Repeated Eigenvalues

Occasionally when we have repeated eigenvalues, we are still able to find the correct number of linearly independent eigenvectors. Take for example

$$\begin{pmatrix} 3 & -1 & 2 \\ 3 & -1 & 6 \\ -2 & 2 & -2 \end{pmatrix}$$

One can verify that the eigenvalues of this matrix are $\lambda = 2, 2, -4$. We can find the eigenvalue corresponding to $\lambda = -4$ using the usual methods, and find $u_4 = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$. For the $\lambda = 2$ case, we must solve the system

$$\begin{pmatrix} 3-2 & -2 & 2 \\ 3 & -1-2 & 6 \\ -2 & 2 & -2-2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

i.e.

$$\begin{pmatrix} 1 & -2 & 2 \\ 3 & -3 & 6 \\ -2 & 2 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The second and third equations are just multiples of the first equation, so we expect to have 2 parameters in our solution. Note that this will not always be the case for a 3x3 matrix.

Let $z = s$ and $y = t$. Then $x = y - 2z = s - 2t$, from the first equation. So our eigenvector is

$$\begin{pmatrix} s-2t \\ s \\ t \end{pmatrix} = s \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

We can see that there are two linearly independent vectors here, and each will be an eigenvector for $\lambda = 2$. In this case the solution to the differential equation

$$\dot{x} = \begin{pmatrix} 3 & -1 & 2 \\ 3 & -1 & 6 \\ -2 & 2 & -2 \end{pmatrix} x$$

is

$$C_1 e^{-4t} \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + e^{2t} \left[C_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + C_3 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right]$$

However, sometimes we can't find two linearly independent eigenvectors in this way. Consider the differential equation

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 10 & -2 \\ 18 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Here we find a repeated eigenvalue of $\lambda = 4$. To find the eigenvector(s), we set up the system

$$\begin{pmatrix} 6 & -2 \\ 18 & -6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

These equations are multiples of each other, so we can set $x = t$ and get $y = 3t$. So there is only one linearly independent eigenvector, $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$. This will give us one solution to the differential equation, but we need to find another one. Instead of just multiplying our previous solution by t, we guess (without much justification) that a solution will be

$$\begin{pmatrix} x \\ y \end{pmatrix} = te^{4t} \begin{pmatrix} 1 \\ 3 \end{pmatrix} + e^{4t} \begin{pmatrix} A \\ B \end{pmatrix}$$

Where now we need to find A and B. We can find these constants by plugging in our solution to the differential equation, and solving the system that we get. Our final solution to the problem will be

$$\begin{pmatrix} x \\ y \end{pmatrix} = C_1 \left[te^{4t} \begin{pmatrix} 1 \\ 3 \end{pmatrix} + e^{4t} \begin{pmatrix} A \\ B \end{pmatrix} \right] + C_2 e^{4t} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$