tilde{A}\int_{0}^{\infty}\frac{1}{k}\cos k\xi e^{-\xi_0 k}\left(\sin k\zeta\frac{1}{1+k^2}e^{-\zeta}\right) && (48)\\
&\times\left(i\cos\tau+\sin\tau\right)\left(\cos k\zeta-ik\cos k\zeta-i\sin k\zeta-k\sin k\zeta\right)dk. && (49)
\end{aligned}
$$

The real part of this is then

$$
\begin{aligned}
&-\beta\tilde{A}\int_{0}^{\infty}\frac{1}{k}\sin k\xi e^{-\xi_0 k}\left(\sin k\zeta\frac{1}{1+k^2}e^{-\zeta}\right) && (50)\\
&\times\left(k\cos\tau\cos k\zeta+\cos\tau\sin k\zeta+\sin\tau\cos k\zeta-k\sin\tau\sin k\zeta\right)dk && (51)\\
&=-\beta\tilde{A}\int_{0}^{\infty}\frac{1}{k}\sin k\xi e^{-\xi_0 k}\left(\sin k\zeta\frac{1}{1+k^2}e^{-\zeta}\right) && (52)\\
&\times\left(k\cos\left(\tau+k\zeta\right)+\sin\left(\tau+k\zeta\right)\right)dk && (53)\\
&=-\beta\tilde{A}\int_{0}^{\infty}\frac{1}{k}\sin k\xi e^{-\xi_0 k}\frac{1}{1+k^2} && (54)\\
&\times\left(\sin k\zeta e^{-\zeta}k\cos\left(\tau+k\zeta\right)+\sin k\zeta e^{-\zeta}\sin\left(\tau+k\zeta\right)\right)dk && (55)
\end{aligned}
$$

Summing (38) and (55) gives

$$
\tilde{\psi}(\xi,\zeta,\tau)=-\beta\tilde{A}\int_{0}^{\infty}\frac{\cos k\xi e^{-\xi_0 k}}{1+k^2}\left(\sin(k\zeta+\tau)-e^{-\zeta}\sin\tau\right)dk \tag{56}
$$

as required!

3 Calculating Potential Temperature

From equation (4) Rotunno (1983) we have

$$
\frac{\partial b}{\partial t}+N^2 w=Q.
$$

The non-dimensional form of this is

$$
\begin{aligned}
&\frac{\partial \tilde{b}}{\partial \tau} h\omega^3 + N^2 \tilde{w} h\omega = \tilde{Q} h\omega^3 \\
&= \frac{\partial \tilde{b}}{\partial \tau} + \left(\frac{N}{\omega}\right)^2 \tilde{w} = \tilde{Q}.
\end{aligned}
$$

Note that $\left(\frac{N}{\omega}\right)^2$ is essentially the Berger number with $H = L$. Thus can solve for $\tilde{b}$ using

$$
\begin{aligned}
&\frac{\tilde{b}_{k+1} - \tilde{b}_k}{\Delta\tau} = \tilde{Q}_k - \left(\frac{N}{\omega}\right)^2 \tilde{w}_k \\
&= \tilde{b}_{k+1} - \tilde{b}_k = \Delta\tau \left(\tilde{Q}_k - \left(\frac{N}{\omega}\right)^2 \tilde{w}_k\right).
\end{aligned}
$$

This produces a linear system of τ_N linearly independent equations in τ_N unknowns. However, in this form the system is singular - so substitute the equation for $\tilde{b}$ for $\tilde{b} = 0$. This produces weird results, with values much too great to be realistic. Thus initial condition flawed? Do we really have an initial condition for the bouyancy?

References

Rotunno, R. (1983), 'On the linear theory of the land and sea breeze', *Journal of the Atmospheric Sciences* **40**(8), 1999–2009.