

Notes concerning “On the Linear Theory of the Land and Sea Breeze” (Rotunno 1983)

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1 Paper Misprints

1. Equation (37) should be

$$-\beta \operatorname{Re} \left\{ \mathcal{F}^{-1} \left[\frac{1}{k} e^{-i \operatorname{sgn}(k) k \zeta} \int_0^\zeta \mathcal{F} \left(\frac{\partial \tilde{Q}}{\partial \xi} \right) \sin k \zeta' d\zeta' + \frac{1}{k} \sin k \zeta \int_\zeta^\infty \mathcal{F} \left(\frac{\partial \tilde{Q}}{\partial \xi} \right) e^{-i \operatorname{sgn}(k) k \zeta'} d\zeta' \right] \right\}.$$

2. \tilde{Q} should have time dependance $e^{-i(\tau - \frac{\pi}{2})} = i e^{-i\tau}$ not simply $\sin \tau$.
3. The expression $\tilde{b} = b h^{-1} \omega^{-3}$ in equation (13) should actually be $\tilde{b} = b h^{-1} \omega^{-2}$ to make the units come out right - assuming b has units m s^{-2} .

2 Mid-Latitude Case

We begin with equation (14)

$$\frac{\partial^2 \tilde{\psi}}{\partial \xi^2} + \frac{\partial^2 \tilde{\psi}}{\partial \zeta^2} = -\beta \frac{\partial \tilde{Q}}{\partial \xi}.$$

Following Duffy (2001), consider the “free-space” Green’s function defined by

$$\frac{\partial^2 g}{\partial \xi^2} + \frac{\partial^2 g}{\partial \zeta^2} = \delta(\xi - \xi') \delta(\zeta - \zeta')$$

where ξ, ζ can vary over all of \mathbb{R} . Integrating over a circle C centred at ξ', ζ' with radius a , where $r = \sqrt{(\xi - \xi')^2 + (\zeta - \zeta')^2}$ is the distance from (ξ', ζ')

$$\begin{aligned} \Rightarrow \iint_C \nabla \cdot \nabla g dV &= 1 \\ \Rightarrow \int_{\partial C} \frac{\partial g}{\partial r} &= 1. \end{aligned}$$

If there are no boundaries, then the response of g to singular forcing at (ξ', ζ') is unaffected by rotation, and is therefore constant on ∂C . Thus

$$\Rightarrow 2\pi a \left. \frac{\partial g}{\partial r} \right|_a = 1.$$

Because a was arbitrary we have therefore have

$$\begin{aligned} \Rightarrow \frac{\partial g}{\partial r} &= \frac{1}{2\pi r} \\ \Rightarrow g &= \frac{\ln(r)}{2\pi} + c = \frac{1}{4\pi} \ln \left((\xi - \xi')^2 + (\zeta - \zeta')^2 \right) + c. \end{aligned}$$

Can assume without loss of generality that $c = 0$, because $c \neq 0$ simply adds a constant to $\tilde{\psi}$, and we only care about the ξ and ζ derivatives of $\tilde{\psi}$. Furthermore, because g represents a fundamental solution for $\tilde{\psi}$ we have $-\frac{\partial g}{\partial \xi} = \tilde{w} = 0$. Using the method of images, we can obtain a solution satisfying this condition from the free space solution by noting that when $\zeta = 0$,

$$\frac{1}{4\pi} \frac{1}{(\xi - \xi')^2 + (\zeta - \zeta')^2} 2\xi - \frac{1}{4\pi} \frac{1}{(\xi - \xi')^2 + (-\zeta - \zeta')^2} 2\xi = 0.$$

This suggests taking

$$\begin{aligned} g &= \frac{1}{4\pi} \ln \left((\xi - \xi')^2 + (\zeta - \zeta')^2 \right) - \frac{1}{4\pi} \ln \left((\xi - \xi')^2 + (-\zeta - \zeta')^2 \right) \\ &= \frac{1}{4\pi} \ln \left(\frac{(\xi - \xi')^2 + (\zeta - \zeta')^2}{(\xi - \xi')^2 + (\zeta + \zeta')^2} \right). \end{aligned}$$

The properties of Green's function's then give the solution given by equation (20) of Rotunno (1983). Note that when calculating the convolution we integrate over the actual domain of $\tilde{\psi}$, i.e. $-\infty < \xi' < \infty$ and $\zeta' \geq 0$, not over free-space, i.e. \mathbb{R}^2 . This is because we are only putting sources in the actual domain.

3 Tropical Case

3.1 Deriving Equation (37)

Equation (36) gives

$$\frac{\partial^2 \tilde{\psi}}{\partial \xi^2} - \frac{\partial^2 \tilde{\psi}}{\partial \zeta^2} = -\beta \frac{\partial^2 \tilde{Q}}{\partial \xi^2}.$$

Take Fourier transform to get

$$\begin{aligned} -k^2 \hat{\tilde{\psi}} - \frac{\partial^2 \hat{\tilde{\psi}}}{\partial \zeta^2} &= -\beta \frac{\partial^2 \hat{\tilde{Q}}}{\partial \xi^2} \\ &= k^2 \hat{\tilde{\psi}} + \frac{\partial^2 \hat{\tilde{\psi}}}{\partial \zeta^2} = \beta \frac{\partial^2 \hat{\tilde{Q}}}{\partial \xi^2}. \end{aligned}$$

Boundary condition becomes $(ik) \hat{\tilde{\psi}}(k, 0) = 0$, and so $\hat{\tilde{\psi}}(k, 0) = 0$ (using the Fourier transform rule for derivatives.) Solve for Green's Function

$$k^2 G + G_{\zeta\zeta} = \delta(\zeta - \zeta').$$

General solution setting RHS to zero is $G = B_1 e^{ik\zeta} + B_2 e^{-ik\zeta}$. For $\zeta < \zeta'$ the boundary condition gives

$$\begin{aligned} 0 &= B_1 + B_2 \Rightarrow B_2 = -B_1 \\ \Rightarrow G &= B_1 2i \sin(k\zeta) = A \sin(k\zeta). \end{aligned}$$

Now consider $\zeta > \zeta'$. For $k > 0$ we have

$$\begin{aligned} Ge^{-i(\tau - \frac{\pi}{2})} &= B_1 e^{ik\zeta - i\tau + i\frac{\pi}{2}} + B_2 e^{-ik\zeta - i\tau + i\frac{\pi}{2}} \\ &= B_1 e^{i(k\zeta - \tau + \frac{\pi}{2})} + B_2 e^{-i(k\zeta + \tau - \frac{\pi}{2})}. \end{aligned}$$

Recall for gravity waves, energy propagates with the group velocity in the *opposite* direction to the phase velocity. As $\zeta > \zeta'$ we require positive group velocity, and therefore negative phase velocity. Thus $B_1 = 0$ and $G = B_2 e^{-ik\zeta}$. Similarly for $k < 0$ we have $G = B_1 e^{ik\zeta}$. Thus $G = B e^{-i \operatorname{sgn}(k) k \zeta}$.

Now, continuity requires that

$$\begin{aligned} \lim_{\zeta \rightarrow \zeta'^+} G &= \lim_{\zeta \rightarrow \zeta'^-} G \\ \lim_{\zeta \rightarrow \zeta'^+} G_\zeta - \lim_{\zeta \rightarrow \zeta'^-} G_\zeta &= 1. \end{aligned}$$

Thus

$$\begin{aligned} B e^{-i \operatorname{sgn}(k) k \zeta'} &= A \sin(k \zeta') \\ -i \operatorname{sgn}(k) k B e^{-i \operatorname{sgn}(k) k \zeta'} - A k \cos(k \zeta') &= 1 \\ \Rightarrow i \operatorname{sgn}(k) k A \sin(k \zeta') + A k \cos(k \zeta') &= -1. \end{aligned}$$

Now, $k > 0$

$$\begin{aligned} \Rightarrow i k A \sin(k \zeta') + A k \cos(k \zeta') &= A k e^{ik\zeta'} = -1 \\ \Rightarrow A &= -\frac{1}{k} e^{-ik\zeta'}. \end{aligned}$$

Also, $k < 0$

$$\begin{aligned} \Rightarrow -i k A \sin(k \zeta') + A k \cos(k \zeta') &= A k e^{-ik\zeta'} = -1 \\ \Rightarrow A &= -\frac{1}{k} e^{ik\zeta'}. \end{aligned}$$

Thus in both cases $A = -\frac{1}{k} e^{-\operatorname{sgn}(k) k \zeta'}$. Thus $B = -\sin(k \zeta')$, and so

$$G = \begin{cases} -\frac{1}{k} e^{-i \operatorname{sgn}(k) k \zeta'} \sin(k \zeta), & \zeta < \zeta', \\ -\frac{1}{k} \sin(k \zeta') e^{-i \operatorname{sgn}(k) k \zeta}, & \zeta > \zeta'. \end{cases}$$

3.2 Working Through Fourier Transform

We are attempting to derive the equation

$$\tilde{\psi}(\xi, \zeta, \tau) = -\beta \tilde{A} \int_0^\infty \frac{\cos k \xi e^{-\xi_0 k}}{1 + k^2} (\sin(k \zeta + \tau) - e^{-\zeta} \sin \tau) dk. \quad (1)$$

Note

$$\tilde{Q} = \beta \tilde{A} \left(\frac{\pi}{2} + \tan^{-1} \frac{\xi}{\xi_0} \right) e^{-\zeta} i e^{-i\tau} \quad (2)$$

$$\Rightarrow \frac{\partial \tilde{Q}}{\partial \xi} = \frac{1}{\xi^2 + \xi_0^2} \beta \tilde{A} \xi_0 e^{-\zeta} i e^{-i\tau} \quad (3)$$

$$\Rightarrow \mathcal{F} \left(\frac{\partial \tilde{Q}}{\partial \xi} \right) = \mathcal{F} \left[\frac{1}{\xi^2 + \xi_0^2} \right] \tilde{A} \xi_0 e^{-\zeta} i e^{-i\tau} \quad (4)$$

$$\Rightarrow \mathcal{F} \left(\frac{\partial \tilde{Q}}{\partial \xi} \right) = \frac{\pi}{\xi_0} e^{-\xi_0 |k|} \tilde{A} \xi_0 e^{-\zeta} e^{-i\tau} = \pi e^{-\xi_0 |k|} \tilde{A} e^{-\zeta} i e^{-i\tau} \quad (5)$$

using the non-unitary, angular frequency form of the Fourier transform. This Fourier transform can be derived by considering the Fourier transform of $\frac{\pi}{\xi_0} e^{-\xi_0 |\xi|}$, applying the inverse Fourier transform, and changing signs.

Now, starting from the corrected form of equation 37 we have,

$$- \beta \mathcal{F}^{-1} \left[\frac{1}{k} e^{-i \operatorname{sgn}(k) k \zeta} \int_0^\zeta \mathcal{F} \left(\frac{\partial \tilde{Q}}{\partial \xi} \right) \sin k \zeta' d\zeta' + \frac{1}{k} \sin k \zeta \int_\zeta^\infty \mathcal{F} \left(\frac{\partial \tilde{Q}}{\partial \xi} \right) e^{-i \operatorname{sgn}(k) k \zeta'} d\zeta' \right] \quad (6)$$

$$= -\frac{\beta}{2} \tilde{A} i e^{-i\tau} \int_{-\infty}^0 e^{ik\xi} \left[\frac{1}{k} e^{ik\zeta} \int_0^\zeta e^{\xi_0 k} e^{-\zeta} \sin k \zeta' d\zeta' + \frac{1}{k} \sin k \zeta \int_\zeta^\infty e^{\xi_0 k} e^{-\zeta} e^{ik\zeta'} d\zeta' \right] dk \quad (7)$$

$$- \frac{\beta}{2} \tilde{A} i e^{-i\tau} \int_0^\infty e^{ik\xi} \left[\frac{1}{k} e^{-ik\zeta} \int_0^\zeta e^{-\xi_0 k} e^{-\zeta} \sin k \zeta' d\zeta' + \frac{1}{k} \sin k \zeta \int_\zeta^\infty e^{-\xi_0 k} e^{-\zeta} e^{-ik\zeta'} d\zeta' \right] dk \quad (8)$$

$$= -\frac{\beta}{2} \tilde{A} i e^{-i\tau} \int_0^\infty e^{-ik\xi} \left[\frac{1}{k} e^{-ik\zeta} \int_0^\zeta e^{-\xi_0 k} e^{-\zeta} \sin k \zeta' d\zeta' + \frac{1}{k} \sin k \zeta \int_\zeta^\infty e^{-\xi_0 k} e^{-\zeta} e^{-ik\zeta'} d\zeta' \right] dk \quad (9)$$

$$- \frac{\beta}{2} \tilde{A} i e^{-i\tau} \int_0^\infty e^{ik\xi} \left[\frac{1}{k} e^{-ik\zeta} \int_0^\zeta e^{-\xi_0 k} e^{-\zeta} \sin k \zeta' d\zeta' + \frac{1}{k} \sin k \zeta \int_\zeta^\infty e^{-\xi_0 k} e^{-\zeta} e^{-ik\zeta'} d\zeta' \right] dk \quad (10)$$

$$= -\beta i e^{-i\tau} \tilde{A} \int_0^\infty \cos(k\xi) e^{-\xi_0 k} \left[\frac{1}{k} e^{-ik\zeta} \int_0^\zeta e^{-\zeta'} \sin k \zeta' d\zeta' + \frac{1}{k} \sin k \zeta \int_\zeta^\infty e^{-\zeta'} e^{-ik\zeta'} d\zeta' \right] dk \quad (11)$$

$$= -\beta \int_0^\infty \cos(k\xi) \left[e^{-ik\zeta} \int_0^\zeta \left(\frac{\partial \tilde{Q}}{\partial \xi} \right) \sin k \zeta' d\zeta' + \sin k \zeta \int_\zeta^\infty \left(\frac{\partial \tilde{Q}}{\partial \xi} \right) e^{-ik\zeta'} d\zeta' \right] \frac{dk}{k}. \quad (12)$$

Note (12) matches what's in Rotunno's notes. From (11) we have

$$\begin{aligned} & -\beta i e^{-i\tau} \tilde{A} \int_0^\infty \cos(k\xi) e^{-\xi_0 k} \left[\frac{1}{k} e^{-ik\zeta} \int_0^\zeta e^{-\zeta'} \sin k \zeta' d\zeta' + \frac{1}{k} \sin k \zeta \int_\zeta^\infty e^{-\zeta'} e^{-ik\zeta'} d\zeta' \right] dk \\ &= -\beta \tilde{A} \int_0^\infty \cos(k\xi) e^{-\xi_0 k} \left[\frac{1}{k} i e^{-i\tau} e^{-ik\zeta} \left(-\frac{e^{-\zeta} (\sin(k\zeta) + k \cos(k\zeta))}{k^2 + 1} + \frac{k}{k^2 + 1} \right) \right. \\ & \quad \left. + i e^{-i\tau} \frac{1}{k} \sin k \zeta \left(\frac{1 - ik}{k^2 + 1} e^{(-ik-1)\zeta} \right) \right] dk \end{aligned}$$

and the real part of this is therefore

$$= -\beta \tilde{A} \int_0^\infty \cos(k\xi) e^{-\xi_0 k} \frac{1}{k^2 + 1} [\sin(\tau + k\zeta) - e^{-\zeta} \sin(\tau)] dk.$$

My attempt at deriving equation (1) - equation (38) in Rotunno's paper - is very messy. It didn't quite work originally as I was using the incorrect version of (37). Now having the correct version,

let's see if it works. We have from (6),

$$- \beta \mathcal{F}^{-1} \left[\frac{1}{k} e^{-i \operatorname{sgn}(k) k \zeta} \int_0^\zeta \left(\pi e^{-\xi_0 |k|} \tilde{A} e^{-\zeta} i e^{-i\tau} \right) \sin k \zeta' d\zeta' \right] \quad (13)$$

$$+ \frac{1}{k} \sin k \zeta \int_\zeta^\infty \left(\pi e^{-\xi_0 |k|} \tilde{A} e^{-\zeta} i e^{-i\tau} \right) e^{-i \operatorname{sgn}(k) k \zeta'} d\zeta' \right] \quad (14)$$

$$= -\beta \tilde{A} \pi i e^{-i\tau} \mathcal{F}^{-1} \left[\frac{1}{k} e^{-i \operatorname{sgn}(k) k \zeta} e^{-\xi_0 |k|} \int_0^\zeta e^{-\zeta'} \sin k \zeta' d\zeta' \right] \quad (15)$$

$$+ \frac{1}{k} \sin k \zeta e^{-\xi_0 |k|} \int_\zeta^\infty e^{-\zeta'} e^{-i \operatorname{sgn}(k) k \zeta'} d\zeta' \right] \quad (16)$$

$$= -\beta \tilde{A} \pi i e^{-i\tau} \mathcal{F}^{-1} \left\{ \frac{1}{k} e^{-i \operatorname{sgn}(k) k \zeta} e^{-\xi_0 |k|} \left[\frac{-e^{-\zeta'}}{k^2 + 1} (k \cos k \zeta' + \sin k \zeta') \right]_0^\zeta \right\} \quad (17)$$

$$- \beta \tilde{A} \pi i e^{-i\tau} \mathcal{F}^{-1} \left\{ \frac{1}{k} \sin k \zeta e^{-\xi_0 |k|} \left[\frac{1}{-i \operatorname{sgn}(k) k - 1} e^{(-i \operatorname{sgn}(k) k - 1) \zeta'} \right]_\zeta^\infty \right\}, \quad (18)$$

$$= -\beta \tilde{A} \pi i e^{-i\tau} \mathcal{F}^{-1} \left\{ \frac{1}{k} e^{-i \operatorname{sgn}(k) k \zeta} e^{-\xi_0 |k|} \left(\frac{-e^{-\zeta}}{k^2 + 1} (k \cos k \zeta + \sin k \zeta) + \frac{k}{k^2 + 1} \right) \right\} \quad (19)$$

$$- \beta \tilde{A} \pi i e^{-i\tau} \mathcal{F}^{-1} \left\{ \frac{1}{k} \sin k \zeta e^{-\xi_0 |k|} \left[\frac{1}{-i \operatorname{sgn}(k) k - 1} e^{(-i \operatorname{sgn}(k) k - 1) \zeta'} \right]_\zeta^\infty \right\}, \quad (20)$$

where the first integral (17) can be calculated by performing integration by parts. Consider the first

term of the sum, i.e. (19). We have

$$- \beta \tilde{A} \pi i e^{-i\tau} \mathcal{F}^{-1} \left\{ \frac{1}{k} e^{-i \operatorname{sgn}(k) k \zeta} e^{-\xi_0 |k|} \left(\frac{-e^{-\zeta}}{k^2 + 1} (k \cos k \zeta + \sin k \zeta) + \frac{k}{k^2 + 1} \right) \right\} \quad (21)$$

$$= -\beta \tilde{A} i e^{-i\tau} \frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{k} e^{ik\xi} e^{-i \operatorname{sgn}(k) k \zeta} e^{-\xi_0 |k|} \left(\frac{-e^{-\zeta}}{k^2 + 1} (k \cos k \zeta + \sin k \zeta) + \frac{k}{k^2 + 1} \right) dk \quad (22)$$

$$= -\beta \tilde{A} i e^{-i\tau} \frac{1}{2} \int_{-\infty}^0 \frac{1}{k} e^{ik\xi} e^{ik\zeta} e^{\xi_0 k} \left(\frac{-e^{-\zeta}}{k^2 + 1} (k \cos k \zeta + \sin k \zeta) + \frac{k}{k^2 + 1} \right) dk \quad (23)$$

$$- \beta \tilde{A} i e^{-i\tau} \frac{1}{2} \int_0^{\infty} \frac{1}{k} e^{ik\xi} e^{-ik\zeta} e^{-\xi_0 k} \left(\frac{-e^{-\zeta}}{k^2 + 1} (k \cos k \zeta + \sin k \zeta) + \frac{k}{k^2 + 1} \right) dk \quad (24)$$

$$= -\beta \tilde{A} i e^{-i\tau} \frac{1}{2} \int_0^{\infty} \frac{1}{-k} e^{-ik\xi} e^{-ik\zeta} e^{-\xi_0 k} \left(\frac{-e^{-\zeta}}{k^2 + 1} (-k \cos k \zeta - \sin k \zeta) - \frac{k}{k^2 + 1} \right) dk \quad (25)$$

$$- \beta \tilde{A} i e^{-i\tau} \frac{1}{2} \int_0^{\infty} \frac{1}{k} e^{ik\xi} e^{-ik\zeta} e^{-\xi_0 k} \left(\frac{-e^{-\zeta}}{k^2 + 1} (k \cos k \zeta + \sin k \zeta) + \frac{k}{k^2 + 1} \right) dk \quad (26)$$

$$= -\beta \tilde{A} i e^{-i\tau} \frac{1}{2} \int_0^{\infty} \frac{1}{k} (e^{ik\xi} + e^{-ik\xi}) e^{-ik\zeta} e^{-\xi_0 k} \left(\frac{-e^{-\zeta}}{k^2 + 1} (k \cos k \zeta + \sin k \zeta) + \frac{k}{k^2 + 1} \right) dk \quad (27)$$

$$= -\beta \tilde{A} (i \cos \tau + \sin \tau) \int_0^{\infty} \frac{1}{k} \cos k \xi e^{-ik\zeta} e^{-\xi_0 k} \frac{1}{k^2 + 1} (-e^{-\zeta} (k \cos k \zeta + \sin k \zeta) + k) dk \quad (28)$$

$$= -\beta \tilde{A} (i \cos \tau + \sin \tau) \int_0^{\infty} \frac{1}{k} \cos k \xi (\cos k \zeta - i \sin k \zeta) \quad (29)$$

$$\times e^{-\xi_0 k} \frac{1}{k^2 + 1} (-e^{-\zeta} (k \cos k \zeta + \sin k \zeta) + k) dk \quad (30)$$

$$= -\beta \tilde{A} \int_0^{\infty} \frac{1}{k} \sin k \xi (i \cos \tau + \sin \tau) (\cos k \zeta - i \sin k \zeta) \quad (31)$$

$$\times e^{-\xi_0 k} \frac{1}{k^2 + 1} (-e^{-\zeta} (k \cos k \zeta + \sin k \zeta) + k) dk \quad (32)$$

$$= -\beta \tilde{A} \int_0^{\infty} \frac{1}{k} \sin k \xi (i \cos \tau \cos k \zeta + \cos \tau \sin k \zeta + \sin \tau \cos k \zeta - i \sin \tau \sin k \zeta) \quad (33)$$

$$\times e^{-\xi_0 k} \frac{1}{k^2 + 1} (-e^{-\zeta} (k \cos k \zeta + \sin k \zeta) + k) dk \quad (34)$$

The real part of this is then

$$- \beta \tilde{A} \int_0^{\infty} \frac{1}{k} \sin k \xi (\cos \tau \cos k \zeta + \sin \tau \sin k \zeta) e^{-\xi_0 k} \frac{1}{k^2 + 1} (-e^{-\zeta} (k \cos k \zeta + \sin k \zeta) + k) dk \quad (35)$$

$$= -\beta \tilde{A} \int_0^{\infty} \frac{1}{k} \sin k \xi \sin (\tau + k \zeta) e^{-\xi_0 k} \frac{1}{k^2 + 1} (-e^{-\zeta} (k \cos k \zeta + \sin k \zeta) + k) dk \quad (36)$$

$$= -\beta \tilde{A} \int_0^{\infty} \frac{1}{k} \sin k \xi \frac{1}{k^2 + 1} e^{-\xi_0 k} \quad (37)$$

$$\times (-\sin (\tau + k \zeta) e^{-\zeta} k \cos k \zeta - \sin (\tau + k \zeta) e^{-\zeta} \sin k \zeta + k \sin (\tau + k \zeta)) dk. \quad (38)$$

Consider now the second term. We have

$$- \beta \tilde{A} \pi i e^{-i\tau} \mathcal{F}^{-1} \left\{ \frac{1}{k} \sin k\zeta e^{-\xi_0|k|} \left[\frac{1}{-i \operatorname{sgn}(k)k - 1} e^{(-i \operatorname{sgn}(k)k - 1)\zeta'} \right]_{\zeta}^{\infty} \right\} \quad (39)$$

$$= -\beta \tilde{A} i e^{-i\tau} \frac{1}{2} \int_{-\infty}^0 \frac{1}{k} e^{ik\xi} \left(\sin k\zeta e^{\xi_0 k} \left[\frac{1}{ik - 1} e^{(ik-1)\zeta'} \right]_{\zeta}^{\infty} \right) dk \quad (40)$$

$$- \beta \tilde{A} i e^{-i\tau} \frac{1}{2} \int_0^{\infty} \frac{1}{k} e^{ik\xi} \left(\sin k\zeta e^{-\xi_0 k} \left[\frac{1}{-ik - 1} e^{(-ik-1)\zeta'} \right]_{\zeta}^{\infty} \right) dk \quad (41)$$

$$= -\beta \tilde{A} i e^{-i\tau} \frac{1}{2} \int_{-\infty}^0 \frac{1}{k} e^{ik\xi} \left(\sin k\zeta e^{\xi_0 k} \frac{1}{1 - ik} e^{(ik-1)\zeta} \right) dk \quad (42)$$

$$- \beta \tilde{A} i e^{-i\tau} \frac{1}{2} \int_0^{\infty} \frac{1}{k} e^{ik\xi} \left(\sin k\zeta e^{-\xi_0 k} \frac{1}{1 + ik} e^{(-ik-1)\zeta} \right) dk \quad (43)$$

$$= -\beta \tilde{A} i e^{-i\tau} \frac{1}{2} \int_0^{\infty} \frac{1}{-k} e^{-ik\xi} \left(-\sin k\zeta e^{-\xi_0 k} \frac{1}{1 + ik} e^{(-ik-1)\zeta} \right) dk \quad (44)$$

$$- \beta \tilde{A} i e^{-i\tau} \frac{1}{2} \int_0^{\infty} \frac{1}{k} e^{ik\xi} \left(\sin k\zeta e^{-\xi_0 k} \frac{1}{1 + ik} e^{(-ik-1)\zeta} \right) dk \quad (45)$$

$$= -\beta \tilde{A} i e^{-i\tau} \int_0^{\infty} \frac{1}{k} \cos k\xi e^{-\xi_0 k} e^{-ik\zeta} \left(\sin k\zeta \frac{1 - ik}{1 + k^2} e^{-\zeta} \right) dk \quad (46)$$

$$= -\beta \tilde{A} i e^{-i\tau} \int_0^{\infty} \frac{1}{k} \cos k\xi e^{-\xi_0 k} (\cos k\zeta - i \sin k\zeta) (1 - ik) \left(\sin k\zeta \frac{1}{1 + k^2} e^{-\zeta} \right) dk \quad (47)$$

$$= -\beta \tilde{A} \int_0^{\infty} \frac{1}{k} \cos k\xi e^{-\xi_0 k} \left(\sin k\zeta \frac{1}{1 + k^2} e^{-\zeta} \right) \quad (48)$$

$$\times (i \cos \tau + \sin \tau) (\cos k\zeta - ik \cos k\zeta - i \sin k\zeta - k \sin k\zeta) dk. \quad (49)$$

The real part of this is then

$$- \beta \tilde{A} \int_0^{\infty} \frac{1}{k} \sin k\xi e^{-\xi_0 k} \left(\sin k\zeta \frac{1}{1 + k^2} e^{-\zeta} \right) \quad (50)$$

$$\times (k \cos \tau \cos k\zeta + \cos \tau \sin k\zeta + \sin \tau \cos k\zeta - k \sin \tau \sin k\zeta) dk \quad (51)$$

$$= -\beta \tilde{A} \int_0^{\infty} \frac{1}{k} \sin k\xi e^{-\xi_0 k} \left(\sin k\zeta \frac{1}{1 + k^2} e^{-\zeta} \right) \quad (52)$$

$$\times (k \cos (\tau + k\zeta) + \sin (\tau + k\zeta)) dk \quad (53)$$

$$= -\beta \tilde{A} \int_0^{\infty} \frac{1}{k} \sin k\xi e^{-\xi_0 k} \frac{1}{1 + k^2} \quad (54)$$

$$\times (\sin k\zeta e^{-\zeta} k \cos (\tau + k\zeta) + \sin k\zeta e^{-\zeta} \sin (\tau + k\zeta)) dk \quad (55)$$

Summing (38) and (55) gives

$$\tilde{\psi}(\xi, \zeta, \tau) = -\beta \tilde{A} \int_0^{\infty} \frac{\cos k\xi e^{-\xi_0 k}}{1 + k^2} (\sin(k\zeta + \tau) - e^{-\zeta} \sin \tau) dk \quad (56)$$

as required!

4 Calculating Potential Temperature

From equation (4) Rotunno (1983) we have

$$\frac{\partial b}{\partial t} + N^2 w = Q.$$

The non-dimensional form of this is

$$\begin{aligned} \frac{\partial \tilde{b}}{\partial \tau} h \omega^3 + N^2 \tilde{w} h \omega &= \tilde{Q} h \omega^3 \\ &= \frac{\partial \tilde{b}}{\partial \tau} + \left(\frac{N}{\omega} \right)^2 \tilde{w} = \tilde{Q}. \end{aligned}$$

Note that $\left(\frac{N}{\omega} \right)^2$ is essentially the Berger number with $H = L$. Thus can solve for \tilde{b} using

$$\begin{aligned} \frac{\tilde{b}_{k+1} - \tilde{b}_k}{\Delta \tau} &= \tilde{Q}_k - \left(\frac{N}{\omega} \right)^2 \tilde{w}_k \\ &= \tilde{b}_{k+1} - \tilde{b}_k = \Delta \tau \left(\tilde{Q}_k - \left(\frac{N}{\omega} \right)^2 \tilde{w}_k \right). \end{aligned}$$

This produces a linear system of τ_N linearly independent equations in τ_N unknowns. However, in this form the system is singular - so substitute the equation for \tilde{b}_1 for $\tilde{b}_1 + \tilde{b}_{\frac{\tau_N}{2}} = 0$ to impose symmetry on the bouyancy. Note can use $\tilde{b}_1 + \tilde{b}_{\lfloor \frac{\tau_N}{2} \rfloor}$! This works - we can solve for bouyancy even without initial conditions! From bouyancy can extract potential temperature!

Note that using this method appears to produce potential temperature perturbations that are two large, i.e. ± 40 K, or even larger! Compare this with the WRF simulation of Vincent & Lane (2016) producing perturbations of ± 4 K. Could it be that the WRF data are composites, and that the signal is being diluted? Note that increasing h decreases θ' . Error possibly due to definition of $\bar{\theta}$ in code?

5 Choosing \tilde{A}

Note that from equation (4) we have at the surface

$$\frac{\partial b}{\partial t} = \frac{g}{\theta_0} \frac{\partial \theta'}{\partial t} = Q,$$

as $w = 0$ at the surface. Thus letting θ_M and θ_m denote the land surface temperature maxima and minima respectively, and noting that $\left(\frac{\pi}{2} + \tan^{-1} \frac{x}{x_0} \right)$ maps onto $(0, \pi)$, we have

$$\frac{g}{\theta_0} \frac{\theta_M - \theta_m}{12 \cdot 60 \cdot 60} = \left(\sin \frac{\pi}{2} - \sin \frac{3\pi}{2} \right) A\pi = 2A\pi,$$

with $A\pi$ the supremum of H at the surface, and it taking 12 hours to go from maximum to minimum temperature. Multiplying both sides by $h^{-1}\omega^{-3}$ gives

$$\tilde{A} = \frac{g}{2\pi\theta_0} \frac{\theta_M - \theta_m}{12 \cdot 60 \cdot 60} h^{-1}\omega^{-3}.$$

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