

Notes concerning “On the Linear Theory of the Land and Sea Breeze” (Rotunno 1983)

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1 Paper Misprints

1. Equation (37) should be

$$-\beta \text{Re} \left\{ \mathcal{F}^{-1} \left[\frac{1}{k} e^{-i \text{sgn}(k)k\zeta} \int_0^\zeta \mathcal{F} \left(\frac{\partial \tilde{Q}}{\partial \xi} \right) \sin k\zeta' d\zeta' + \frac{1}{k} \sin k\zeta \int_\zeta^\infty \mathcal{F} \left(\frac{\partial \tilde{Q}}{\partial \xi} \right) e^{-i \text{sgn}(k)k\zeta'} d\zeta' \right] \right\}.$$

2. \tilde{Q} should have time dependance $e^{-i(\tau - \frac{\pi}{2})} = ie^{-i\tau}$ not simply $\sin \tau$.
3. The expression $\tilde{b} = bh^{-1}\omega^{-3}$ in equation (13) should actually be $\tilde{b} = bh^{-1}\omega^{-2}$ to make the units come out right - assuming b has units m s^{-1} .

2 Tropical Case

We are attempting to derive the equation

$$\tilde{\psi}(\xi, \zeta, \tau) = -\beta \tilde{A} \int_0^\infty \frac{\cos k\xi e^{-\xi_0 k}}{1 + k^2} (\sin(k\zeta + \tau) - e^{-\zeta} \sin \tau) dk. \quad (1)$$

Note

$$\tilde{Q} = \beta \tilde{A} \left(\frac{\pi}{2} + \tan^{-1} \frac{\xi}{\xi_0} \right) e^{-\zeta} ie^{-i\tau} \quad (2)$$

$$\Rightarrow \frac{\partial \tilde{Q}}{\partial \xi} = \frac{1}{\xi^2 + \xi_0^2} \beta \tilde{A} \xi_0 e^{-\zeta} ie^{-i\tau} \quad (3)$$

$$\Rightarrow \mathcal{F} \left(\frac{\partial \tilde{Q}}{\partial \xi} \right) = \mathcal{F} \left[\frac{1}{\xi^2 + \xi_0^2} \right] \tilde{A} \xi_0 e^{-\zeta} ie^{-i\tau} \quad (4)$$

$$\Rightarrow \mathcal{F} \left(\frac{\partial \tilde{Q}}{\partial \xi} \right) = \frac{\pi}{\xi_0} e^{-\xi_0 |k|} \tilde{A} \xi_0 e^{-\zeta} ie^{-i\tau} = \pi e^{-\xi_0 |k|} \tilde{A} e^{-\zeta} ie^{-i\tau} \quad (5)$$

using the non-unitary, angular frequency form of the Fourier transform. This Fourier transform can be derived by considering the Fourier transform of $\frac{\pi}{\xi_0} e^{-\xi_0 |\xi|}$, applying the inverse Fourier transform, and changing signs.

Now, starting from the corrected form of equation 37 we have,

$$- \beta \mathcal{F}^{-1} \left[\frac{1}{k} e^{-i \operatorname{sgn}(k) k \zeta} \int_0^\zeta \mathcal{F} \left(\frac{\partial \tilde{Q}}{\partial \xi} \right) \sin k \zeta' d\zeta' + \frac{1}{k} \sin k \zeta \int_\zeta^\infty \mathcal{F} \left(\frac{\partial \tilde{Q}}{\partial \xi} \right) e^{-i \operatorname{sgn}(k) k \zeta'} d\zeta' \right] \quad (6)$$

$$= -\frac{\beta}{2} \tilde{A} i e^{-i\tau} \int_{-\infty}^0 e^{ik\xi} \left[\frac{1}{k} e^{ik\zeta} \int_0^\zeta e^{\xi_0 k} e^{-\zeta} \sin k \zeta' d\zeta' + \frac{1}{k} \sin k \zeta \int_\zeta^\infty e^{\xi_0 k} e^{-\zeta} e^{ik\zeta'} d\zeta' \right] dk \quad (7)$$

$$- \frac{\beta}{2} \tilde{A} i e^{-i\tau} \int_0^\infty e^{ik\xi} \left[\frac{1}{k} e^{-ik\zeta} \int_0^\zeta e^{-\xi_0 k} e^{-\zeta} \sin k \zeta' d\zeta' + \frac{1}{k} \sin k \zeta \int_\zeta^\infty e^{-\xi_0 k} e^{-\zeta} e^{-ik\zeta'} d\zeta' \right] dk \quad (8)$$

$$= -\frac{\beta}{2} \tilde{A} i e^{-i\tau} \int_0^\infty e^{-ik\xi} \left[\frac{1}{k} e^{-ik\zeta} \int_0^\zeta e^{-\xi_0 k} e^{-\zeta} \sin k \zeta' d\zeta' + \frac{1}{k} \sin k \zeta \int_\zeta^\infty e^{-\xi_0 k} e^{-\zeta} e^{-ik\zeta'} d\zeta' \right] dk \quad (9)$$

$$- \frac{\beta}{2} \tilde{A} i e^{-i\tau} \int_0^\infty e^{ik\xi} \left[\frac{1}{k} e^{-ik\zeta} \int_0^\zeta e^{-\xi_0 k} e^{-\zeta} \sin k \zeta' d\zeta' + \frac{1}{k} \sin k \zeta \int_\zeta^\infty e^{-\xi_0 k} e^{-\zeta} e^{-ik\zeta'} d\zeta' \right] dk \quad (10)$$

$$= -\beta i e^{-i\tau} \tilde{A} \int_0^\infty \cos(k\xi) e^{-\xi_0 k} \left[\frac{1}{k} e^{-ik\zeta'} \int_0^\zeta e^{-\zeta'} \sin k \zeta' d\zeta' + \frac{1}{k} \sin k \zeta \int_\zeta^\infty e^{-\zeta'} e^{-ik\zeta'} d\zeta' \right] dk \quad (11)$$

$$= -\beta \int_0^\infty \cos(k\xi) \left[e^{-ik\zeta'} \int_0^\zeta \left(\frac{\partial \tilde{Q}}{\partial \xi} \right) \sin k \zeta' d\zeta' + \sin k \zeta \int_\zeta^\infty \left(\frac{\partial \tilde{Q}}{\partial \xi} \right) e^{-ik\zeta'} d\zeta' \right] \frac{dk}{k}. \quad (12)$$

Note (12) matches what's in Rotunno's notes. From (12) the derivation is easy.

My attempt at deriving equation (1) - equation (38) in Rotunno's paper - is very messy. It didn't quite work originally as I was using the incorrect version of (37). Now having the correct version, let's see if it works. We have from (6),

$$- \beta \mathcal{F}^{-1} \left[\frac{1}{k} e^{-i \operatorname{sgn}(k) k \zeta} \int_0^\zeta \left(\pi e^{-\xi_0 |k|} \tilde{A} e^{-\zeta} i e^{-i\tau} \right) \sin k \zeta' d\zeta' \right] \quad (13)$$

$$+ \frac{1}{k} \sin k \zeta \int_\zeta^\infty \left(\pi e^{-\xi_0 |k|} \tilde{A} e^{-\zeta} i e^{-i\tau} \right) e^{-i \operatorname{sgn}(k) k \zeta'} d\zeta' \right] \quad (14)$$

$$= -\beta \tilde{A} \pi i e^{-i\tau} \mathcal{F}^{-1} \left[\frac{1}{k} e^{-i \operatorname{sgn}(k) k \zeta} e^{-\xi_0 |k|} \int_0^\zeta e^{-\zeta'} \sin k \zeta' d\zeta' \right] \quad (15)$$

$$+ \frac{1}{k} \sin k \zeta e^{-\xi_0 |k|} \int_\zeta^\infty e^{-\zeta'} e^{-i \operatorname{sgn}(k) k \zeta'} d\zeta' \right] \quad (16)$$

$$= -\beta \tilde{A} \pi i e^{-i\tau} \mathcal{F}^{-1} \left\{ \frac{1}{k} e^{-i \operatorname{sgn}(k) k \zeta} e^{-\xi_0 |k|} \left[\frac{-e^{-\zeta'}}{k^2 + 1} (k \cos k \zeta' + \sin k \zeta') \right]_0^\zeta \right\} \quad (17)$$

$$- \beta \tilde{A} \pi i e^{-i\tau} \mathcal{F}^{-1} \left\{ \frac{1}{k} \sin k \zeta e^{-\xi_0 |k|} \left[\frac{1}{-i \operatorname{sgn}(k) k - 1} e^{(-i \operatorname{sgn}(k) k - 1) \zeta'} \right]_\zeta^\infty \right\}, \quad (18)$$

$$= -\beta \tilde{A} \pi i e^{-i\tau} \mathcal{F}^{-1} \left\{ \frac{1}{k} e^{-i \operatorname{sgn}(k) k \zeta} e^{-\xi_0 |k|} \left(\frac{-e^{-\zeta}}{k^2 + 1} (k \cos k \zeta + \sin k \zeta) + \frac{k}{k^2 + 1} \right) \right\} \quad (19)$$

$$- \beta \tilde{A} \pi i e^{-i\tau} \mathcal{F}^{-1} \left\{ \frac{1}{k} \sin k \zeta e^{-\xi_0 |k|} \left[\frac{1}{-i \operatorname{sgn}(k) k - 1} e^{(-i \operatorname{sgn}(k) k - 1) \zeta'} \right]_\zeta^\infty \right\}, \quad (20)$$

where the first integral (17) can be calculated by performing integration by parts. Consider the first

term of the sum, i.e. (19). We have

$$- \beta \tilde{A} \pi i e^{-i\tau} \mathcal{F}^{-1} \left\{ \frac{1}{k} e^{-i \operatorname{sgn}(k) k \zeta} e^{-\xi_0 |k|} \left(\frac{-e^{-\zeta}}{k^2 + 1} (k \cos k \zeta + \sin k \zeta) + \frac{k}{k^2 + 1} \right) \right\} \quad (21)$$

$$= -\beta \tilde{A} i e^{-i\tau} \frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{k} e^{ik\xi} e^{-i \operatorname{sgn}(k) k \zeta} e^{-\xi_0 |k|} \left(\frac{-e^{-\zeta}}{k^2 + 1} (k \cos k \zeta + \sin k \zeta) + \frac{k}{k^2 + 1} \right) dk \quad (22)$$

$$= -\beta \tilde{A} i e^{-i\tau} \frac{1}{2} \int_{-\infty}^0 \frac{1}{k} e^{ik\xi} e^{ik\zeta} e^{\xi_0 k} \left(\frac{-e^{-\zeta}}{k^2 + 1} (k \cos k \zeta + \sin k \zeta) + \frac{k}{k^2 + 1} \right) dk \quad (23)$$

$$- \beta \tilde{A} i e^{-i\tau} \frac{1}{2} \int_0^{\infty} \frac{1}{k} e^{ik\xi} e^{-ik\zeta} e^{-\xi_0 k} \left(\frac{-e^{-\zeta}}{k^2 + 1} (k \cos k \zeta + \sin k \zeta) + \frac{k}{k^2 + 1} \right) dk \quad (24)$$

$$= -\beta \tilde{A} i e^{-i\tau} \frac{1}{2} \int_0^{\infty} \frac{1}{-k} e^{-ik\xi} e^{-ik\zeta} e^{-\xi_0 k} \left(\frac{-e^{-\zeta}}{k^2 + 1} (-k \cos k \zeta - \sin k \zeta) - \frac{k}{k^2 + 1} \right) dk \quad (25)$$

$$- \beta \tilde{A} i e^{-i\tau} \frac{1}{2} \int_0^{\infty} \frac{1}{k} e^{ik\xi} e^{-ik\zeta} e^{-\xi_0 k} \left(\frac{-e^{-\zeta}}{k^2 + 1} (k \cos k \zeta + \sin k \zeta) + \frac{k}{k^2 + 1} \right) dk \quad (26)$$

$$= -\beta \tilde{A} i e^{-i\tau} \frac{1}{2} \int_0^{\infty} \frac{1}{k} (e^{ik\xi} + e^{-ik\xi}) e^{-ik\zeta} e^{-\xi_0 k} \left(\frac{-e^{-\zeta}}{k^2 + 1} (k \cos k \zeta + \sin k \zeta) + \frac{k}{k^2 + 1} \right) dk \quad (27)$$

$$= -\beta \tilde{A} (i \cos \tau + \sin \tau) \int_0^{\infty} \frac{1}{k} \cos k \xi e^{-ik\zeta} e^{-\xi_0 k} \frac{1}{k^2 + 1} (-e^{-\zeta} (k \cos k \zeta + \sin k \zeta) + k) dk \quad (28)$$

$$= -\beta \tilde{A} (i \cos \tau + \sin \tau) \int_0^{\infty} \frac{1}{k} \cos k \xi (\cos k \zeta - i \sin k \zeta) \quad (29)$$

$$\times e^{-\xi_0 k} \frac{1}{k^2 + 1} (-e^{-\zeta} (k \cos k \zeta + \sin k \zeta) + k) dk \quad (30)$$

$$= -\beta \tilde{A} \int_0^{\infty} \frac{1}{k} \sin k \xi (i \cos \tau + \sin \tau) (\cos k \zeta - i \sin k \zeta) \quad (31)$$

$$\times e^{-\xi_0 k} \frac{1}{k^2 + 1} (-e^{-\zeta} (k \cos k \zeta + \sin k \zeta) + k) dk \quad (32)$$

$$= -\beta \tilde{A} \int_0^{\infty} \frac{1}{k} \sin k \xi (i \cos \tau \cos k \zeta + \cos \tau \sin k \zeta + \sin \tau \cos k \zeta - i \sin \tau \sin k \zeta) \quad (33)$$

$$\times e^{-\xi_0 k} \frac{1}{k^2 + 1} (-e^{-\zeta} (k \cos k \zeta + \sin k \zeta) + k) dk \quad (34)$$

The real part of this is then

$$- \beta \tilde{A} \int_0^{\infty} \frac{1}{k} \sin k \xi (\cos \tau \cos k \zeta + \sin \tau \sin k \zeta) e^{-\xi_0 k} \frac{1}{k^2 + 1} (-e^{-\zeta} (k \cos k \zeta + \sin k \zeta) + k) dk \quad (35)$$

$$= -\beta \tilde{A} \int_0^{\infty} \frac{1}{k} \sin k \xi \sin (\tau + k \zeta) e^{-\xi_0 k} \frac{1}{k^2 + 1} (-e^{-\zeta} (k \cos k \zeta + \sin k \zeta) + k) dk \quad (36)$$

$$= -\beta \tilde{A} \int_0^{\infty} \frac{1}{k} \sin k \xi \frac{1}{k^2 + 1} e^{-\xi_0 k} \quad (37)$$

$$\times (-\sin (\tau + k \zeta) e^{-\zeta} k \cos k \zeta - \sin (\tau + k \zeta) e^{-\zeta} \sin k \zeta + k \sin (\tau + k \zeta)) dk. \quad (38)$$

Consider now the second term. We have

$$- \beta \tilde{A} \pi i e^{-i\tau} \mathcal{F}^{-1} \left\{ \frac{1}{k} \sin k\zeta e^{-\xi_0|k|} \left[\frac{1}{-i \operatorname{sgn}(k)k - 1} e^{(-i \operatorname{sgn}(k)k - 1)\zeta'} \right]_{\zeta}^{\infty} \right\} \quad (39)$$

$$= -\beta \tilde{A} i e^{-i\tau} \frac{1}{2} \int_{-\infty}^0 \frac{1}{k} e^{ik\xi} \left(\sin k\zeta e^{\xi_0 k} \left[\frac{1}{ik - 1} e^{(ik-1)\zeta'} \right]_{\zeta}^{\infty} \right) dk \quad (40)$$

$$- \beta \tilde{A} i e^{-i\tau} \frac{1}{2} \int_0^{\infty} \frac{1}{k} e^{ik\xi} \left(\sin k\zeta e^{-\xi_0 k} \left[\frac{1}{-ik - 1} e^{(-ik-1)\zeta'} \right]_{\zeta}^{\infty} \right) dk \quad (41)$$

$$= -\beta \tilde{A} i e^{-i\tau} \frac{1}{2} \int_{-\infty}^0 \frac{1}{k} e^{ik\xi} \left(\sin k\zeta e^{\xi_0 k} \frac{1}{1 - ik} e^{(ik-1)\zeta} \right) dk \quad (42)$$

$$- \beta \tilde{A} i e^{-i\tau} \frac{1}{2} \int_0^{\infty} \frac{1}{k} e^{ik\xi} \left(\sin k\zeta e^{-\xi_0 k} \frac{1}{1 + ik} e^{(-ik-1)\zeta} \right) dk \quad (43)$$

$$= -\beta \tilde{A} i e^{-i\tau} \frac{1}{2} \int_0^{\infty} \frac{1}{-k} e^{-ik\xi} \left(-\sin k\zeta e^{-\xi_0 k} \frac{1}{1 + ik} e^{(-ik-1)\zeta} \right) dk \quad (44)$$

$$- \beta \tilde{A} i e^{-i\tau} \frac{1}{2} \int_0^{\infty} \frac{1}{k} e^{ik\xi} \left(\sin k\zeta e^{-\xi_0 k} \frac{1}{1 + ik} e^{(-ik-1)\zeta} \right) dk \quad (45)$$

$$= -\beta \tilde{A} i e^{-i\tau} \int_0^{\infty} \frac{1}{k} \cos k\xi e^{-\xi_0 k} e^{-ik\zeta} \left(\sin k\zeta \frac{1 - ik}{1 + k^2} e^{-\zeta} \right) dk \quad (46)$$

$$= -\beta \tilde{A} i e^{-i\tau} \int_0^{\infty} \frac{1}{k} \cos k\xi e^{-\xi_0 k} (\cos k\zeta - i \sin k\zeta) (1 - ik) \left(\sin k\zeta \frac{1}{1 + k^2} e^{-\zeta} \right) dk \quad (47)$$

$$= -\beta \tilde{A} \int_0^{\infty} \frac{1}{k} \cos k\xi e^{-\xi_0 k} \left(\sin k\zeta \frac{1}{1 + k^2} e^{-\zeta} \right) \quad (48)$$

$$\times (i \cos \tau + \sin \tau) (\cos k\zeta - ik \cos k\zeta - i \sin k\zeta - k \sin k\zeta) dk. \quad (49)$$

The real part of this is then

$$- \beta \tilde{A} \int_0^{\infty} \frac{1}{k} \sin k\xi e^{-\xi_0 k} \left(\sin k\zeta \frac{1}{1 + k^2} e^{-\zeta} \right) \quad (50)$$

$$\times (k \cos \tau \cos k\zeta + \cos \tau \sin k\zeta + \sin \tau \cos k\zeta - k \sin \tau \sin k\zeta) dk \quad (51)$$

$$= -\beta \tilde{A} \int_0^{\infty} \frac{1}{k} \sin k\xi e^{-\xi_0 k} \left(\sin k\zeta \frac{1}{1 + k^2} e^{-\zeta} \right) \quad (52)$$

$$\times (k \cos(\tau + k\zeta) + \sin(\tau + k\zeta)) dk \quad (53)$$

$$= -\beta \tilde{A} \int_0^{\infty} \frac{1}{k} \sin k\xi e^{-\xi_0 k} \frac{1}{1 + k^2} \quad (54)$$

$$\times (\sin k\zeta e^{-\zeta} k \cos(\tau + k\zeta) + \sin k\zeta e^{-\zeta} \sin(\tau + k\zeta)) dk \quad (55)$$

Summing (38) and (55) gives

$$\tilde{\psi}(\xi, \zeta, \tau) = -\beta \tilde{A} \int_0^{\infty} \frac{\cos k\xi e^{-\xi_0 k}}{1 + k^2} (\sin(k\zeta + \tau) - e^{-\zeta} \sin \tau) dk \quad (56)$$

as required!

References

Rotunno, R. (1983), ‘On the linear theory of the land and sea breeze’, *Journal of the Atmospheric Sciences* **40**(8), 1999–2009.