Notes concerning "On the Linear Theory of the Land and Sea Breeze" (Rotunno 1983)

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1 Paper Misprints

1. Equation (37) should be

$$-\beta \mathrm{Re}\left\{\mathcal{F}^{-1}\left[\frac{1}{k}e^{-i\,\mathrm{sgn}(k)k\zeta}\int_0^\zeta \mathcal{F}\left(\frac{\partial \tilde{Q}}{\partial \xi}\right)\sin k\zeta' d\zeta' + \frac{1}{k}\sin k\zeta \int_\zeta^\infty \mathcal{F}\left(\frac{\partial \tilde{Q}}{\partial \xi}\right)e^{-i\,\mathrm{sgn}(k)k\zeta'}d\zeta'\right]\right\}.$$

2. $\tilde{Q}$ should have time dependance $e^{-i\left(\tau-\frac{\pi}{2}\right)} = ie^{-i\tau}$ not simply $\sin\tau$.

3. The expression $\tilde{b} = bh^{-1}\omega^{-3}$ in equation (13) should actually be $\tilde{b} = bh^{-1}\omega^{-2}$ to make the units come out right - assuming b has units m s^{-2}.

4. Note formula (13) is incorrect for $\tilde{w}$. Note that

$$\tilde{w} = -\frac{\partial \tilde{\psi}}{\partial \xi} = -\frac{\partial \psi}{\partial \xi}h^{-2}\omega^{-1} = -\frac{\partial \psi}{\partial x}\frac{\partial x}{\partial \xi}h^{-2}\omega^{-1} = wN(f^2-\omega^2)^{-\frac{1}{2}}h^{-1}\omega^{-1} \tag{1}$$

and so

$$w = \tilde{w}h\omega(f^2-\omega^2)^{\frac{1}{2}}N^{-1} \tag{2}$$

2 Mid-Latitude Case

We begin with equation (14)

$$\frac{\partial^2\tilde{\psi}}{\partial \xi^2} + \frac{\partial^2\tilde{\psi}}{\partial \zeta^2} = -\beta\frac{\partial \tilde{Q}}{\partial \xi}.$$

Following Duffy (2001), consider the "free-space" Green's function defined by

$$\frac{\partial^2 g}{\partial \xi^2} + \frac{\partial^2 g}{\partial \zeta^2} = \delta(\xi-\xi')\delta(\zeta-\zeta')$$

where ξ, ζ can vary over all of $\mathbb{R}$. Integrating over a circle C centred at ξ', ζ' with radius a, where $r = \sqrt{(\xi-\xi')^2 + (\zeta-\zeta')^2}$ is the distance from (ξ', ζ')

$$\Rightarrow \iint_C \nabla\cdot\nabla g dV = 1$$

$$\Rightarrow \int_{\partial C}\frac{\partial g}{\partial r} = 1.$$

If there are no boundaries, then the response of g to singular forcing at (ξ', ζ') is unaffected by rotation, and is therefore constant on ∂C. Thus

$$\Rightarrow 2\pi a \left.\frac{\partial g}{\partial r}\right|_a = 1.$$

Because a was arbitrary we have therefore have

$$\begin{aligned}
&\Rightarrow \frac{\partial g}{\partial r} = \frac{1}{2\pi r} \\
&\Rightarrow g = \frac{\ln(r)}{2\pi} + c = \frac{1}{4\pi}\ln\left((\xi - \xi')^2 + (\zeta - \zeta')^2\right) + c.
\end{aligned}$$

Can assume without loss of generality that $c = 0$, because $c \neq 0$ simply adds a constant to $\tilde{\psi}$, and we only care about the ξ and ζ derivatives of $\tilde{\psi}$. Furthermore, because g represents a fundamental solution for $\tilde{\psi}$ we have $-\frac{\partial g}{\partial \xi} = \tilde{w} = 0$. Using the method of images, we can obtain a solution satisfying this condition from the free space solution by noting that when $\zeta = 0$,

$$\frac{1}{4\pi}\frac{1}{(\xi - \xi')^2 + (\zeta - \zeta')^2} 2(\xi - \xi') - \frac{1}{4\pi}\frac{1}{(\xi - \xi')^2 + (-\zeta - \zeta')^2} 2(\xi - \xi') = 0.$$

This suggests taking

$$\begin{aligned}
g &= \frac{1}{4\pi}\ln\left((\xi - \xi')^2 + (\zeta - \zeta')^2\right) - \frac{1}{4\pi}\ln\left((\xi - \xi')^2 + (-\zeta - \zeta')^2\right) \\
&= \frac{1}{4\pi}\ln\left(\frac{(\xi - \xi')^2 + (\zeta - \zeta')^2}{(\xi - \xi')^2 + (\zeta + \zeta')^2}\right).
\end{aligned}$$

The properties of Green's function's then give the solution given by equation (20) of Rotunno (1983). Note that when calculating the convolution we integrate over the actual domain of $\tilde{\psi}$, i.e. $-\infty < \xi' < \infty$ and $\zeta' \geq 0$, not over free-space, i.e. $\mathbb{R}^2$. This is because we are only putting real sources in the actual domain.

3 Tropical Case

3.1 Deriving Equation (37)

Equation (36) gives

$$\frac{\partial^2 \tilde{\psi}}{\partial \xi^2} - \frac{\partial^2 \tilde{\psi}}{\partial \zeta^2} = -\beta \frac{\partial \tilde{Q}}{\partial \xi}.$$

Take Fourier transform to get

$$\begin{aligned}
&-k^2 \widehat{\tilde{\psi}} - \frac{\partial^2 \widehat{\tilde{\psi}}}{\partial \zeta^2} = -\beta \frac{\partial \widehat{\tilde{Q}}}{\partial \xi} \\
&= k^2 \widehat{\tilde{\psi}} + \frac{\partial^2 \widehat{\tilde{\psi}}}{\partial \zeta^2} = \beta \frac{\partial \widehat{\tilde{Q}}}{\partial \xi}.
\end{aligned}$$

Boundary condition becomes $(ik)\,\widehat{\tilde{\psi}}(k, 0) = 0$, and so $\widehat{\tilde{\psi}}(k, 0) = 0$ (using the Fourier transform rule for derivatives.) Solve for Green's Function

$$k^2 G + G_{\zeta\zeta} = \delta(\zeta - \zeta'). \tag{3}$$

General solution setting RHS to zero is $G = B_1 e^{ik\zeta} + B_2 e^{-ik\zeta}$. For $\zeta < \zeta'$ the boundary condition gives

$$\begin{aligned} 0 = B_1 + B_2 &\Rightarrow B_2 = -B_1 \\ &\Rightarrow G = B_1 2i \sin(k\zeta) = A \sin(k\zeta). \end{aligned}$$

Now consider $\zeta > \zeta'$. For $k > 0$ we have

$$\begin{aligned} Ge^{-i\left(\tau - \frac{\pi}{2}\right)} &= B_1 e^{ik\zeta - i\tau + i\frac{\pi}{2}} + B_2 e^{-ik\zeta - i\tau + i\frac{\pi}{2}} \\ &= B_1 e^{i\left(k\zeta - \tau + \frac{\pi}{2}\right)} + B_2 e^{-i\left(k\zeta + \tau - \frac{\pi}{2}\right)}. \end{aligned}$$

Recall for gravity waves, energy propagates with the group velocity in the *opposite* direction to the phase velocity. As $\zeta > \zeta'$ we require positive group velocity, and therefore negative phase velocity. Thus $B_1 = 0$ and $G = B_2 e^{-ik\zeta}$. Similarly for $k < 0$ we have $G = B_1 e^{ik\zeta}$. Thus $G = Be^{-i\,\mathrm{sgn}(k)k\zeta}$.

Now, continuity requires that

$$\begin{aligned} \lim_{\zeta \to \zeta'^+} G &= \lim_{\zeta \to \zeta'^-} G \\ \lim_{\zeta \to \zeta'^+} G_\zeta - \lim_{\zeta \to \zeta'^-} G_\zeta &= 1 \end{aligned}$$

where the second condition follows from integrating (3) from $\zeta' - \epsilon$ to $\zeta' + \epsilon$ and taking the limit as $\epsilon \to 0$, noting that G is continuous at ζ' so the $k^2 G$ integral goes to zero in the limit. Thus

$$\begin{aligned} &Be^{-i\,\mathrm{sgn}(k)k\zeta'} = A\sin(k\zeta') \\ &- i\,\mathrm{sgn}(k)kBe^{-i\,\mathrm{sgn}(k)k\zeta'} - Ak\cos(k\zeta') = 1 \\ \Rightarrow\ &i\,\mathrm{sgn}(k)kA\sin(k\zeta') + Ak\cos(k\zeta') = -1. \end{aligned}$$

Now, $k > 0$

$$\begin{aligned} &\Rightarrow ikA\sin(k\zeta') + Ak\cos(k\zeta') = Ake^{ik\zeta'} = -1 \\ &\Rightarrow A = -\frac{1}{k}e^{-ik\zeta'}. \end{aligned}$$

Also, $k < 0$

$$\begin{aligned} &\Rightarrow -ikA\sin(k\zeta') + Ak\cos(k\zeta') = Ake^{-ik\zeta'} = -1 \\ &\Rightarrow A = -\frac{1}{k}e^{ik\zeta'}. \end{aligned}$$

Thus in both cases $A = -\frac{1}{k}e^{-\,\mathrm{sgn}(k)k\zeta'}$. Thus $B = -\sin(k\zeta')$, and so

$$G = \begin{cases} -\frac{1}{k}e^{-i\,\mathrm{sgn}(k)k\zeta'}\sin(k\zeta), & \zeta < \zeta', \\ -\frac{1}{k}\sin(k\zeta')e^{-i\,\mathrm{sgn}(k)k\zeta}, & \zeta > \zeta'. \end{cases}$$

3.2 Working Through Fourier Transform

We are attempting to derive the equation

$$\tilde{\psi}(\xi, \zeta, \tau) = -\beta\tilde{A}\int_0^\infty \frac{\cos k\xi e^{-\xi_0 k}}{1 + k^2}\left(\sin(k\zeta + \tau) - e^{-\zeta}\sin\tau\right)dk. \tag{4}$$

Note

$$\tilde{Q} = \beta \tilde{A} \left(\frac{\pi}{2} + \tan^{-1} \frac{\xi}{\xi_0} \right) e^{-\zeta} i e^{-i\tau} \tag{5}$$

$$\Rightarrow \frac{\partial \tilde{Q}}{\partial \xi} = \frac{1}{\xi^2 + \xi_0^2} \beta \tilde{A} \xi_0 e^{-\zeta} i e^{-i\tau} \tag{6}$$

$$\Rightarrow \mathcal{F} \left(\frac{\partial \tilde{Q}}{\partial \xi} \right) = \mathcal{F} \left[\frac{1}{\xi^2 + \xi_0^2} \right] \tilde{A} \xi_0 e^{-\zeta} i e^{-i\tau} \tag{7}$$

$$\Rightarrow \mathcal{F} \left(\frac{\partial \tilde{Q}}{\partial \xi} \right) = \frac{\pi}{\xi_0} e^{-\xi_0 |k|} \tilde{A} \xi_0 e^{-\zeta} e^{-i\tau} = \pi e^{-\xi_0 |k|} \tilde{A} e^{-\zeta} i e^{-i\tau} \tag{8}$$

using the non-unitary, angular frequency form of the Fourier transform. This Fourier transform can be derived by considering the Fourier transform of $\frac{\pi}{\xi_0} e^{-\xi_0 |\xi|}$, applying the inverse Fourier transform, and changing signs.

Now, starting from the corrected form of equation 37 we have,

$$-\beta \mathcal{F}^{-1} \left[\frac{1}{k} e^{-i \operatorname{sgn}(k) k \zeta} \int_0^\zeta \mathcal{F} \left(\frac{\partial \tilde{Q}}{\partial \xi} \right) \sin k\zeta' d\zeta' + \frac{1}{k} \sin k\zeta \int_\zeta^\infty \mathcal{F} \left(\frac{\partial \tilde{Q}}{\partial \xi} \right) e^{-i \operatorname{sgn}(k) k \zeta'} d\zeta' \right] \tag{9}$$

$$= -\frac{\beta}{2} \tilde{A} i e^{-i\tau} \int_{-\infty}^0 e^{ik\xi} \left[\frac{1}{k} e^{ik\zeta} \int_0^\zeta e^{\xi_0 k} e^{-\zeta} \sin k\zeta' d\zeta' + \frac{1}{k} \sin k\zeta \int_\zeta^\infty e^{\xi_0 k} e^{-\zeta} e^{ik\zeta'} d\zeta' \right] dk \tag{10}$$

$$-\frac{\beta}{2} \tilde{A} i e^{-i\tau} \int_0^\infty e^{ik\xi} \left[\frac{1}{k} e^{-ik\zeta} \int_0^\zeta e^{-\xi_0 k} e^{-\zeta'} \sin k\zeta' d\zeta' + \frac{1}{k} \sin k\zeta \int_\zeta^\infty e^{-\xi_0 k} e^{-\zeta'} e^{-ik\zeta'} d\zeta' \right] dk \tag{11}$$

$$= -\frac{\beta}{2} \tilde{A} i e^{-i\tau} \int_0^\infty e^{-ik\xi} \left[\frac{1}{k} e^{-ik\zeta} \int_0^\zeta e^{-\xi_0 k} e^{-\zeta'} \sin k\zeta' d\zeta' + \frac{1}{k} \sin k\zeta \int_\zeta^\infty e^{-\xi_0 k} e^{-\zeta'} e^{-ik\zeta'} d\zeta' \right] dk \tag{12}$$

$$-\frac{\beta}{2} \tilde{A} i e^{-i\tau} \int_0^\infty e^{ik\xi} \left[\frac{1}{k} e^{-ik\zeta} \int_0^\zeta e^{-\xi_0 k} e^{-\zeta'} \sin k\zeta' d\zeta' + \frac{1}{k} \sin k\zeta \int_\zeta^\infty e^{-\xi_0 k} e^{-\zeta'} e^{-ik\zeta'} d\zeta' \right] dk \tag{13}$$

$$= -\beta i e^{-i\tau} \tilde{A} \int_0^\infty \cos(k\xi) e^{-\xi_0 k} \left[\frac{1}{k} e^{-ik\zeta} \int_0^\zeta e^{-\zeta'} \sin k\zeta' d\zeta' + \frac{1}{k} \sin k\zeta \int_\zeta^\infty e^{-\zeta'} e^{-ik\zeta'} d\zeta' \right] dk \tag{14}$$

$$= -\beta \int_0^\infty \cos(k\xi) \left[e^{-ik\zeta} \int_0^\zeta \left(\frac{\partial \tilde{Q}}{\partial \xi} \right) \sin k\zeta' d\zeta' + \sin k\zeta \int_\zeta^\infty \left(\frac{\partial \tilde{Q}}{\partial \xi} \right) e^{-ik\zeta'} d\zeta' \right] \frac{dk}{k}. \tag{15}$$

Note (15) matches what's in Rotunno's notes. Before integrating ζ', note that

$$\frac{\partial}{\partial \zeta'} \left(-\frac{e^{-\zeta'} \left(\sin(k\zeta') + k \cos(k\zeta') \right)}{k^2 + 1} \right) = \sin(k\zeta') e^{-\zeta'} \tag{16}$$

From (14) we have

$$-\beta i e^{-i\tau} \tilde{A} \int_0^\infty \cos(k\xi) e^{-\xi_0 k} \left[\frac{1}{k} e^{-ik\zeta} \int_0^\zeta e^{-\zeta'} \sin k\zeta' d\zeta' + \frac{1}{k} \sin k\zeta \int_\zeta^\infty e^{-\zeta'} e^{-ik\zeta'} d\zeta' \right] dk$$

$$= -\beta \tilde{A} \int_0^\infty \cos(k\xi) e^{-\xi_0 k} \left[\frac{1}{k} i e^{-i\tau} e^{-ik\zeta} \left(-\frac{e^{-\zeta} \left(\sin(k\zeta) + k \cos(k\zeta) \right)}{k^2 + 1} + \frac{k}{k^2 + 1} \right) \right.$$
$$\left. + i e^{-i\tau} \frac{1}{k} \sin k\zeta \left(\frac{1 - ik}{k^2 + 1} e^{(-ik-1)\zeta} \right) \right] dk$$

and the real part of this is therefore

$$= -\beta \tilde{A} \int_0^\infty \cos(k\xi) e^{-\xi_0 k} \frac{1}{k^2 + 1} \left[\sin(\tau + k\zeta) - e^{-\zeta} \sin(\tau) \right].$$

However, note that getting the real part is actually subtle, as we must use $e^{ik\zeta} = \cos(k\zeta) + i\sin(k\zeta)$ from the first and third terms. This leaves us with our two terms.

My attempt at deriving equation (4) - equation (38) in Rotunno's paper - is very messy. It didn't quite work originally as I was using the incorrect version of (37). Now having the correct version, let's see if it works. We have from (9),

$$
\begin{aligned}
&-\beta\mathcal{F}^{-1}\left[\frac{1}{k}e^{-i\,\mathrm{sgn}(k)k\zeta}\int_0^\zeta\left(\pi e^{-\xi_0|k|}\tilde{A}e^{-\zeta}ie^{-i\tau}\right)\sin k\zeta' d\zeta'\right. && (17)\\
&\left.+\frac{1}{k}\sin k\zeta\int_\zeta^\infty\left(\pi e^{-\xi_0|k|}\tilde{A}e^{-\zeta}ie^{-i\tau}\right)e^{-i\,\mathrm{sgn}(k)k\zeta'}d\zeta'\right] && (18)\\
&= -\beta\tilde{A}\pi ie^{-i\tau}\mathcal{F}^{-1}\left[\frac{1}{k}e^{-i\,\mathrm{sgn}(k)k\zeta}e^{-\xi_0|k|}\int_0^\zeta e^{-\zeta'}\sin k\zeta' d\zeta'\right. && (19)\\
&\left.+\frac{1}{k}\sin k\zeta e^{-\xi_0|k|}\int_\zeta^\infty e^{-\zeta'}e^{-i\,\mathrm{sgn}(k)k\zeta'}d\zeta'\right] && (20)\\
&= -\beta\tilde{A}\pi ie^{-i\tau}\mathcal{F}^{-1}\left\{\frac{1}{k}e^{-i\,\mathrm{sgn}(k)k\zeta}e^{-\xi_0|k|}\left[\frac{-e^{-\zeta'}}{k^2+1}\left(k\cos k\zeta'+\sin k\zeta'\right)\right]_0^\zeta\right\} && (21)\\
&-\beta\tilde{A}\pi ie^{-i\tau}\mathcal{F}^{-1}\left\{\frac{1}{k}\sin k\zeta e^{-\xi_0|k|}\left[\frac{1}{-i\,\mathrm{sgn}(k)k-1}e^{(-i\,\mathrm{sgn}(k)k-1)\zeta'}\right]_\zeta^\infty\right\}, && (22)\\
&= -\beta\tilde{A}\pi ie^{-i\tau}\mathcal{F}^{-1}\left\{\frac{1}{k}e^{-i\,\mathrm{sgn}(k)k\zeta}e^{-\xi_0|k|}\left(\frac{-e^{-\zeta}}{k^2+1}\left(k\cos k\zeta+\sin k\zeta\right)+\frac{k}{k^2+1}\right)\right\} && (23)\\
&-\beta\tilde{A}\pi ie^{-i\tau}\mathcal{F}^{-1}\left\{\frac{1}{k}\sin k\zeta e^{-\xi_0|k|}\left[\frac{1}{-i\,\mathrm{sgn}(k)k-1}e^{(-i\,\mathrm{sgn}(k)k-1)\zeta'}\right]_\zeta^\infty\right\}, && (24)
\end{aligned}
$$

where the first integral (21) can be calculated by performing integration by parts. Consider the first term of the sum, i.e. (23). We have

$$
\begin{aligned}
&-\beta\tilde{A}\pi i e^{-i\tau}\mathcal{F}^{-1}\left\{\frac{1}{k}e^{-i\operatorname{sgn}(k)k\zeta}e^{-\xi_0|k|}\left(\frac{-e^{-\zeta}}{k^2+1}\left(k\cos k\zeta+\sin k\zeta\right)+\frac{k}{k^2+1}\right)\right\} && (25)\\
&=-\beta\tilde{A}ie^{-i\tau}\frac{1}{2}\int_{-\infty}^{\infty}\frac{1}{k}e^{ik\xi}e^{-i\operatorname{sgn}(k)k\zeta}e^{-\xi_0|k|}\left(\frac{-e^{-\zeta}}{k^2+1}\left(k\cos k\zeta+\sin k\zeta\right)+\frac{k}{k^2+1}\right)dk && (26)\\
&=-\beta\tilde{A}ie^{-i\tau}\frac{1}{2}\int_{-\infty}^{0}\frac{1}{k}e^{ik\xi}e^{ik\zeta}e^{\xi_0 k}\left(\frac{-e^{-\zeta}}{k^2+1}\left(k\cos k\zeta+\sin k\zeta\right)+\frac{k}{k^2+1}\right)dk && (27)\\
&-\beta\tilde{A}ie^{-i\tau}\frac{1}{2}\int_{0}^{\infty}\frac{1}{k}e^{ik\xi}e^{-ik\zeta}e^{-\xi_0 k}\left(\frac{-e^{-\zeta}}{k^2+1}\left(k\cos k\zeta+\sin k\zeta\right)+\frac{k}{k^2+1}\right)dk && (28)\\
&=-\beta\tilde{A}ie^{-i\tau}\frac{1}{2}\int_{0}^{\infty}\frac{1}{-k}e^{-ik\xi}e^{-ik\zeta}e^{-\xi_0 k}\left(\frac{-e^{-\zeta}}{k^2+1}\left(-k\cos k\zeta-\sin k\zeta\right)-\frac{k}{k^2+1}\right)dk && (29)\\
&-\beta\tilde{A}ie^{-i\tau}\frac{1}{2}\int_{0}^{\infty}\frac{1}{k}e^{ik\xi}e^{-ik\zeta}e^{-\xi_0 k}\left(\frac{-e^{-\zeta}}{k^2+1}\left(k\cos k\zeta+\sin k\zeta\right)+\frac{k}{k^2+1}\right)dk && (30)\\
&=-\beta\tilde{A}ie^{-i\tau}\frac{1}{2}\int_{0}^{\infty}\frac{1}{k}\left(e^{ik\xi}+e^{-ik\xi}\right)e^{-ik\zeta}e^{-\xi_0 k}\left(\frac{-e^{-\zeta}}{k^2+1}\left(k\cos k\zeta+\sin k\zeta\right)+\frac{k}{k^2+1}\right)dk && (31)\\
&=-\beta\tilde{A}\left(i\cos\tau+\sin\tau\right)\int_{0}^{\infty}\frac{1}{k}\cos k\xi e^{-ik\zeta}e^{-\xi_0 k}\frac{1}{k^2+1}\left(-e^{-\zeta}\left(k\cos k\zeta+\sin k\zeta\right)+k\right)dk && (32)\\
&=-\beta\tilde{A}\left(i\cos\tau+\sin\tau\right)\int_{0}^{\infty}\frac{1}{k}\cos k\xi\left(\cos k\zeta-i\sin k\zeta\right) && (33)\\
&\times e^{-\xi_0 k}\frac{1}{k^2+1}\left(-e^{-\zeta}\left(k\cos k\zeta+\sin k\zeta\right)+k\right)dk && (34)\\
&=-\beta\tilde{A}\int_{0}^{\infty}\frac{1}{k}\sin k\xi\left(i\cos\tau+\sin\tau\right)\left(\cos k\zeta-i\sin k\zeta\right) && (35)\\
&\times e^{-\xi_0 k}\frac{1}{k^2+1}\left(-e^{-\zeta}\left(k\cos k\zeta+\sin k\zeta\right)+k\right)dk && (36)\\
&=-\beta\tilde{A}\int_{0}^{\infty}\frac{1}{k}\sin k\xi\left(i\cos\tau\cos k\zeta+\cos\tau\sin k\zeta+\sin\tau\cos k\zeta-i\sin\tau\sin k\zeta\right) && (37)\\
&\times e^{-\xi_0 k}\frac{1}{k^2+1}\left(-e^{-\zeta}\left(k\cos k\zeta+\sin k\zeta\right)+k\right)dk && (38)
\end{aligned}
$$

The real part of this is then

$$
\begin{aligned}
&-\beta\tilde{A}\int_{0}^{\infty}\frac{1}{k}\sin k\xi\left(\cos\tau\cos k\zeta+\sin\tau\sin k\zeta\right)e^{-\xi_0 k}\frac{1}{k^2+1}\left(-e^{-\zeta}\left(k\cos k\zeta+\sin k\zeta\right)+k\right)dk && (39)\\
&=-\beta\tilde{A}\int_{0}^{\infty}\frac{1}{k}\sin k\xi\sin\left(\tau+k\zeta\right)e^{-\xi_0 k}\frac{1}{k^2+1}\left(-e^{-\zeta}\left(k\cos k\zeta+\sin k\zeta\right)+k\right)dk && (40)\\
&=-\beta\tilde{A}\int_{0}^{\infty}\frac{1}{k}\sin k\xi\frac{1}{k^2+1}e^{-\xi_0 k} && (41)\\
&\times\left(-\sin\left(\tau+k\zeta\right)e^{-\zeta}k\cos k\zeta-\sin\left(\tau+k\zeta\right)e^{-\zeta}\sin k\zeta+k\sin\left(\tau+k\zeta\right)\right)dk. && (42)
\end{aligned}
$$

Consider now the second term. We have

$$- \beta \tilde{A} \pi i e^{-i\tau} \mathcal{F}^{-1} \left\{ \frac{1}{k} \sin k\zeta e^{-\xi_0 |k|} \left[\frac{1}{-i \operatorname{sgn}(k) k - 1} e^{(-i \operatorname{sgn}(k) k - 1)\zeta'} \right]_\zeta^\infty \right\} \tag{43}$$

$$= -\beta \tilde{A} i e^{-i\tau} \frac{1}{2} \int_{-\infty}^{0} \frac{1}{k} e^{ik\xi} \left(\sin k\zeta e^{\xi_0 k} \left[\frac{1}{ik - 1} e^{(ik-1)\zeta'} \right]_\zeta^\infty \right) dk \tag{44}$$

$$- \beta \tilde{A} i e^{-i\tau} \frac{1}{2} \int_{0}^{\infty} \frac{1}{k} e^{ik\xi} \left(\sin k\zeta e^{-\xi_0 k} \left[\frac{1}{-ik - 1} e^{(-ik-1)\zeta'} \right]_\zeta^\infty \right) dk \tag{45}$$

$$= -\beta \tilde{A} i e^{-i\tau} \frac{1}{2} \int_{-\infty}^{0} \frac{1}{k} e^{ik\xi} \left(\sin k\zeta e^{\xi_0 k} \frac{1}{1 - ik} e^{(ik-1)\zeta} \right) dk \tag{46}$$

$$- \beta \tilde{A} i e^{-i\tau} \frac{1}{2} \int_{0}^{\infty} \frac{1}{k} e^{ik\xi} \left(\sin k\zeta e^{-\xi_0 k} \frac{1}{1 + ik} e^{(-ik-1)\zeta} \right) dk \tag{47}$$

$$= -\beta \tilde{A} i e^{-i\tau} \frac{1}{2} \int_{0}^{\infty} \frac{1}{-k} e^{-ik\xi} \left(- \sin k\zeta e^{-\xi_0 k} \frac{1}{1 + ik} e^{(-ik-1)\zeta} \right) dk \tag{48}$$

$$- \beta \tilde{A} i e^{-i\tau} \frac{1}{2} \int_{0}^{\infty} \frac{1}{k} e^{ik\xi} \left(\sin k\zeta e^{-\xi_0 k} \frac{1}{1 + ik} e^{(-ik-1)\zeta} \right) dk \tag{49}$$

$$= -\beta \tilde{A} i e^{-i\tau} \int_{0}^{\infty} \frac{1}{k} \cos k\xi e^{-\xi_0 k} e^{-ik\zeta} \left(\sin k\zeta \frac{1 - ik}{1 + k^2} e^{-\zeta} \right) dk \tag{50}$$

$$= -\beta \tilde{A} i e^{-i\tau} \int_{0}^{\infty} \frac{1}{k} \cos k\xi e^{-\xi_0 k} \left(\cos k\zeta - i \sin k\zeta \right) (1 - ik) \left(\sin k\zeta \frac{1}{1 + k^2} e^{-\zeta} \right) dk \tag{51}$$

$$= -\beta \tilde{A} \int_{0}^{\infty} \frac{1}{k} \cos k\xi e^{-\xi_0 k} \left(\sin k\zeta \frac{1}{1 + k^2} e^{-\zeta} \right) \tag{52}$$

$$\times \left(i \cos \tau + \sin \tau \right) \left(\cos k\zeta - ik \cos k\zeta - i \sin k\zeta - k \sin k\zeta \right) dk. \tag{53}$$

The real part of this is then

$$- \beta \tilde{A} \int_{0}^{\infty} \frac{1}{k} \sin k\xi e^{-\xi_0 k} \left(\sin k\zeta \frac{1}{1 + k^2} e^{-\zeta} \right) \tag{54}$$

$$\times \left(k \cos \tau \cos k\zeta + \cos \tau \sin k\zeta + \sin \tau \cos k\zeta - k \sin \tau \sin k\zeta \right) dk \tag{55}$$

$$= -\beta \tilde{A} \int_{0}^{\infty} \frac{1}{k} \sin k\xi e^{-\xi_0 k} \left(\sin k\zeta \frac{1}{1 + k^2} e^{-\zeta} \right) \tag{56}$$

$$\times \left(k \cos \left(\tau + k\zeta \right) + \sin \left(\tau + k\zeta \right) \right) dk \tag{57}$$

$$= -\beta \tilde{A} \int_{0}^{\infty} \frac{1}{k} \sin k\xi e^{-\xi_0 k} \frac{1}{1 + k^2} \tag{58}$$

$$\times \left(\sin k\zeta e^{-\zeta} k \cos \left(\tau + k\zeta \right) + \sin k\zeta e^{-\zeta} \sin \left(\tau + k\zeta \right) \right) dk \tag{59}$$

Summing (42) and (59) gives

$$\tilde{\psi}(\xi, \zeta, \tau) = -\beta \tilde{A} \int_{0}^{\infty} \frac{\cos k\xi e^{-\xi_0 k}}{1 + k^2} \left(\sin(k\zeta + \tau) - e^{-\zeta} \sin \tau \right) dk \tag{60}$$

as required!

4 Calculating Potential Temperature

From equation (4) Rotunno (1983) we have

$$\frac{\partial b}{\partial t} + N^2 w = Q.$$

The non-dimensional form of this is

$$\frac{\partial \tilde{b}}{\partial \tau} h\omega^3 + N^2 \tilde{w} h\omega = \tilde{Q} h\omega^3$$
$$= \frac{\partial \tilde{b}}{\partial \tau} + \left(\frac{N}{\omega}\right)^2 \tilde{w} = \tilde{Q}.$$

Note that $\left(\frac{N}{\omega}\right)^2$ is essentially the Berger number with $H = L$. Thus can solve for $\tilde{b}$ using

$$\frac{\tilde{b}_{k+1} - \tilde{b}_k}{\Delta \tau} = \tilde{Q}_k - \left(\frac{N}{\omega}\right)^2 \tilde{w}_k$$
$$= \tilde{b}_{k+1} - \tilde{b}_k = \Delta \tau \left(\tilde{Q}_k - \left(\frac{N}{\omega}\right)^2 \tilde{w}_k\right).$$

This produces a linear system of τ_N linearly independent equations in τ_N unknowns. However, in this form the system is singular - so substitute the equation for $\tilde{b_1}$ for $\tilde{b_1} + \tilde{b_{\frac{\tau_n}{2}}} = 0$ to impose symmetry on the bouyancy. Note can use $\tilde{b_1} + \tilde{b}_{\lfloor \frac{\tau_n}{2} \rfloor}$! This works - we can solve for bouyancy even without initial conditions! From bouyancy can extract potential temperature!

Note that using this method appears to produce potential temperature perturbations that are two large, i.e. ± 40 K, or even larger! Compare this with the WRF simulation of Vincent & Lane (2016) producing perturbations of ± 4 K. Could it be that the WRF data are composites, and that the signal is being diluted? Note that increasing h decreases θ'. Error possibly due to definition of $\bar{\theta}$ in code?

5 Choosing $\tilde{A}$

Note that from equation (4) we have at the surface

$$\frac{\partial b}{\partial t} = \frac{g}{\theta_0} \frac{\partial \theta'}{\partial t} = Q,$$

as $w = 0$ at the surface. Thus letting θ_M and θ_m denote the land surface temperature maxima and minima respectively, and noting that $\left(\frac{\pi}{2} + \tan^{-1} \frac{x}{x_0}\right)$ maps onto $(0, \pi)$, we have

$$\frac{g}{\theta_0} \frac{\theta_M - \theta_m}{12 \cdot 60 \cdot 60} = \left(\sin \frac{\pi}{2} - \sin \frac{3\pi}{2}\right) A\pi = 2A\pi,$$

with $A\pi$ the supremum of H at the surface, and it taking 12 hours to go from maximum to minimum temperature. Multiplying both sides by $h^{-1}\omega^{-3}$ gives

$$\tilde{A} = \frac{g}{2\pi\theta_0} \frac{\theta_M - \theta_m}{12 \cdot 60 \cdot 60} h^{-1} \omega^{-3}.$$

6 Notes for Boat Seminar

Equations of motion linearised about a hydrostatic, stationary background atmosphere.

$$\text{Acceleration} = \frac{\text{Force}}{\text{Mass}} \qquad \text{(Following a blob of fluid)}$$

References

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