**Reading** Please finish reading Chapter 2 by the time this assignment is due.

**Problems**

1. Let $F = \forall x((\forall y Q(y, x)) \rightarrow \forall x R(z, x))$.
   (a) Transform $F$ into an equivalent RPF formula, $F'$. Please show your work. (See section 2.2 of the text and the Prenex-Skolem Handout.)
   (b) Now Skolemize $F'$.

2. Do Exercise 64, p. 59: dual of Skolemization.

3. [15 pts.] Let $\mathcal{A}, \mathcal{A}'$ be logical structures of the same vocabulary. We say that $\mathcal{A}'$ is a **substructure** of $\mathcal{A}$ ($\mathcal{A}' \leq \mathcal{A}$) iff $|\mathcal{A}'| \subseteq |\mathcal{A}|$ and $\mathcal{A}'$ interprets all predicate symbols and function symbols the same way that $\mathcal{A}$ does, i.e., for each predicate symbol $P$ of arity $i$, $P^{\mathcal{A}'} = P^{\mathcal{A}} \cap |\mathcal{A}|^i$, and for each function symbol $f$ of arity $i$, and for each $e_1, \ldots e_i \in |\mathcal{A}'|$, $f^{\mathcal{A}'}(e_1, \ldots, e_i) = f^{\mathcal{A}}(e_1, \ldots, e_i)$. Note in particular that since $A'$ is a structure it must be closed under all defined functions. Example: for graphs, $G' \leq G$ iff G' is an induced subgraph of G, i.e., G' has a subset of $G$'s vertices and all the edges between them that $G$ has.

   A first-order formula is **universal** if it is in prenex normal form and all of it's quantifiers are $\forall$'s. Similarly it is **existential** if it is in prenex normal form and all of it's quantifiers are $\exists$'s. Suppose that $\mathcal{A} \leq \mathcal{B}$, i.e. $\mathcal{A}$ is a substructure of $\mathcal{B}$. Prove the following:
   (a) If $\varphi$ is universal and $\mathcal{B} \models \varphi$ then $\mathcal{A} \models \varphi$.
   (b) If $\varphi$ is existential and $\mathcal{A} \models \varphi$ then $\mathcal{B} \models \varphi$.

   [Hint: (a) and (b) imply each other. Just prove one and show why the other follows. Your proof should be by induction on the number of quantifiers. You may assume that the two structures interpret the same free variables the same way, i.e., if one of them inteprets $x$ then they both do and $x^{\mathcal{A}} = x^{\mathcal{B}}$.]

4. Prove the lemma that we needed for the fundamental theorem of Ehrenfeucht-Fraïssé games, i.e., that if our vocabulary $\tau$ is finite and has no function symbols of arity greater than 0, then for each $n$ there are only finitely many sentences up to equivalence in $\mathcal{L}^k_n(\tau)$, i.e., the quantifier-depth is at most $n$ and number of different variables occuring is at most $n$. [Hint: show this by induction on $n$. First come up with an upper bound for how many terms there are and then for how many atomic formulas there are, and then how many quantifier-free formulas there are. This gives the base case. For the inductive stpe give an upper bound for how much this number can increase each time you increase the quantifier depth by one.]

$5^\star$ (a) Give an example to show that the lemma from problem 4 is not true if there are infinitely many relation symbols of arity one.

(b) Give an example to show that the lemma is not true if there is a function symbol of arity one.

(c) Find a counter-example to the fundamental theorem of EF games for the vocabulary of part a. You should come up with two structures $\mathcal{A}, \mathcal{B}$ in this vocabulary that agree on all sentences of qr 1, but for which Samson wins the one-move game.

(d) Similarly find a counter-example to the fundamental theorem of EF games for the vocabulary of part b.

$6^\star$ Do exercise 73, p. 77: show that the notions of "semi-decidablity" and recursive enumerability (r.e.) are equivalent. A set, $M$, is semi-decidable if there is an algorithm that halts and answers, "yes", on exactly all the elements of $M$. On non-elements of $M$ it may halt and say "no" or never halt. A set $M$ is r.e. if it is empty or if it can be written as $M = \{f(1), f(2), \ldots\}$ where $f$ is a total and computable function. By total, I mean that it halts and computes an answer on all inputs $1, 2, \ldots$.