Notes concerning "On the Linear Theory of the Land and Sea Breeze" (Rotunno 1983)

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November 22, 2019

1 Paper Misprints

1. Equation (37) should be

$$-\beta\mathrm{Re}\left\{\mathcal{F}^{-1}\left[\frac{1}{k}e^{-i\,\mathrm{sgn}(k)k\zeta}\int_0^{\zeta}\mathcal{F}\left(\frac{\partial\tilde{Q}}{\partial\xi}\right)\sin k\zeta' d\zeta' + \frac{1}{k}\sin k\zeta\int_{\zeta}^{\infty}\mathcal{F}\left(\frac{\partial\tilde{Q}}{\partial\xi}\right)e^{-i\,\mathrm{sgn}(k)k\zeta'}d\zeta'\right]\right\}.$$

2. $\tilde{Q}$ should have time dependance $e^{-i\left(\tau-\frac{\pi}{2}\right)} = ie^{-i\tau}$ not simply $\sin\tau$.

3. The expression $\tilde{b} = bh^{-1}\omega^{-3}$ in equation (13) should actually be $\tilde{b} = bh^{-1}\omega^{-2}$ to make the units come out right - assuming b has units m s^{-2}.

4. Note formula (13) is incorrect for $\tilde{w}$. Note that

$$\tilde{w} = -\frac{\partial\tilde{\psi}}{\partial\xi} = -\frac{\partial\psi}{\partial\xi}h^{-2}\omega^{-1} = -\frac{\partial\psi}{\partial x}\frac{\partial x}{\partial\xi}h^{-2}\omega^{-1} = wN(f^2-\omega^2)^{-\frac{1}{2}}h^{-1}\omega^{-1} \tag{1}$$

   and so

$$w = \tilde{w}h\omega(f^2-\omega^2)^{\frac{1}{2}}N^{-1} \tag{2}$$

5. Note my (and Rotunno's) expressions for Fourier transforms are missing a factor of $\frac{1}{\pi}$, which explains the additional missing factor of π when converting from Q_0 in **?** to $\tilde{A}$! Now fixed in code. Rotunno code results now match my Qian code results.

2 Mid-Latitude Case

We begin with equation (14)

$$\frac{\partial^2\tilde{\psi}}{\partial\xi^2} + \frac{\partial^2\tilde{\psi}}{\partial\zeta^2} = -\beta\frac{\partial\tilde{Q}}{\partial\xi}.$$

Following Duffy (2001), consider the "free-space" Green's function defined by

$$\frac{\partial^2 g}{\partial\xi^2} + \frac{\partial^2 g}{\partial\zeta^2} = \delta(\xi-\xi')\delta(\zeta-\zeta')$$

where ξ, ζ can vary over all of $\mathbb{R}$. Integrating over a circle C centred at ξ', ζ' with radius a, where $r = \sqrt{(\xi - \xi')^2 + (\zeta - \zeta')^2}$ is the distance from (ξ', ζ')

$$\Rightarrow \iint_C \nabla \cdot \nabla g dV = 1$$
$$\Rightarrow \int_{\partial C} \frac{\partial g}{\partial r} = 1.$$

If there are no boundaries, then the response of g to singular forcing at (ξ', ζ') is unaffected by rotation, and is therefore constant on ∂C. Thus

$$\Rightarrow 2\pi a \left. \frac{\partial g}{\partial r} \right|_a = 1.$$

Because a was arbitrary we have therefore have

$$\Rightarrow \frac{\partial g}{\partial r} = \frac{1}{2\pi r}$$
$$\Rightarrow g = \frac{\ln(r)}{2\pi} + c = \frac{1}{4\pi} \ln\left((\xi - \xi')^2 + (\zeta - \zeta')^2\right) + c.$$

Can assume without loss of generality that $c = 0$, because $c \neq 0$ simply adds a constant to $\tilde{\psi}$, and we only care about the ξ and ζ derivatives of $\tilde{\psi}$. Furthermore, because g represents a fundamental solution for $\tilde{\psi}$ we have $-\frac{\partial g}{\partial \xi} = \tilde{w} = 0$. Using the method of images, we can obtain a solution satisfying this condition from the free space solution by noting that when $\zeta = 0$,

$$\frac{1}{4\pi} \frac{1}{(\xi - \xi')^2 + (\zeta - \zeta')^2} 2\xi - \frac{1}{4\pi} \frac{1}{(\xi - \xi')^2 + (-\zeta - \zeta')^2} 2\xi = 0.$$

This suggests taking

$$\begin{aligned} g &= \frac{1}{4\pi} \ln\left((\xi - \xi')^2 + (\zeta - \zeta')^2\right) - \frac{1}{4\pi} \ln\left((\xi - \xi')^2 + (-\zeta - \zeta')^2\right) \\ &= \frac{1}{4\pi} \ln\left(\frac{(\xi - \xi')^2 + (\zeta - \zeta')^2}{(\xi - \xi')^2 + (\zeta + \zeta')^2}\right). \end{aligned}$$

The properties of Green's function's then give the solution given by equation (20) of Rotunno (1983). Note that when calculating the convolution we integrate over the actual domain of $\tilde{\psi}$, i.e. $-\infty < \xi' < \infty$ and $\zeta' \geq 0$, not over free-space, i.e. $\mathbb{R}^2$. This is because we are only putting real sources in the actual domain.

3 Tropical Case

3.1 Deriving Equation (37)

Equation (36) gives

$$\frac{\partial^2 \tilde{\psi}}{\partial \xi^2} - \frac{\partial^2 \tilde{\psi}}{\partial \zeta^2} = -\beta \frac{\partial^2 \tilde{Q}}{\partial \xi^2}.$$

Take Fourier transform to get

$$
\begin{aligned}
& -k^2\widehat{\widetilde{\psi}} - \frac{\partial^2\widehat{\widetilde{\psi}}}{\partial\zeta^2} = -\beta\frac{\partial^2\widehat{\widetilde{Q}}}{\partial\xi^2} \\
&= k^2\widehat{\widetilde{\psi}} + \frac{\partial^2\widehat{\widetilde{\psi}}}{\partial\zeta^2} = \beta\frac{\partial^2\widehat{\widetilde{Q}}}{\partial\xi^2}.
\end{aligned}
$$

Boundary condition becomes $(ik)\,\widehat{\widetilde{\psi}}(k,0) = 0$, and so $\widehat{\widetilde{\psi}}(k,0) = 0$ (using the Fourier transform rule for derivatives.) Solve for Green's Function

$$k^2G + G_{\zeta\zeta} = \delta\left(\zeta - \zeta'\right).$$

General solution setting RHS to zero is $G = B_1e^{ik\zeta} + B_2e^{-ik\zeta}$. For $\zeta < \zeta'$ the boundary condition gives

$$
\begin{aligned}
0 = B_1 + B_2 &\Rightarrow B_2 = -B_1 \\
&\Rightarrow G = B_1 2i\sin(k\zeta) = A\sin(k\zeta).
\end{aligned}
$$

Now consider $\zeta > \zeta'$. For $k > 0$ we have

$$
\begin{aligned}
Ge^{-i\left(\tau - \frac{\pi}{2}\right)} &= B_1e^{ik\zeta - i\tau + i\frac{\pi}{2}} + B_2e^{-ik\zeta - i\tau + i\frac{\pi}{2}} \\
&= B_1e^{i\left(k\zeta - \tau + \frac{\pi}{2}\right)} + B_2e^{-i\left(k\zeta + \tau - \frac{\pi}{2}\right)}.
\end{aligned}
$$

Recall for gravity waves, energy propagates with the group velocity in the *opposite* direction to the phase velocity. As $\zeta > \zeta'$ we require positive group velocity, and therefore negative phase velocity. Thus $B_1 = 0$ and $G = B_2e^{-ik\zeta}$. Similarly for $k < 0$ we have $G = B_1e^{ik\zeta}$. Thus $G = Be^{-i\,\mathrm{sgn}(k)k\zeta}$.

Now, continuity requires that

$$
\begin{aligned}
\lim_{\zeta\to\zeta'^+} G &= \lim_{\zeta\to\zeta'^-} G \\
\lim_{\zeta\to\zeta'^+} G_\zeta - \lim_{\zeta\to\zeta'^-} G_\zeta &= 1.
\end{aligned}
$$

Thus

$$
\begin{aligned}
& Be^{-i\,\mathrm{sgn}(k)k\zeta'} = A\sin(k\zeta') \\
& -i\,\mathrm{sgn}(k)kBe^{-i\,\mathrm{sgn}(k)k\zeta'} - Ak\cos(k\zeta') = 1 \\
\Rightarrow\ & i\,\mathrm{sgn}(k)kA\sin(k\zeta') + Ak\cos(k\zeta') = -1.
\end{aligned}
$$

Now, $k > 0$

$$
\begin{aligned}
&\Rightarrow ikA\sin(k\zeta') + Ak\cos(k\zeta') = Ake^{ik\zeta'} = -1 \\
&\Rightarrow A = -\frac{1}{k}e^{-ik\zeta'}.
\end{aligned}
$$

Also, $k < 0$

$$
\begin{aligned}
&\Rightarrow -ikA\sin(k\zeta') + Ak\cos(k\zeta') = Ake^{-ik\zeta'} = -1 \\
&\Rightarrow A = -\frac{1}{k}e^{ik\zeta'}.
\end{aligned}
$$

Thus in both cases $A = -\frac{1}{k}e^{-\,\mathrm{sgn}(k)k\zeta'}$. Thus $B = -\sin(k\zeta')$, and so

$$
G = \begin{cases}
-\frac{1}{k}e^{-i\,\mathrm{sgn}(k)k\zeta'}\sin(k\zeta), & \zeta < \zeta', \\
-\frac{1}{k}\sin(k\zeta')e^{-i\,\mathrm{sgn}(k)k\zeta}, & \zeta > \zeta'.
\end{cases}
$$

3.2 Working Through Fourier Transform

We are attempting to derive the equation

$$\tilde{\psi}(\xi,\zeta,\tau) = -\beta\tilde{A}\int_0^\infty \frac{\cos k\xi e^{-\xi_0 k}}{1+k^2}\left(\sin(k\zeta+\tau) - e^{-\zeta}\sin\tau\right)dk. \tag{3}$$

Note

$$\tilde{Q} = \beta\tilde{A}\left(\frac{\pi}{2} + \tan^{-1}\frac{\xi}{\xi_0}\right)e^{-\zeta}ie^{-i\tau} \tag{4}$$

$$\Rightarrow \frac{\partial\tilde{Q}}{\partial\xi} = \frac{1}{\xi^2+\xi_0^2}\beta\tilde{A}\xi_0 e^{-\zeta}ie^{-i\tau} \tag{5}$$

$$\Rightarrow \mathcal{F}\left(\frac{\partial\tilde{Q}}{\partial\xi}\right) = \mathcal{F}\left[\frac{1}{\xi^2+\xi_0^2}\right]\tilde{A}\xi_0 e^{-\zeta}ie^{-i\tau} \tag{6}$$

$$\Rightarrow \mathcal{F}\left(\frac{\partial\tilde{Q}}{\partial\xi}\right) = \frac{\pi}{\xi_0}e^{-\xi_0|k|}\tilde{A}\xi_0 e^{-\zeta}e^{-i\tau} = \pi e^{-\xi_0|k|}\tilde{A}e^{-\zeta}ie^{-i\tau} \tag{7}$$

using the non-unitary, angular frequency form of the Fourier transform. This Fourier transform can be derived by considering the Fourier transform of $\frac{\pi}{\xi_0}e^{-\xi_0|\xi|}$, applying the inverse Fourier transform, and changing signs.

Now, starting from the corrected form of equation 37 we have,

$$-\beta\mathcal{F}^{-1}\left[\frac{1}{k}e^{-i\,\mathrm{sgn}(k)k\zeta}\int_0^\zeta \mathcal{F}\left(\frac{\partial\tilde{Q}}{\partial\xi}\right)\sin k\zeta' d\zeta' + \frac{1}{k}\sin k\zeta\int_\zeta^\infty \mathcal{F}\left(\frac{\partial\tilde{Q}}{\partial\xi}\right)e^{-i\,\mathrm{sgn}(k)k\zeta'}d\zeta'\right] \tag{8}$$

$$= -\frac{\beta}{2}\tilde{A}ie^{-i\tau}\int_{-\infty}^0 e^{ik\xi}\left[\frac{1}{k}e^{ik\zeta}\int_0^\zeta e^{\xi_0 k}e^{-\zeta}\sin k\zeta' d\zeta' + \frac{1}{k}\sin k\zeta\int_\zeta^\infty e^{\xi_0 k}e^{-\zeta}e^{ik\zeta'}d\zeta'\right]dk \tag{9}$$

$$-\frac{\beta}{2}\tilde{A}ie^{-i\tau}\int_0^\infty e^{ik\xi}\left[\frac{1}{k}e^{-ik\zeta}\int_0^\zeta e^{-\xi_0 k}e^{-\zeta}\sin k\zeta' d\zeta' + \frac{1}{k}\sin k\zeta\int_\zeta^\infty e^{-\xi_0 k}e^{-\zeta}e^{-ik\zeta'}d\zeta'\right]dk \tag{10}$$

$$= -\frac{\beta}{2}\tilde{A}ie^{-i\tau}\int_0^\infty e^{-ik\xi}\left[\frac{1}{k}e^{-ik\zeta}\int_0^\zeta e^{-\xi_0 k}e^{-\zeta}\sin k\zeta' d\zeta' + \frac{1}{k}\sin k\zeta\int_\zeta^\infty e^{-\xi_0 k}e^{-\zeta}e^{-ik\zeta'}d\zeta'\right]dk \tag{11}$$

$$-\frac{\beta}{2}\tilde{A}ie^{-i\tau}\int_0^\infty e^{ik\xi}\left[\frac{1}{k}e^{-ik\zeta}\int_0^\zeta e^{-\xi_0 k}e^{-\zeta}\sin k\zeta' d\zeta' + \frac{1}{k}\sin k\zeta\int_\zeta^\infty e^{-\xi_0 k}e^{-\zeta}e^{-ik\zeta'}d\zeta'\right]dk \tag{12}$$

$$= -\beta ie^{-i\tau}\tilde{A}\int_0^\infty \cos(k\xi)e^{-\xi_0 k}\left[\frac{1}{k}e^{-ik\zeta}\int_0^\zeta e^{-\zeta'}\sin k\zeta' d\zeta' + \frac{1}{k}\sin k\zeta\int_\zeta^\infty e^{-\zeta'}e^{-ik\zeta'}d\zeta'\right]dk \tag{13}$$

$$= -\beta\int_0^\infty \cos(k\xi)\left[e^{-ik\zeta}\int_0^\zeta\left(\frac{\partial\tilde{Q}}{\partial\xi}\right)\sin k\zeta' d\zeta' + \sin k\zeta\int_\zeta^\infty\left(\frac{\partial\tilde{Q}}{\partial\xi}\right)e^{-ik\zeta'}d\zeta'\right]\frac{dk}{k}. \tag{14}$$

Note (14) matches what's in Rotunno's notes. From (13) we have

$$-\beta ie^{-i\tau}\tilde{A}\int_0^\infty \cos(k\xi)e^{-\xi_0 k}\left[\frac{1}{k}e^{-ik\zeta}\int_0^\zeta e^{-\zeta'}\sin k\zeta' d\zeta' + \frac{1}{k}\sin k\zeta\int_\zeta^\infty e^{-\zeta'}e^{-ik\zeta'}d\zeta'\right]dk$$

$$= -\beta\tilde{A}\int_0^\infty \cos(k\xi)e^{-\xi_0 k}\left[\frac{1}{k}ie^{-i\tau}e^{-ik\zeta}\left(-\frac{e^{-\zeta}(\sin(k\zeta)+k\cos(k\zeta))}{k^2+1} + \frac{k}{k^2+1}\right)\right.$$
$$\left. +ie^{-i\tau}\frac{1}{k}\sin k\zeta\left(\frac{1-ik}{k^2+1}e^{(-ik-1)\zeta}\right)\right]dk$$

and the real part of this is therefore

$$= -\beta\tilde{A}\int_0^\infty \cos(k\xi)e^{-\xi_0 k}\frac{1}{k^2+1}\left[\sin(\tau+k\zeta) - e^{-\zeta}\sin(\tau)\right].$$

My attempt at deriving equation (3) - equation (38) in Rotunno's paper - is very messy. It didn't quite work originally as I was using the incorrect version of (37). Now having the correct version, let's see if it works. We have from (8),

$$-\beta\mathcal{F}^{-1}\left[\frac{1}{k}e^{-i\,\mathrm{sgn}(k)k\zeta}\int_0^\zeta\left(\pi e^{-\xi_0|k|}\tilde{A}e^{-\zeta}ie^{-i\tau}\right)\sin k\zeta' d\zeta'\right. \tag{15}$$

$$\left.+\frac{1}{k}\sin k\zeta\int_\zeta^\infty\left(\pi e^{-\xi_0|k|}\tilde{A}e^{-\zeta}ie^{-i\tau}\right)e^{-i\,\mathrm{sgn}(k)k\zeta'}d\zeta'\right] \tag{16}$$

$$= -\beta\tilde{A}\pi ie^{-i\tau}\mathcal{F}^{-1}\left[\frac{1}{k}e^{-i\,\mathrm{sgn}(k)k\zeta}e^{-\xi_0|k|}\int_0^\zeta e^{-\zeta'}\sin k\zeta' d\zeta'\right. \tag{17}$$

$$\left.+\frac{1}{k}\sin k\zeta e^{-\xi_0|k|}\int_\zeta^\infty e^{-\zeta'}e^{-i\,\mathrm{sgn}(k)k\zeta'}d\zeta'\right] \tag{18}$$

$$= -\beta\tilde{A}\pi ie^{-i\tau}\mathcal{F}^{-1}\left\{\frac{1}{k}e^{-i\,\mathrm{sgn}(k)k\zeta}e^{-\xi_0|k|}\left[\frac{-e^{-\zeta'}}{k^2+1}\left(k\cos k\zeta'+\sin k\zeta'\right)\right]_0^\zeta\right\} \tag{19}$$

$$-\beta\tilde{A}\pi ie^{-i\tau}\mathcal{F}^{-1}\left\{\frac{1}{k}\sin k\zeta e^{-\xi_0|k|}\left[\frac{1}{-i\,\mathrm{sgn}(k)k-1}e^{(-i\,\mathrm{sgn}(k)k-1)\zeta'}\right]_\zeta^\infty\right\}, \tag{20}$$

$$= -\beta\tilde{A}\pi ie^{-i\tau}\mathcal{F}^{-1}\left\{\frac{1}{k}e^{-i\,\mathrm{sgn}(k)k\zeta}e^{-\xi_0|k|}\left(\frac{-e^{-\zeta}}{k^2+1}\left(k\cos k\zeta+\sin k\zeta\right)+\frac{k}{k^2+1}\right)\right\} \tag{21}$$

$$-\beta\tilde{A}\pi ie^{-i\tau}\mathcal{F}^{-1}\left\{\frac{1}{k}\sin k\zeta e^{-\xi_0|k|}\left[\frac{1}{-i\,\mathrm{sgn}(k)k-1}e^{(-i\,\mathrm{sgn}(k)k-1)\zeta'}\right]_\zeta^\infty\right\}, \tag{22}$$

where the first integral (19) can be calculated by performing integration by parts. Consider the first term of the sum, i.e. (21). We have

$$
\begin{aligned}
& -\beta\tilde{A}\pi i e^{-i\tau}\mathcal{F}^{-1}\left\{\frac{1}{k}e^{-i\,\mathrm{sgn}(k)k\zeta}e^{-\xi_0|k|}\left(\frac{-e^{-\zeta}}{k^2+1}\left(k\cos k\zeta+\sin k\zeta\right)+\frac{k}{k^2+1}\right)\right\} && (23)\\
&= -\beta\tilde{A}ie^{-i\tau}\frac{1}{2}\int_{-\infty}^{\infty}\frac{1}{k}e^{ik\xi}e^{-i\,\mathrm{sgn}(k)k\zeta}e^{-\xi_0|k|}\left(\frac{-e^{-\zeta}}{k^2+1}\left(k\cos k\zeta+\sin k\zeta\right)+\frac{k}{k^2+1}\right)dk && (24)\\
&= -\beta\tilde{A}ie^{-i\tau}\frac{1}{2}\int_{-\infty}^{0}\frac{1}{k}e^{ik\xi}e^{ik\zeta}e^{\xi_0 k}\left(\frac{-e^{-\zeta}}{k^2+1}\left(k\cos k\zeta+\sin k\zeta\right)+\frac{k}{k^2+1}\right)dk && (25)\\
& -\beta\tilde{A}ie^{-i\tau}\frac{1}{2}\int_{0}^{\infty}\frac{1}{k}e^{ik\xi}e^{-ik\zeta}e^{-\xi_0 k}\left(\frac{-e^{-\zeta}}{k^2+1}\left(k\cos k\zeta+\sin k\zeta\right)+\frac{k}{k^2+1}\right)dk && (26)\\
&= -\beta\tilde{A}ie^{-i\tau}\frac{1}{2}\int_{0}^{\infty}\frac{1}{-k}e^{-ik\xi}e^{-ik\zeta}e^{-\xi_0 k}\left(\frac{-e^{-\zeta}}{k^2+1}\left(-k\cos k\zeta-\sin k\zeta\right)-\frac{k}{k^2+1}\right)dk && (27)\\
& -\beta\tilde{A}ie^{-i\tau}\frac{1}{2}\int_{0}^{\infty}\frac{1}{k}e^{ik\xi}e^{-ik\zeta}e^{-\xi_0 k}\left(\frac{-e^{-\zeta}}{k^2+1}\left(k\cos k\zeta+\sin k\zeta\right)+\frac{k}{k^2+1}\right)dk && (28)\\
&= -\beta\tilde{A}ie^{-i\tau}\frac{1}{2}\int_{0}^{\infty}\frac{1}{k}\left(e^{ik\xi}+e^{-ik\xi}\right)e^{-ik\zeta}e^{-\xi_0 k}\left(\frac{-e^{-\zeta}}{k^2+1}\left(k\cos k\zeta+\sin k\zeta\right)+\frac{k}{k^2+1}\right)dk && (29)\\
&= -\beta\tilde{A}\left(i\cos\tau+\sin\tau\right)\int_{0}^{\infty}\frac{1}{k}\cos k\xi e^{-ik\zeta}e^{-\xi_0 k}\frac{1}{k^2+1}\left(-e^{-\zeta}\left(k\cos k\zeta+\sin k\zeta\right)+k\right)dk && (30)\\
&= -\beta\tilde{A}\left(i\cos\tau+\sin\tau\right)\int_{0}^{\infty}\frac{1}{k}\cos k\xi\left(\cos k\zeta-i\sin k\zeta\right) && (31)\\
&\quad\times e^{-\xi_0 k}\frac{1}{k^2+1}\left(-e^{-\zeta}\left(k\cos k\zeta+\sin k\zeta\right)+k\right)dk && (32)\\
&= -\beta\tilde{A}\int_{0}^{\infty}\frac{1}{k}\sin k\xi\left(i\cos\tau+\sin\tau\right)\left(\cos k\zeta-i\sin k\zeta\right) && (33)\\
&\quad\times e^{-\xi_0 k}\frac{1}{k^2+1}\left(-e^{-\zeta}\left(k\cos k\zeta+\sin k\zeta\right)+k\right)dk && (34)\\
&= -\beta\tilde{A}\int_{0}^{\infty}\frac{1}{k}\sin k\xi\left(i\cos\tau\cos k\zeta+\cos\tau\sin k\zeta+\sin\tau\cos k\zeta-i\sin\tau\sin k\zeta\right) && (35)\\
&\quad\times e^{-\xi_0 k}\frac{1}{k^2+1}\left(-e^{-\zeta}\left(k\cos k\zeta+\sin k\zeta\right)+k\right)dk && (36)
\end{aligned}
$$

The real part of this is then

$$
\begin{aligned}
& -\beta\tilde{A}\int_{0}^{\infty}\frac{1}{k}\sin k\xi\left(\cos\tau\cos k\zeta+\sin\tau\sin k\zeta\right)e^{-\xi_0 k}\frac{1}{k^2+1}\left(-e^{-\zeta}\left(k\cos k\zeta+\sin k\zeta\right)+k\right)dk && (37)\\
&= -\beta\tilde{A}\int_{0}^{\infty}\frac{1}{k}\sin k\xi\sin\left(\tau+k\zeta\right)e^{-\xi_0 k}\frac{1}{k^2+1}\left(-e^{-\zeta}\left(k\cos k\zeta+\sin k\zeta\right)+k\right)dk && (38)\\
&= -\beta\tilde{A}\int_{0}^{\infty}\frac{1}{k}\sin k\xi\frac{1}{k^2+1}e^{-\xi_0 k} && (39)\\
&\quad\times\left(-\sin\left(\tau+k\zeta\right)e^{-\zeta}k\cos k\zeta-\sin\left(\tau+k\zeta\right)e^{-\zeta}\sin k\zeta+k\sin\left(\tau+k\zeta\right)\right)dk. && (40)
\end{aligned}
$$

Consider now the second term. We have

$$
\begin{aligned}
&-\beta\tilde{A}\pi i e^{-i\tau}\mathcal{F}^{-1}\left\{\frac{1}{k}\sin k\zeta e^{-\xi_0|k|}\left[\frac{1}{-i\,\mathrm{sgn}(k)k-1}e^{(-i\,\mathrm{sgn}(k)k-1)\zeta'}\right]_\zeta^\infty\right\} && (41)\\
&=-\beta\tilde{A}ie^{-i\tau}\frac{1}{2}\int_{-\infty}^{0}\frac{1}{k}e^{ik\xi}\left(\sin k\zeta e^{\xi_0 k}\left[\frac{1}{ik-1}e^{(ik-1)\zeta'}\right]_\zeta^\infty\right)dk && (42)\\
&-\beta\tilde{A}ie^{-i\tau}\frac{1}{2}\int_{0}^{\infty}\frac{1}{k}e^{ik\xi}\left(\sin k\zeta e^{-\xi_0 k}\left[\frac{1}{-ik-1}e^{(-ik-1)\zeta'}\right]_\zeta^\infty\right)dk && (43)\\
&=-\beta\tilde{A}ie^{-i\tau}\frac{1}{2}\int_{-\infty}^{0}\frac{1}{k}e^{ik\xi}\left(\sin k\zeta e^{\xi_0 k}\frac{1}{1-ik}e^{(ik-1)\zeta}\right)dk && (44)\\
&-\beta\tilde{A}ie^{-i\tau}\frac{1}{2}\int_{0}^{\infty}\frac{1}{k}e^{ik\xi}\left(\sin k\zeta e^{-\xi_0 k}\frac{1}{1+ik}e^{(-ik-1)\zeta}\right)dk && (45)\\
&=-\beta\tilde{A}ie^{-i\tau}\frac{1}{2}\int_{0}^{\infty}\frac{1}{-k}e^{-ik\xi}\left(-\sin k\zeta e^{-\xi_0 k}\frac{1}{1+ik}e^{(-ik-1)\zeta}\right)dk && (46)\\
&-\beta\tilde{A}ie^{-i\tau}\frac{1}{2}\int_{0}^{\infty}\frac{1}{k}e^{ik\xi}\left(\sin k\zeta e^{-\xi_0 k}\frac{1}{1+ik}e^{(-ik-1)\zeta}\right)dk && (47)\\
&=-\beta\tilde{A}ie^{-i\tau}\int_{0}^{\infty}\frac{1}{k}\cos k\xi e^{-\xi_0 k}e^{-ik\zeta}\left(\sin k\zeta\frac{1-ik}{1+k^2}e^{-\zeta}\right)dk && (48)\\
&=-\beta\tilde{A}ie^{-i\tau}\int_{0}^{\infty}\frac{1}{k}\cos k\xi e^{-\xi_0 k}\left(\cos k\zeta-i\sin k\zeta\right)(1-ik)\left(\sin k\zeta\frac{1}{1+k^2}e^{-\zeta}\right)dk && (49)\\
&=-\beta\tilde{A}\int_{0}^{\infty}\frac{1}{k}\cos k\xi e^{-\xi_0 k}\left(\sin k\zeta\frac{1}{1+k^2}e^{-\zeta}\right) && (50)\\
&\times(i\cos\tau+\sin\tau)\left(\cos k\zeta-ik\cos k\zeta-i\sin k\zeta-k\sin k\zeta\right)dk. && (51)
\end{aligned}
$$

The real part of this is then

$$
\begin{aligned}
&-\beta\tilde{A}\int_{0}^{\infty}\frac{1}{k}\sin k\xi e^{-\xi_0 k}\left(\sin k\zeta\frac{1}{1+k^2}e^{-\zeta}\right) && (52)\\
&\times\left(k\cos\tau\cos k\zeta+\cos\tau\sin k\zeta+\sin\tau\cos k\zeta-k\sin\tau\sin k\zeta\right)dk && (53)\\
&=-\beta\tilde{A}\int_{0}^{\infty}\frac{1}{k}\sin k\xi e^{-\xi_0 k}\left(\sin k\zeta\frac{1}{1+k^2}e^{-\zeta}\right) && (54)\\
&\times\left(k\cos(\tau+k\zeta)+\sin(\tau+k\zeta)\right)dk && (55)\\
&=-\beta\tilde{A}\int_{0}^{\infty}\frac{1}{k}\sin k\xi e^{-\xi_0 k}\frac{1}{1+k^2} && (56)\\
&\times\left(\sin k\zeta e^{-\zeta}k\cos(\tau+k\zeta)+\sin k\zeta e^{-\zeta}\sin(\tau+k\zeta)\right)dk && (57)
\end{aligned}
$$

Summing (40) and (57) gives

$$
\tilde{\psi}(\xi,\zeta,\tau)=-\beta\tilde{A}\int_0^\infty\frac{\cos k\xi e^{-\xi_0 k}}{1+k^2}\left(\sin(k\zeta+\tau)-e^{-\zeta}\sin\tau\right)dk \qquad (58)
$$

as required!

4 Calculating Potential Temperature

From equation (4) Rotunno (1983) we have

$$
\frac{\partial b}{\partial t}+N^2w=Q.
$$

The non-dimensional form of this is

$$\frac{\partial \tilde{b}}{\partial \tau} h\omega^3 + N^2 \tilde{w} h\omega = \tilde{Q} h\omega^3$$
$$= \frac{\partial \tilde{b}}{\partial \tau} + \left(\frac{N}{\omega}\right)^2 \tilde{w} = \tilde{Q}.$$

Note that $\left(\frac{N}{\omega}\right)^2$ is essentially the Berger number with $H = L$. Thus can solve for $\tilde{b}$ using

$$\frac{\tilde{b}_{k+1} - \tilde{b}_k}{\Delta\tau} = \tilde{Q}_k - \left(\frac{N}{\omega}\right)^2 \tilde{w}_k$$
$$= \tilde{b}_{k+1} - \tilde{b}_k = \Delta\tau \left(\tilde{Q}_k - \left(\frac{N}{\omega}\right)^2 \tilde{w}_k\right).$$

This produces a linear system of τ_N linearly independent equations in τ_N unknowns. However, in this form the system is singular - so substitute the equation for $\tilde{b_1}$ for $\tilde{b_1} + \tilde{b_{\frac{\tau_n}{2}}} = 0$ to impose symmetry on the bouyancy. Note can use $\tilde{b_1} + \tilde{b}_{\lfloor \frac{\tau_n}{2} \rfloor}$! This works - we can solve for bouyancy even without initial conditions! From bouyancy can extract potential temperature!

Note that using this method appears to produce potential temperature perturbations that are two large, i.e. ± 40 K, or even larger! Compare this with the WRF simulation of Vincent & Lane (2016) producing perturbations of ± 4 K. Could it be that the WRF data are composites, and that the signal is being diluted? Note that increasing h decreases θ'. Error possibly due to definition of $\bar{\theta}$ in code?

5 Choosing $\tilde{A}$

Note that from equation (4) we have at the surface

$$\frac{\partial b}{\partial t} = \frac{g}{\theta_0}\frac{\partial \theta'}{\partial t} = Q,$$

as $w = 0$ at the surface. Thus letting θ_M and θ_m denote the land surface temperature maxima and minima respectively, and noting that $\left(\frac{\pi}{2} + \tan^{-1}\frac{x}{x_0}\right)$ maps onto $(0, \pi)$, we have

$$\frac{g}{\theta_0}\frac{\theta_M - \theta_m}{12 \cdot 60 \cdot 60} = \left(\sin\frac{\pi}{2} - \sin\frac{3\pi}{2}\right) A\pi = 2A\pi,$$

with $A\pi$ the supremum of H at the surface, and it taking 12 hours to go from maximum to minimum temperature. Multiplying both sides by $h^{-1}\omega^{-3}$ gives

$$\tilde{A} = \frac{g}{2\pi\theta_0}\frac{\theta_M - \theta_m}{12 \cdot 60 \cdot 60} h^{-1}\omega^{-3}.$$

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