DRAFT VERSION OCTOBER 19, 2021
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1. THE QUESTION: SCHWARZSCHILD OR LEDOUX?

The fundamental question behind this project is not, "If you look at an instantaneous profile of a star, which stability criterion should you use?" We know the answer to that question: Ledoux. The question here is, "is the Ledoux criterion *fragile*?" Or, given time, will a Ledoux-stable (but Schwarzschild-unstable) region which is adjacent to a convection zone remain stable over dynamical timescales?

2. THE QUASI-BOUSSINESQ EQUATIONS OF MOTION

2.1. *Dimensional equations*

We will solve the Boussinesq equations of motion in the incompressible limit. We will solve for the velocity ($\mathbf{u}$), the temperature (T), and the composition (μ). In this limit, density variations are ignored except in the buoyant term in the momentum equation, where they follow

$$\frac{\rho}{\rho_0} = \alpha T + \beta\mu, \qquad \alpha \equiv \frac{\partial \ln \rho}{\partial T} \qquad \text{and} \qquad \beta \equiv \frac{\partial \ln \rho}{\partial \mu} \tag{1}$$

We will use values of $\beta > 0$ and $\alpha < 0$ (higher concentration $\rightarrow$ denser, lower temperature $\rightarrow$ denser). The Boussinesq momentum equation (in which ρ can only vary on the gravitational term) is,

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \frac{\nabla p}{\rho_0} = \frac{\rho}{\rho_0}\mathbf{g} + \nu\nabla^2\boldsymbol{u}. \tag{2}$$

Removing ρ under the Boussinesq approximation, we retrieve our evolution equations,

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \frac{\nabla p}{\rho_0} = (\alpha T + \beta\mu)\mathbf{g} + \nu\nabla^2\mathbf{u} \tag{3}$$

$$\partial_t T + \mathbf{u}\cdot(\nabla T - \nabla T_{\rm ad}) = \nabla\cdot[k_0\nabla\overline{T}] + \chi\nabla^2 T' + Q \tag{4}$$

$$\partial_t \mu + \mathbf{u}\cdot\nabla\mu = \chi_{\mu,0}\nabla^2\overline{\mu} + \chi_\mu\nabla^2\mu'. \tag{5}$$

Here, k_0 and $\chi_{\mu,0}$ are diffusivities which act on the horizontally averaged mode while χ and χ_μ act on the fluctuations; ν is a viscosity which acts on all flows. Assuming that a vertical energy flux F is carried through the system, this allows us to define an adiabatic and radiative gradient,

$$\nabla \equiv -\nabla T, \qquad \nabla_{\rm ad} \equiv -\nabla T_{\rm ad}, \qquad \nabla_{\rm rad} \equiv \frac{F}{k_0}. \tag{6}$$

Note that in this system, if you linearize and idealize and solve for the dispersion relation, the squared brunt frequencies (assuming $\mathbf{g} = -g\hat{z}$) are

$$N^2_{\rm therm} = \alpha g(\nabla - \nabla_{\rm ad}), \qquad N^2_{\rm comp} = -\beta g\nabla\mu, \qquad N^2 = N^2_{\rm therm} + N^2_{\rm comp}, \tag{7}$$

and the stability criterion is to have $N^2 > 0$.

2.2. *Nondimensional Equations*

The nondimensional equations are then (with ϖ the dynamical pressure),

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla\varpi = (-T + \mathrm{R_d^{-1}Y})\hat{\mathrm{z}} + \frac{1}{\mathrm{Re}}\nabla^2\mathbf{u}, \tag{8}$$

$$\partial_t T - \boldsymbol{u}\cdot(\nabla - \nabla_{\rm ad}) = -\nabla\cdot[k_0\overline{\nabla}] + \frac{1}{\mathrm{Pe}}\nabla^2 T' + Q, \tag{9}$$

$$\partial_t \mu + \boldsymbol{u}\cdot\nabla\mu = \frac{1}{\mathrm{Re}_{\mu,0}}\nabla^2\overline{\mu} + \frac{1}{\mathrm{Re}_\mu}\nabla^2\mu'. \tag{10}$$

The nondimensional brunt frequencies are

$$N^2_{\rm therm} = (\nabla_{\rm ad} - \nabla), \qquad N^2_{\rm comp} = -\mathrm{R_d^{-1}}\nabla\mu, \equiv \nabla_\mu \qquad \mathrm{N^2 = N^2_{therm} + N^2_{comp}} = (\nabla_\mu + \nabla_{\rm ad} - \nabla). \tag{11}$$

3. THREE-LAYER EXPERIMENT

We want to set up an experiment where there are three layers, characterized by:

1. (CZ) $\nabla = \nabla_{\rm ad}$, $\nabla_\mu = 0$, $\nabla_{\rm rad} > \nabla_{\rm ad}$.
2. (semiconvection) $\nabla = \nabla_{\rm rad}$, $\nabla_{\rm rad} - \nabla_\mu < \nabla_{\rm ad}$, $\nabla_{\rm rad} > \nabla_{\rm ad}$.
3. (RZ) $\nabla = \nabla_{\rm rad}$, $\nabla_\mu = 0$, $\nabla_{\rm rad} < \nabla_{\rm ad}$.

As in our previous work, we will set the convective flux $F_{\rm conv} = Q\delta_H = 0.2$ with $Q = 1$ and $\delta_H = 0.2$. We will set the flux entering the bottom of the domain, and thus we will set k_0, so that $F_{\rm bot} = k_0\nabla_{\rm ad} = \eta F_{\rm conv}$. By our choice of Q and by setting the convective domain size to $L = 1$, we implicitly set $f_{\rm conv} \sim 1$. To set the stiffness $\mathcal{S}$, we set the value of $N^2 = \nabla_{\rm ad} - \nabla_{\rm rad} = \mathcal{S}$ in the RZ. We have still not chosen a value of $\nabla_{\rm ad}$ (just its relation to $\nabla_{\rm rad}$ in the RZ and $\nabla_{\rm rad}$, or k_0, in the CZ). So we will choose a value of the penetration parameter $\mathcal{P}$ that is either realistic of stellar values (1) or which has no penetration ($\ll 1$). This parameter is

$$\mathcal{P} = -\frac{k_{\rm CZ}(\nabla_{\rm rad} - \nabla_{\rm ad})_{\rm CZ}}{k_{\rm RZ}(\nabla_{\rm rad} - \nabla_{\rm ad})_{\rm RZ}}. \tag{12}$$

We furthermore want to set it so that $\Delta\mu = 1$ (that's the magnitude of the μ decrease over zone 2 above). We additionally want to ensure $N^2 > 0$ in zone 2, so $\mathrm{R_d}$ must be set so that $\nabla_\mu > \nabla_{\rm rad} - \nabla_{\rm ad}$ in the CZ. Under these constraints, we are free to choose whatever values of Re, Re_μ, and Pe we want. We want to choose them, probably, so that the semiconvective layer is *not unstable* to any classical semiconvective instabilities. We should prove that it is stable to these instabilities using a 1-layer model which just encompasses zone (2) above. Once we know it's stable, then we should do the 3-layer model.