

# **Studies of Quantum Dots using Machine Learning**

by

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## **Abstract**

This thesis aims the study of circular quantum dots using machine learning to reduce the need of physical intuition



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After 5 years at Blindern, 5 years at Blindern

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I have always believed that everything has a reason, which I used to point out whenever people talked about things that could not easily be explained. First when I was introduced to quantum mechanics, I realized that it might not be that simple, which is one of the reasons why this beautiful theory caught my attention. Even though my philosophy has changed to a less deterministic direction, I still like to think that everything can be explained out of some first principles.



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# Chapter 1

## Introduction

The properties and behavior of quantum many-body systems are determined by the laws of quantum physics which have been known since the 1930s. The time-dependent Schrödinger equation describes the binding energy of atoms and molecules, as well as the interaction between particles in a gas. In addition, it has been used to determine the energy of artificial structures like quantum dots, nanowires and ultracold condensates. As the quantum theory is the most precisely tested theory in the history of science, computer experiments are capable of obtaining the energy as precise as laboratory experiments, and can in that sense replace laboratory experiments.

Even though we know the laws of quantum mechanics, many challenges are encountered when calculating real-world problems. First, interesting systems often involve large number of particles, which causes expensive calculations. Second, the correct wave functions are seldomly known for a complex system, which is vital for measuring the observable accurately. They are both covered by the many-body problem.

### 1.1 The many-body problem

In quantum mechanics, a many-body system contains three or more particles interacting with each other. These interactions create so-called quantum correlations, which makes the wave function of the system a complicated object holding a large amount of information. As a consequence, exact or analytical calculations become impractical or even impossible, which is known as the many-body problem. Indeed, Paul Dirac recognized those problems already in 1929:

“ The general theory of quantum mechanics is now almost complete... ...The underlying physical laws necessary for the mathematical theory of a large part of physics and the whole of chemistry are thus completely known, and the difficulty is only that the exact application of these laws leads to equations much too complicated to be soluble.

Paul M. Dirac, *Quantum Mechanics of Many-electron Systems*, [8]. ”

There are numerous approaches to solve this problem where approximative methods often are used to reduce the sometimes extreme computational cost. Popular methods include the Hartree-Fock method, which replaces the interaction by a mean field, and methods like Full Configuration Interaction (FCI) and Coupled Cluster which seek to solve the problem by approximating the wave function with expansions. Lastly, quantum Monte Carlo (QMC) methods are totally different approaches attempting to solve the problem directly using a stochastic evaluation of the integrals occurring from the Schrödinger equation.

What all these methods have in common, is that they require significant amounts of physical intuition to work. In general, prior knowledge of the wave function, covering all the interactions and the cusps, is required. This knowledge is often unavailable, especially for complex systems, which also make accurate estimates of the observable unavailable. In this thesis we will try to bypass this problem by inventing more flexible and robust methods which allow a relatively bad wave function guess. A natural base for this are the machine learning algorithms, as they can "learn" themselves and thus hopefully find good estimates through a training process. The obvious link between machine learning and quantum mechanics are the QMC methods, since they both are based on minimizing a *cost function* in order to obtain optimal configurations. The connection between machine learning and QMC methods, in particular variational Monte Carlo (VMC), will be discussed thoroughly throughout this thesis.

## 1.2 Machine learning

Machine learning has recently achieved immense popularity in fields such as computer vision, economics, autonomy - and science, due to its ability of learning without being explicitly programmed. As a branch of artificial intelligence, machine learning is based on studies of the human brain and attempts to recreate the way neurons in the brain process information.

Especially the artificial neural networks have experienced a significant progress over the past decade, which can be attributed to an array of innovations. Most notably, the convolutional neural network (CNN) AlexNet managed to increase the top-5 test error rate of image recognition with a remarkable 11.1% compared to the second best back in 2012 [55]! Today, the CNNs have been further improved, and they are even able to beat humans in recognizing images [75]. Also speech recognition algorithms have lately been revolutionized, thanks to recurrent neural networks (RNNs), and especially long short-term memory (LSTM) networks. Their ability to recognize sequential (time-dependent) data made the technology good enough for an entry to millions of peoples everyday-life through services such as Google Translate [71], Apple's Siri [70] and Amazon Alexa [90]. It is also interesting to see how machine learning has made computers eminent tacticians using reinforcement learning. The Google developed program AlphaGo demonstrated this by beating the 9-dan professional L. Sedol in the board game Go [89], before an improved version, AlphaZero, beat the at that time highest rated chess computer, StockFish, in chess [95]. Both these scenarios were unbelievable just a couple of decades ago.

Even though all these branches are both exciting and promising, they will not be discussed further in this work, since they will simply not work for our purposes. The reason is that they initially require a data set with known outputs in order to be trained, they obey so-called *supervised* learning. For our quantum mechanical systems, we do not have those targets and need to rely on *unsupervised* learning with the focus on restricted Boltzmann machines (RBMs). Lately, some effort has been put into this field, known as quantum machine learning. Giuseppe Carleo and Matthias Troyer demonstrated the link between RBMs and QMC and named the states *neural-network quantum states* (NQS). They used the technique to study the Ising model and the Heisenberg model [73]. Vilde Flugsrud went further and investigated circular quantum dots with the same method [91]. We will extend the work she did to larger quantum dots and double quantum dots.

## 1.3 Quantum dots

Quantum dots are often called artificial atoms because of their common features to real atoms, and their popularity is increasing due to their applications in semiconductor technology. For instance, quantum dots are expected to be the next big thing in display technology due to their

ability of emitting photons of specific wavelength in addition to being 30% more energy efficient than today's LED displays [67]. Samsung already claim that they use this technology in their displays [88].

Another reason why we are interested in simulating quantum dots, is the fact that there exist experiments which can be used as benchmarks. Due to very strong confinement in the  $z$ -direction, the experimental dots, made by patterning GaAs/AlGaAs heterostructures, become essentially two-dimensional [27, 29]. For that reason, our main focus in this work is on two-dimensional dots, but also dots of three dimensions will be investigated. Quantum dots also allow the study of Wigner crystals, as their strength can be decreased such that the interaction energy dominates the kinetic energy.

## 1.4 Computer experiments

The advent of computer technology has offered us a new window of opportunity for studies of quantum (and many other) problems. It serves as a third way of doing science which is based on simulations, in contrast to laboratory experiments and analytical approaches. In a broad sense, by simulations we mean computational models of reality based on physical laws, such as the fundamental Schrödinger equation. Such models have value when they enable to make predictions or to provide new information which is otherwise impossible or too costly to obtain. In this respect, QMC methods represent an illustration and an example of what is the potential of such methodologies.

The use and popularity of QMC methods has increased as personal computers and computer clusters have been more powerful. With today's strong computers, we see those methods as a natural choice when ground state properties of quantum mechanical systems are investigated. Even the simplest method, VMC, does typically yield excellent results, and the more complicated diffusion Monte Carlo (DMC) is in principle capable of employing exact results. They both appears to have among the highest performance-to-cost ratios out of all the quantum many-body methods.

Albeit the fact that the QMC methods relatively recently have been applied on large scale calculations, some of the ideas go back to the time before the invention of the electronic computer. Already in the 1940's, Enrico Fermi revealed the similarities between the imaginary time Schrödinger equation and stochastic processes in statistical mechanics. The first attempt of using this link on actual calculations, was performed by a group of scientists at Los Alamos National laboratory in the early 1950's when they tried to compute the ground state energy of the hydrogen molecule using a simple version of VMC [51]. Around the same time, Metropolis et.al. introduced the original Metropolis algorithm which estimates the integrals by moving particles randomly in an ergodic scheme and rejecting inappropriate moves [13]. This method was further improved in the late 1950's when Kalos laid down the statistical and mathematical framework for the Green's function in QMC methods, and Hastings et. al. developed an efficient algorithm based on the theory, where the particles are moved after an educated guess [16]. The use of the QMC methods on many-fermion systems was first done by Ceperly and Alder in the 1980's, and started a new era of stochastic methods applied to electronic structure problems [21].

## 1.5 Ethics in science

In science as an entirety, there are some general guidelines that we all should follow in order to maintain an ethical behavior. Firstly, one should always have respect for others work and the authors should be credited whenever one uses others work, no matter the scope. This includes illustrations, text, code, methods, algorithms and so on, which are covered by the Copyright Act

(in Norway, åndsverkloven). Secondly, all the research that one does should always be detailed in a such way that the experiments and results can be reproduced by others. This means that all the factors which possibly have a significant impact on the experiments should be described, and computer experiments is no exception. Lastly, there are unfortunately many examples on misuse of knowledge throughout the history, which obviously has to be avoided.

In our specific work, most of the ethical aspects are related to the use of machine learning, which can cause fatal consequences if it is not used correctly and carefully. The fact that machine learning allows the computers to learn things themselves have made profiled people like Stephen Hawking and Elon Musk warn us that they can be greatly misused if they are set on learning the wrong things [61, 68]. Every person who develop machine learning algorithms should take this warning seriously, remember that in the end there is one of us who end up creating the multi-headed monster *Hydra*.

## 1.6 Our goals and contributions

The goals of this thesis are mainly related to the development and investigation of a quantum many-body method that requires less physical intuition. In order to do this, the software implementation is essential, which obviously is a significant part of the work and thereby the goals. This also includes a VMC code for benchmark purposes, in addition to the RBM code. Further, the next goal is naturally to obtain some results, which will mostly come from the quantum dots. In the end, we will provide a thorough comparison of the VMC and RBM results, inclusive wave function studies. We can summarize the goals in a list

- Develop a VMC code for the study of large fermionic systems.
- Implementing RBMs as flexible trial wave function guesses.
- Study ground state properties of quantum dots, including energy, variance and electron densities.
- Make a critical evaluation of the RBMs compared with VMC studies.

We believe that machine learning based methods is the next big thing in many-body quantum mechanics, and our contribution to the field is therefore to investigate one of the many such approaches. There are also some very interesting similarities between the typical sampling process behind Boltzmann machines and the QMC methods, and by revealing the actual mechanisms, we might be able to develop more robust methods.

## 1.7 Our developed code

There exist plenty of commercial software programs for solving the quantum many-body problem, and they are often efficiency optimized. In general, it is wise to use already existing code and not try to reinvent the wheel, but in our case we will investigate a new approach, which forces us to write the code from scratch.

Our VMC solver is written in object-orientated C++ inspired by Morten Ledum's example implementation [69], and our basis set is assumed to be given in Cartesian coordinates. The goal is not to compete with the performance of commercial software, but we will still make significant efforts to develop an efficient code. As far it is possible, all operations are vectorized using the open source template library Eigen, which again is based on the tremendously fast packages BLAS and LAPACK. We used the profiling tool Valgrind to analyze which functions that require most computer power and thus which parts of the code that make it slow. Additionally, the message

passing interface (MPI) lets us parallelize the code and thereby run on computer clusters. For implementation details, see chapter 9 and 10. The VMC code can in its entirety be found on <https://github.com/evenmn> under MIT license.

## 1.8 Structure of the thesis

Fundamental theory, including many-body quantum physics and basic machine learning, is given in chapter 2-4. The theory behind the RBMs and VMC is given in chapter 6-7 as our advanced theory, and their optimization schemes and implementation is presented in chapter 9-10. The remaining chapters include presentation and discussion of the results and a brief conclusion.



# **Part I**

# **Fundamental Theory**



## Chapter 2

# Preliminary Quantum Mechanics

I do not like it, and I am sorry I ever had anything to do with it.

---

Erwin Schrödinger, [39]



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**Figure 2.1:** The first ever image of quantum entanglement, where two groups of particles are sharing the physical state  $\Psi(\mathbf{r})$ , was published by Moreau et.al. under the title *Imaging Bell-type nonlocal behavior* in July 2019 [83].

The quantum theory is a fundamental theory which describes the nature at the smallest scales, typically used to explain the behavior of atoms and subatomic particles. Although the theory in principle holds for systems consisting of a large number of particles as well, the calculations become both comprehensive and expensive as the system size increases, and exact computations are therefore reserved small systems. For larger systems, we in practice need to rely on approximative estimates of the observable, which also fast becomes infeasible when the system size increase.

In the most general form, the theory is based on the time-dependent Schrödinger equation,

$$i\hbar \frac{\partial}{\partial t} |\Psi(\mathbf{r}, t)\rangle = \hat{\mathcal{H}} |\Psi(\mathbf{r}, t)\rangle \quad (2.1)$$

which describes the state, or the wave function,  $\Psi(\mathbf{r}, t)$ , of a system with Hamilton operator  $\hat{\mathcal{H}}$ , known as the Hamiltonian.  $\hbar$  is the reduced Planck's constant and  $i$  is the solution of the equation

$x^2 = -1$ . As we later will see, the quantum mechanical system is completely specified by the wave function, so by solving the equation we will in principle know anything about the system.

Erwin Schrödinger introduced the now-called Schrödinger equation in a 1926 paper as one of many contributions to the quantum mechanics at that time [6]. Other contributors include Max Born who suggested now-standard interpretation of the probability density function as  $|\Psi(\mathbf{r}, t)|^2$  in 1926 [5] and his companion Werner Heisenberg who formulated the matrix mechanics representation in 1925 [4]. However, many physicists were skeptical to the new theory, including Albert Einstein and Schrödinger himself. The theory differed from the classical mechanics in the sense that it was totally based on a statistical interpretation, with some strange consequences. For instance, the theory allowed negative kinetic energies to occur with the consequence that a particle could *go through* a potential wall, known as quantum tunneling. Later, Einstein also pointed out that when pairs or groups of particles are generated in ways such that the wave function of each particle cannot be described independently of the other state, they will be affected by each other even at large distances [11]. He called this "*spooky action at a distance*", and used the observation to argue that the quantum theory had to be incorrect. However, the observation was later proven to be correct and as late as in July 2019 the first image of quantum entanglement, as the phenomenon is called, was captured [83]. See figure (2.1).

Today, there is wide agreement in the physics society on the quantum theory. Actually, the theory is the most precisely tested theory in the history of science, where computed observable of atoms agree perfectly with experiments. Most notably, quantum electrodynamical calculations of the fine structure constant  $\alpha$  are found to agree with experiments within ten parts in a billion,  $10^{-8}$  [98].

In this chapter, will first state the postulates of quantum mechanics, and thereafter we discuss the time-independent Schrödinger equation and challenges related to solving it. As we will see, every observable in quantum mechanics is associated with an operator, and the consequence is that all obtained values are identified with standard errors which are also important to specify in order to present the accuracy of the value. In that manner, quantum mechanics is based on statistics and in section (2.3) we discuss the statistical interpretation of the theory. The variational principle states that the ground state energy is the lowest possible energy calculated, so by minimizing the energy in a variational scheme, a ground state energy estimation can be obtained. Other things discussed are the quantum numbers, which are the values that give acceptable solutions to the Schrödinger equation, and the virial theorem which relates kinetic and potential energy.

Albeit the quantum mechanics will make up the framework for this work, we will in this chapter only discuss the theory needed for the work and this is therefore not meant as an encyclopedia to quantum mechanics. For a complete introduction to the topic, Ref. [40] serves as an excellent read. Before we get started, we make a few assumptions in order to simplify our problem. The most important ones are specified below with an explanation why they are valid.

- **Point-like particles:** First, all particles involved will be assumed to be point-like, i.e., they lack spatial extension. For electrons this makes sense, since they, as far as we know, do not extent. The assumption is also applied on the nucleus in atomic systems, but it still makes sense since the distance from the nucleus to the electrons is known to be much larger than the nucleus extension.
- **Non-relativistic spacetime:** Second, we operate in the non-relativistic spacetime, which is an extremely good approximation as long as we do not approach the speed of light and we do not involve strong forces. Applying classical physics, we can find that the speed of the electron in a hydrogen atom is about 1% of the speed of light. Even though the electrons

achieve higher velocities in heavy atoms, we do not need to worry about it as we will stick to the lighter atoms. In the quantum dots, this assumption hold even for large systems, as the velocity and density will still be sufficiently low.

- For specific systems we might make new assumptions and approximations. For instance, for atomic systems we will assume that the nucleus is at rest. Those approximations will be discussed consecutively.

## 2.1 The postulates of Quantum Mechanics

Quantum mechanics is characterized by a set of fundamental axioms that make up the base of the theory. As we assume that they always hold, every statement in quantum mechanics is based on them and it is therefore naturally to list them now in the beginning. They can be formulated by six postulates, in the following way:

1. *"The state of a quantum mechanical system is completely specified by the wave function  $\Psi(\mathbf{r}, t)$ ."*
2. *"To every observable in classical mechanics, there corresponds a linear, Hermitian operator in quantum mechanics."*
3. *"In any measurement of the observable associated with an operator  $\hat{O}$ , the only values that will ever be observed are the eigenvalues  $o$  which satisfy  $\hat{O}\Psi(\mathbf{r}, t) = o\Psi(\mathbf{r}, t)$ ."*
4. *"The expectation value of the observable corresponding to operator  $\hat{O}$  is given by*

$$\langle \hat{O} \rangle = \frac{\int d\tau \Psi^*(\mathbf{r}, t) \hat{O} \Psi(\mathbf{r}, t)}{\int d\tau \Psi^*(\mathbf{r}, t) \Psi(\mathbf{r}, t)}. "$$

5. *"The wave function evolves in time according to the time-dependent Schrödinger equation,*

$$\hat{\mathcal{H}}\Psi(\mathbf{r}, t) = i\hbar \frac{\partial \Psi(\mathbf{r}, t)}{\partial t}. "$$

6. *"The total wavefunction must be antisymmetric with respect to the interchange of all coordinates of one fermion with those of another. Electronic spin must be included in this set of coordinates."*

All the postulates will be used in this work, and they will be described in detail when we need them. Albeit we will look at stationary states only, even the time-dependent Schrödinger from the fifth postulate will be discussed due to the description of the diffusion Monte Carlo method in section 7.5. The other will be covered in the current and the next chapter. The formulation of the presented postulates are taken from [38].

## 2.2 The Schrödinger equation

We have already presented the time-dependent Schrödinger equation on several occasions, but as mentioned above, we will in this work study stationary systems only. If we also recall that the particles are assumed to be non-relativistic, the focus will be on solving the time-independent

non-relativistic Schrödinger equation. By defining  $\Psi_n(\mathbf{r})$  as the wave function of a state  $n$  with energy  $\epsilon_n$ , the equation can be expressed as

$$\hat{\mathcal{H}}\Psi_n(\mathbf{r}) = \epsilon_n\Psi_n(\mathbf{r}) \quad (2.2)$$

where  $\hat{\mathcal{H}}$  is the aforementioned Hamiltonian, the total energy operator. By analogy with the classical mechanics, the total mechanical energy is the kinetic and potential energy summarized,

$$\hat{H} = \hat{T} + \hat{V} \quad (2.3)$$

with  $\hat{T}$  and  $\hat{V}$  as the kinetic and potential energy operators respectively.

Again from classical mechanics, the kinetic energy of a moving particle of mass  $m$  yields  $T = p^2/2m$  where  $p$  is the (linear) momentum, such that the kinetic energy operator can be represented as

$$\hat{T} = \frac{\hat{p}^2}{2m} \quad (2.4)$$

according to Ehrenfest's theorem. Further, the momentum operator is  $\hat{p} = -i\hbar\hat{\nabla}$  with  $\hat{\nabla}$  as the differential operator and the factor  $i\hbar$  arising from the canonical commutator relation between the position operator and the momentum operator,

$$[\hat{x}, \hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar \quad (2.5)$$

which indicates that the momentum and the position do not *commute*. In other words, the order of the operators in an equation is not arbitrary. The potential, on the other hand, is obviously dependent on the system we want to study. For atomic systems, the potential as a function of the distance from the nucleus can be found from Coulomb's law and reads

$$V(r) = \frac{1}{4\pi\epsilon_0} \frac{Ze^2}{r} \quad (2.6)$$

where the nucleus is assumed to be at rest at the origin,  $k_e = 1/4\pi\epsilon_0$  is the Coulomb constant,  $Z$  is the atomic number of the nucleus and  $e$  is the elementary charge. For a general potential, the Hamiltonian can be expressed as

$$\hat{\mathcal{H}} = -\frac{\hbar^2}{2m}\nabla^2 + V(r) \quad (2.7)$$

which is the farthest we can go without specifying the external potential  $V(r)$ .

By setting up equation (2.2) with respect to the energies, we obtain an integral,

$$\epsilon_n = \frac{\int d\mathbf{r}\Psi_n^*(\mathbf{r})\hat{\mathcal{H}}\Psi_n(\mathbf{r})}{\int d\mathbf{r}\Psi_n^*(\mathbf{r})\Psi_n(\mathbf{r})}, \quad (2.8)$$

which not necessarily is trivial to solve. For almost<sup>1</sup> all many-electron systems, this becomes analytically infeasible due to a two-body interaction term. This will be covered in chapter 3.

As suggested by Max Born, we get the probability density function if we take the dot product between the complex conjugate wave function and the wave function itself,

$$P(\mathbf{r}) = \Psi_n^*(\mathbf{r})\Psi_n(\mathbf{r}) = |\Psi_n(\mathbf{r})|^2, \quad (2.9)$$

so the denominator in equation (2.8) is basically the integral over all the probabilities. If the wave function is normalized correctly, this should always give 1.

As a very brief example, we will below calculate the bounding energy of the Hydrogen atom.

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<sup>1</sup>Exceptions include quantum dots of two electrons in two and three dimensions, where M.Taut has presented semi-analytical energies for a few oscillator frequencies [26, 30].

### 2.2.1 The Hydrogen atom

We presented the one-particle Hamiltonian in equation (2.7), and the potential from the nucleus in equation (2.6). By introducing the Bohr radius

$$a_0 = \frac{4\pi\epsilon_0\hbar^2}{m_e Ze^2} \quad (2.10)$$

we can set up the total Hamiltonian as

$$\hat{\mathcal{H}} \cdot \frac{(4\pi\epsilon_0)^2\hbar^2}{m_e Z^2 e^4} = -\frac{1}{2}a_0^2\nabla^2 - \frac{a_0}{r} \quad (2.11)$$

where we have multiplied all the terms by the factor  $(4\pi\epsilon_0)^2\hbar^2/m_e Z^2 e^4$  in order to make the equation dimensionless. We can further scale the energy as  $E \leftarrow E \cdot (4\pi\epsilon_0)^2\hbar^2/m_e Z^2 e^4$  and the distance as  $r \leftarrow r/a_0$ , which both are dimensionless, and obtain the Hamiltonian

$$\hat{\mathcal{H}} = -\frac{1}{2}\nabla^2 - \frac{1}{r} \quad (2.12)$$

where all quantities are in atomic units.

The Hydrogen ground state wave function is well-known and reads

$$\Psi(r) \propto e^{-r} \quad (2.13)$$

in atomic units. The binding energy in Hydrogen is found from the Schrödinger equation in equation (2.2), and gives

$$\hat{\mathcal{H}}\Psi(r) = \left( -\frac{1}{2}\nabla^2 - \frac{1}{r} \right)\Psi(r) = -\frac{1}{2}\Psi(r) \quad (2.14)$$

which indicates that  $\varepsilon = -0.5$ .

## 2.3 Statistical interpretation

In equation (2.8), we found the expectation value of the energy using the Hamiltonian, which is the energy operator. By the second postulate of quantum mechanics, every observable in the classical mechanics is associated with such a Hermitian operator  $\hat{O}$  related to an expectation value  $\langle \hat{O} \rangle$  is the same way,

$$\langle \hat{O} \rangle = \frac{\int d\mathbf{r} \Psi_n^*(\mathbf{r}) \hat{O} \Psi_n(\mathbf{r})}{\int d\mathbf{r} \Psi_n^*(\mathbf{r}) \Psi_n(\mathbf{r})}. \quad (2.15)$$

As a consequence, there is always an uncertainty associated with a quantum mechanical calculation, such that we can only tell the probability of measuring something. The uncertainty is usually described by the variance, which for a set of independent measurements is given by

$$\sigma^2 = \langle \hat{O}^2 \rangle - \langle \hat{O} \rangle^2. \quad (2.16)$$

If there is correlations between the measurements, this expression underestimates the actual sampling variance as the *covariance* is not taken into account, which is detailed in section 7.6. If we take the square root of the variance, we obtain the standard deviation which is the quantity given as the standard error in the results. From the standard deviation, a variety of mathematical inequalities follows, where Heisenberg's uncertainty principle is the most famous. It states that

the more precisely the position of some particle is determined, the less precisely its momentum can be known, and is mathematically presented as

$$\sigma_x \sigma_p \geq \frac{\hbar}{2} \quad (2.17)$$

where  $\sigma_x$  is the standard deviation of the position and  $\sigma_p$  is the standard deviation of the momentum.

The general expected value in equation (2.15) is expressed using the Schrödinger wave mechanics picture, but it can also be written in terms of for instance the Heisenberg picture applying matrices. However, the standard formalism today in most branches of the quantum mechanics is the Dirac notation, which relates the Schrödinger picture and the Heisenberg picture in an elegant way. Using this, the expectation value from equation (2.15) reads

$$\langle \hat{O} \rangle = \frac{\langle \Psi | \hat{O} | \Psi \rangle}{\langle \Psi | \Psi \rangle} \quad (2.18)$$

where  $\langle \Psi |$  is called the *bra* and  $|\Psi \rangle$  is called the *ket*. For that reason, the formalism is also called braket formalism. Usually, the wave function is assumed to be normalized, which further simplifies the expected value to

$$\langle \hat{O} \rangle = \langle \Psi | \hat{O} | \Psi \rangle, \quad (2.19)$$

and will henceforth be assumed. More information about the Dirac formalism can be found in Appendix A.

## 2.4 The variational principle

In the equations above, the presented wave functions are assumed to be the exact eigenfunctions of the Hamiltonian. But often we do not know the exact wave functions, and we need to guess what the wave functions might be. In those cases we apply the variational principle, which states that only the exact ground state wave function is able to give the ground state energy. All other wave functions that fulfill the required properties (see section 3.2) give higher energies, and mathematically we can express the statement as

$$\varepsilon_0 \leq \langle \Psi | \hat{\mathcal{H}} | \Psi \rangle \quad (2.20)$$

where  $\Psi$  is an arbitrary wave function. The variational method is a way of finding approximations to the energy ground state, which is based on the variational principle. Most notably, the variational Monte Carlo method is named after the principle and attempts to solve the integrals directly by varying a trial wave function. The variational principle ensures that the obtained energy never goes below the ground state energy, as further described in chapter 7. Other methods that apply the variational method are the famous Hartree-Fock method and the infamous coupled cluster method.

## 2.5 Quantum numbers

Unlike in the classical mechanics, all the observable in quantum mechanics are discrete or *quantized*, which means that the  $n$  associated with  $\varepsilon_n$  above cannot take any number. In fact,  $n$  can only take positive integers, and is named the principal quantum number. We also have other quantum numbers identified with the angular momentum and spin as described below.

## Principal

The **principal** quantum number describes the electron shell, and can take the numbers  $n \in [1, 2, 3, \dots]$ . As  $n$  increases, the electron excites to a higher shells such that also the energy increases. In general,  $\varepsilon_1 < \varepsilon_2 < \varepsilon_3 \dots$  as long as all other quantum numbers are fixed. The electron shells can again be split up in subshells, requiring more quantum numbers.

## Angular

An electron shell can possibly have more than one subshell, described by the **angular** quantum number  $l$ .  $l$  can take the values  $0, 1, \dots, n - 1$ , such that the degeneracy of subshells in a shell is simply  $n$ . In atoms, the angular quantum number describes the shape of the shell, where  $l = 0$  gives a spherical shape,  $l = 1$  gives a polar shape while  $l = 2$  gives a cloverleaf shape.

## Magnetic

We also have a **magnetic** quantum number  $m_l$ , which has the range  $-l, -l + 1, \dots, l - 1, l$ . If  $l$  describes the shape of a shell,  $m_l$  specifies its orientation in space. This quantum number was first observed under presence of a magnetic field, hence the name.

## Spin

The **spin** quantum number  $s$  gives the spin of a particle, which can just be seen as a particle's property. Particles are often divided in two groups dependent on the spin because of their different behavior: **bosons** have integer spin, while **fermions** have half-integer spin. Electrons and protons have spin  $s = 1/2$ , which makes them fermions.

## Spin projection

The last number we will discuss is the **spin projection** quantum number  $m_s$ . It has the range  $-s, -s + 1, \dots, s - 1, s$ , and is therefore related to the spin quantum number in the same way as the magnetic number  $m_l$  is related to the angular number  $l$ . Electrons can for that reason take the values  $m_s = +1/2$  or  $m_s = -1/2$ , such that there are two groups of electrons. The consequences will be discussed in the section (3.2).

## 2.6 The virial theorem

The virial theorem relates the kinetic energy to the potential energy, and makes it possible to find the average kinetic energy even for complex systems. The classical statement of the theorem was formulated during the 19th century and named by Rudolf Clausius in 1870 [3]. It is in the most general form given by

$$\langle \hat{T} \rangle = -\frac{1}{2} \sum_{i=1}^N \langle \mathbf{F}_i \cdot \mathbf{r}_i \rangle \quad (2.21)$$

where  $\mathbf{F}_i$  represents the force on particle  $i$  at position  $\mathbf{r}_i$ . If we further assume that all the sources of potential are in the form of  $V_i = ar^{n_i}$ , we can use the relation  $\mathbf{F}_i = -\nabla V_i$  to express the virial theorem in a simpler fashion

$$2\langle \hat{T} \rangle = \sum_i n_i \langle \hat{V}_i \rangle, \quad (2.22)$$

which by Vladimir Fock was proven to be true for quantum mechanics in a 1930 paper [9]. The assumption raised above holds for our situation as the interaction energy goes as  $r^{-1}$ . We will also later see that the potential used in quantum dots goes as  $r^2$  and the potential used in atoms goes as  $r^{-1}$  as well.

The expectation value is in principle the time average of the operator. However, if the ergodic hypothesis holds for the system, i.e., the ensemble average is equal to the time average [23], an ensemble average can also be taken.

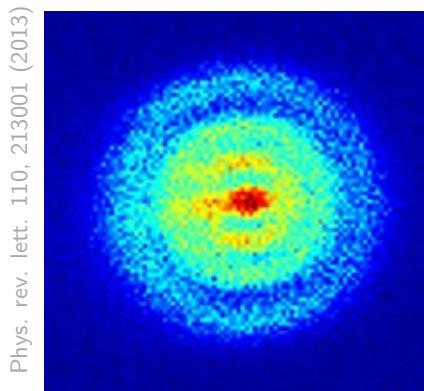
## Chapter 3

# Many-body Quantum Mechanics

We have to remember that what we observe is not nature in itself but nature exposed to our method of questioning.

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Werner Heisenberg, [25]



**Figure 3.1:** The first photograph of a Hydrogen atom was captured by an ultra sensitive camera in 2013. One can actually see the probability distribution  $|\Psi(\mathbf{r})|^2$  with the naked eye. Published by Stodolna et.al. under the title *Hydrogen atoms under magnification* [57].

In the previous chapter, the quantum mechanics of single particles was discussed. We presented the time-independent Schrödinger equation and from that we obtained the general expression from the energy,

$$E_n = \langle \Psi_n(\mathbf{r}) | \hat{\mathcal{H}} | \Psi_n(\mathbf{r}) \rangle \quad (3.1)$$

where the Hamiltonian  $\hat{\mathcal{H}}$  defines the system and  $\Psi_n(\mathbf{r})$  defines the state  $n$ . What is wonderful, is that the same expression can be used for a system containing an arbitrary number of particles as long as the many-body wave function is defined. However, as noted before, the wave function for large systems needs to store an immense amount of information and is therefore expensive or impossible to deal with. In this section we will first look at how the many-body Hamiltonian is set up, before we have an extensive discussion of how the many-body wave function is constructed.

### 3.1 The electronic Hamiltonian

We have already seen what the one-body Hamiltonian looks like, and the many-body Hamiltonian is not very different. We recall that it can be split in a kinetic and a potential term,

$$\hat{H} = \hat{T} + \hat{V} \quad (3.2)$$

where  $\hat{T}$  is the kinetic energy and  $\hat{V}$  is the potential energy. Nevertheless, as we study electrons, they are charged and will therefore interact with each other. For that reason, we need to add an interaction term to the Hamiltonian, which in general is included in the potential term  $\hat{V} = V_{\text{ext}} + V_{\text{int}}$  with  $V_{\text{ext}}$  as the external potential and  $V_{\text{int}}$  as the interaction potential. In the same way as the nucleus potential, the interaction potential is given by Coulomb's law, for two electrons given by

$$V_{\text{int}} = k_e \frac{e^2}{r_{12}} \quad (3.3)$$

where  $r_{12}$  is the distance between the electrons. For a general system containing  $N$  electrons, the total Hamiltonian can therefore be expressed as

$$\hat{\mathcal{H}} = - \sum_i^N \frac{\hbar^2}{2m_e} \nabla_i^2 + \sum_i^N V_i + \sum_i^N \sum_{j>i}^N k_e \frac{e^2}{r_{ij}} \quad (3.4)$$

which is the farthest we can go without specifying the external potential  $V_i$ .  $r_{ij}$  is the relative distance between particle  $i$  and  $j$ , defined by  $r_{ij} \equiv |\mathbf{r}_i - \mathbf{r}_j|$ . From now on, we will use atomic units setting  $\hbar = m_e = k_e = e = 1$ , see appendix B for details.

By putting this Hamiltonian into equation (3.1), the integral can be split in three parts,

$$\varepsilon_n = - \sum_{i=1}^N \left[ \langle \Psi_n(\mathbf{r}) | \nabla_i^2 | \Psi_n(\mathbf{r}) \rangle + \langle \Psi_n(\mathbf{r}) | V_i | \Psi_n(\mathbf{r}) \rangle + \sum_{j>i}^N \langle \Psi_n(\mathbf{r}) | \frac{1}{r_{ij}} | \Psi_n(\mathbf{r}) \rangle \right] \quad (3.5)$$

where the two former ones are the one-body integrals, or *matrix elements*, which we in many cases easily can solve. However, the last term is very tricky to solve, and in fact there are analytical solutions available for the two-particle case only. With other words, a precise evaluation of this integral can usually only be found using numerical methods, which we will have a closer look at in chapter 7 in conjunction with quantum Monte Carlo methods.

### 3.2 The wave function

By the first postulate of quantum mechanics presented in section 2.1, the wave function contains all the information specifying the state of the system. This means that all observable in classical mechanics can also be estimated from the wave function, which makes finding the wave function our aim. We will in this section discuss the symmetry properties of the wave function for bosonic and fermionic systems, and see how the wave function can be set up as a Slater determinant in order to meet these properties. Also Jastrow factors will be touched, as they are used in quantum Monte Carlo methods to handle the correlations. With that in mind, we approximate the wave function by a *trial wave function*,

$$\Psi_T(\mathbf{r}; \boldsymbol{\theta}) = |\hat{D}(\mathbf{r}; \boldsymbol{\theta})| J(\mathbf{r}; \boldsymbol{\theta}). \quad (3.6)$$

which is an educated guess of the form of the wave function. Here the first part,  $|\hat{D}(\mathbf{r}; \boldsymbol{\theta})|$ , is the Slater determinant and  $J(\mathbf{r}; \boldsymbol{\theta})$  is a Jastrow factor with  $\boldsymbol{\theta}$  as some variational parameters. Other

quantum many-body methods, such as full configuration interaction and coupled cluster use a linear expansion of Slater determinants to approximate the wave function, while the Hartree-Fock method relies on one single Slater determinant (without any Jastrow factor). Nevertheless, as we will focus on quantum Monte Carlo methods, this chapter will be tailored to the method.

The trial wave function needs to satisfy some requirements in order to be used in the variational principle, and we thus need to make an educated guess on the wave function where the requirements are fulfilled. The requirements are the following:

1. **Normalizability:** The wave function needs to be normalizable in order to make physical sense. The total probability should always be 1, and a wave function that cannot be normalized will not have a finite total probability. The consequence is that the wave function goes to zero when the positions get large,  $\Psi(x \rightarrow \pm\infty) \rightarrow 0$ .
2. **Cusp condition:** The cusp condition (also called the Kato theorem) states that the wave function should have a cusp where the potential explodes. An example on this is when electrons come close to each other, i.e., the electron-electron cusp.
3. **Symmetry and anti-symmetry:** The wave function needs to be either symmetric or anti-symmetric under exchange of two coordinates, dependent on whether the particles are fermions or bosons. This is the statement of the sixth postulate, which will be further explained in the next section.

### 3.2.1 Anti-symmetry and the Pauli principle

Symmetry and anti-symmetry are central concepts in quantum mechanics, and often one can use symmetry arguments to simplify expressions and calculations. In this section we will look at the symmetry and anti-symmetry properties of the wave function and discuss how it can be used to define a wave function in the next section.

Assume that we have a permutation operator  $\hat{P}$  which exchanges two coordinates in the wave function,

$$\hat{P}(i \rightarrow j)\Psi_n(\mathbf{r}_1, \dots, \mathbf{r}_i, \dots, \mathbf{r}_j, \dots, \mathbf{r}_M) = p\Psi_n(\mathbf{r}_1, \dots, \mathbf{r}_j, \dots, \mathbf{r}_i, \dots, \mathbf{r}_M), \quad (3.7)$$

where  $p$  is just a factor which comes from the transformation. If we again apply the  $\hat{P}$  operator, we should switch the same coordinates back, and we expect to end up with the initial wave function. For that reason,  $p$  must be either +1 or -1.<sup>1</sup>

The particles that have an anti-symmetric wave function under exchange of two coordinates are called fermions, named after Enrico Fermi, and as discussed before they have half-integer spin. On the other hand, the particles that have a symmetric wave function under exchange of two coordinates are called bosons, named after Satyendra Nath Bose, and have integer spin. A consequence of the anti-symmetric wave function, is that two identical fermion cannot occupy the same state at the same time, known as the Pauli principle. This means that identical fermions even in the ground state (at zero temperature) spread over multiple states, and in the next section we will see how this principle is baked into the wave function through a Slater determinant.

### 3.2.2 The Slater determinant

For a system of many particles we can define a total wave function, which is a composition of all the single-particle wave functions (SPF) and contains all the information about the system as the

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<sup>1</sup>Actually, in two-dimensional systems a third possibility is allowed which gives an *anyon*. The theory on this was developed by J.M. Leinaas and J. Myrheim during the 1970's [18].

first postulate requires. For fermions, we need to combine the SPF's such that the Pauli principle is fulfilled at all times, which can be accomplished by a determinant.

Consider a system of two identical fermions with SPF's  $\psi_1(\mathbf{r}, \sigma)$  and  $\psi_2(\mathbf{r}, \sigma)$  at position and spin  $\mathbf{r}_1, \sigma_1$  and  $\mathbf{r}_2, \sigma_2$  respectively. The way we define the wavefunction of the system is then

$$\begin{aligned}\Psi(\mathbf{r}_1, \mathbf{r}_2, \sigma_1, \sigma_2) &= \frac{1}{\sqrt{2}} \begin{vmatrix} \psi_1(\mathbf{r}_1, \sigma_1) & \psi_2(\mathbf{r}_1, \sigma_2) \\ \psi_1(\mathbf{r}_2, \sigma_1) & \psi_2(\mathbf{r}_2, \sigma_2) \end{vmatrix} \\ &= \frac{1}{\sqrt{2}} [\psi_1(\mathbf{r}_1, \sigma_1)\psi_2(\mathbf{r}_2, \sigma_2) - \psi_2(\mathbf{r}_1, \sigma_2)\psi_1(\mathbf{r}_2, \sigma_1)],\end{aligned}\quad (3.8)$$

which is automatically set to zero if the particles happen to be at the same position and have the same spin at the same time. If the particles, on the other hand, have different spins, they are allowed to be at the same position at the same time and the determinant will not cancel. For larger systems, the Slater determinant is constructed in the same way as above, and any pair of identical particles located in the same state will make the determinant collapse. A Slater determinant containing  $N$  electrons reads

$$\Psi(\mathbf{r}, \sigma) = \frac{1}{\sqrt{N!}} \begin{vmatrix} \psi_1(\mathbf{r}_1, \sigma_1) & \psi_2(\mathbf{r}_1, \sigma_2) & \dots & \psi_N(\mathbf{r}_1, \sigma_N) \\ \psi_1(\mathbf{r}_2, \sigma_1) & \psi_2(\mathbf{r}_2, \sigma_2) & \dots & \psi_N(\mathbf{r}_2, \sigma_N) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_1(\mathbf{r}_N, \sigma_1) & \psi_2(\mathbf{r}_N, \sigma_2) & \dots & \psi_N(\mathbf{r}_N, \sigma_N) \end{vmatrix}\quad (3.9)$$

where the  $\psi(\mathbf{r}, \sigma)$  is the tensor product between the radial part  $\phi(\mathbf{r})$  and the spin part  $\xi(\sigma)$ ,

$$\psi(\mathbf{r}, \sigma) = \phi(\mathbf{r}) \otimes \xi(\sigma).\quad (3.10)$$

In section 9.3.4, it is shown that the Slater determinant can be split in a spin-up part and a spin-down part such that the spin-dependency  $\xi(\sigma)$  can be omitted. For that reason, we can define a basis set consisting of the spatial parts only, discussed in the next section.

As a note, we will in the rest of this thesis use  $\Psi$  as the many-particle wave function,  $\psi$  are the single-particle functions and  $\phi$  as the spatial part of the single-particle function.  $\xi$  will be reserved the spin part of the single-particle functions, but will often be omitted as the spin-part can be factorized out. Sometimes it is appropriate to split up the many-particle wave function, and we will in that case denote each part by  $\Psi_i$  where  $i$  is an index associated with the respective element. Lastly, the basis functions will be called  $\varphi$ , which we will discuss in the next section.

### 3.2.3 Basis set

In quantum chemistry, a basis set is a set of basis functions, which often are called the atomic orbitals. Generally, a basis function should not be confused with a single-particle function, as a single-particle function is a linear combination of basis functions, and are thus often called the molecular orbitals. Commonly used atomic orbitals are Pople basis sets (in the form of x-yz G), correlation-consistent basis sets (in the form of cc-pVNZ) and Slater-type orbitals (in the form of STO-nG), where all are built on Gaussian functions. Gaussian functions are preferred as they allow efficient implementations of post-Hartree-Fock methods.

In our work, however, the single-particle functions are often the basis functions, as we in most of the cases do not expand our single-particle functions  $\psi(\mathbf{r}, \sigma)$  in a basis. Our single-particle functions are typically the solution of the non-interacting system, and as the Jastrow factor is supposed to deal with the interaction, we use the same functions for the interacting case as well.

The standard notation is to use the Greek letter  $\varphi$  for the atomic orbitals and  $\psi$  for the molecular orbitals, such that single-particle functions can be obtained from an expansion of the

$N$  basis functions  $\{\varphi_1(\mathbf{r}), \varphi_2(\mathbf{r}), \dots, \varphi_N(\mathbf{r})\}$  in the manner of

$$\psi_i(\mathbf{r}) = \sum_{j=1}^N c_{ji} \varphi_j(\mathbf{r}). \quad (3.11)$$

where  $c_{ij}$  are the coefficients to be found. There are different approaches to obtain these coefficients, where the popular Hartree-Fock algorithm generates  $c_{ij}$ 's in order to find the optimal single Slater determinant. The larger basis, the more accurate results we will get. The actual functions used in this work are presented in the next chapter, 4, and linked to their respective systems.

### 3.2.4 Jastrow factors

From electrostatics we know that identical, charged particles will repulse each other. This means that the probability of finding two particles close to each other should be low, which needs to be baked into the wave function. One way to do this is to simply multiply the wave function with the distance between the particles; the smaller distance the lower probability. However, since we are going to work in the logarithmic space, dealing with exponential functions will be much easier. This is the main idea behind the simple Jastrow factor.

#### 3.2.4.1 Simple Jastrow

The simple Jastrow factor is just an exponential function with the sum over all particle distances. In addition, each distance  $r_{ij}$  is weighted by a parameter  $\beta_{ij}$ , and the factor becomes

$$J(\mathbf{r}; \beta) = \exp \left( \sum_{i=1}^N \sum_{j>i}^N \beta_{ij} r_{ij} \right). \quad (3.12)$$

All the  $\beta_{ij}$  are free variational parameters, which are expected to be symmetric since the distance matrix is symmetric. One problem with this Jastrow factor, is that it does not create the cusp around each particle correctly. Basically, the Jastrow factor increases faster than it should when a particle is moved away from another. To solve this, we need to introduce a more complex Jastrow factor, the Padé-Jastrow.

#### 3.2.4.2 Padé-Jastrow

The Padé-Jastrow factor is closely related to the simple Jastrow above, but a denominator is added to make the cusp correct. It reads

$$J(\mathbf{r}; \beta) = \exp \left( \sum_{i=1}^N \sum_{j>i}^N \frac{a_{ij} r_{ij}}{1 + \beta r_{ij}} \right). \quad (3.13)$$

where  $\beta$  is a variational parameter. In addition, the fractions are multiplied with constants  $a_{ij}$  which depend on the particles  $i$  and  $j$  in the following way:

$$a_{ij} = \begin{cases} e^2/(D+1) & \text{if } i, j \text{ are particles of same spin} \\ e^2/(D-1) & \text{if } i, j \text{ are particles of opposite spin,} \end{cases} \quad (3.14)$$

for dimensions  $D \in [2, 3]$  where  $e$  is the elementary charge. We will later use natural and atomic units, and set  $e = 1$ , which for two dimensions gives  $a_{ij} = 1/3$  (same spin) or  $a_{ij} = 1$  (opposite spin) and for three dimensions  $a_{ij} = 1/4$  (same spin) and  $a_{ij} = 1/2$  (opposite spin) [56, 79].

This Jastrow factor is known to give accurate results for fermions and bosons because it gives the correct cusp condition, and it is the one we gonna use in the standard variational Monte-Carlo simulations.

### 3.3 Electron density

In quantum many-body computations, the electron density is frequently calculated, and there are several reasons for that. Firstly, the electron density can be found experimentally, such that the calculations can be benchmarked. Secondly, the electron density is very informative, since information about all particles can be gathered in one plot.

The  $P$ -body electron density is defined by the multi-dimensional integral over the probability density function of all the particles but  $P$ , reading

$$\rho_P(\mathbf{r}_1, \dots, \mathbf{r}_P) = \frac{1}{N - P} \int_{-\infty}^{\infty} d\mathbf{r}_{P+1} \dots d\mathbf{r}_N |\Psi(\mathbf{r}_1, \dots, \mathbf{r}_N)|^2. \quad (3.15)$$

where  $P \leq N$ . This integral is in general really tricky to solve, and it can be found analytically for just a few realistic systems. For other systems, we need to solve the integral numerically, and we will in chapter 7 describe how this can be done using Monte Carlo integration.

For non-interacting systems, the wave function is trivially separable with respect to the different particles. In those cases one can easily find the electron density analytically, since the functions of coordinates which we do not integrate over can be factorized out of the integral,

$$\begin{aligned} \rho_P(\mathbf{r}_1, \dots, \mathbf{r}_P) &= \int_{-\infty}^{\infty} d\mathbf{r}_{P+1} \dots d\mathbf{r}_N |\Psi(\mathbf{r}_1, \dots, \Psi(\mathbf{r}_N)|^2 \\ &= \int_{-\infty}^{\infty} d\mathbf{r}_{P+1} \dots d\mathbf{r}_N |\Psi(\mathbf{r}_1)|^2 \dots |\Psi(\mathbf{r}_N)|^2 \\ &= |\Psi(\mathbf{r}_1)|^2 \dots |\Psi(\mathbf{r}_P)|^2 \int_{-\infty}^{\infty} d\mathbf{r}_{P+1} |\Psi(\mathbf{r}_{P+1})|^2 \dots \int_{-\infty}^{\infty} d\mathbf{r}_N |\Psi(\mathbf{r}_N)|^2 \\ &= |\Psi(\mathbf{r}_1)|^2 \dots |\Psi(\mathbf{r}_P)|^2 \end{aligned} \quad (3.16)$$

where we have assumed that the wave functions are normalized. This result has no scientific importance, but will be used to validate the implementation of the electron density in the code, see section 11.3.

As we cannot pinpoint a particle in quantum mechanics, i.e., it is impossible to distinguish two identical particles in any way, the electron density is the same no matter which particles we decide to leave out. In equations, we always leave out the first  $P$  particles, but we could also for instance have left out the  $P$  last particles and so forth.

So, what does the electron density tell us? The two most commonly used electron densities are the one-body density and the two-body density. The former is the most applied electron density, and is sometimes simply referred to as the electron density. It gives the probability of finding a particle at a relative distance  $r$  from another, and give insight about how the particles are distributed in the system. The two-body density becomes a two-dimensional function, and is therefore often expressed as a matrix. It gives the probability of finding a particle pair at certain positions. For closed-shell circular quantum dots the radial electron density profile is often preferred as the density then also is circular and independent of the angle.

#### 3.3.1 Wigner crystals

A Wigner crystal is a solid phase where electrons maximize the distance to each other in order to minimize the potential energy. As the interaction potential is given by Coulomb's law, the poten-

tial is minimized when the distances between the electrons are maximized. In one-dimensional systems, the electrons are thus found at discrete locations, such that they form an evenly spaced lattice. In two-dimensional systems, the Wigner crystals form triangular lattices which are known to be the configuration that maximized the distance in a limited space, and in three-dimensional systems the electrons form so-called body-centered cubics. The phenomenon occurs only when the potential energy dominates the kinetic energy, since the electrons then are almost "at rest" and external forces are not strong enough to push the electrons close to each other.

If the concept still is unclear, imagine Norwegians waiting for the metro. In the nature of the people, the Norwegians always want to maximize the distance to each other, meaning that they form triangular lattices on the metro station. In the extreme case, this is equivalent with the Wigner crystals.



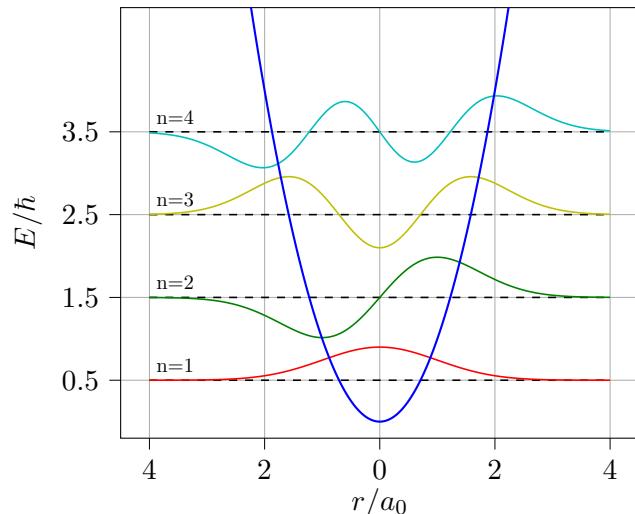
# Chapter 4

## Systems

We must be clear that when it comes to atoms, language can be used only as in poetry.

---

Niels Bohr, [17]



**Figure 4.1:** The quantum harmonic oscillator, with the Hermite functions represented up to 4th order. As in classical mechanics, the harmonic oscillator is able to describe various quantum systems, such as lattice vibration (phonons) and quantum fields.

In quantum physics, as in classical physics, there are many different systems with various complexities. If a system gets very complex, it is often beneficial to approximate it with a simpler system which hopefully catches the most important features in the system. This is especially advantageous in quantum mechanics, as only a few systems can be solved analytically. We have already seen that the atomic system can be solved analytically for the non-interacting case, and additionally harmonic systems can be solved analytically. Our circular quantum dots form a harmonic potential, and for non-interacting quantum dots we can also find analytical expressions.

We will in this chapter give explicit expressions of the quantum dot systems and the atomic system. This includes defining the potential and thereby the Hamiltonian, and thereafter we present an appropriate set of single-particle functions. Also double quantum dots will be covered, as one can find analytical solutions by an expansion of single quantum dot functions.

## 4.1 Quantum dots

Quantum dots are very small particles, and consist of fermions or bosons held together by an external potential which is not created by a nucleus. Similar to atoms, these dots have discrete electronic states with a well-defined shell structure, and are therefore often called artificial atoms.

In this thesis we will study circular quantum dots with electrons affected by a harmonic oscillator potential. For an electron  $i$ , the potential reads

$$V_i(r) = \frac{1}{2}m\omega^2r_i^2, \quad (4.1)$$

where  $m$  is the electron mass,  $\omega$  is the oscillator frequency and  $r_i$  is the relative distance from particle  $i$  to the center of the dot. Using natural units as described in Appendix B, we can write the Hamiltonian as

$$\hat{\mathcal{H}} = \sum_{i=1}^N \left( -\frac{1}{2}\nabla_i^2 + \frac{1}{2}\omega^2r_i^2 \right) + \sum_{i < j} \frac{1}{r_{ij}} \quad (4.2)$$

where the energy is given in Hartrees and the lengths are scaled with respect to  $\sqrt{\hbar/m}$ .

The exact solutions of the non-interacting Hamiltonian are the Hermite functions,

$$\phi_n(x) = H_n(\sqrt{\omega}x) \exp(-\omega x^2/2) \quad (4.3)$$

which are natural choices of single particle functions also for systems with interaction.  $H_n(x)$  is the Hermite polynomial of  $n$ 'th degree, and the first four Hermite functions are illustrated in figure (4.1). The energy of a particle with principal quantum number  $n$  in a certain dimension is given by

$$E_n = \omega \left( n + \frac{1}{2} \right) \quad \forall \quad n \in \{0, 1, 2, \dots\} \quad (4.4)$$

where we in multi-dimensional potentials need to summarize the contribution from all the dimensions.

We will study closed-shell systems only, since the Slater determinant in that case is unambiguous. The number of particles in closed-shell systems are called magic numbers, which in two dimensions are  $N = 2, 6, 12, \dots$ . In general, the magic numbers are given by

$$N = s \binom{n+D}{D} \quad \forall \quad n \in \{0, 1, 2, \dots\} \quad (4.5)$$

where  $s$  is the number of spin configurations (2),  $n$  is the principal quantum number and  $D$  is the number of dimensions. This is a direct consequence of the Pauli principle, where we in the ground state can have two particles with spatial wave functions  $\phi_{n_x=0, n_y=0}$ , in the next energy level we can have 4 particles with spatial wave functions  $\phi_{n_x=1, n_y=0}$  and  $\phi_{n_x=0, n_y=1}$  with degeneracy 2 and so on. See figure (4.2) for a schematic of the first few states in a two-dimensional quantum dot.



**Figure 4.2:** The possible states of a two-dimensional quantum dot for  $n = n_x + n_y = 0, 1, 2, 3$ . Recalling that the electrons can have spin  $\pm 1$ , one can use this schematic to determine how many electrons there are in each closed shell and thus find the magic numbers.

## 4.2 Quantum double dots

Another historically important quantum system is the double dot, which similarly to the single dot can be solved analytically. For the same reason as the single dot often is called an artificial atom, the double dots are called artificial molecules.

The potential of symmetrical quantum dots can in principle have a variety of different shapes, but the most used one-dimensional potentials can be derived from

$$V_i(x) = \frac{1}{2}\omega^2 \left[ |x_i|^a - \left(\frac{b}{2}\right)^a \right]^2 \quad (4.6)$$

with  $b$  as the distance between the bottoms of the wells and  $a$  as an arbitrary integer [54]. Setting  $a = 1$  gives two parabolic wells with a sharp local maximum at  $x = 0$ , while  $a = 2$  gives a smoother but steeper well. In figure (4.3) the potential is plotted for  $a = 1, 2$  and 3.

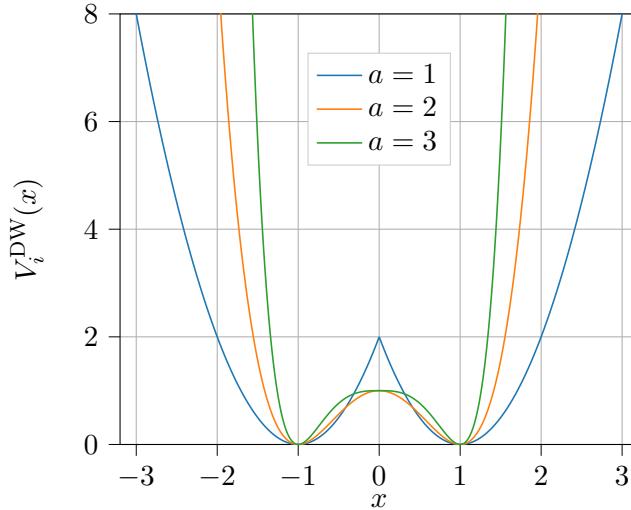
For reference and benchmark purposes, we will focus on the case with  $a = 1$  and  $b = 2$ , which can be written out as

$$V_i(x) = \frac{1}{2}\omega^2 \left[ x_i^2 + \frac{1}{4}b^2 - b|x_i| \right], \quad (4.7)$$

still in one dimension. For more than one dimension, we assume that the double dot expands in the  $x$ -direction, which gives us the general expression

$$V_i^{\text{DW}} = \frac{1}{2}\omega^2 \left[ r_i^2 + \frac{1}{4}b^2 - b|x_i| \right] = V_i^{\text{HO}} + \frac{1}{2}\omega^2 \left[ \frac{1}{4}b^2 - b|x_i| \right] \quad (4.8)$$

where HO means harmonic oscillator potential and DW means double-well potential. What we actually observe, is that the potential separates in a single-dot part and a double-dot part, which



**Figure 4.3:** One-dimensional double-well potentials plotted with  $a = 1, 2$  and  $3$ , and  $b = 2$ . For  $a = 1$  the potential was multiplied with 2 to make it comparable to the others.

makes the double-dot Hamiltonian similar to the single-dot Hamiltonian,

$$\hat{\mathcal{H}} = \sum_{i=1}^N \left( -\frac{1}{2}\nabla_i^2 + \frac{1}{2}\omega^2 r_i^2 + \frac{1}{2}\omega^2 \left( \frac{1}{4}b^2 - b|x_i| \right) \right) + \sum_{i < j} \frac{1}{r_{ij}}. \quad (4.9)$$

What is remaining is to find an appropriate set of single-particle functions. Based on the observations above, a reasonable set should be similar to the Hermite functions which we found to be the analytical single-particle functions of the single dot. For that reason, we will use the single-particle functions

$$|\phi_n^{\text{DW}}(x)\rangle = \sum_{\lambda=1}^L C_{n\lambda} |\phi_{\lambda}^{\text{HO}}(x)\rangle, \quad (4.10)$$

where  $L \in [1, \infty)$  is the number of basis functions used and  $C_{n\lambda}$  is the coefficient associated with the double-dot function  $n$  and the single-dot function  $\lambda$ , which is what we want to find. Inserting this into the double-dot Schrödinger equation and multiplying with  $\langle \phi_{\nu}^{\text{HO}}(x) |$  on the left-hand-side gives

$$\sum_{\lambda=1}^L C_{n\lambda} \langle \phi_{\nu}^{\text{HO}}(x) | \hat{\mathcal{H}}_{\text{DW}} | \phi_{\lambda}^{\text{HO}}(x) \rangle = \epsilon_n \sum_{\lambda=1}^L C_{n\lambda} \langle \phi_{\nu}^{\text{HO}}(x) | \phi_{\lambda}^{\text{HO}}(x) \rangle \quad (4.11)$$

where the right-hand-side sum collapses because the overlap is just the Kronecker delta,  $\langle \phi_{\nu}^{\text{HO}}(x) | \phi_{\lambda}^{\text{HO}}(x) \rangle = \delta_{\nu\lambda}$ . By defining the matrix elements

$$\hat{h}_{\nu\lambda} \equiv \langle \phi_{\nu}^{\text{HO}}(x) | \hat{\mathcal{H}}_{\text{DW}} | \phi_{\lambda}^{\text{HO}}(x) \rangle, \quad (4.12)$$

we can set up equation (4.11) as an eigenvalue problem in the form of

$$\hat{h}\hat{C} = \epsilon\hat{C} \quad (4.13)$$

where our targets,  $\hat{C}$ , are just the eigenvectors of the  $\hat{h}$ -matrix. Now recall that the double dot Hamiltonian is just an extension of the single dot Hamiltonian, such that we can rewrite

$$\hat{h}_{\nu\lambda} = \langle \phi_{\nu}^{\text{HO}}(x) | \hat{\mathcal{H}}_{\text{HO}} | \phi_{\lambda}^{\text{HO}}(x) \rangle + \langle \phi_{\nu}^{\text{HO}}(x) | \hat{\mathcal{H}}_{+} | \phi_{\lambda}^{\text{HO}}(x) \rangle \quad (4.14)$$

with  $\hat{\mathcal{H}}_+ = (1/2)\omega \sum_{i=1}^N ((1/4)b^2 - b|x_i|)$  as the extension. The former integrals are just the harmonic oscillator energies, presented in (4.4), while the latter integrals are trivial to calculate. However, since we need to calculate a relatively large number of matrix elements, we decided to do it using numerical integration on the computer. The script doing this is given the creative name `doublewell_functions.py`, found on <http://www.github.com/evenmn/VMC/scripts>.

### 4.3 Atoms

We will also investigate real atoms, where we freeze out the nucleonic degrees of freedom known as the Born-Oppenheimer approximation. The electrons will in fact affect the nucleus, but due to the mass difference this effect will be negligible.

We again have Coulomb interaction between the electrons and the nucleus, and since we assume the latter to be at rest at the origin, the external potential affecting particle  $i$  is

$$V_i(r) = -\frac{1}{2}k_e \frac{Ze^2}{r_i}, \quad (4.15)$$

where  $Z$  is the atomic number (number of protons in the nucleus). The total Hamiltonian is given in (Hartree) atomic units,

$$\hat{\mathcal{H}} = \sum_{i=1}^N \left( -\frac{1}{2}\nabla_i^2 - \frac{Z}{r_i} \right) + \sum_{i < j} \frac{1}{r_{ij}}, \quad (4.16)$$

which also is discussed in Appendix B. For the non-interacting case, the energy of a particle in shell  $n$  is given by the Bohr formula

$$E_n = -\frac{Z^2}{2n^2}, \quad (4.17)$$

which means that to find the energy of an atom, we need to summarize the energy of all the electrons.

Also for this system, we need to specify a basis set to use. For atoms where the electrons do not interact, the wave functions are given by the *Hydrogen-like orbitals*, which in spherical coordinates can be split in a radial part and an angular part,

$$\psi_{nlm_l}(r, \theta, \phi) = R_{nl}(r)Y_l^{m_l}(\theta, \phi). \quad (4.18)$$

where  $n$  again is the principal quantum number,  $l$  is the angular quantum number and  $m_l$  is the magnetic quantum number. Henceforth  $m_l$  will simply be displayed as  $m$  to make the expressions neater. Since we are physicists, we use  $\theta$  as the polar angle and  $\phi$  as the azimuthal angle.

The radial part can be presented as a function of the *associated Laguerre polynomials* or *generalized Laguerre polynomials*,  $L_q^p(x)$  and the Hydrogen wave function presented in section 2.2.1, and reads

$$R_{nl}(r) \propto r^l e^{-Zr/n} \left[ L_{n-l-1}^{2l+1} \left( \frac{2r}{n} Z \right) \right]. \quad (4.19)$$

The angular part is given by the *spherical harmonics*,

$$Y_l^m(\theta, \phi) \propto P_l^{|m|}(\cos \theta) e^{im\phi} \quad (4.20)$$

where  $P_l^m(x)$  are the *associated Legendre polynomials*. The complex part in the spherical harmonics can be avoided by introducing the real solid harmonics instead

$$S_l^m(\theta, \phi) \propto P_l^{|m|}(\cos \theta) \begin{cases} \cos(m\phi) & \text{if } m \geq 0 \\ \sin(|m|\phi) & \text{if } m < 0, \end{cases} \quad (4.21)$$

such that

$$\psi_{nlm}(r, \theta, \phi) = R_{nl}(r)S_l^m(\theta, \phi). \quad (4.22)$$

The real solid harmonics do not alter any physical quantities that are degenerate in the subspace consisting of opposite magnetic quantum numbers, and for that reason they will give the same energies as the spherical harmonics as long as there is no magnetic field present [84].

For atomic systems it is also common to use a Hartree-Fock basis, based on Gaussian functions as discussed in section 3.2.3. Actually, more or less all electronic structure studies of atoms nowadays use these expansions, as they provide very precise results and high performance due to the evaluation properties of the Gaussian functions. Even though the Gaussian functions do not have the correct shape, an expansion can be fitted pretty well [15].

As for the quantum dot systems we will study closed shells only, but for atoms we will introduce subshells as well which are dependent on  $l$  and  $m$  in addition to the principal quantum number  $n$ . Traditionally, the first few subshells are denoted by  $s, p, d$  and  $f$ , and the meaning can be found in table (4.1), together with number of electrons in each subshell.

**Table 4.1:** Degeneracy and naming conventions of the first subshells.

Subshell label	$l$	Max electrons	Name
$s$	0	2	sharp
$p$	1	6	principal
$d$	2	10	diffuse
$f$	3	14	fundamental
$g$	4	18	alphabetic hereafter

For Helium, we have two electrons with  $n = 1$ , which means that both have  $l = 0$  and both electrons are in the  $s$ -subshell, or the so-called  $s$ -wave. We can thus write the electron configuration as  $1s^2$ .

Similarly to the principal quantum number  $n$ , we can use the rule of thumb that the lower  $l$  the lower energy, such that for Beryllium all four electrons are still in the  $s$ -subshell. Beryllium therefore has electron configuration  $1s^22s^2$  or  $[\text{He}] 2s^2$ . Since both subshells are fully occupied, Beryllium can be included in our closed-shell calculations.

If we continue with the same rules, we see that the next closed-shell atom has a fully occupied  $p$ -subshell as well, which is Neon with 10 electrons. This is a noble gas, and we can write the electron configuration as  $[\text{Be}] 2p^6$ . All noble gases have endings  $Xs^2Xp^6$ , which is the reason why they always have 8 valence electrons.

We can now compare this to the periodic table, and observe that the first two rows agree with the theory presented above: The first row has two elements and the second has eight. However, the third one also has eight elements, which does not fit our theory. It must be something we have overlooked.

The reason is that the angular momentum contribution is not taken into account, i.e., we need to include the Hamiltonian term

$$V_L = \frac{l(l+1)}{2r^2} \quad (4.23)$$

as well. If we do so, we see that the rule of thumb defined above not always holds. Sometimes a low  $l$  in a higher  $n$  causes lower energy than a high  $l$  in a lower  $n$ .

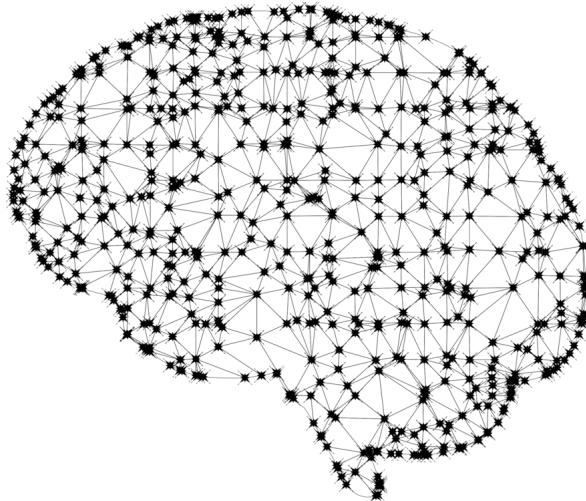
# Chapter 5

## Machine Learning

People worry that computers will get too smart and take over the world, but the real problem is that they're too stupid and they've already taken over the world.

---

Pedro Domingos, [66]



**Figure 5.1:** Artificial neural networks are inspired by neural networks in the brain.  
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The use of the term *machine learning* has exploded over the past years, and sometimes it sounds like it is a totally new field. However, the truth is that many of the methods are relatively old, where for instance *linear regression* was known early in the 19th century [1, 2]. Those methods have just recently been taken under the machine learning umbrella, which is one of the reasons why the term is used more frequently than before. Another important contributor to the booming popularity is the dramatically improvement of a majority of the machine learning algorithms. Some important milestones were discussed in the introduction.

Unlike traditional algorithms, machine learning algorithms are not told what to do explicitly, but they use optimization tools to find patterns in a data set with or without prior knowledge. Based on this, we land at the following definition:

“ Machine learning is the science of getting computers to act without being explicitly programmed.”

As a consequence, we often do not know exactly what the algorithm does and why it behaves as is does. Because of this behavior and the fact that artificial neural networks are inspired by the human brain, the processing is often called artificial intelligence. In our search for a technique to solve quantum mechanical problems where less physical intuition is needed, machine learning appears as a natural tool.

We typically classify the machine learning methods as either supervised or unsupervised learning, where the former methods require targets which it can be trained on. However, in this thesis we want to study a robust method which also works in the cases where we do not have any targets to train the models on. For that reason, supervised learning does not qualify and we need to look at unsupervised methods. However, we will first discuss supervised learning as an introduction to unsupervised learning, since they have many common concepts. Supervised learning and optimization methods are discussed in this chapter, and unsupervised learning and restricted Boltzmann machines are discussed in chapter 6.

As hinted above, in machine learning we want to fit a model to a data set in the best possible way. In supervised learning, we have prior knowledge about what kind of results the model should give in some specific cases, which we can use to train our model. After training, we want the model to

1. be able to reproduce the *targets* (the prior known results),
2. be able to fit future observations.

In this section we will take a closer look at how to find a model which satisfies both these goals. If the first one is satisfied, the second is not necessarily satisfied. Let us first look at polynomial regression.

## 5.1 Polynomial regression

Polynomial regression is perhaps the most intuitive example on this, where we want to find the line that fits some data points in the best possible way. In two dimensions, the data set consists of some  $n$  number of x- and y coordinates,

$$\begin{aligned}\mathbf{x} &= (x_1, x_2, \dots, x_n) \\ \mathbf{y} &= (y_1, y_2, \dots, y_n)\end{aligned}$$

which we for instance could try to fit to a second order polynomial,

$$f(x) = ax^2 + bx + c, \quad (5.1)$$

where the parameters  $a$ ,  $b$  and  $c$  are our estimators. The polynomial is now our model. By inserting the  $x$ -values into the polynomial, we obtain a set of equations

$$\begin{aligned}\tilde{y}_1 &= ax_1^2 + bx_1 + c \\ \tilde{y}_2 &= ax_2^2 + bx_2 + c \\ \vdots &\quad \vdots \quad \vdots \quad \vdots \\ \tilde{y}_n &= ax_n^2 + bx_n + c\end{aligned} \quad (5.2)$$

where  $\tilde{y}_i = f(x_i)$  is the  $y$ -value of the polynomial at  $x = x_i$ . What we want to do is to determine the estimators  $a$ ,  $b$  and  $c$  such that the mean squared error (MSE) of all these equations is minimized,

$$\min_{a,b,c} \frac{1}{n} \sum_{i=0}^{n-1} (y_i - f(x_i; a, b, c))^2. \quad (5.3)$$

There are several ways to do this, but they all are based on the *cost function* (also called the loss function), which can simply be defined as the MSE,

$$\mathcal{C}(a, b, c) = \frac{1}{n} \sum_{i=0}^{n-1} \left( y_i - (ax_i^2 + bx_i + c) \right)^2, \quad (5.4)$$

and which we seek to minimize. Before we start minimizing this, we will introduce a more general notation, where the estimators are collected in a column vector

$$\boldsymbol{\theta} = (a, b, c)^T$$

and the  $x_i^j$ 's also are collected in a row vector

$$\mathbf{X}_i = (x_i^2, x_i, 1).$$

Using this, the cost function can be written as

$$\begin{aligned} \mathcal{C}(\boldsymbol{\theta}) &= \frac{1}{n} \sum_{i=0}^{n-1} \left( y_i - \sum_{j=0}^2 X_{ij} \theta_j \right)^2 \\ &= \frac{1}{n} \sum_{i=0}^{n-1} \left( y_i - \mathbf{X}_i \boldsymbol{\theta} \right)^2 \\ &= \frac{1}{n} (\mathbf{y} - \mathbf{X} \boldsymbol{\theta})^T (\mathbf{y} - \mathbf{X} \boldsymbol{\theta}) \end{aligned} \quad (5.5)$$

where we in the last step have collected all the vectors  $\mathbf{X}_i$  in a matrix. As the minimum of the cost function with respect to a estimator  $\theta_j$  is found when the derivative is zero, we need to solve

$$\begin{aligned} \frac{\partial \mathcal{C}(\boldsymbol{\theta})}{\partial \theta_j} &= \frac{\partial}{\partial \theta_j} \left( \frac{1}{n} \sum_{i=0}^{n-1} \left( y_i - \sum_{j=0}^2 X_{ij} \theta_j \right)^2 \right) \\ &= \frac{2}{n} \sum_{i=0}^{n-1} X_{ij} \left( y_i - \sum_{j=0}^2 X_{ij} \theta_j \right) = 0. \end{aligned}$$

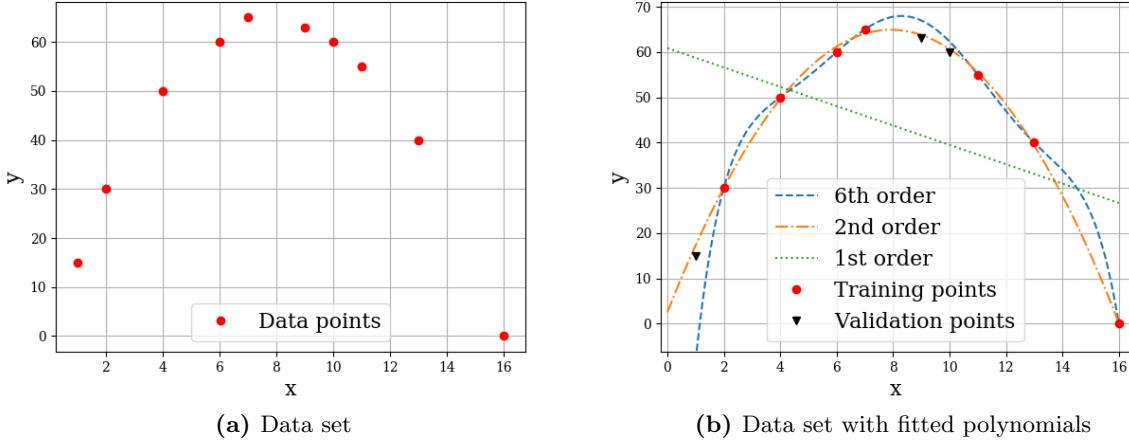
We can go further and write it on matrix-vector form as

$$\frac{\partial \mathcal{C}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = \frac{2}{n} \mathbf{X}^T (\mathbf{y} - \mathbf{X} \boldsymbol{\theta}) = 0$$

where differentiating with respect to a vector here means that each component is  $\partial \mathcal{C}(\boldsymbol{\theta}) / \partial \theta_j$ . This is satisfied if

$$\boldsymbol{\theta} = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{y} \quad (5.6)$$

which is the equation we seek to solve to find the best fitting polynomial. Before we proceed to the general case, let us have a quick look at an example.



**Figure 5.2:** Figure (a) presents the data points given in (5.7), while the figure (b) illustrates how a first-, second- and third order polynomial can be fitted to the training set in the best possible way.

**Table 5.1:** Best fitting polynomials of 1st, 2nd and 6th order degree to the data set in equation (5.7).  $f(x)$  gives the actual form of the polynomial, the training error gives the MSE to the training data set and the prediction error gives the MSE to the validation set.

Order	$f(x)$	Training error	Prediction error
1st	$-2.14x + 60.87$	327.22	927.87
2nd	$-x^2 + 15.74x + 2.51$	0.47	2.04
6th	$-0.001x^6 + 0.04x^5 - 0.90x^4 + 9.04x^3 - 47.52x^2 + 129.74x - 98.67$	2.54E-11	187.53

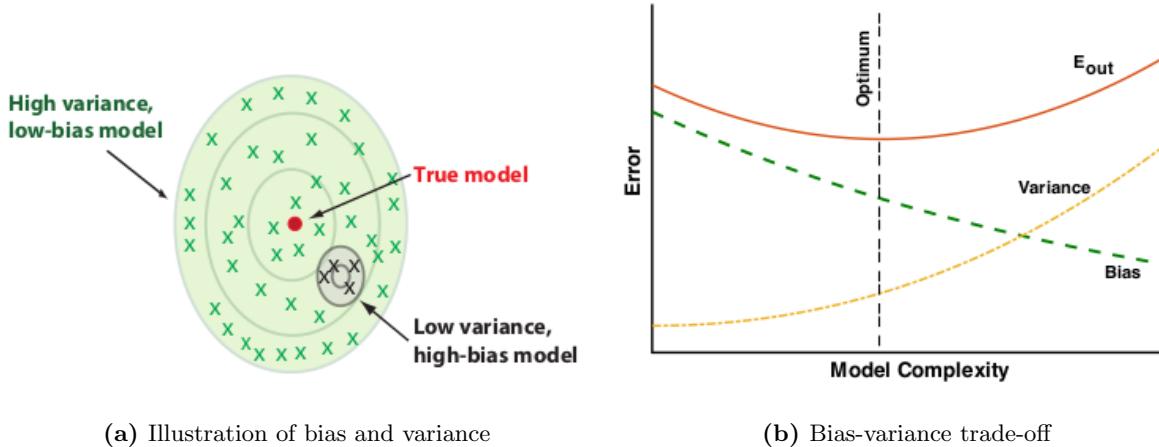
### 5.1.1 Example

In this example, we will see how we in practice fit a polynomial to a data set. Suppose we have a tiny data set consisting of 10 points on a plane

$$\begin{aligned} \mathbf{x} &= (1, 2, 4, 6, 7, 9, 10, 11, 13, 16) \\ \mathbf{y} &= (15, 30, 50, 60, 65, 63, 60, 55, 40, 0) \end{aligned} \quad (5.7)$$

to which we want to fit a polynomial of degree  $p$ . The data points can be seen in figure (5.2 a). The first thing we need to realize, is that in order to validate our models, we cannot use all points for the training. There is no strict rules on how much of the data set that should be used for training and validation, but at least the training data set should be larger than the validation data set. For this particular problem, we decide to leave out  $\{(1, 15), (9, 63), (10, 60)\}$  from the training, which we later will use for validation.

Furthermore, we use equation (5.6) to find the best fitting first-, second- and sixth order polynomials, and obtain the functions presented in table (5.1) with the respective training and prediction errors. The polynomials are also plotted in figure (5.2 b) together with the actual data points.



**Figure 5.3:** Examples of high variance, low-bias and low variance high-bias (a) and illustration of the bias-variance trade-off (b). Figures are taken from Mehta et.al., [82].

What we immediately observe, is that the more complex model (higher degree polynomial) the lower training error. In fact, the polynomial of sixth order reproduces the points perfectly. The first order polynomial is quite bad, while the second order polynomial is intermediate.

However, what really makes sense is the prediction error, and for that we can see that the sixth order polynomial performs terribly. When a model can reproduce the training set very well, but is not able to reproduce the validation set, we say that it overfits the data set. This means that the model is too complex for the purpose.

On the other hand, we see that the first order polynomial has also a large prediction error, which means that it is not able to reproduce the validation set either. We say that it is underfitted, and we are in need of a more complex model.

Finally, we have the second order polynomial, which is miles ahead its competitors when it comes to the prediction error. It turns out that the second order model has an appropriate complexity, which we could have guessed just by looking at the data points.

The natural question now is "*How do we find a correct model complexity?*". The answer is that one should try various complexities and calculate the prediction error for each model. To find the prediction error precisely, the standard is to use  $K$  cross-validation resampling, which tries  $K$  choices of validation set to make the most use of our data. More about resampling analysis can be found in section (7.6.2). A deeper understanding of the prediction error will hopefully be gained in the next section, on bias-variance decomposition.

## 5.2 Bias-variance tradeoff

Up to this point, we have skipped some important terms in the statistics behind machine learning. First, we have the *bias*, which describes the best our model could do if we had an infinite amount of training data. We also have the *variance*, which is a measure on the fluctuations in the predictions. In figure (5.3 a), an example on high variance low-bias and a low variance high bias models are presented. What we actually want is a low variance low-bias model, but this model is normally infeasible and we need to find the optimal tradeoff between bias and variance. This is known as the bias-variance tradeoff.

In figure (5.3 b), the bias-variance tradeoff is illustrated as a function of the model complexity. We observe that the prediction error is large when the model complexity is too low, which

corresponds to a low variance. This substantiates what we discussed in the example in (5.1.1), where we claimed that a too low model complexity underfits the data set. Therefore, a too low variance is associated with underfitting.

On the other side of the plot, we can see that also a too complex model causes a large prediction error, which corresponds to a low bias. As discussed before, a too complex model overfits the model, which is associated with low bias.

To minimize the prediction error, we should therefore neither minimize the bias nor the variance. Instead, we should find the bias and variance which corresponds to the lowest error. The prediction error can also be decomposed into bias and variance, given by

$$E[(y - \tilde{y})^2] = \text{bias}[\tilde{y}]^2 + \sigma^2[\tilde{y}] \quad (5.8)$$

where

$$\text{bias}[\tilde{y}] = E[\tilde{y}] - y \quad (5.9)$$

and

$$\sigma^2[\tilde{y}] = E[\tilde{y}^2] - E[\tilde{y}]^2. \quad (5.10)$$

### 5.3 Linear regression

Polynomial regression, as already discussed, is an example on a linear regression method, and was meant as motivation before we study linear regression in general. Instead of fitting a polynomials to a set of points, we can fit a general function in the form of

$$f(x_i) = \sum_{j=0}^p X_{ij}(x_i)\theta_j \quad (5.11)$$

where we have  $p$  estimators  $\theta_j$ . The matrix  $\mathbf{X}$  is called the *design matrix*, and the case where  $X_{ij}(x_i) = x_i^j$  corresponds to polynomial regression, but it can in principle be an arbitrary function of  $x_i$ .

The cost function for *the ordinary least square regression* (OLS) case is already found in equation (5.5), and we can recall it as

$$\mathcal{C}(\boldsymbol{\theta}) = \sum_{i=1}^n \left( y_i - \sum_{j=0}^p X_{ij}\theta_j \right)^2, \quad \text{OLS} \quad (5.12)$$

which is minimized when

$$\boldsymbol{\theta} = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{y}. \quad (5.13)$$

To solve this equation, we need to find the inverse of the matrix  $\mathbf{X}^T \mathbf{X}$ , which is typically done by *lower-upper* decomposition (LU) or *singular values decomposition* (SVD). Quite often when we deal with large data sets, the matrix above is singular, which means that the determinant of the matrix is zero. In those cases, we cannot find the inverse, and LU decomposition does not work. Fortunately, SVD *always* works, and in cases where the matrix is singular, it turns out to be a good idea.

#### 5.3.1 Singular value decomposition

Singular value decomposition is a method which decomposes a matrix into a product of three matrices, written as

$$\mathbf{X} = \mathbf{U} \boldsymbol{\Sigma} \mathbf{V}^T. \quad (5.14)$$

This might sounds like a bad idea, but especially for singular matrices this often makes life easier. The reason for this, is that only  $\Sigma$  is singular after the decomposition. For our case, we can thus write the matrix  $\mathbf{X}^T \mathbf{X}$  as

$$\mathbf{X}^T \mathbf{X} = \mathbf{V} \Sigma^T \Sigma \mathbf{V}^T = \mathbf{V} \mathbf{D} \mathbf{V}^T \quad (5.15)$$

which is non-singular. By multiplying with  $\mathbf{V}$  on the right-hand-side, we obtain

$$(\mathbf{X}^T \mathbf{X}) \mathbf{V} = \mathbf{V} \mathbf{D} \quad (5.16)$$

and similarly

$$(\mathbf{X} \mathbf{X}^T) \mathbf{U} = \mathbf{U} \mathbf{D} \quad (5.17)$$

when transposing the matrix. Using those expressions, one can show that

$$\mathbf{X} \boldsymbol{\theta} = \mathbf{U} \mathbf{U}^T \mathbf{y}. \quad (5.18)$$

### 5.3.2 Ridge regression

So, how can we avoid non-singular values in our matrix  $\mathbf{X}^T \mathbf{X}$ ? We can remove them by introduce a penalty  $\lambda$  to ensure that all the diagonal values are non-zero, which can be accomplished by adding a small value to all diagonal elements. Doing this, all diagonal elements will get a non-zero value and the matrix is guaranteed to be non-singular. Still using the matrix-vector form, this can be written as

$$\boldsymbol{\theta} = (\mathbf{X}^T \mathbf{X} + \lambda \mathbf{I})^{-1} \mathbf{X}^T \mathbf{y} \quad (5.19)$$

where  $\mathbf{I}$  is the identity matrix. The penalty  $\lambda$  is also a *hyper parameter*, which is a parameter that is set before the training begins, in contrast to the estimators which are determined throughout training. This method is called Ridge regression, and has a cost function given by

$$\mathcal{C}(\boldsymbol{\theta}) = \sum_{i=1}^n \left( y_i - \sum_{j=0}^p X_{ij} \theta_j \right)^2 + \lambda \sum_{j=1}^p |\theta_j|^2 \quad \text{Ridge} \quad (5.20)$$

where we in principle just add the L2-norm of the estimator vector to the OLS cost function. This can easiest be proven going from the cost function to the matrix-vector expression of  $\boldsymbol{\theta}$ , as we did for ordinary least squares.

### 5.3.3 LASSO regression

Finally, we introduce the *least absolute shrinkage and selection operator* (LASSO) regression, which in the same way as Ridge regression is based on a regularization. Instead of adding the L2-norm of the estimator matrix, we add the the L1-norm  $|\theta_j|$ , and the cost function expresses

$$\mathcal{C}(\boldsymbol{\theta}) = \sum_{i=1}^n \left( y_i - \sum_{j=0}^p X_{ij} \theta_j \right)^2 + \lambda \sum_{j=1}^p |\theta_j|. \quad \text{Lasso} \quad (5.21)$$

For LASSO regression, we cannot set  $\partial \mathcal{C}(\boldsymbol{\theta}) / \partial \theta_j = 0$  and find a closed-form expression of  $\boldsymbol{\theta}$ , which means that we need to use an iterative optimization algorithm in order to obtain the optimal estimators. Such optimization methods are essential in non-linear problems such as deep neural networks and variational Monte-Carlo. They will therefore be familiar to the reader throughout this thesis.

In section 5.6, we will present various optimization schemes, but for now on we will stick to one of the most basic methods, *gradient descent*, which can be written as

$$\theta_j^+ = \theta_j - \eta \frac{\partial \mathcal{C}(\boldsymbol{\theta})}{\partial \theta_j}, \quad (5.22)$$

where  $\theta_j^+$  is the updated version of  $\theta_j$  and  $\mathcal{C}(\boldsymbol{\theta})$  is an arbitrary cost function. Here we are introduced to a new hyperparameter,  $\eta$ , known as the *learning rate*, which controls how much the estimators should be changed for each iteration. This should be set carefully, where a too large  $\eta$  will make the cost function diverge and a too small  $\eta$  will make the training too slow. Typically, to choose a  $\eta \in 0.01 - 0.0001$  is a good choice. For ordinary least squares, the parameter update can be written as

$$\boldsymbol{\theta}^+ = \boldsymbol{\theta} - \eta \mathbf{X}^T (\mathbf{y} - \mathbf{X}^T \boldsymbol{\theta}) \quad (5.23)$$

## 5.4 Logistic regression

Up to this point, we have discussed regression with continuous outputs. But what do we do if we want a discrete output, for example in form of classification? This is what logistic regression is about, where the name comes from the logistic function (sigmoid function) which is used to fire or not fire the neurons. As for the linear regression, we also need a cost function in logistic regression, which we will define in the following.

Consider a system that can take two possible energies  $\varepsilon_0$  and  $\varepsilon_1$ . From elementary statistical mechanics, we know that the probability of finding a system in a state of a certain energy is given by the Boltzmann distribution, such that

$$P(y_i = 0) = \frac{\exp(-\varepsilon_0/k_B T)}{\exp(-\varepsilon_0/k_B T) + \exp(-\varepsilon_1/k_B T)} \quad (5.24)$$

$$= \frac{1}{1 + \exp(-(\varepsilon_1 - \varepsilon_0)/k_B T)} \quad (5.25)$$

which is the *sigmoid function*, in the most general given by

$$f(x) = \frac{1}{1 + \exp(-x)}. \quad (5.26)$$

The first denominator is known as the *partition function*,

$$Z = \sum_{i=0}^1 \exp(-\varepsilon_i/k_B T) \quad (5.27)$$

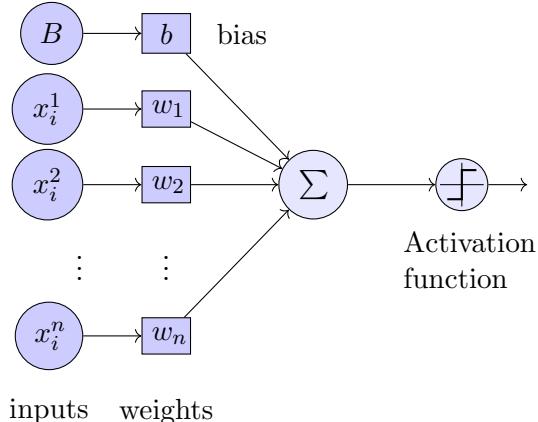
where  $k_B$  is Boltzmann's constant and  $T$  is the system temperature. The probability of finding the system in the second state is given by

$$P(y_i = 1) = 1 - P(y_i = 0) \quad (5.28)$$

$$= \frac{1}{1 + \exp(-(\varepsilon_0 - \varepsilon_1)/k_B T)}. \quad (5.29)$$

Notice that the only thing we need is the difference in energy between those two systems, not the energy itself. This is often the case in physics. If we now assume that the difference in energy can be written as a function of the coordinates that specify the state  $i$ ,  $\mathbf{X}_i$  and a matrix of parameters,  $\mathbf{w}$ , known as the *weights*, the difference can be written as

$$\varepsilon_1 - \varepsilon_0 = \mathbf{X}_i^T \mathbf{w} \equiv \tilde{y}_i, \quad (5.30)$$



**Figure 5.4:** Logistic regression model with  $n$  inputs. Each input  $X_i^j$  is multiplied with a weight  $w_j$ , and the contribution from all elements is summarized. The output is obtained after the sum is activated by an activation function.

giving the conditional probability

$$P(\mathbf{X}_i, y_i | \mathbf{w}) = (f(\mathbf{X}_i^T \mathbf{w}))^{y_i} (1 - f(\mathbf{X}_i^T \mathbf{w}))^{1-y_i}. \quad (5.31)$$

where  $\mathbf{w}$  needs to be *a priori* known. If we have a set of multiple states stored in a  $\mathcal{D} = \{(\mathbf{X}_i, y_i)\}$ , the joint probability yields

$$P(\mathcal{D} | \mathbf{w}) = \prod_{i=1}^n (f(\mathbf{X}_i^T \mathbf{w}))^{y_i} (1 - f(\mathbf{X}_i^T \mathbf{w}))^{1-y_i} \quad (5.32)$$

which is known as the *likelihood*. The *log-likelihood* function is simply the log of the likelihood, and is given by

$$l(\mathbf{w}) = \sum_{i=1}^n \left[ y_i \log f(\mathbf{X}_i^T \mathbf{w}) + (1 - y_i) \log (1 - f(\mathbf{X}_i^T \mathbf{w})) \right]. \quad (5.33)$$

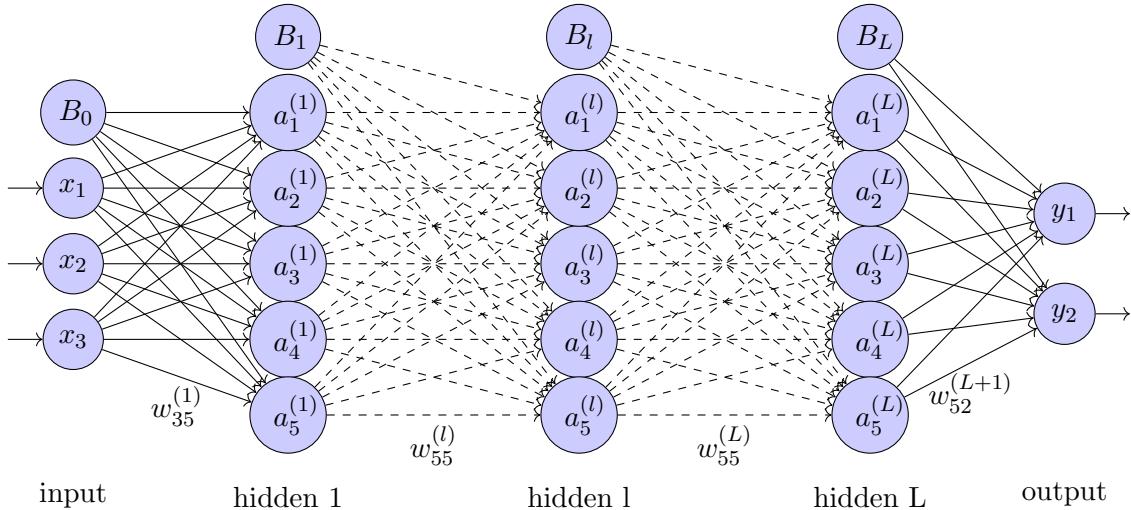
As in linear regression, we want to find a cost function which we can minimize in order to fit the model to the data set. Since the log-likelihood function is maximized where the highest probability is, a natural choice is to set the cost function as the negative log-likelihood function,

$$\mathcal{C}(\mathbf{w}) = -l(\mathbf{w}) = - \sum_{i=1}^n \left[ y_i \log f(\mathbf{X}_i^T \mathbf{w}) + (1 - y_i) \log (1 - f(\mathbf{X}_i^T \mathbf{w})) \right], \quad (5.34)$$

which is the *cross entropy*. To clarify things, we will try to illustrate how this works. In figure (5.4), we have a input set  $\mathbf{X}_i$  where each element is multiplied with a parameter from  $\mathbf{w}$  and summarized. This corresponds to the inner product  $\mathbf{X}_i^T \mathbf{w}$ . Further, the sum (or the inner product) is *activated* by an *activation function*, which we above have assumed to be the sigmoid function. The output is then given by

$$z_i = f(\mathbf{X}_i^T \mathbf{w}). \quad (5.35)$$

where we have assumed that the bias node is included in the  $\mathbf{X}$ 's and the bias weight is included in the  $\mathbf{w}$ 's. The bias node is added in order to shift the activation function to the left or right, and works in the same way as a constant term in a function.



**Figure 5.5:** Neural network with 3 input units,  $L$  hidden layers with 5 hidden units each and two outputs.  $B_0, B_1, B_l$  and  $B_L$  are bias units for their respective layers, and the dashed lines indicate that it might be more layers between the two layers. We have labeled a few of the lines to relate them to the weights.

The output from the activation is used further in the cost function to calculate the cost. As for LASSO regression, the cost function is then minimized in an iterative scheme, where for example the gradient descent method gives the weight update

$$\mathbf{w}^+ = \mathbf{w} - \eta \mathbf{X}^T [\mathbf{y} - f(\mathbf{X}^T \mathbf{w})]. \quad (5.36)$$

which is extremely similar, not to say identical to the estimator update for ordinary least square presented in equation (5.23).

## 5.5 Neural networks

Now we know enough to dive into the field of artificial neural networks. Neural networks can give either continuous or discrete outputs, and is therefore a competitor to both linear and logistic regression. The big strength of neural networks is that one can add multiple *layers*, which potentially makes the model extremely flexible. According to **the universal approximation theorem**, a neural network with only one hidden layer can approximate any continuous function [24]. However, often multiple layers are used since this tends to give fewer units in total, and is known to give better results. Neural networks of more than one layer are called *deep* networks, and as more layers are added the network gets *deeper*.

In figure (5.5), a multi-layer neural network is illustrated. It has some similarities with the logistic regression model in figure (5.4), but a hidden layer and multiple outputs are added. We have also dropped the representation of the weights (apart from a few labeled ones), but each line corresponds to a weight. Each node represents a neuron in the brain.

Without a hidden layer, we have seen that the update of weights is quite straight forward. For a neural network consisting of multiple layers, the question is: how do we update the weights when we do not know the values of the hidden units? And how do we know which layer causing the error? This will be explained in section 5.5.3, where backward propagation, the most popular technique for weight update in a neural network, is discussed. Before that, we will generalize the forward phase presented in logistic regression.

### 5.5.1 Forward phase

In the previous section, we saw how the output is found for a single perceptron. For a neural network, the net output to the first layer is similar, and given by

$$z_j^{(1)} = \sum_{i=1}^{N_0} x_i w_{ij}^{(1)} = \mathbf{x}^T \mathbf{w}_j^{(1)}$$

where  $N_0$  is the number of units in layer 0 (the input layer) and we again have assumed that the bias node is included in the input vector  $\mathbf{x}$  and the first set of bias weights are included in the first weight matrix  $\mathbf{w}^{(1)}$ . The same applies for the other layers as well. If we let the activation function,  $f(x)$ , act on the net output, we get the real output given by

$$a_j^{(1)} = f(z_j^{(1)}) = f\left(\sum_{i=1}^{N_0} x_i w_{ij}^{(1)}\right).$$

This is then again the input to the next layer with  $N_1$  units, so the output from the second layer is simply

$$a_j^{(2)} = f\left(\sum_{i=1}^{N_1} a_i^{(1)} w_{ij}^{(2)}\right).$$

For a neural network of multiple layers, the same procedure applies for all the layers and we can find a general formula for the output at a layer  $l$ . The net output to a node  $z_j^{(l)}$  in layer  $l$  can be found to be

$$z_j^{(l)} = \sum_{i=1}^{N_{l-1}} a_i^{(l-1)} w_{ij}^{(l)} \quad (5.37)$$

where layer  $l - 1$  has  $N_{l-1}$  units and we need to be aware that  $a_j^{(0)} = x_j$ . After activation, the output is obviously found to be

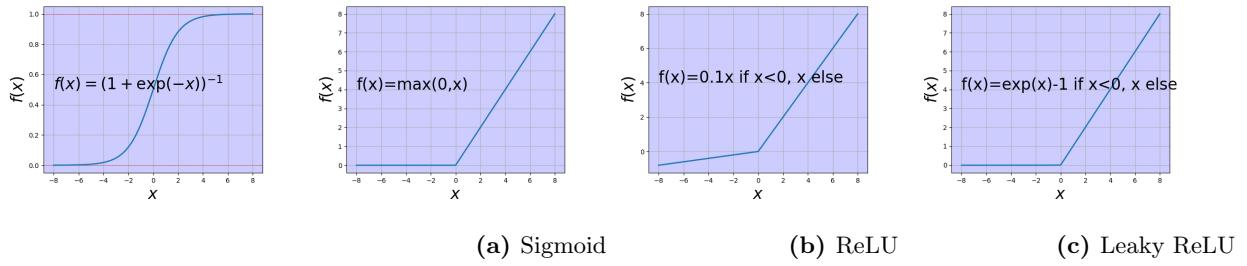
$$a_j^{(l)} = f\left(\sum_{i=1}^{N_{l-1}} a_i^{(l-1)} w_{ij}^{(l)}\right) \quad (5.38)$$

which is the only formula needed for the forward phase. The activation function  $f(x)$  is not explicitly defined, because it is often expedient having the chance to experiment with multiple activation functions.

### 5.5.2 Activation function

Until now, we have mentioned the sigmoid function as the only activation function. However, there are plenty of other activation functions that one can use. In fact, the sigmoid function has lost its popularity, and is today superseded by the more modern functions based on *rectified linear units* (ReLU). Some popular choices are the *leaky* ReLU and *exponential linear units* (ELU), which are linear for positive numbers. The pure linear activation function is still widely used, especially on the output layer.

In figure (5.6), standard ReLU, leaky ReLU and ELU are plotted along with the sigmoid function.



**Figure 5.6:** Some well-known activation functions. The sigmoid function stands out from the others since it maps between 0 and 1, and it is not linear for positive numbers.

### 5.5.3 Backward propagation

Backward propagation is the most robust technique for updating the weights in a neural network, and is actually again based on the weight update presented for linear and logistic regression. The algorithm for this was presented 1986, which made the deep neural networks able to solve relatively complicated problems for the first time [22]. To update the weights, one starts with the outputs and updates the weights layer-wise until one gets to the inputs, hence the name backward propagation.

As observed above, a node is dependent on all the units in the previous layers, and so are the weights. This means that the units are dependent on a large number of parameters, which makes the training scheme quite complex. Nevertheless, there is possible to generalize this to express the updating formulas on a relatively simple form.

From the linear and logistic regression, we know that we need the derivative of the cost function in order to implement the weight update regime. Again, we define the cost function as the mean square error,

$$\mathcal{C}(\mathbf{w}) = \frac{1}{2} \sum_{i=1}^{N_L} (y_i - a_i^{(L)})^2 \quad (5.39)$$

where we have  $L + 1$  layers ( $L$  is the last layer) and  $N_L$  output units. The derivative of this with respect to one of the weights between the  $L - 1$ 'th and  $L$ 'th layer can be written as a sum using the chain rule

$$\frac{\partial \mathcal{C}(\mathbf{w})}{\partial w_{jk}^{(L)}} = \frac{\partial \mathcal{C}(\mathbf{w})}{\partial a_j^{(L)}} \frac{\partial a_j^{(L)}}{\partial z_j^{(L)}} \frac{\partial z_j^{(L)}}{\partial w_{jk}^{(L)}}. \quad (5.40)$$

where  $z_j^{(L)}$  and  $a_j^{(L)}$  are found from equations (5.37) and (5.38) respectively. If we start with the first factor, it can easily be found to be

$$\frac{\partial \mathcal{C}(\mathbf{w})}{\partial a_j^{(L)}} = -(y_j - a_j^{(L)}) \quad (5.41)$$

using the definition of the cost function. The second factor is the derivative of the activation function with respect to its argument, and is for the sigmoid function given by

$$\frac{\partial a_j^{(L)}}{\partial z_j^{(L)}} = a_j^{(L)}(1 - a_j^{(L)}). \quad (5.42)$$

Finally, the last factor is found from equation (5.37), and we obtain

$$\frac{\partial z_j^{(L)}}{\partial w_{jk}^{(L)}} = a_k^{(L-1)}. \quad (5.43)$$

Collecting all the factors, the update of the last set of weights can be found by

$$\frac{\partial \mathcal{C}(\mathbf{w})}{\partial w_{jk}^{(L)}} = -(y_j - a_j^{(L)}) a_j^{(L)} (1 - a_j^{(L)}) a_k^{(L-1)}. \quad (5.44)$$

when the sigmoid function is used in the activation. In the next step, we can define

$$\delta_j^{(L)} = -a_j^{(L)} (1 - a_j^{(L)}) (y_j - a_j^{(L)}) = f'(a_j^{(L)}) \frac{\partial \mathcal{C}(\mathbf{w})}{\partial a_j^{(L)}} = \frac{\partial \mathcal{C}(\mathbf{w})}{\partial z_j^{(L)}} \quad (5.45)$$

such that the weight update can be expressed on a neater form

$$\frac{\partial \mathcal{C}(\mathbf{w})}{\partial w_{jk}^{(L)}} = \delta_j^{(L)} a_k^{(L-1)}. \quad (5.46)$$

For a general layer  $l$ , the derivative of the cost function with respect to a weight  $w_{jk}^{(l)}$  is similar, and given by

$$\frac{\partial \mathcal{C}(\mathbf{w})}{\partial w_{jk}^{(l)}} = \delta_j^{(l)} a_k^{(l-1)}. \quad (5.47)$$

Our goal is to find the general relation between layer  $l$  and  $l + 1$ , and therefore we use the chain rule and sum over all the net outputs in layer  $l + 1$ ,

$$\delta_j^{(l)} = \frac{\partial \mathcal{C}(\mathbf{w})}{\partial z_j^{(l)}} = \sum_k \frac{\partial \mathcal{C}(\mathbf{w})}{\partial z_k^{(l+1)}} \frac{\partial z_k^{(l+1)}}{\partial z_j^{(l)}}. \quad (5.48)$$

We now recognize that the first factor in the sum is just  $\delta_k^{(l+1)}$  and the last factor can be found from equation (5.37). We obtain the final expression,

$$\delta_j^{(l)} = \sum_k \delta_k^{(l+1)} w_{kj}^{(l+1)} f'(z_j^{(l)}) \quad (5.49)$$

where we use the expression of  $\delta_j^{(L)}$  as our initial condition. As for several of the methods discussed above, a solution of the weight update does not exist in closed form and we need to rely on iterative optimization methods. Using gradient descent, a new weight  $w_{ij}^{(l)+}$  is found from

$$w_{ij}^{(l)+} = w_{ij}^{(l)} - \eta \frac{\partial \mathcal{C}(\mathbf{w})}{\partial w_{ij}^{(l)}}, \quad (5.50)$$

where other optimization methods will be discussed in the following section.

## 5.6 Optimization algorithms

We have above discussed the gradient descent optimization algorithm, which is among the most basic optimization methods available. That method is based on the gradient, which is the slope of the cost function, but many methods are also in need of the Hessian matrix, which gives the curvature of the cost function. We will barely scratch the surface of this field, limiting us to the gradient methods.

To have the method fresh in mind, we will start with reintroducing the gradient descent method before we move on to its stochastic brother. We will then have a look at how momentum can be added, and finally we examine the stochastic and momentum based ADAM optimizer. Given a cost function  $\mathcal{C}(\boldsymbol{\theta})$ , the gradient with respect to a parameter  $\theta$  can be found from

$$\nabla_{\theta}\mathcal{C}(\boldsymbol{\theta}) \equiv \frac{\partial\mathcal{C}(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}}, \quad (5.51)$$

where we henceforth use the short-hand notation with  $\nabla$  representing the gradient. Differentiating with respect to the vector implies that the operation shall be done element-wise.

### 5.6.1 Gradient descent

Perhaps the simplest and most intuitive method for finding the minimum is the gradient descent method, which reads

$$\boldsymbol{\theta}_t = \boldsymbol{\theta}_{t-1} - \eta \nabla_{\theta}\mathcal{C}(\boldsymbol{\theta}_{t-1}) \quad (5.52)$$

where  $\boldsymbol{\theta}_t$  is the parameter vector at time step (iteration)  $t$  and  $\eta$  is the learning rate.  $\nabla_{\theta}\mathcal{C}(\boldsymbol{\theta}_{t-1})$  is the gradient of the cost function with respect to all the parameters  $\theta$  at time  $t-1$ .

The idea is to find the direction where the cost function  $\mathcal{C}(\boldsymbol{\theta})$  has the steepest slope, and move in the direction which minimizes the cost function. For every time step, the cost function is thus minimized, and when the gradient approaches zero the minimum is found. A possible, but basic, stop criterion is

$$\nabla_{\theta}\mathcal{C}(\boldsymbol{\theta}_t) < \varepsilon. \quad (5.53)$$

where  $\varepsilon$  is a tolerance. More robust methods are based on comparing the value of the cost function for several past iterations.

In cases where the cost function is not strictly decreasing, we will have both local and global minima. Often, it is hard to say whether we are stuck in a local or global minimum, and this is where the stochasticity enters the game.

### 5.6.2 Stochastic gradient descent

Stochastic gradient descent is closely related to the gradient descent method, but the method uses randomly selected batches to evaluate the gradients, hence the stochasticity. By introducing this randomness, the parameters will not always be updated in order to minimize the energy, which makes us less likely to be stuck in a local minimum.

In practice, one splits the data set in  $n$  batches, and select one of them to be used in the parameter update. Our hope is that this batch is representative for the entire data set, such that the new parameters provide a lower cost function. If that is the case, we have reduced the cost of an iteration significantly, since we only need to care about a batch. After each batch in the data set has had an opportunity to update the internal parameters, we say that we have gone through an *epoch*.

We are not guaranteed that updating the parameters with respect to a batch gives a lower cost function, and when it is not, we need to run more batches in order to minimize the cost

function. Since each iteration is faster than for standard gradient descent, this is acceptable. As long as the batch is slightly representative for the entire data set, the cost function will be minimized in the end.

Mathematically, the method can be expressed as

$$\boldsymbol{\theta}_t = \boldsymbol{\theta}_{t-1} - \eta \nabla_{\boldsymbol{\theta}} \mathcal{C}_i(\boldsymbol{\theta}_{t-1}) \quad (5.54)$$

where we use the  $i$ 'th batch in the parameter update. Standard gradient descent is actually just a special case of this, where we only have one batch ( $i$  includes the whole data set). If we still get stuck in local minima after adding the stochasticity, it might be a good idea to add momentum as well.

### 5.6.3 Adding momentum

If we go back to an introductory mechanics course, you might remember that momentum is a quantity that maintains the motion of a body. Imagine a ball that rolls down a steep hill, but then there is a local minimum that it needs to escape to keep rolling. If it has enough momentum, it will be able to escape.

Exactly the same idea lies behind the momentum used in optimization algorithms; the momentum will try to maintain the motion towards the global minimum, which makes the system less likely to be stuck in a local minimum. Momentum can be added to most optimization algorithms, also gradient descent and stochastic gradient descent. The way we do it is to save the direction we were moving during the previous iteration, and use it as a contribution to the next gradient update. A typical implementation of the first-order momentum applied on gradient descent looks like

$$\begin{aligned} \mathbf{m}_t &= \gamma \mathbf{m}_{t-1} + \eta \nabla_{\boldsymbol{\theta}} \mathcal{C}_i(\boldsymbol{\theta}_{t-1}) \\ \boldsymbol{\theta}_t &= \boldsymbol{\theta}_{t-1} - \mathbf{m}_t \end{aligned} \quad (5.55)$$

where  $\gamma$  is the momentum parameter, which is just another hyper-parameter usually initialized to a small number.  $\mathbf{m}_t$  is the momentum vector, and can be initialized as the zero vector.

The optimization algorithm can be modified further in unlimited ways. A common improvement is to add higher order momentum, another is to make the learning rate adaptive. We have implemented the most basic version of this, with monotonic adaptivity. Many algorithms, such as the conjugate gradient method, also make use of the Hessian as discussed in the introduction, but that is another level of complexity.

We will end this section with setting up the algorithm of a stochastic gradient descent optimization with momentum and monotonic adaptivity. The algorithm is found in algorithm (1).

### 5.6.4 ADAM

ADAM is a first-order stochastic optimization method which is widely used in machine learning. It was discovered by D.P. Kingma and J. Ba, and published in a 2014 paper. The article has already more than 25000 citations [64]! So what makes this method so popular?

The main reason why it is widely used, is obviously that it provides good performance. The fact that it only requires the gradient makes it efficient, and the way the momentum is implemented still makes it capable of handle a large number of parameters. The optimization algorithm

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**Algorithm 1:** Adaptive stochastic gradient descent with momentum. See sections (5.6.2-5.6.3) for details. Robust default settings for the hyper-parameters are  $\eta = 0.001$ ,  $\gamma = 0.01$  and  $\lambda = 0.1$ . All the operations are element-wise.

---

**Parameter:**  $\eta$ : Learning rate  
**Parameter:**  $\gamma$ : Momentum parameter  
**Parameter:**  $\lambda$ : Monotonic decay rate  
**Require** :  $\mathcal{C}(\theta)$ : Cost function  
**Data** :  $\theta_0$ : Initial parameters

- 1  $\mathbf{m}_0 \leftarrow 0$  (Initialize momentum vector);
- 2  $t \leftarrow 0$  (Initialize time step);
- 3 **while**  $\theta_t$  not converged **do**
- 4   |  $t \leftarrow t + 1$  (Increase time for each iteration);
- 5   |  $\mathbf{g}_t \leftarrow \nabla_{\theta} \mathcal{C}_t(\theta_{t-1})$  (Get gradients from a given batch at time  $t$ );
- 6   |  $\mathbf{m}_t \leftarrow \gamma \mathbf{m}_{t-1} + \eta \cdot \mathbf{g}_t$  (Update first momentum estimate);
- 7   |  $\theta_t = \theta_{t-1} - \eta \cdot \mathbf{m}_t / \lambda^t$  (Update parameters);
- 8 **end**

**Result:** Updated parameters  $\theta_t$  after convergence

---

can be expressed as a set of equations

$$\begin{aligned}
 \mathbf{g}_t &= \nabla_{\theta} \mathcal{C}_t(\theta_{t-1}) \\
 \mathbf{m}_t &= \gamma_1 \mathbf{m}_{t-1} + (1 - \gamma_1) \mathbf{g}_t \\
 \mathbf{v}_t &= \gamma_2 \mathbf{v}_{t-1} + (1 - \gamma_2) \mathbf{g}_t^2 \\
 \hat{\mathbf{m}}_t &= \mathbf{m}_t / (1 - \gamma_1^t) \\
 \hat{\mathbf{v}}_t &= \mathbf{v}_t / (1 - \gamma_2^t) \\
 \theta_t &= \theta_{t-1} - \eta \hat{\mathbf{m}}_t / (\sqrt{\hat{\mathbf{v}}_t} + \epsilon)
 \end{aligned} \tag{5.56}$$

where  $\mathbf{m}_t$  is the biased first momentum estimate of the parameter vector  $\theta$  and  $\mathbf{v}_t$  is the biased second raw moment estimate. The momentum parameters need to be in the range  $\gamma_1, \gamma_2 \in [0, 1]$ , and are often set to values close to 1. This makes the optimization adaptive: as time goes, the factors  $1 - \gamma_1^t$  and  $1 - \gamma_2^t$  approach 1 from below.  $\eta$  corresponds to the learning rate, and should be a small number. Finally, the parameter  $\epsilon$  is added to avoid division by zero.

We can set up the algorithm in a similar manner to the adaptive stochastic gradient descent algorithm from above, which gives the algorithm (2).

---

**Algorithm 2:** ADAM optimizer. Robust default settings for the hyper-parameters are  $\eta = 0.001$ ,  $\gamma = 0.01$  and  $\lambda = 0.1$ . All the operations are element-wise, and for in-depth information see the original paper, [64].

---

**Parameter:**  $\eta$ : Learning rate  
**Parameter:**  $\gamma_1, \gamma_2 \in [0, 1]$ : Momentum parameters  
**Parameter:**  $\varepsilon$ : Division parameter  
**Require** :  $\mathcal{C}(\theta)$ : Cost function  
**Data** :  $\theta_0$ : Initial parameters

- 1  $m_0 \leftarrow 0$  (Initialize 1<sup>st</sup> momentum vector);
- 2  $v_0 \leftarrow 0$  (Initialize 2<sup>nd</sup> momentum vector);
- 3  $t \leftarrow 0$  (Initialize time step);
- 4 **while**  $\theta_t$  not converged **do**
- 5   |  $t \leftarrow t + 1$  (Increase time for each iteration);
- 6   |  $\mathbf{g}_t \leftarrow \nabla_{\theta} \mathcal{C}_t(\theta_{t-1})$  (Get gradients from a given batch at time  $t$ );
- 7   |  $\mathbf{m}_t \leftarrow \gamma_1 \mathbf{m}_{t-1} + (1 - \gamma_1) \cdot \mathbf{g}_t$  (Update first momentum estimate);
- 8   |  $\mathbf{v}_t \leftarrow \gamma_2 \mathbf{v}_{t-1} + (1 - \gamma_2) \cdot \mathbf{g}_t^2$  (Update second raw momentum estimate);
- 9   |  $\hat{\mathbf{m}}_t \leftarrow \mathbf{m}_t / (1 - \gamma_1^t)$  (Bias-corrected first momentum estimate);
- 10   |  $\hat{\mathbf{v}}_t \leftarrow \mathbf{v}_t / (1 - \gamma_2^t)$  (Bias-corrected second momentum estimate) ;
- 11   |  $\theta_t \leftarrow \theta_{t-1} - \eta \cdot \hat{\mathbf{m}}_t / (\sqrt{\hat{\mathbf{v}}_t} + \varepsilon)$  (Update parameters) ;
- 12 **end**

**Result:** Updated parameters  $\theta_t$  after convergence

---



## **Part II**

# **Advanced Theory**



# Chapter 6

## Boltzmann Machines

Available energy is the main object at stake in the struggle for existence and the evolution of the world.

---

Ludwig Boltzmann, [42]

Boltzmann machines are generative, energy based neural network models that fall under the category unsupervised learning. In unsupervised learning, unlike supervised learning which was discussed in the previous chapter, the network is fed with an input data set only, i.e., we do not have any targets to supervise the network during the training. The task is then to find structures in the data, comparing data sets to each other and categorize the data sets with respect to their similarities and differences.

They were invented in 1985 by Geoffrey Hinton [20], often referred to as "The Godfather of Deep Learning"<sup>1</sup> and are based on the Boltzmann distribution, hence the name. Boltzmann machines with constrained connectivity, known as restricted Boltzmann machines, have found applications in classification [49], feature learning [52] and many-body quantum mechanics [73] as discussed before. In this chapter, we will discuss the Boltzmann machines in general, and leave out all the quantum mechanics. The actual connection between quantum mechanics and the Boltzmann machines will be presented in chapter 10, which is the implementation chapter for the Boltzmann machines.

We have previously seen how the weights in supervised learning can be adjusted using the backward propagation algorithm, but it does not work when we do not have prior known targets. Instead, the weights are controlled by a set of probabilities, and we let the cost function be defined by the log-likelihood function. This is known as Bayesian statistics, and will be presented in the next section.

### 6.1 Statistical foundation

In this section, we will use Bayesian statistics to exploit the link between some data  $\mathbf{x}$ , called the *hypothesis*, and some other data  $\mathbf{y}$ , called the *evidence*. We will first do it in a general way, before we link it to machine learning in the next section.

Bayesian statistics appear in many fields of science, as it is a basic and often useful probability theory. It is based on Bayes' theorem, which gives rise to some marginal and conditional

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<sup>1</sup>Hinton's contribution to machine learning can hardly be overstated. He was co-author of the paper popularizing the backpropagation algorithm, [22] supervisor of Alex Krizhevsky who designed AlexNet [55] and the main author of the paper introducing the regularization technique *dropout* [53].

distributions. The expressions can either be set up in the continuous space or the discrete space, but here we will stick to the latter as we in practice will deal with discrete data.

We start expressing the joint probability distribution of measuring both  $\mathbf{x}$  and  $\mathbf{y}$  using the general relation,

$$P(\mathbf{x}, \mathbf{y}) = P(\mathbf{x}|\mathbf{y})P(\mathbf{y}) = P(\mathbf{y}|\mathbf{x})P(\mathbf{x}), \quad (6.1)$$

which basically states that the probability of observing  $\mathbf{x}$  and  $\mathbf{y}$  is just the probability of observing  $\mathbf{x}$  multiplied with the probability of observing  $\mathbf{y}$  given  $\mathbf{x}$ .  $p(\mathbf{x}|\mathbf{y})$  is the conditional distribution of  $\mathbf{x}$  and gives the probability of  $\mathbf{x}$  given that  $\mathbf{y}$  is true. The opposite applies for  $P(\mathbf{y}|\mathbf{x})$ .  $P(\mathbf{x})$  and  $P(\mathbf{y})$  are called the marginal probabilities for  $\mathbf{x}$  and  $\mathbf{y}$ , and by reordering equation (6.1), we obtain Bayes' theorem

$$P(\mathbf{x}|\mathbf{y}) = \frac{P(\mathbf{y}|\mathbf{x})P(\mathbf{x})}{P(\mathbf{y})}. \quad (6.2)$$

The marginal probability of  $\mathbf{y}$ ,  $P(\mathbf{y})$ , is given by the sum over all the possible joint probabilities when  $\mathbf{y}$  is fixed,

$$P(\mathbf{y}) = \sum_i P(x_i, \mathbf{y}) = \sum_i P(\mathbf{y}|x_i)P(x_i), \quad (6.3)$$

and from this we observe that Bayes' theorem gives us the *posterior* probability,  $P(\mathbf{x}|\mathbf{y})$ , given the *prior* probability,  $P(\mathbf{x})$ , and the *likelihood*,  $P(\mathbf{y}|\mathbf{x})$ , seen from

$$P(\mathbf{x}|\mathbf{y}) = \frac{P(\mathbf{y}|\mathbf{x})P(\mathbf{x})}{\sum_i P(\mathbf{y}|x_i)P(x_i)}. \quad (6.4)$$

However, the summation gets extremely expensive quickly, and is intractable even for small systems. This was a big problem for a long time, but with the advent of powerful computers, algorithms like Markov chain Monte-Carlo can be used to estimate the posterior without knowing the *normalization constant*,  $P(\mathbf{y})$ . More about that in chapter 7.

In the section on supervised learning, the cost function was an important concept, and so is the case in unsupervised learning. But how do we define a cost function when we do not have any targets? We find the answer by revealing the similarities between the logistic regression and the Bayesian statistics. In logistic regression, we find the probability that a system is in a particular state, and define the cost function as the log-likelihood. We can do the same in unsupervised learning, and define the cost function as

$$\mathcal{C}(\mathbf{y}) = \ln \prod_{i=1}^l P(\mathbf{x}_i|\mathbf{y}) = \sum_{i=1}^l \ln P(\mathbf{x}_i|\mathbf{y}) \quad (6.5)$$

which is the log-likelihood. Maximizing the likelihood is the same as maximizing the log-likelihood, which again corresponds to minimizing the distance between the unknown distribution  $Q$  underlying  $\mathbf{x}$  and the distribution  $P$  of the Markov random field  $\mathbf{y}$ . This distance is expressed in terms of the Kullback-Leibler divergence (KL divergence), which for a finite state space  $\Omega$  is given by

$$\text{KL}(Q||P) = \sum_{\mathbf{x} \in \Omega} Q(\mathbf{x}) \frac{Q(\mathbf{x})}{P(\mathbf{x})}. \quad (6.6)$$

The KL divergence is a measure of the difference between two *probability density functions* (PDFs), and is zero for two identical PDFs. The divergence is often called a distance, but that is an unsatisfying description as it is non-symmetric ( $\text{KL}(Q||P) \neq \text{KL}(P||Q)$ ) in general.

To proceed further, we will introduce latent variables in form of hidden units. Suppose we want to model an  $m$ -dimensional unknown probability distribution  $Q$ . Typically, not all the

variables  $\mathbf{s}$  are observed components, they can also be latent variables. If we split  $\mathbf{s}$  into *visible* variables  $\mathbf{x}$  and hidden variables  $\mathbf{h}$ , and under the assumption that  $\mathbf{x}$  and  $\mathbf{h}$  are variables in an energy function  $E(\mathbf{x}, \mathbf{h})$ , we can express the joint probability as the Boltzmann distribution

$$P(\mathbf{x}, \mathbf{h}) = \frac{\exp(-E(\mathbf{x}, \mathbf{h}))}{Z} \quad (6.7)$$

where  $Z$  is the partition function, which is the sum of the probability of all possible states, which was already introduced in equation (5.27). We have ignored the factor  $k_B T$  by setting it to 1. Where the visible units correspond to components of an observation, the hidden units introduce the system to more degrees of freedom. This allows us to describe complex distributions over the visible variables by means of simple conditional distributions [63]. Those conditional distributions will be described later, but let us first take a look at the marginal distributions.

### 6.1.1 Marginal distributions

We have already used the term marginal distribution, which means that we get rid of a set of variables by integrating the joint probability over all of them. The marginal probability of  $\mathbf{x}$  is given by

$$P(\mathbf{x}) = \sum_{\mathbf{h}} P(\mathbf{x}, \mathbf{h}) = \frac{1}{Z} \sum_{\mathbf{h}} \exp(-E(\mathbf{x}, \mathbf{h})). \quad (6.8)$$

The sum over the  $\mathbf{h}$  vector is just a short-hand notation where we sum over all the possible values of all the variables in  $\mathbf{h}$ . Further, the marginal probability of  $\mathbf{h}$  is expressed similarly, with

$$P(\mathbf{h}) = \sum_{\mathbf{x}} P(\mathbf{x}, \mathbf{h}) = \frac{1}{Z} \sum_{\mathbf{x}} \exp(-E(\mathbf{x}, \mathbf{h})). \quad (6.9)$$

$P(\mathbf{x})$  is important as it gives the probability of a particular set of visible units  $\mathbf{x}$ , while  $P(\mathbf{h})$  will not be used in the same scope in this work.

### 6.1.2 Conditional distributions

The conditional distributions can be found from Bayes' theorem, and read

$$P(\mathbf{h}|\mathbf{x}) = \frac{P(\mathbf{x}, \mathbf{h})}{P(\mathbf{x})} = \frac{\exp(-E(\mathbf{x}, \mathbf{h}))}{\sum_{\mathbf{h}} \exp(-E(\mathbf{x}, \mathbf{h}))} \quad (6.10)$$

and

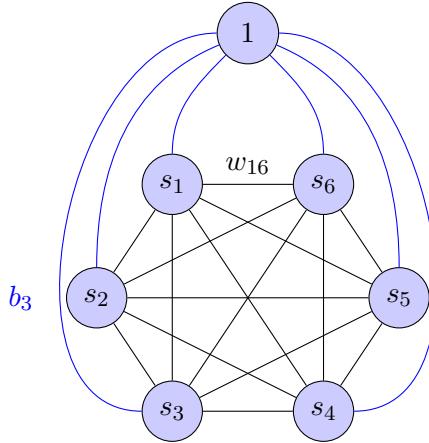
$$P(\mathbf{x}|\mathbf{h}) = \frac{P(\mathbf{x}, \mathbf{h})}{P(\mathbf{h})} = \frac{\exp(-E(\mathbf{x}, \mathbf{h}))}{\sum_{\mathbf{x}} \exp(-E(\mathbf{x}, \mathbf{h}))}. \quad (6.11)$$

The conditional probabilities are especially important in Gibbs sampling, where we want to update the  $\mathbf{x}$ 's given a  $\mathbf{h}$  and vice versa.

### 6.1.3 Maximum log-likelihood estimate

Now suppose that the energy function also is a function of some parameters  $\boldsymbol{\theta}$ . We have already expressed the log-likelihood function,

$$\ln P(\mathbf{x}|\boldsymbol{\theta}) = \ln \left[ \frac{1}{Z} \sum_{\mathbf{h}} \exp(-E(\mathbf{x}, \mathbf{h})) \right] = \ln \sum_{\mathbf{h}} \exp(-E(\mathbf{x}, \mathbf{h})) - \ln \sum_{\mathbf{x}, \mathbf{h}} \exp(-E(\mathbf{x}, \mathbf{h})) \quad (6.12)$$



**Figure 6.1:** Unrestricted Boltzmann machine. Black lines are connections between all the units, where for instance the line between  $s_1$  and  $s_6$  is related to the weight  $w_{16}$ . The blue lines are related to the bias weights, and, for instance, the line going from the bias unit to  $s_3$  is related to  $b_3$ .

and by maximizing this we find the maximum log-likelihood estimate. This estimate is important in neural networks since we always seek to maximize the likelihood in the training process. The function is maximized when

$$\begin{aligned} \frac{\partial \ln P(\mathbf{x}|\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} &= \frac{\partial}{\partial \boldsymbol{\theta}} \left( \ln \sum_{\mathbf{h}} \exp(-E(\mathbf{x}, \mathbf{h})) \right) - \frac{\partial}{\partial \boldsymbol{\theta}} \left( \ln \sum_{\mathbf{x}, \mathbf{h}} \exp(-E(\mathbf{x}, \mathbf{h})) \right) \\ &= - \sum_{\mathbf{h}} P(\mathbf{h}|\mathbf{x}) \frac{\partial E(\mathbf{x}, \mathbf{h})}{\partial \boldsymbol{\theta}} + \sum_{\mathbf{x}, \mathbf{h}} P(\mathbf{x}, \mathbf{h}) \frac{\partial E(\mathbf{x}, \mathbf{h})}{\partial \boldsymbol{\theta}} = 0. \end{aligned} \quad (6.13)$$

Similar to neural networks presented in chapter 5, we cannot find a closed-form expression for this, and we need to solve it iteratively.

## 6.2 Unrestricted Boltzmann Machines

Unrestricted Boltzmann machines, or simply Boltzmann machines, are energy-based, generative neural networks based on the more primitive Hopfield network. They consist of a set of units where a unit is connected to all other units through weights, similar to the neural network already presented. It is also common to add bias units. In figure (6.1), a simple Boltzmann machine consisting of 6 units and 1 bias unit is illustrated.

By multiplying each unit with all the other units and the weight connecting them, one obtains the system energy. For the simplest case, the energy reads

$$E(\mathbf{s}) = - \sum_{i=1}^N s_i b_i - \sum_{i=1}^N \sum_{j=i}^N s_i w_{ij} s_j \quad (6.14)$$

where  $\mathbf{s}$  are the units and  $w_{ij}$  is the weight between unit  $s_i$  and  $s_j$ . The bias unit is fixed to 1, as always, and the weight between the bias unit and the unit  $s_i$  is denoted by  $b_i$ . In its most simple form, the units can only take binary values, and we therefore call it a binary-unit Boltzmann machine. Also other architectures are possible.

The energy formula is identical to the system energy of Hopfield networks, but what distinguishes a Boltzmann machine from a Hopfield network is that the units are *stochastic*. This means that

their values are randomly determined, introducing some randomness to the system. Also the energy of an Ising model takes the same form as equation (6.14).

You might already have foreseen the next step, which is to use the Boltzmann distribution to define the probability of finding the system in a particular state  $E(\mathbf{s}; \mathbf{w}, \mathbf{b})$ , as discussed in the previous section. The probability distribution function (PDF) is then given by

$$P(\mathbf{s}) = \frac{1}{Z} \exp(-E(\mathbf{s})), \quad (6.15)$$

where  $Z$  again is the partition function. The PDF contains weights, which can be adjusted to change the distribution. In a supervised scheme, one can update the parameters in order to minimize the Kullback-Leibler divergence to a prior known distribution and in that manner reproduce the known distribution. In unsupervised learning, we cannot do this, but we can hope that a reasonable distribution is obtained by minimizing the system energy.

A Boltzmann machine is also a Markov random field, as the stochastic processes satisfy the Markov property. Loosely speaking, this means that all the probabilities of going from one state to another are known, making it possible to predict future of the process based solely on its present state. It is also determined by a "memorylessness", meaning that the next state of the system depends only on the current state and not on the sequence of events that preceded it [63]. Markov chain is an important part of the sampling methods that will be discussed later.

## 6.3 Restricted Boltzmann Machines

When there is an unrestricted guy, a restricted guy must exist as well. What the term restricted means in this context, is that we ignore all the connections between units in the same layer, and keep only the inter-layer ones. Only the units in the first layer are the observable, while the units in the next layer are latent or hidden. In the same manner as in equation (6.14), we can look at the linear case, where each unit is multiplied with the corresponding weight, but now we need to distinguish between a visible unit  $x_i$  and a hidden unit  $h_j$ . For the same reason, all the bias weights need to be divided into a group connected to the visible units,  $a_i$  and a group connected to the hidden units,  $b_j$ . The system energy then reads

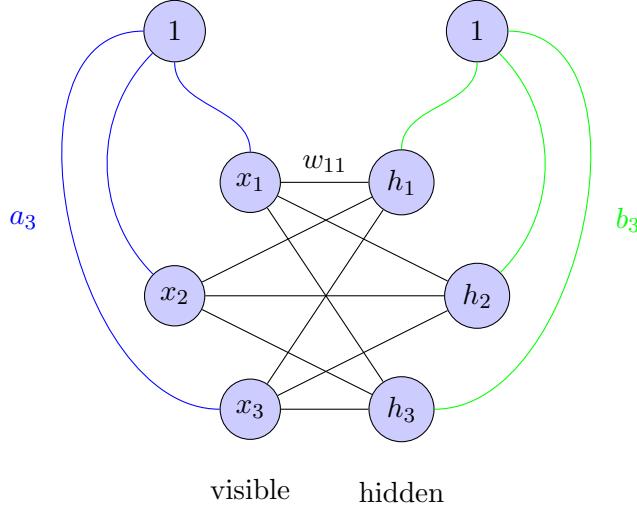
$$E(\mathbf{x}, \mathbf{h}) = - \sum_{i=1}^F x_i a_i - \sum_{j=1}^H h_j b_j - \sum_{i=1}^F \sum_{j=i}^H x_i w_{ij} h_j \quad (6.16)$$

which is called binary-binary units or Bernoulli-Bernoulli units.  $F$  is the number of visible units and  $H$  is number of hidden units. In figure (6.2), a restricted Boltzmann machine with three visible units and three hidden units is illustrated.

### 6.3.1 Gaussian-binary units

Until now we have discussed the linear models only, but as for feed-forward neural networks, we need non-linear models to solve non-linear problems. A natural next step is the model with Gaussian-binary units, which has a Gaussian mapping between the visible unit bias and the visible units. The simplest such structure gives the following system energy:

$$E(\mathbf{x}, \mathbf{h}) = \sum_{i=1}^F \frac{(x_i - a_i)^2}{2\sigma_i^2} - \sum_{j=1}^H h_j b_j - \sum_{i=1}^F \sum_{j=i}^H \frac{x_i w_{ij} h_j}{\sigma_i^2} \quad (6.17)$$



**Figure 6.2:** Restricted Boltzmann machine. Black lines represent the inter-layer connections, where for instance the line between  $x_1$  and  $h_1$  is related to the weight  $w_{11}$ . The blue lines are related to the input bias weights, and, for instance, the line going from the bias unit to  $x_3$  is called  $a_3$ . Similarly, the green lines are connections between the hidden units and the bias, and, for instance, the line going from the bias unit to  $h_3$  is called  $b_3$ .

where  $\sigma_i$  is the width of the Gaussian distribution, which can be set to an arbitrary number. Inserting the energy expression into equation (6.15), we obtain the general expression

$$P(\mathbf{x}, \mathbf{h}) \propto \exp \left( -\sum_{i=1}^F \frac{(x_i - a_i)^2}{2\sigma^2} \right) \prod_{j=1}^H \exp \left( h_j b_j + \sum_{i=1}^F \frac{h_j w_{ij} x_i}{\sigma^2} \right) \quad (6.18)$$

which is the Gaussian-binary joint probability distribution. Generative sampling algorithms, as Gibbs' sampling, use this distribution directly, while other sampling tools, as Metropolis sampling, need the marginal distribution. Since the hidden units are binary, we just need to sum the joint probability distribution over  $h = 0$  and  $h = 1$  to find the marginal distribution of the visible units,

$$P(\mathbf{x}) \propto \exp \left( -\sum_{i=1}^F \frac{(x_i - a_i)^2}{2\sigma^2} \right) \prod_{j=1}^H \left( 1 + \exp \left( b_j + \sum_{i=1}^F \frac{w_{ij} x_i}{\sigma^2} \right) \right), \quad (6.19)$$

which is shown thoroughly in appendix C. Since the visible units take continuous values, we need to integrate to find the marginal distribution of the hidden units, but since we never will use that distribution in this work, we will ignore the marginal distribution of the hidden units.

The conditional distributions are important in Gibbs sampling, and read

$$P(\mathbf{h}|\mathbf{x}) = \frac{P(\mathbf{x}, \mathbf{h})}{P(\mathbf{x})} = \prod_{j=1}^H \frac{\exp \left( h_j b_j + \sum_{i=1}^F x_i w_{ij} h_j / \sigma^2 \right)}{1 + \exp \left( b_j + \sum_{i=1}^F x_i w_{ij} / \sigma^2 \right)} \quad (6.20)$$

and

$$P(\mathbf{x}|\mathbf{h}) = \mathcal{N}(\mathbf{x}; \mathbf{a} + \mathbf{w}^T \mathbf{h}, \sigma^2) \quad (6.21)$$

where the latter is assumed to be given by a normal distribution in a Bayesian scheme. Note that the mean is  $\boldsymbol{\mu} = \mathbf{a} + \mathbf{w}^T \mathbf{h}$ , which is the vector obtained when going backwards in the restricted Boltzmann machine (multiplying the hidden units with the weights).

In Metropolis sampling, we only use the marginal distribution of the visible units use the weights to the hidden units are additional variational parameters. For completeness reasons, we will discuss the Gibbs sampling, but we will in practice stick to the Metropolis sampling. More about the different sampling tools can be found in chapter 7.

We also need the gradient of the log-likelihood function in order to train the network. The likelihood function is defined as the probability of  $\mathbf{x}$  given a set of parameters  $\theta$ , which relate to our problem as  $P(\mathbf{x}|\mathbf{a}, \mathbf{b}, \mathbf{w})$ . We therefore get three maximum log-likelihood estimates,

$$\frac{\partial \ln P(\mathbf{x}|\mathbf{a}, \mathbf{b}, \mathbf{w})}{\partial \mathbf{a}} = \frac{\mathbf{x} - \mathbf{a}}{\sigma^2} \quad (6.22)$$

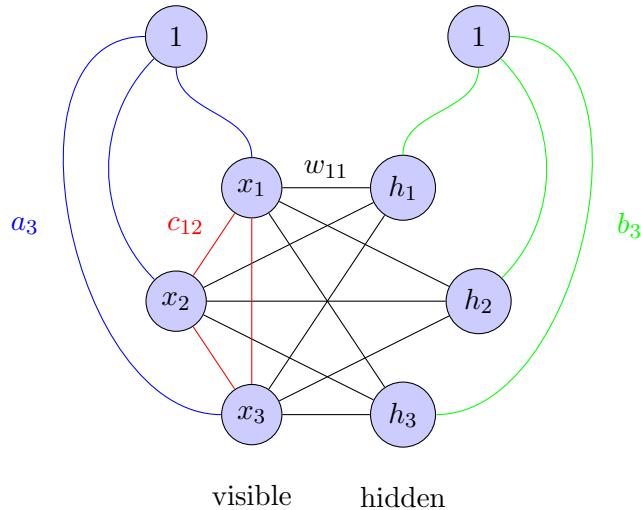
$$\frac{\partial \ln P(\mathbf{x}|\mathbf{a}, \mathbf{b}, \mathbf{w})}{\partial \mathbf{b}} = \mathbf{n} \quad (6.23)$$

$$\frac{\partial \ln P(\mathbf{x}|\mathbf{a}, \mathbf{b}, \mathbf{w})}{\partial \mathbf{w}} = \frac{\mathbf{x}\mathbf{n}^T}{\sigma^2} \quad (6.24)$$

which will be used later to maximize the likelihood with respect to the respective set of parameters.

## 6.4 Partly Restricted Boltzmann Machines

One can also imagine a partly restricted architecture, where we have intern connections between the visible units, but not the hidden units. This is what we have decided to call a partly restricted Boltzmann machine. A such neural network with three visible units and three hidden units is illustrated in figure (6.3).



**Figure 6.3:** Partly restricted Boltzmann machine. Black lines represent inter-layer connections, where for instance the line between  $x_1$  and  $h_1$  is related to the weight  $w_{11}$ . The blue lines are related to the input bias weights, and, for instance, the line going from the bias unit to  $x_3$  is related to  $a_3$ . Similarly, the green lines are related to the hidden units bias weights, and, for instance, the line going from the bias unit to  $h_3$  is related to  $b_3$ . Finally, the red lines are the intra-layer connections related to the intra-layer weights. The weight between unit  $x_1$  and  $x_2$  is called  $c_{12}$ .

Compared to a standard restricted Boltzmann machine, we get an extra term in the energy expression where the visible units are connected. It is easy to find that the expression should be

$$E(\mathbf{x}, \mathbf{h}) = \sum_{i=1}^F \frac{(x_i - a_i)^2}{2\sigma_i^2} - \sum_{i=1}^F \sum_{j>i}^F x_i c_{ij} x_j - \sum_{j=1}^H h_j b_j - \sum_{i=1}^F \sum_{j=i}^H \frac{x_i w_{ij} h_j}{\sigma_i^2} \quad (6.25)$$

with  $c_{ij}$  as the weights between the visible units. For the later calculations, we are interested in the marginal distribution only, which reads

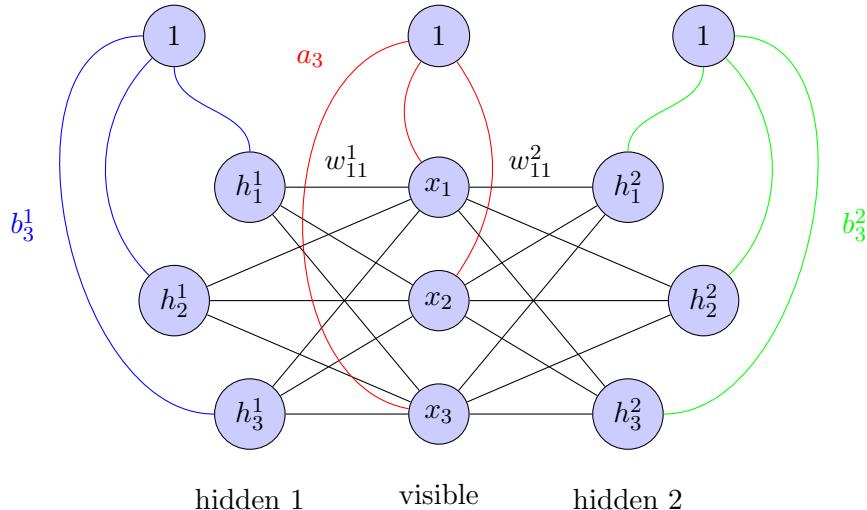
$$P(\mathbf{x}) = \exp \left( - \sum_{i=1}^F \frac{(x_i - a_i)^2}{2\sigma^2} + \sum_{i=1}^F \sum_{j>i}^F x_i c_{ij} x_j \right) \prod_{j=1}^H \left( 1 + \exp \left( b_j + \sum_{i=1}^F \frac{w_{ij} x_i}{\sigma^2} \right) \right). \quad (6.26)$$

In chapter 10, we utilize that this marginal distribution can be split in a Gaussian part, a *partly restricted* part and a product part. Then we see that the expression of the gradient of the log-likelihood function becomes the same with respect to  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{w}$  compared to the restricted Boltzmann machine, which means that we only need to calculate the expression of the gradient of the log-likelihood with respect to  $\mathbf{c}$ . This is given by the outer product

$$\frac{\partial \ln P(\mathbf{x}|\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{w})}{\partial \mathbf{c}} = \mathbf{x}\mathbf{x}^T. \quad (6.27)$$

## 6.5 Deep Boltzmann Machines

We can also construct deep Boltzmann machines, also known as deep belief networks, where we just stack single-layer Boltzmann machines. There are many ways to construct those networks, where the number of layers, unit types, number of units and the degree of restriction can be chosen as the constructor wants. The number of combinations is endless, but in order to make use of the dept, all the layer should have different configurations. Otherwise, the deep network can be reduced to a shallower network. In figure (6.4), a restricted Boltzmann machine of two hidden layers is illustrated. We have chosen three hidden units in each layer, and three visible units. It should be trivial to imagine how the network can be expanded to more layers.



**Figure 6.4:** Deep restricted Boltzmann machine. Black lines represent the inter-layer connections, where for instance the line between  $x_1$  and  $h_1^1$  is related to the weight  $w_{11}$ . The blue lines are related to the input bias weights, and, for instance, the line going from the bias unit to  $x_3$  is related to  $a_3$ . Similarly, the green lines are related to the hidden units bias weights, and, for instance, the line going from the bias unit to  $h_3^2$  is related to  $b_3^2$ .

As the main focus so far has been restricted Boltzmann machines, also the deep networks will be assumed to be restricted, although both partly restricted and unrestricted can be constructed. The system energy of a deep restricted Boltzmann machine of  $L$  layers can be expressed as

$$E(\mathbf{x}, \mathbf{h}) = \sum_{i=1}^F \frac{(x_i - a_i)^2}{2\sigma_i^2} - \sum_{l=1}^L \sum_{j=1}^{H_L} h_j^l b_j^l - \sum_{l=1}^L \sum_{i=1}^F \sum_{j=i}^{H_L} \frac{x_i w_{ij}^l h_j^l}{\sigma_i^2} \quad (6.28)$$

where  $H_L$  is the number of hidden units in layer  $L$ . The marginal probability distribution of the visible units read

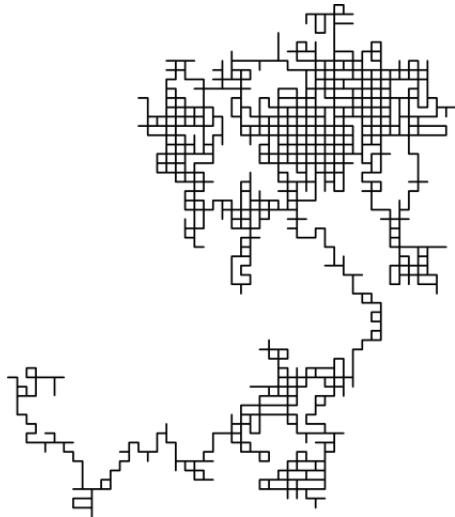
$$P(\mathbf{x}) = \exp \left( - \sum_{i=1}^F \frac{(x_i - a_i)^2}{2\sigma^2} \right) \prod_{l=1}^L \prod_{j=1}^{H_L} \left( 1 + \exp \left( b_j^l + \sum_{i=1}^F \frac{w_{ij}^l x_i}{\sigma^2} \right) \right). \quad (6.29)$$

which again can be obtained from the general expressions in appendix C.



## Chapter 7

# Quantum Monte Carlo Methods



**Figure 7.1:** Random walk on a two-dimensional grid.  
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Quantum Monte Carlo methods (QMC) is a bunch of *ab initio* methods that attempt to solve the Schrödinger equation using stochastic Monte Carlo integration. *Ab initio* reads "from first principles", which implies that the methods constitute a fundamental approach to the problem. They all seek to evaluate the multi-dimensional integral arising from the Schrödinger equation,

$$E_0 = \langle \Psi_0 | \hat{\mathcal{H}} | \Psi_0 \rangle = \frac{\int d\mathbf{r} \Psi_0(\mathbf{r})^* \hat{\mathcal{H}} \Psi_0(\mathbf{r})}{\int d\mathbf{r} \Psi_0(\mathbf{r})^* \Psi_0(\mathbf{r})}, \quad (7.1)$$

which provides the ground state energy expectation value for the exact ground state wave function  $\Psi_0$ . As aforementioned, this integral is analytically infeasible for more or less all systems of interest, rising the need of numerical methods like QMC.

In Monte Carlo integration, we use random numbers to evaluate integrals numerically. Typically, we want to estimate an expectation value  $\langle \mathcal{O} \rangle$  by approximating the definition integral with a dense sum,

$$\langle \mathcal{O} \rangle \equiv \int_{-\infty}^{\infty} d\mathbf{r} P(\mathbf{r}) \hat{\mathcal{O}}(\mathbf{r}) \approx \frac{1}{M} \sum_{i=1}^M \hat{\mathcal{O}}(\mathbf{r}_i) \quad (7.2)$$

where  $M$  is the number of *Monte Carlo cycles* and the coordinates  $\mathbf{r}_i$  are drawn randomly from the probability density function  $P(\mathbf{r})$ .

A great advantage of the QMC methods, is that we obtain approximative ground state wave functions when solving equation (7.1), which by the fourth postulate of quantum mechanics allows estimations of the ground state expectation values associated with other operators as well.

Two widely popular QMC methods are the variational Monte Carlo method (VMC) and the diffusion Monte Carlo method (DMC), where the former is arguably the simplest of all the QMC methods. It attempts to directly solve the integrals in equation (7.1) by varying parameters, with the support of the variational principle presented in section 2.4. This makes VMC a comparably computationally cheap method, but the performance is not in the league of the best methods.

DMC, on the other hand, is computationally expensive, but is also potentially numerically exact, making it a preferred method when high accuracy is needed. At first glance it might seem like a tradeoff where VMC is used when computational time is more important than the accuracy and DMC is used when the opposite is true. However, DMC requires a wave function input which is close to the exact wave function, forcing us to first run VMC to obtain this wave function before the DMC machinery can be started.

As VMC is our main focus in this work, it will be explained thoroughly in this chapter together with common sampling techniques. Thereafter we will briefly explain the idea behind the DMC method, but since this method is not implemented it will not be our main priority. To reveal the uncertainty of our results, we will also discuss some methods to estimate the errors, in particular the blocking method.

## 7.1 Variational Monte Carlo

The variational Monte Carlo method (hereafter VMC) is today widely used when it comes to the study of ground state properties of quantum many-body systems. It makes use of Markov chain Monte Carlo methods, often abbreviated MCMC, where the particles are assumed to be moving in Markov chains controlled by Monte Carlo simulations. Going back to the variational principle in equation (2.20), one observes that by choosing an approved wave function, one gets an energy larger or equal to the ground state energy.

Before we present the mathematical framework of the method, we will restate the two big problems in many-body physics, mentioned in the introduction:

1. The many-body energy expectation value is analytically infeasible for most systems.
2. The correct many-body wave function is generally unavailable.

### 7.1.1 Approaching the first problem

Let us first address the first problem, which often is considered as the root of all evil in many-body physics. In chapter 2, we saw that the two-body interaction term makes the integral impossible to solve for many particles, such that we need to rely on numerical methods.

In VMC, we start with a trial wave function guess  $\Psi_T(\mathbf{r}; \boldsymbol{\theta})$  where the parameters  $\boldsymbol{\theta}$  are varied to minimize the energy. According to the variational principle, the obtained energy will always be higher or equal to the true ground state energy, where the equality is the case if and only if the wave function is the exact ground state wave function. Denoting the exact ground state energy by  $E_0$  and the obtained energy as  $E$ , we can summarize this by

$$E_0 \leq E = \frac{\int d\mathbf{r} \Psi_T(\mathbf{r})^* \hat{\mathcal{H}} \Psi_T(\mathbf{r})}{\int d\mathbf{r} \Psi_T(\mathbf{r})^* \Psi_T(\mathbf{r})}. \quad (7.3)$$

Furthermore, the integral can be written in the form of a general expectation value

$$E = \int d\mathbf{r} E_L(\mathbf{r}) P(\mathbf{r}) \quad (7.4)$$

defining the local energy as

$$E_L(\mathbf{r}) \equiv \frac{1}{\Psi_T(\mathbf{r})} \hat{\mathcal{H}} \Psi_T(\mathbf{r}). \quad (7.5)$$

$P(\mathbf{r})$  is called the probability density function (PDF) and was first introduced in equation (2.9). In a more general scheme the PDF reads

$$P(\mathbf{r}) = \frac{|\Psi_T(\mathbf{r})|^2}{\int d\mathbf{r} |\Psi_T(\mathbf{r})|^2}. \quad (7.6)$$

Since the energy expectation value now is in the form of a general expectation value, we can use the approximation set up by Monte Carlo integration in equation (7.2), yielding

$$E \approx \frac{1}{M} \sum_{i=1}^M E_L(\mathbf{r}_i). \quad (7.7)$$

We recall that the local energies  $E_L(\mathbf{r}_i)$  are evaluated with particle configurations  $\mathbf{r}_i$  drawn from  $P(\mathbf{r})$ . In this manner the obtained energy is guaranteed to approach the exact energy as the number of Monte Carlo cycles,  $M$ , increases. Actually, the standard error goes as  $\mathcal{O}(1/\sqrt{M})$ , making the method pretty accurate for large numbers of cycles. In the limit  $M \rightarrow \infty$ , the error goes to zero,

$$\langle E_L \rangle = \lim_{M \rightarrow \infty} \frac{1}{M} \sum_{i=1}^M E_L(\mathbf{r}_i), \quad (7.8)$$

indicating the desire of largest number of cycles. This is associated with a zero-variance property governing the VMC method, stating that the variance in the true ground state should be zero. For more statistical details, see Ref.[62]. Various sampling regimes are described in section 7.2.

### 7.1.2 Approaching the second problem

So far so good, but how do we attack the second problem stated above? One of the big strengths of VMC, as aforesaid, is that we obtain an approximative wave function meanwhile the energy expectation value is estimated. If we go back to equation (7.5), we see that a PDF is required in order to estimate the energy, which again is based on a wave function.

In VMC, we need to "guess" an initial trial wave function,  $\Psi_T(\mathbf{r}, \boldsymbol{\theta})$ , which hopefully is close to the exact wave function. The parameters,  $\boldsymbol{\theta}$ , are then updated in order to minimize  $\langle E_L \rangle$ , such that the energy gets lower for each *iteration*. If the estimated energy approaches the ground state energy, we also assume that the wave function approaches the ground state wave function. Consequently, we will end up with a more or less accurate wave function approximation when the energy has converged.

As a notice, the term *iterations* should not be confused with the term *cycles*. For each iteration, we run  $M$  Monte Carlo cycles and then update the parameters. The parameter update is discussed in section 7.3.

### 7.1.3 Common extensions

This finalizes the essential theory behind the VMC method. However, a sampling algorithm is needed to draw samples randomly from  $P(\mathbf{r})$ , and in section 7.2 some popular sampling techniques are described. Before we jump into this field, we will discuss some usual extensions and improvements to the VMC method.

Initially, the particle configuration  $\mathbf{r}$  might not be representative for a configuration in equilibrium, making the first cycles a poor estimate of the expectation value. An easy fix to this problem is to basically ignore the first sampling period, known as the *burn-in period*. With this in mind, we implement an equilibration fraction describing the fraction of the total number of cycles that is used in the burn-in period. When running multiple parallel processes, the burn-in period should be the same for all the processes.

We also have a technique called *thinning*, which means picking every  $n$ 'th sample in the chain and ignore the rest. This is shown to give a more or less identical expectation value and makes the program less memory-requiring, but due to worse statistics, this should be avoided as long as there is no lack of memory.

## 7.2 The Metropolis algorithm

Metropolis sampling is a Markov chain Monte Carlo method, which generates a Markov chain using a proposal density for new steps, and the method also rejects unsatisfying moves. In its most simple form, a particle is moved in a random direction, which was the original method invented by Metropolis et.al. back in 1953 [14]. Later, the method was improved by Hastings et.al., giving rise to the more general Metropolis-Hastings algorithm [16]. The genius of the Metropolis algorithms, is that the acceptance of a move is not based on the probabilities themselves, but the ratio between the new and the old probabilities. In that way, we avoid calculating the normalizing factor, which is often computational intractable.

We will first discuss Markov chains in general, before we connect them to the original Metropolis algorithm, henceforth called *brute-force sampling*, and then the Metropolis-Hastings algorithm, also called *importance sampling*.

If we denote the current state by  $\mathbf{r}$ , and the proposed state by  $\mathbf{r}'$ , we have a transition rule  $P(\mathbf{r}'|\mathbf{r})$  for going from  $\mathbf{r}$  to  $\mathbf{r}'$  and a transition rule  $P(\mathbf{r}|\mathbf{r}')$  for going the opposite way. If we then assume that the rules satisfy *ergodicity* and *detailed balance*, we have the following relationship:

$$P(\mathbf{r}'|\mathbf{r})P(\mathbf{r}) = P(\mathbf{r}|\mathbf{r}')P(\mathbf{r}'), \quad (7.9)$$

which is actually a restatement of Bayes' theorem presented in section 6.1.

The next step is to rewrite the transition rules in terms of a proposal distribution  $T(\mathbf{r}'|\mathbf{r})$  and an acceptance probability  $A(\mathbf{r}', \mathbf{r})$ ,

$$P(\mathbf{r}'|\mathbf{r}) = T(\mathbf{r}'|\mathbf{r})A(\mathbf{r}', \mathbf{r}). \quad (7.10)$$

In order to satisfy the detailed balance, we need to choose  $A(\mathbf{r}', \mathbf{r})$  such that

$$A(\mathbf{r}', \mathbf{r}) = \min \left[ 1, \frac{T(\mathbf{r}'|\mathbf{r})P(\mathbf{r}')}{T(\mathbf{r}|\mathbf{r}')P(\mathbf{r})} \right], \quad (7.11)$$

where we limit  $A$  to maximum 1 as the probability should not exceed 1. We want to accept a move with probability  $A(\mathbf{r}', \mathbf{r})$ . One way to do that is to draw a number from a uniform distribution between 0 and 1. If this number is lower than the acceptance, the move should be accepted and rejected otherwise.

### 7.2.1 Brute-force sampling

In its simplest form, the move is proposed randomly both in magnitude and direction. Mathematically, we can write this as

$$\mathbf{r}' = \mathbf{r} + \Delta x d\mathbf{r} \quad (7.12)$$

where  $\Delta x$  is the step length and  $d\mathbf{r}$  has a random magnitude and direction (typically which particle to move). We obtain the naïve acceptance probability when requiring  $T(\mathbf{r}'|\mathbf{r}) = T(\mathbf{r}|\mathbf{r}')$ , such that it simplifies to

$$A(\mathbf{r}', \mathbf{r}) = \min \left[ 1, \frac{P(\mathbf{r}')}{P(\mathbf{r})} \right]. \quad (7.13)$$

However, with this approach an unsatisfying number of moves will be rejected, which results in a significant waste of computing power. A better method is **importance sampling**, since the particles are moved according to the quantum force.

### 7.2.2 Importance sampling

Importance sampling is a more intelligent sampling method than the brute-force sampling, since the new position is based on an educated guess. To understand how it works, we need to take a quick look at diffusion processes. We start from the Fokker-Planck equation,

$$\frac{\partial P(\mathbf{r}, t)}{\partial t} = D \nabla (\nabla - \mathbf{F}) P(\mathbf{r}, t) \quad (7.14)$$

which describes how a probability distribution  $P(\mathbf{r}, t)$  evolves in appearance of a drift force  $\mathbf{F}$ . In the case  $\mathbf{F} = 0$ , the equation reduces to the diffusion equation with  $D$  as the diffusion constant. This simplifies to  $D = 1/2$  in natural units.

The Langevin equation states that a diffusion particle tends to move parallel to the drift force in the coordinate space, with  $\boldsymbol{\eta}$  introducing some random noise. The equation reads

$$\frac{\partial \mathbf{r}(t)}{\partial t} = D \mathbf{F}(\mathbf{r}(t)) + \boldsymbol{\eta}. \quad (7.15)$$

Given a position  $\mathbf{r}$ , the new position  $\mathbf{r}'$  can be found by applying forward-Euler on the Langevin equation, obtaining

$$\mathbf{r}' = \mathbf{r} + D \Delta t \mathbf{F}(\mathbf{r}) + \boldsymbol{\xi} \sqrt{\Delta t} \quad (7.16)$$

where  $\Delta t$  is a fictive time step and  $\boldsymbol{\xi}$  is a Gaussian random variable. Further, we want to find an expression of the drift force  $\mathbf{F}$  which makes the system converge to a stationary state. A stationary state is found when the probability density function,  $P(\mathbf{r})$ , is constant in time, i.e., when the left-hand-side of the Fokker-Planck equation is zero. In that case, we can rewrite the equation as

$$\nabla^2 P(\mathbf{r}) = P(\mathbf{r}) \nabla \mathbf{F}(\mathbf{r}) + \mathbf{F}(\mathbf{r}) \nabla P(\mathbf{r}). \quad (7.17)$$

where the parenthesis are written out and we have moved the term independent of the force  $\mathbf{F}$  to the left-hand-side. Moreover, we assume that the drift force takes the form  $\mathbf{F}(\mathbf{r}) = g(\mathbf{r}) \nabla P(\mathbf{r})$  based on the fact that the force should point to a higher probability. We can then go ahead and write

$$\nabla^2 P(\mathbf{r}) (1 - P(\mathbf{r})g(\mathbf{r})) = \nabla(g(\mathbf{r})P(\mathbf{r})) \nabla P(\mathbf{r}) \quad (7.18)$$

where the quantity  $\nabla^2 P(\mathbf{r})$  is factorized out from two of the terms. The equation is satisfied when  $g(\mathbf{r}) = 1/P(\mathbf{r})$ , such that the drift force evolves to the well-known form

$$\mathbf{F}(\mathbf{r}) = \frac{\nabla P(\mathbf{r})}{P(\mathbf{r})} = 2 \frac{\nabla \Psi_T(\mathbf{r})}{\Psi_T(\mathbf{r})} = 2 \nabla \ln \Psi_T(\mathbf{r}), \quad (7.19)$$

which is also known as the *quantum force*.

The remaining part concerns how to decide if a proposed move should be accepted or not. For this, we need to find the sampling distributions  $T(\mathbf{r}'|\mathbf{r})$  from equation (7.11), which are just the solutions of the Fokker-Planck equation. The solutions read

$$G(\mathbf{r}', \mathbf{r}, \Delta t) \propto \exp\left(-(\mathbf{r}' - \mathbf{r} - D\Delta t \mathbf{F}(\mathbf{r}))^2 / 4D\Delta t\right) \quad (7.20)$$

which is categorized as a Green's function. In general, the essential property of any Green's function is that it provides a way to describe the response of an arbitrary differential equation solution. For our particular case, it corresponds to the normal distribution centered around  $\mu = \mathbf{r} + D\Delta t \mathbf{F}(\mathbf{r})$  with variance  $\sigma = 2D\Delta t$ ,  $\mathcal{N}(\mathbf{r}'|\mathbf{r} + D\Delta t \mathbf{F}(\mathbf{r}), 2D\Delta t)$ . By utilizing this, the acceptance probability for the importance sampling can finally be written as

$$A(\mathbf{r}'|\mathbf{r}) = \min \left[ 1, \frac{G(\mathbf{r}, \mathbf{r}', \Delta t)P(\mathbf{r}')}{G(\mathbf{r}', \mathbf{r}, \Delta t)P(\mathbf{r})} \right], \quad (7.21)$$

where the marginal probabilities are still given by equation (7.6).

### 7.2.3 Gibbs sampling

Gibbs sampling has throughout the years gained high popularity in the machine learning community when it comes to training Boltzmann machines. There are probably many factors that contribute to this, where the performance obviously is one of them. Another relevant motivation is the absent of tuning parameters, which makes the method easier to deal with compared to many of its competitors.

The method is an instance of the Metropolis-Hastings algorithm, and is therefore classified as another Markov chain Monte Carlo method. It differs from the Metropolis methods discussed above by the fact that all the moves are accepted, such that we do not waste computational time on rejected moves. Nevertheless, we should not use this argument alone to motivate the use of Gibbs sampling, as less than 1% of the moves in importance sampling is usually rejected.

We will in the following briefly describe the mathematical foundation of the method, before we, for the sake of clarity, connect it to the restricted Boltzmann machines. The method is built on the concept that, given a multivariate distribution, it is simpler to sample from a conditional distribution than to marginalize by integrating over a joint distribution. This is the reason why we do not need the marginal distributions in Gibbs sampling, but rather the conditional distributions. In the most general, Gibbs sampling proposes a rule for going from a sample  $\mathbf{x}^{(i)} = (x_1^{(i)}, x_2^{(i)}, \dots, x_n^{(i)})$  to another sample  $\mathbf{x}^{(i+1)} = (x_1^{(i+1)}, x_2^{(i+1)}, \dots, x_n^{(i+1)})$ , similar to the Metropolis algorithm. However, in Gibbs sampling the transition from  $x_j^{(i)}$  to  $x_j^{(i+1)}$  is performed according to the conditional distribution specified by

$$P(x_j^{(i+1)} | x_1^{(i+1)}, \dots, x_{j-1}^{(i+1)}, x_{j+1}^{(i)}, \dots, x_n^{(i)}). \quad (7.22)$$

The marginal distribution of a subset of variables can then be approximated by simply considering the samples for that subset of variables, ignoring the rest.

For a restricted Boltzmann machine, this becomes a two-step sampling process due to the latent variables, such that we use the conditional probabilities  $P(\mathbf{r}|\mathbf{h})$  and  $P(\mathbf{h}|\mathbf{r})$  to update the visible and hidden units respectively. For the restricted Boltzmann machine with Gaussian-binary units presented in section 6.3, the conditional probability of  $h_j = 0$  given a set of visible nodes  $\mathbf{x}$  is

$$P(h_j = 0 | \mathbf{x}) = \frac{1}{1 + \exp\left(+b_j + \sum_{i=1}^N x_i w_{ij} / \sigma^2\right)} \quad (7.23)$$

and the corresponding conditional probability of  $h_j = 1$  is

$$P(h_j = 1|\mathbf{x}) = \frac{1}{1 + \exp\left(-b_j - \sum_{i=1}^N x_i w_{ij}/\sigma^2\right)}, \quad (7.24)$$

which is by the way again our friend sigmoid. Note that  $P(h_j = 0|\mathbf{x}) + P(h_j = 1|\mathbf{x}) = 1$ , indicating that a hidden node  $h_j$  can only take 0 or 1, hence binary. When updating the hidden node, one typically calculate the sigmoid  $P(h_j = 1|\mathbf{x})$  and set  $h_j$  to 1 if this probability is larger than 1/2, and set it to 0 otherwise.

For the update of the visible nodes, we see from equation (6.21) that the visible nodes are chosen after the normal distribution,

$$P(x_i|\mathbf{h}) = \mathcal{N}\left(a_i + \sum_{j=1}^H w_{ij} h_j, \sigma^2\right) \quad (7.25)$$

which introduce some stochasticity to the system. By going back and forth in the Boltzmann machine multiple times (a round trip corresponds to an iteration), the hope is that the expectation values can be approximated by averaging over all the iterations.

As pointed out earlier, the Gibbs sampling will not be implemented in this work, but is described for completeness purposes. The reason for this, is that the method has not shown promising results for our specific problem, and we will instead rely on Metropolis sampling. We have already tested the Gibbs sampling in another similar project on small quantum dots [80], and so have others like Vilde Flugsrud [91]. The results are matching and show poor performance compared to the Metropolis-Hastings algorithm.

However, the Gibbs sampling method itself should not be underestimated. Carleo and Troyer showed its importance when solving the Ising model using a restricted Boltzmann machine and Gibbs sampling, and in traditional Boltzmann machines the Gibbs sampling is the preferred tool [73].

### 7.3 Parameter update

Above we have discussed the sampling in detail, which is a central part of the VMC method. Another important part is obviously the *variation* of the parameters, which is needed in order to finding an approximative wave function. In section 5.6, we discussed various gradient based optimization algorithms for iterative minimization of the cost function  $\mathcal{C}(\boldsymbol{\theta})$  in machine learning, and the same methods can be used in VMC.

However, we need to define a cost function for VMC which is minimized when the ground state energy is obtained. According to the variational principle, the ground state energy is the lowest energy we can obtain, so an obvious cost function is the energy expectation value. We therefore set

$$\mathcal{C}(\boldsymbol{\theta}) = \langle E_L \rangle \quad (7.26)$$

since we get the expectation value of the local energy from the Metropolis sampling. Further, we use the definition of the gradient of an expectation value from Ref.[84] and obtain

$$\nabla_{\theta_j} \langle E_L \rangle = 2 \left( \langle E_L \nabla_{\theta_j} \ln \Psi_T \rangle - \langle E_L \rangle \langle \nabla_{\theta_j} \ln \Psi_T \rangle \right) \quad (7.27)$$

which means that we need to calculate the expectation values  $\langle E_L \nabla_{\theta_j} \ln \Psi_T \rangle$  and  $\langle \nabla_{\theta_j} \ln \Psi_T \rangle$  in addition to the expectation value of the local energy. Those expectation values are found from the integrals

$$\langle \nabla_{\theta_j} \ln \Psi_T \rangle = \int_{-\infty}^{\infty} d\mathbf{r} \nabla_{\theta_j} \ln \Psi_T(\mathbf{r}) P(\mathbf{r}) \quad (7.28)$$

and

$$\langle E_L \nabla_{\theta_j} \ln \Psi_T \rangle = \int_{-\infty}^{\infty} d\mathbf{r} E_L(\mathbf{r}) \nabla_{\theta_j} \ln \Psi_T(\mathbf{r}) P(\mathbf{r}), \quad (7.29)$$

which can be found by Monte Carlo integration in the same way as the local energy:

$$\langle \nabla_{\theta_j} \ln \Psi_T \rangle \approx \frac{1}{M} \sum_{i=1}^M \nabla_{\theta_j} \ln \Psi_T(\mathbf{r}_i) \quad (7.30)$$

and

$$\langle E_L \nabla_{\theta_j} \ln \Psi_T \rangle \approx \frac{1}{M} \sum_{i=1}^M E_L(\mathbf{r}_i) \nabla_{\theta_j} \ln \Psi_T(\mathbf{r}_i). \quad (7.31)$$

Note that the only closed-form expression needed, in addition to the local energy, is the expression of  $\nabla_{\theta_j} \ln \Psi_T(\mathbf{r}_i)$ . This will later be found for the various wave functions.

We want to stress that the local energy is not the only possible choice of cost function. By taking advantage of the zero-variance property of the expectation value of the local energy in the minimum, one can also minimize the variance. This requires the calculation of a few additional expectation values, but it is a fully manageable task to do. See for instance Ref.[51] for more information.

## 7.4 Electron density

In section 3.3, we introduced the electron density, and looked at the integral representation,

$$\rho_P(\mathbf{r}_1, \dots, \mathbf{r}_P) = \frac{1}{N - P} \int_{-\infty}^{\infty} d\mathbf{r}_{P+1} \dots d\mathbf{r}_N |\Psi(\mathbf{r}_1, \dots, \mathbf{r}_N)|^2. \quad (7.32)$$

Not surprisingly, also this integral will be solved using Monte Carlo integration, but in a slightly different way than the integrals above since this integral returns a distribution rather than an expectation value. First, we divide the space into bins with equal sizes or at least known sizes. Thereafter, we move the particles around and count how many times a particle is found in each bin. We want to find the relative number of particles in each bin with respect to the size of the bin, so if the bins are of different sizes we need to divide them all by their respective size. In the end, we normalize the density such that the sum of all densities become equal to 1. This method and the respective implementation is detailed in section 9.8.

## 7.5 Diffusion Monte Carlo

The second and last quantum Monte Carlo method we will discuss, is the diffusion Monte Carlo (DMC) method. DMC belongs to a class of projection and Green's function approaches, and provides in principle an exact solution of the Schrödinger equation. However, for fermionic systems the method is plagued by the occurrence of a sign-problem, known as the fermion sign problem, and we need to use approximations to bypass the problem. The most common fix is the fixed-node approximation, which introduces errors to the computations. The problem is also attempted to solve using shadow wave functions, most notably by Ref.[59], which have many commonalities with our restricted Boltzmann machine (RBM) approach. An idea is that also the RBMs can contribute to the search for a solution to the problem, but we leave this for later studies.

Since the DMC method is used as a reference for more or less all the produced results, it is naturally to give a brief discussion of the idea behind the method. By the fifth postulate of

quantum mechanics, the motion of a particle that moves in a potential  $V(\mathbf{r})$  is described by the time-dependent Schrödinger equation,

$$i \frac{\partial \Psi(\mathbf{r}, t)}{\partial t} = \hat{\mathcal{H}} \Psi(\mathbf{r}, t) \quad (7.33)$$

where the Hamiltonian is given by

$$\hat{\mathcal{H}} = -\frac{1}{2} \nabla^2 + V(\mathbf{r}) \quad (7.34)$$

in natural units. Further, we assume that the potential becomes infinite as  $x \rightarrow \pm\infty$ , such that the particle motion is confined to a finite spatial domain. The formal solution of the time-dependent Schrödinger equation is then given by an expansion of the eigenfunctions of  $\hat{\mathcal{H}}$ ,  $\phi_n(\mathbf{r})$ ,

$$\Psi(\mathbf{r}, t) = \sum_{n=0}^{\infty} c_n \phi_n(\mathbf{r}) e^{-\frac{E_n}{\hbar} t} \quad (7.35)$$

where the coefficients can be obtained by

$$c_n = \int_{-\infty}^{\infty} d\mathbf{r} \phi_n(\mathbf{r}) \Psi(\mathbf{r}, 0) \quad (7.36)$$

under the assumption that the eigenfunctions are orthonormal and that the energies  $E_n$  are in increasing order. The DMC method is based on the time-imaginary Schrödinger equation, which is obtained by a **shift of energy scale**, setting  $V(\mathbf{r}) \rightarrow V(\mathbf{r}) - E_R$  and  $E_n \rightarrow E_n - E_R$ , which does not change any physical quantity. By further using the **Wick rotation of time** setting  $\tau = it$ , we obtain the time-imaginary Schrödinger equation [31]

$$-\frac{\partial \Psi(\mathbf{r}, \tau)}{\partial \tau} = -\frac{1}{2} \nabla^2 \Psi(\mathbf{r}, \tau) + [V(x) - E_R] \Psi(\mathbf{r}, \tau) \quad (7.37)$$

and the expansion now reads

$$\Psi(\mathbf{r}, \tau) = \sum_{n=0}^{\infty} c_n \phi_n(\mathbf{r}) e^{-(E_n - E_R)\tau}. \quad (7.38)$$

The time-imaginary Schrödinger equation is similar to the Fokker-Planck equation presented in equation (7.14), which was noticed by Fermi already around 1945 [13, 21]. Since we have solutions to the Fokker-Planck equation, the idea is to apply the same solutions to equation 7.37. To clarify this, consider the time-imaginary Schrödinger equation for non-interacting particles,

$$\frac{\partial \Psi(\mathbf{r}, \tau)}{\partial \tau} = \frac{1}{2} \nabla^2 \Psi(\mathbf{r}, \tau) \quad (7.39)$$

which is really similar to the diffusion equation,

$$\frac{\partial \phi(\mathbf{r}, t)}{\partial t} = D \nabla^2 \phi(\mathbf{r}, t) \quad (7.40)$$

where  $D$  is the diffusion constant set to 1/2 in our calculations, no wonder why. We end our motivation of the DMC method here, and relegate the reader to Ref.[31] for a thorough and comprehensive explanation of the method. The last section of this chapter is about error estimation, which is very important in experimental physics, also when it comes to computer experiments.

## 7.6 Error estimation

In experiments, we have two main classes of errors, systematical errors and statistical errors. The former is a result of external factors such as uncertainties in the apparatus or a person constantly takes an incorrect measurement, which is an error that is hard to estimate. The latter, however, can be found by estimating the variance of the sample mean, which we want to find accurately and efficiently. Monte Carlo simulations can be treated as computer experiments, and therefore we can use the same analyzing tools as we do for real experiments.

There are several ways to estimate the variance, where the cheapest ones also are the less accurate ones. To make the most use of our data, we use resampling methods to estimate the statistical errors. Some well-known resampling algorithms are blocking, bootstrap and jackknife. We will cover the blocking method only, since that is the only one we use in our particular implementations. To save computational time, we resample the final iteration only, for the others we use the simple estimation method.

### 7.6.1 Basic concepts of statistics

Before we go through the methods, we will give a brief introduction to some useful statistical quantities. We start with the *moments*, which are given by

$$\langle x^n \rangle = \int dx p(x) x^n$$

where  $p(x)$  is the true probability density function. In order to make physical sense, this function needs to be normalized such that the integral over all possible outcomes gives a total probability of 1. This is associated with the zero'th moment, where we get  $\int dx p(x) = 1$ . The first moment is the *mean* of  $p(x)$ , and is often denoted by the letter  $\mu$ ,

$$\langle x \rangle = \mu = \int dx p(x) x. \quad (7.41)$$

Moreover, we can define the *central moments* given by

$$\langle (x - \langle x \rangle)^n \rangle = \int dx (x - \langle x \rangle)^n p(x), \quad (7.42)$$

which is centered around the mean. With  $n = 0$  and  $n = 1$ , this is easy to find, but what is the central moment with  $n = 2$ ? The central moment with  $n = 2$  is what we call the *variance*, and is often denoted as  $\sigma^2$  as we did in the equation (2.16). One can show that

$$\sigma^2 = \langle (x - \langle x \rangle)^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2 \quad (7.43)$$

which was already stated in chapter 2. The *standard deviation* is given by the square-root of the variance,  $\sigma$ , and is the quantity we will present as the measure of the uncertainty in the last digit of a numerical result.

Using the expressions above, we can calculate the mean and the variance of a probability density function  $p(x)$ , but what do we do if we have a set of data drawn from an unknown  $p(x)$ ?

If the probability density function is unavailable, we cannot find the exact sample mean and sample mean variance. However, we can make an estimate, writing

$$\langle X \rangle \approx \frac{1}{n} \sum_{i=1}^n X_i \equiv \bar{X}, \quad (7.44)$$

for the sample mean assuming that  $X$  is our sample containing  $n$  points. Further, the variance of the sample mean, assuming that the measurements are *independent*, can be found from

$$\text{var}(\bar{X}) \equiv \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2 = \frac{\sigma^2}{n} \quad (7.45)$$

where  $\sigma^2$  is the variance of the probability density function. The law of large numbers states that the estimated sample mean approaches the true sample mean as  $n$  goes to infinity. This is closely related to the central limit theorem, which says that the probability distribution of  $X$ ,  $p_X(x)$ , can be approximated as a normal distribution  $\mathcal{N}(\mu, \sigma)$  with  $\mu = \bar{X}_n$  and  $\sigma = \text{var}(\bar{X})$  as the number of sampling points  $n$  goes to infinity.

On the other hand, if the samples are *correlated*, equation (7.45) becomes an underestimation of the *sample error*  $\text{err}_X^2$ , and is thus more a guideline for the size of the uncertainty, more than an actual estimate of it. In that case, we need to calculate the more general *covariance*, given by

$$\text{cov}(X_i, X_j) = (X_i - \bar{X})(X_j - \bar{X}) \quad (7.46)$$

which gives a measure on the correlations between  $X_i$  and  $X_j$ . Note that the variance is just the special case where  $X_i$  and  $X_j$  are independent. We define the sample error as

$$\text{err}_X^2 = \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n \text{cov}(X_i, X_j) \quad (7.47)$$

which can be further expressed as a function of the *autocovariance*, defined as

$$\gamma(h) = \text{cov}(X_i, X_{i+h}) \quad (7.48)$$

and is thus just a measure of the correlation of a part of a time series with another part of the same time series. If  $h = 0$ , this is just the variance,  $\gamma(0) = \sigma^2/n$ . Using this, the sample error reads

$$\text{err}_X^2 = \frac{\sigma^2}{n} + \frac{2}{n} \sum_{h=1}^{n-1} \left(1 - \frac{h}{n}\right) \gamma(h). \quad (7.49)$$

A problem with this definition of the sample error, is that it turns out to be very expensive to calculate as the number of samples gets large. We are in need of a cheaper method, which is the task of the resampling methods. We will in the following discuss the blocking algorithm, which is the resampling method of choice in this work.

### 7.6.2 Blocking

Above, we have described the need of a proper estimation of the uncertainty in computational simulations, where the covariance was included in the calculation of  $\sigma^2$ . A quick and easy way to get a proper estimate of the uncertainty, is by using the blocking method.

When the blocking method was made popular by Flyvbjerg and Peterson in 1989, the method required hyper-parameters which had to be carefully adjusted for each particular data set [23]. In 2018, Marius Jonsson reinvented the algorithm and made it automated, with no need for external parameters. Despite this, no compromise was made on performance. The method scales as  $12n + \mathcal{O}(\log_2 n)$  for small data sets, but reduces to  $n + \mathcal{O}(1)$  for large data sets, which makes it preferred over bootstrap and jackknife for large data sets [78]. We will now go through the idea behind the blocking method.

Consider a time series  $\{x_1, x_2, \dots, x_n\}$  with  $n = 2^d$  data point for some integer  $d > 1$ . For this series, an autocovariance function  $\gamma(h)$  is guaranteed to exist. We arrange the data in a vector

$$X = (x_i, x_2, \dots, x_n), \quad (7.50)$$

which we assume to be asymptotically uncorrelated. The idea is to take the mean of subsequent pair of elements from  $X$ , and form a new vector  $X_1$ . We then repeat the operation on  $X_1$  and form a new vector  $X_2$  and so on. This is the reason why we require  $n = 2^d$ . If  $k$  denotes an element in vector  $X_i$ , we can write the procedure recursively as

$$\begin{aligned} (X_0)_k &\equiv (X)_k \\ (X_{i+1})_k &\equiv \frac{1}{2} \left( (X_i)_{2k-1} + (X_i)_{2k} \right) \end{aligned} \quad (7.51)$$

where  $1 \leq i \leq d - 1$  and  $1 \leq k \leq n/2^i$ , which are known as the *blocking transformations*. According to equation (7.49), we can express the sample mean variance of  $X_k$  as

$$\text{var}(\bar{X}_k) = \frac{\sigma_k^2}{n_k} + \frac{2}{n_k} \sum_{h=1}^{n_k-1} \left( 1 - \frac{h}{n_k} \right) \gamma_k(h), \quad (7.52)$$

where we define the last term as the *truncation error*,  $e_k$ , as it is intractable. It can be shown that the sample variance of all pairs  $(X_i)_k$  and  $(X_j)_k$  after a while will be identical [23],

$$\text{var}(\bar{X}_i) = \text{var}(\bar{X}_j) \quad \forall \quad i, j \in \{0, 1, \dots, d - 1\}, \quad (7.53)$$

with the consequence that the sample mean variance of the entire set of samples is also given by

$$\text{var}(\bar{X}) = \frac{\sigma_k^2}{n_k} + e_k \quad \forall \quad k \in \{0, 1, \dots, d - 1\}. \quad (7.54)$$

In the original (manual) blocking method, we had to know exactly where to stop the procedure in order to equation (7.53) to hold. If we do not continue long enough, the sample variance has not converged, while if we keep on going for too long, the standard error of  $\text{var}(\hat{\sigma}_k^2/n_k)$  get very large. If one plots the sample variance as a function of the iterations, one will see that the curve forms a plateau before it gets very noisy, but with Jonsson's automated method, we do not need to worry about this.

# Part III

# Implementation



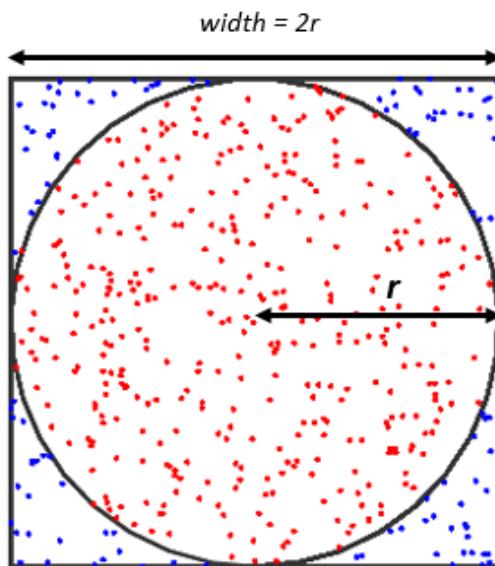
## Chapter 8

# Scientific Programming

Always code as if the guy who ends up maintaining your code will be a violent psychopath who knows where you live.

---

John F. Woods, [101]



**Figure 8.1:** One can estimate the value of  $\pi$  using Monte-Carlo integration. Here we approximate the ratio between the area of the circle and the area of the square by counting the number of random points occurring inside the circle and inside the square. The estimate of  $\pi$  is found from  $\pi = 4 \cdot A_{\text{circle}}/A_{\text{square}}$ .

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Since this scientific work is based on solving equations numerically on the computer, software development is a major part of the work. We will in this chapter go through the main concepts of scientific programming, and directly relate them to our code in the next chapters on implementation. As a motivation, we will provide a brief recap of some historical milestones in scientific computing.

Computers have long been used for solving complex scientific problems. Actually, for a long time science was the main motivation of developing computers. Already in 1929, Egil Hylleraas

used his mechanical desk calculator to calculate the ground state energy of the Helium atom, which remains a milestone in computational quantum chemistry as well as quantum chemistry in general [35]. Later, pioneers like Alan Turing and John von Neumann contributed to the invention of the first electronic computer, which made numerical solutions of ordinary differential equations possible [76].

Also on the software side, big breakthroughs have been made since the advent of the electronic computer. The programming language Fortran was released in 1957, which provided a new basis of scientific programming [19]. Later, Ole Johan Dahl and Kristen Nygaard developed the language SIMULA, which is considered as the first object orientated language [28]. The language that we mostly will rely on in this work, C++, is an extension of the language C which was developed in the early 1970's and is, together with Fortran, one of few languages that have survived the ravages of time.

The computers used today, included both the hardware and the software, have become possible due to heroic effort of an array of scientists, engineers and mechanics over a long period. The computers are stronger than ever, have more memory than ever, are compacter than ever and, last but not least, they are more user-friendly than ever. This has made heavy computations that was unthinkable just decades ago possible, and has contributed to the development of essentially all branches of science. Due to the user-friendliness, advanced simulations have also been available to young and curious students like the author, which we should not take for granted.

The computer's language itself is binary, and is the lowest level. To translate commands to this language, we need a "translator", which is a language that fills the gap between the binary language and human commands. This language is categorized in levels based on how similar they are to the binary language. Low-level languages are more similar to the binary language than a high-level language. The result is that low-level languages in general are more demanding to deal with, but they are typically faster than high-level languages. In our work, we have used the low-level language C++ for the main code, but for scripting, including plotting and solving less expensive problems, the high-level language Python has been used.

In this chapter, we will explain the essentials behind scientific programming, using examples from both C++ and Python. We will begin from the very basic, so if the reader is an experienced programmer this chapter will probably be perceived as old news. Programs are often either classified as *procedural programming* or *object oriented programming*, based on whether the code is written in the same order as the program flow goes (procedural) or is based on objects (object-orientated).

## 8.1 Object orientated programming in Python

In the everyday life, we are surrounded by objects all the time which can be placed in different categories. For instance, a *circle* is an object with properties like area, circumference and so on, and can be placed in the class *shapes*. Other members of the class shapes might be square and triangle. In object orientated programming, we create similar objects, or classes, to make the program flow more intuitive for the reader. The main class is called the parent class, while the sub-classes are called the children. The reason for this analogy, is that the sub-classes always inherit from their main class, but not the other way around.

The example with the shapes can therefore be implemented with `Shapes` being the parent, and `Circle`, `Square` and `Triangle` being its children. In the high-level language Python, this can be implemented as

```
import numpy as np
class Shapes:
```

```

def __init__(self, r):
    self.r = r

def getArea(self):
    raise NotImplementedError("Class {} has no instance 'getArea'."
        .format(self.__class__.__name__))

class Circle(Shapes):
    def getArea(self):
        return np.pi * self.r**2

    def getCircumference(self):
        return 2 * np.pi * self.r

    def getExtent(self, x, y):
        if np.linalg.norm([x, y]) < self.r:
            return True
        return False

class Square(Shapes):
    def getArea(self):
        return 4 * self.r**2

    def getCircumference(self):
        return 8 * self.r

    def getExtent(self, x, y):
        if abs(x) < self.r and abs(y) < self.r:
            return True
        return False

class Triangle(Shapes):
    def getCircumference(self):
        return 6 * self.r

```

where we give the children the properties area, circumference and extent. The length  $r$  means the radius of a circle, half the side length of a square and also half a side length of the equilateral triangle, see figure (8.1). We can implement a circle, square and triangle of  $r=1$ , respectively named "circ", "sqr" and "tria" by these three lines

```

circ = Circle(1)
sqr = Square(1)
tria = Triangle(1)

```

Moreover, we can easily get their areas by calling the `getArea()` function,

```

>>> circ.getArea()
3.141592653589793

>>> sqr.getArea()
4

>>> tria.getArea()
NotImplementedError: Class Triangle has no instance 'getArea'.

```

As we can see, the area of the circle and the square were calculated successfully, giving  $\pi r^2$  and  $4r$  respectively. On the other hand, the area of the triangle raised an error, which is obviously because we do not have defined the area of the triangle! This example illustrates that properties of the children overwrites the properties of the parent, but if the children does not have the called property, it will instead return the parent's property.

### 8.1.1 An example based on Monte-Carlo simulations

Up to this point, we have talked a lot about Monte-Carlo simulations without giving an illustrating example on what it is. Methods where we draw numbers randomly from a function to reveal properties of the function are in general categorized as Monte-Carlo simulations.

Suppose we did not know the value of  $\pi$ , what could we do to approximate its value? There are many ways to do this, where the most intuitive might be to measure the ratio between the diameter and the circumference in a circle. It is not possible to do this very accurate manually. A more accurate and robust solution would be to use Monte-Carlo simulations, where we for instance can take the starting point on the relation

$$\pi = 4 \frac{A_{\text{circle}}}{A_{\text{square}}} \quad (8.1)$$

which is derived from the ratio between the circle area and the square area. By drawing random numbers from a uniform distribution and count the number of points which are in the square and in the circle, we can approximate  $A_{\text{circle}}/A_{\text{square}}$ . We can do this by using the `getExtent()` function of the square class and circle class,

```
for i in range(1,10):                      # Iteration loop
    M = 10**i
    A_square = 0
    A_circle = 0
    for _ in range(int(M)):                  # Monte-Carlo loop
        x = np.random.uniform(-1,1)
        y = np.random.uniform(-1,1)
        if circ.getExtent(x,y):
            A_circle += 1
        if sqr.getExtent(x,y):
            A_square += 1
    print("Pi: ", 4 * A_circle/A_square)
```

which gives a output similar to

```
Pi: 2.4
Pi: 3.32
Pi: 3.156
Pi: 3.1296
Pi: 3.14168
Pi: 3.140044
Pi: 3.1420708
Pi: 3.1416844
Pi: 3.141615628
```

Considering the fact that  $\pi = 3.141592653589793\dots$ , the obtained result is quite acceptable. However, to run the program with  $M = 10^9$  Monte-Carlo cycles, the program is quite slow. Is it possible to do this faster? The answer is yes, by switching entirely or partly to a low-level language.

## 8.2 Low-level programming with C++

Above, we looked at how inheritance can be implemented in the high-level language Python, and we observed the main weakness of high-level languages: the computational time. In this section, we will repeat the implementation and example above using C++. This will, hopefully, provide a significantly speed-up. A similar class structure in C++ looks like

```

class Shapes {
public:
    Shapes();
    virtual double getArea() = 0;
    virtual double getCircumference() = 0;
    virtual bool getExtent(double x, double y) = 0;
    virtual ~Shapes();
};

class Circle : public Shapes {
public:
    Circle(double r);
    double getArea() {
        return M_PI * m_r * m_r;
    }
    double getCircumference() {
        return 2 * M_PI * m_r;
    }
    bool getExtent(double x, double y) {
        if(sqrt(x*x+y*y) < m_r) {
            return true;
        }
        return false;
    }
private:
    double m_r = 1;
};

Circle::Circle(double r) : Shapes() {
    m_r = r;
}

class Square: public Shapes {
public:
    Square(double r);
    double getArea() {
        return 4 * m_r * m_r;
    }
    double getCircumference() {
        return 8 * m_r;
    }
    bool getExtent(double x, double y) {
        if(fabs(x) < m_r && fabs(y) < m_r) {
            return true;
        }
        return false;
    }
private:
    double m_r = 1;
};
Square::Square(double r) : Shapes() {
m_r = r;
}

```

which looks quite different from the implementation in Python. We will go thoroughly through the different part of the code.

One of the first thing we observe, is that we here need to *declare* all the variables, which means that we need to specify what kind of variable it is. In the following we will have a look at some standard *data types*.

### 8.2.1 Built-in data types

In low-level languages we need to specify the data types manually, giving us more control and flexibility. The most fundamental data types are `int` representing an integer number, `float`

**Table 8.1:** Built-in data types in C++, with their memory occupation and range. Parenthesis () means that the extension is optional. Numbers are taken from [72].

Data types	Size [bytes]	Range
(signed) short (int)	2	$-2^{15}$ to $2^{15} - 1$
(signed) int / long (int)	4	$-2^{31}$ to $2^{31} - 1$
(signed) long long (int)	8	$-2^{63}$ to $2^{63} - 1$
unsigned short (int)	2	0 to $2^{16}$
unsigned int / unsigned long (int)	4	0 to $2^{32}$
unsigned long long (int)	8	0 to $2^{64}$
float	4	$\sim \pm 3.4E38$ ( $\sim 7$ digits)
(long) double	8	$\sim \pm 1.7E308$ ( $\sim 15$ digits)
(signed) char	1	$-2^7$ to $2^7 - 1$
unsigned char	1	0 to $2^8$

representing a floating number, `bool` representing a Boolean number and a `char` representing a character.

Often, these four data types are not sufficient for the task we want to implement and we need to introduce more data types. An especially common error is *overflow*, meaning that the number computed is out of the range of our data type. To avoid this, we can switch to data types of longer range, replacing `int` with a `long long int` (or just a `long long`), and replacing `float` with a `double`.

Some languages also deal with `long int` and `long double`, but in C++ an `int` and a `long int` is the same. Additionally, a `long double` is the same as a `double`. As lack of memory seldomly is a problem on modern computers, it is common to declare all floating numbers as `doubles`. In table (8.1), the most common data types are listed with their range and memory occupation in C++.

For integers it is also possible to "move" the range from its zero-centered default to only include positive numbers. This is done by declaring a `unsigned int` or `unsigned long`. This doubles the range in the positive direction, but should not be done if one, by any chance, can get negative numbers, as one will get *underflow*.

### 8.2.2 Access modifiers

Another thing that we observe from the code example above, is that we define the functions as `private` or `public`, which are called access modifiers. The access modifiers specify how much access the environment should have to the function: `private` means that the function can only be called from the particular class, while `public` means that also sub-classes (children) and other classes have access to the function.

We also have a third access modifier in C++, called `protected`. `protected` members of a class A are not accessible outside of A's code, but any class that inherit from A has access to the protected member.

### 8.2.3 Pure virtual functions

In Python, we saw how a parent's function was overwritten by a function of the child. This work as long as the child's function has identical name and identical arguments as the parent's

function, otherwise it all will fail.

In C++, a much safer method exists based on pure virtual functions, where a child is *forced* to have the same virtual functions as its parent. The virtual functions of the parent therefore serves as a template of its children, such that all the children have the same features.

This is often very useful, as the children tend to have the same features. Taking it back to the `Shapes` class, all the shapes ought to have features area and circumference, which can be forced by using virtual functions, as done in the example above.

#### 8.2.4 Constructors and destructors

In the example above, we observe that the class `Shapes` has a function with the same name, `Shapes` and another virtual function `~Shapes`. The former one is used to create the object, and is hence named a *constructor*. This function gets called automatically when the object of a class is created. Similarly, the latter one destructs an object when we are done evaluating the object. It is therefore named a *destructor*, and is also called automatically when an object is deleted or goes out of scope.

#### 8.2.5 Pointers

In the example above, we have not directly used any pointer. However, pointers are important in low-level languages, and we will here give a very brief introduction to pointers.

For a C++ program, the memory of a computer is like a succession of memory cells, each with a unique address and of size one byte. When declaring a variable of size larger than one byte, the object will occupy memory cells that have consecutive addresses. Often, it is useful to know this address, which allows you to change the object directly in the memory. The address of a variable is given by the address-of operator, `&`, working as

```
address=&variable
```

Doing this, we say that we *point to the address*, and a variable that stores the address of another variable is thereby called a pointer.

Furthermore, the dereference operator, `*`, gives the value of the address to where a pointer points. Thus, the following is true

```
variable==*address
```

#### 8.2.6 Back to the Monte-Carlo example

We can now call the class above from the `main` function. We start defining the children `circ` and `sqr`, and use the `getExtent()` function to approximate the area ratio between them. The code looks like

```
int main() {
    class Shapes* circ = new Circle(1);
    class Shapes* sqr = new Square(1);

    for(int i=1; i<10; i++) {
        int M = int(pow(10, i));
        long A_square = 0;
        long A_circle = 0;
        for(int m=0; i<M; i++) {
            double x = uniform(gen);
            double y = uniform(gen);
```

```
    if(circ->getExtent(x, y)) {
        A_circle++;
    }
    if(sqr->getExtent(x, y)) {
        A_square++;
    }
}
std::cout << std::fixed;
std::cout << std::setprecision(10);
std::cout << "Pi: " << 4 * double(A_circle)/A_square << std::endl;
}
return 0;
}
```

which gives the same output as the Python script, but is much faster.

## Chapter 9

# Implementation: Variational Monte Carlo

There are only two hard things in Computer Science: cache invalidation and naming things.

---

Phil Karlton, [92]

In this chapter we will describe the implemented variational Monte Carlo (VMC) code, which was developed from scratch in C++. As the code itself is around 7000 significant<sup>1</sup> lines of code, we will just go through selected and often not obvious parts. As often said, *good planning is half the battle*, which largely relates to writing VMC code. The code was rewritten and restructured several times before we ended on the final version. As a starting point, we used Morten Ledum's VMC framework [69], which was meant as an example implementation in the course *FYS4411 - Computational Physics II: Quantum Mechanical Systems*. The entire source code can be found on the authors github, [86].

For all matrix operations, the open source template library for linear algebra Eigen was used throughout the code. Eigen provides an elegant interface, with support for all the needed matrix and vector operations. In addition, Eigen is built on the standard software libraries for numerical linear algebra, BLAS and LAPACK, which are incredibly fast. These contribute greatly to the performance of the code.

The code was developed in regards to three principal aims:

- fast,
- flexible,
- readable.

It needs to be flexible in order to support the Boltzmann machines as our trial wave function guess, and since we will try out various Jastrow factors it should be easy to add and remove wave function elements. Since quantum mechanical simulations in general are very expensive, it is important to develop efficient code to be able to study systems of some size. Lastly, we aim to write readable code such that others can reuse the code in its entirety or parts of it later.

How we work to achieve the goals will be illustrated by code mainly picked from the `WaveFunction` class, which is the heart of the code.

---

<sup>1</sup>Significant lines of code in this sense means lines that are not blank or commented. Counted by the cloc program [81].

## 9.1 Flexibility and legibility

We have done several things in order to keep the code as legible as possible. Firstly, the code was written in an object orientated scheme which makes it more intuitive to a human as discussed in chapter 8. Actually, the Hamiltonians, optimizers, wave functions, sampling methods and even the random number generator were treated as objects, making the code more or less as object orientated as possible. This makes it also easier to define a system, since we simply can set the preferred object. Below, we define a two-dimensional quantum dot system of 6 electrons with frequency  $\omega = 1.0$ , learning rate  $\eta = 0.1$ , number of Metropolis cycles  $M = 2^{20} = 1048576$  and max number of iterations set to 1000.

```
System *QD = new System();
QD->setNumberOfDimensions(2);
QD->setNumberOfParticles(6);
QD->setNumberOfMetropolisSteps(int(pow(2, 20)));
QD->setFrequency(1.0);
QD->setLearningRate(0.1);

QD->setBasis(new Hermite(QD));
QD->setHamiltonian(new HarmonicOscillator(QD));

QD->setWaveFunctionElement(new Gaussian(QD));
QD->setWaveFunctionElement(new SlaterDeterminant(QD));
QD->setWaveFunctionElement(new PadeJastrow(QD));

QD->runIterations(1000);
```

**Listing 9.1:** Example on how a quantum dot system can be initialized.

We observe that one first needs to define an object `System`, and then the various settings can be connected to this object. As you might notice, we use the **lowerCamelCase** naming convention for function and variable names, which means that each word begins with a capital letter except the initial word. For classes, we use the **UpperCamelCase** to distinguish from function names. This is known to be easy to read, and apart from for example the popular `snake_case`, we do not need delimiters between the words, which saves some space. After the naming convention is decided, we are still responsible for giving reasonable names, which is not always an easy task, as Phil Karlton points out. When one sees the name, one should know exactly what the variable/function/class is or does. More about naming conventions can be read in reference [85]. In addition, as a code format we use the ClangFormat, which provides a consequent way of formating the code.

The snippet above also demonstrates how the code is made flexible when it comes to the wave function. One can simply construct a wave function consisting of various elements by calling the `setWaveFunctionElement` multiple times. This creates a wave function vector, `m_waveFunctionVector`, in the background containing all the elements, which makes it easy to compose the wave function in whatever way you want, all the elements can be combined. The reader might stubs on the use of the element `Gaussian`, isn't the trial wave function defined by the Slater determinant multiplied with a Padé-Jastrow factor? It is, but as we will see later in section 9.3.2, the Gaussian part can be factorized out of the Slater determinant when using a Hermite basis. However, we will now start from the fundamental assumption that the trial wave function consists of a Slater determinant and a Jastrow factor, and take it from there.

## 9.2 Splitting the wave function in elements

In a standard VMC computation, the trial wave function is assumed to consist of a single Slater determinant, and a Jastrow factor to take care of the repulsive interactions. Mathematically it can be expressed as

$$\Psi_T(\mathbf{r}) = |\hat{D}(\mathbf{r})| J(\mathbf{r}_{ij}) \quad (9.1)$$

where the Slater determinant is the determinant of the matrix  $\hat{D}(\mathbf{r})$ , henceforth the Slater matrix. To convince the reader that the Slater determinant and the Jastrow factor can be treated separately, we will consider a general trial wave function consisting of  $p$  *wave function elements*  $\{\Psi_1, \Psi_2 \dots \Psi_p\}$ ,

$$\Psi_T(\mathbf{r}) = \prod_{i=1}^p \Psi_i(\mathbf{r}), \quad (9.2)$$

where  $\mathbf{r}$  contains all the positions at a certain time. In order to treat each such element independently, we need to assume that a parameter  $\theta_j$  only appears in an element.

### 9.2.1 Kinetic energy computations

The local energy, defined in equation (7.5), is

$$\begin{aligned} E_L &= \frac{1}{\Psi_T(\mathbf{r})} \hat{\mathcal{H}} \Psi_T(\mathbf{r}) \\ &= \sum_{k=1}^F \left[ -\frac{1}{2} \left( \frac{1}{\Psi_T(\mathbf{r})} \nabla_k^2 \Psi_T(\mathbf{r}) \right) + \mathcal{V} \right]. \end{aligned} \quad (9.3)$$

where we have  $F = ND$  degrees of freedom. The first term, which is the kinetic energy term, is the only wave function-dependent one, and we will in this section split it up with respect to the elements. The potential energy term,  $\mathcal{V}$ , is not directly dependent on the wave function and will therefore not be further touched here.

From the definition of differentiation of a logarithm, we have that

$$\frac{1}{\Psi_T(\mathbf{r})} \nabla_k \Psi_T(\mathbf{r}) = \nabla_k \ln \Psi_T(\mathbf{r}), \quad (9.4)$$

which provides the following useful relation

$$\frac{1}{\Psi_T(\mathbf{r})} \nabla_k^2 \Psi_T(\mathbf{r}) = \nabla_k^2 \ln \Psi_T(\mathbf{r}) + (\nabla_k \ln \Psi_T(\mathbf{r}))^2. \quad (9.5)$$

Using the fact that the trial wave function is a product of all the elements, the term above is calculated by

$$\frac{1}{\Psi_T(\mathbf{r})} \nabla_k^2 \Psi_T(\mathbf{r}) = \sum_{i=1}^p \nabla_k^2 \ln \Psi_i(\mathbf{r}) + \left( \sum_{i=1}^p \nabla_k \ln \Psi_i(\mathbf{r}) \right)^2 \quad (9.6)$$

such that the total kinetic energy is given by

$$-\frac{1}{2} \frac{1}{\Psi_T(\mathbf{r})} \nabla^2 \Psi_T(\mathbf{r}) = -\frac{1}{2} \left[ \sum_{i=1}^p \nabla^2 \ln \Psi_i(\mathbf{r}) + \sum_{k=1}^F \left( \sum_{i=1}^p \nabla_k \ln \Psi_i(\mathbf{r}) \right)^2 \right]. \quad (9.7)$$

This can be found when all local derivatives  $\nabla^2 \ln \Psi_i(\mathbf{r})$  and  $\nabla_k \ln \Psi_i(\mathbf{r})$  are given. By assuming that the former is returned by a function `computeLaplacian()` and the latter is returned by a function `computeGradient(k)`, we compute the kinetic energy using the following function

```

double System::getKineticEnergy()
{
    double kineticEnergy = 0;
    for (auto &i : m_waveFunctionElements) {
        kineticEnergy += i->computeLaplacian();
    }
    for (int k = 0; k < m_degreesOfFreedom; k++) {
        double nablaLnPsi = 0;
        for (auto &i : m_waveFunctionElements) {
            nablaLnPsi += i->computeGradient(k);
        }
        kineticEnergy += nablaLnPsi * nablaLnPsi;
    }
    return -0.5 * kineticEnergy;
}

```

**Listing 9.2:** From system.cpp.

Note that some of the variables that are declared globally, here the vector `m_waveFunctionElements` and the integer `m_degreesOfFreedom` are denoted by an `m_` to distinguish them from the variables declared locally.

### 9.2.2 Parameter gradients

In section 7.3, we presented how the parameters can be updated by minimizing the energy expectation value. We recall that the only closed-form expression needed in addition to the local energy is  $\nabla_{\theta_j} \ln \Psi_T(\mathbf{r}_i)$ , which needs to be found. By applying equation (9.2), we find that

$$\nabla_{\theta_j} \ln \Psi_T(\mathbf{r}) = \sum_{i=1}^p \nabla_{\theta_j} \ln \Psi_i(\mathbf{r}) = \nabla_{\theta_j} \ln \Psi_{\theta_j}(\mathbf{r}), \quad (9.8)$$

where  $\Psi_{\theta_j}$  is the only element which contains parameter  $\theta_j$ . With this in mind, we need to find closed-form expressions of  $\nabla_{\theta_j} \ln \Psi_{\theta_j}(\mathbf{r})$  for all wave function elements  $\Psi_{\theta_j}(\mathbf{r})$  that are associated with a variational parameter  $\theta_j$ .

In the code, we store all the parameters in a *parameter matrix* where each element has its own row of parameters. Similarly, we create a *gradient matrix* of the same dimensions to store the gradient  $\nabla_{\theta_j} \ln \Psi_i(\mathbf{r})$  for each variational parameter. The implementation to get all the gradients are straight-forward, and given by

```

Eigen::MatrixXd System::getAllParameterGradients()
{
    for (int i = 0; i < m_numberOfElements; i++) {
        m_gradients.row(i) = m_waveFunctionElements[i]->computeParameterGradient();
    }
    return m_gradients;
}

```

**Listing 9.3:** From system.cpp.

where `m_gradients` has the same number of rows as number of elements and the same number of columns as the maximum number of parameters in an element. The function `computeParameterGradient()` returns a vector with all the gradients  $\nabla_{\theta_j} \ln \Psi_i(\mathbf{r})$  of the respective element. Even though the gradients are used to update the parameters, we will postpone the discussion of the parameter update to section 9.6.

### 9.2.3 Probability ratio

How much a wave function element can be optimized heavily depends on the specific form of the element. For instance, sometimes the previous and present  $\nabla_k \ln \Psi_i$  are closely related, and only differ from each other by a factor, while for some other elements they are not related at all. Those subjective optimizations will therefore be described when presenting each wave function element.

However, there are still optimizations that apply to all elements and give great speed-up. An example is when calculating the ratio between the previous and present wave functions squared for all wave function elements instead of the wave function itself. Firstly, this is usually cheaper to calculate than the wave function itself because we are working in the logarithm space. Secondly, the ratio is actually what we use in the sampling, so it is a natural thing to calculate. The total probability ratio is just the product of all the probability ratios

$$\frac{P(\mathbf{r}_{\text{new}})}{P(\mathbf{r}_{\text{old}})} = \frac{|\Psi_T(\mathbf{r}_{\text{new}})|^2}{|\Psi_T(\mathbf{r}_{\text{old}})|^2} = \prod_{i=1}^p \frac{|\Psi_i(\mathbf{r}_{\text{new}})|^2}{|\Psi_i(\mathbf{r}_{\text{old}})|^2}, \quad (9.9)$$

and below we will calculate this ratio for all the elements since we are going to use that directly in the sampling. We name the function returning the ratio for a particular element `evaluateRatio()`, and we obtain the total probability ratio in the following way

```
double System::evaluateProbabilityRatio()
{
    double ratio = 1;
    for (auto &i : m_waveFunctionElements) {
        ratio *= i->evaluateRatio();
    }
    return ratio;
}
```

**Listing 9.4:** From `system.cpp`.

With this, we have introduced the four central functions of the wave function elements: `computeLaplacian()`, `computeGradient(k)`, `computeParameterGradient()` and `evaluateRatio()`. We will now proceed further and see how the trial wave function is split in various elements, and then find closed-form expressions to be used in these four functions for each element. Initially, the Slater determinant will be evaluated.

## 9.3 Slater determinant

As we have seen above, the Slater determinant is the fundamental part of the trial wave function, in addition to the Jastrow factor. The main problem with the Slater determinant, is that it is very expensive to deal with as the number of particles increases. To find the gradient of the Slater determinant, as requested by equation (9.7), we need to compute the inverse of the Slater matrix, which by standard LU decomposition scales as  $\sim \mathcal{O}(N^3)$  for an  $N \times N$  matrix [43]. Fortunately, there exist algorithms that let us obtain the inverse of the matrix by recursive relations, scaling as  $\sim \mathcal{O}(N^2)$ . This will be detailed in section 9.3.4.1.

Additionally, the Slater determinant can be split in a spin-up part and a spin-down part, which reduces the Slater matrix to two  $N/2 \times N/2$  matrices. Also factorizing common functions from the Slater determinant will give some speed-up. In this section, we will mostly discuss the various methods to make the update of the Slater matrix more efficient.

The Slater determinant was first introduced in section 3.2.2, and looks like

$$\Psi_{sd}(\mathbf{r}, \sigma) = |\hat{D}(\mathbf{r}, \sigma)| \propto \begin{vmatrix} \psi_1(\mathbf{r}_1, \sigma_1) & \psi_2(\mathbf{r}_1, \sigma_2) & \dots & \psi_N(\mathbf{r}_1, \sigma_N) \\ \psi_1(\mathbf{r}_2, \sigma_1) & \psi_2(\mathbf{r}_2, \sigma_2) & \dots & \psi_N(\mathbf{r}_2, \sigma_N) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_1(\mathbf{r}_N, \sigma_1) & \psi_2(\mathbf{r}_N, \sigma_2) & \dots & \psi_N(\mathbf{r}_N, \sigma_N) \end{vmatrix} \quad (9.10)$$

where  $\psi_j(\mathbf{r}_i, \sigma_j)$  is the single particle function occupying spot  $i, j$  in the matrix.

### 9.3.1 Splitting up the Slater determinant

In order to reduce the computational cost of the Slater determinant, we will split it up in a spin-up part and a spin-down part. In real life, we cannot immediately tell if an electron has spin-up or spin-down, but since we in the program need to know which particles that have spin-up and spin-down, we simply decide that the first  $N_\uparrow$  particle have spin up and the remaining particles have spin down.

In addition to reduce the computational cost, splitting up the Slater determinant also makes it possible to factorize out the spin-part from the spin-orbitals. Only the spatial part of the single particle functions contribute to the energies, such that we with advantage can get rid of the spin parts. Recall that the spin-orbitals can be written as

$$\psi(\mathbf{r}, \sigma) = \Psi(\mathbf{r}) \otimes \chi(\sigma) \quad (9.11)$$

where only the spatial part,  $\Psi(\mathbf{r})$ , is dependent on the coordinates. As we in the ground state have double degeneracy, the spatial part will be the same for pairwise spin-up and spin-down particles, and we arrange them as

$$\psi_j(\mathbf{r}_i, \sigma_i) = \begin{cases} \Psi_j(\mathbf{r}_i)\chi_\uparrow(\sigma_i) & \text{if } j < N_\uparrow \\ \Psi_j(\mathbf{r}_i)\chi_\downarrow(\sigma_i) & \text{if } j \geq N_\uparrow. \end{cases} \quad (9.12)$$

Since the first  $N_\uparrow$  particles have spin up,  $\sigma_i = \uparrow \quad \forall i \in \{1, 2, \dots, N_\uparrow\}$  and the remaining have spin down, we can now set up the Slater determinant in equation (9.10) where each single particle function is split in a spatial part and a spin part,

$$\Psi_{sd}(\mathbf{r}, \sigma) \propto \begin{vmatrix} \Psi_1(\mathbf{r}_1)\chi_\uparrow(\uparrow) & \dots & \Psi_{N_\uparrow}(\mathbf{r}_1)\chi_\uparrow(\uparrow) & \Psi_1(\mathbf{r}_1)\chi_\downarrow(\uparrow) & \dots & \Psi_{N_\downarrow}(\mathbf{r}_1)\chi_\downarrow(\uparrow) \\ \vdots & & \vdots & \vdots & & \vdots \\ \Psi_1(\mathbf{r}_{N_\uparrow})\chi_\uparrow(\uparrow) & \dots & \Psi_{N_\uparrow}(\mathbf{r}_{N_\uparrow})\chi_\uparrow(\uparrow) & \Psi_1(\mathbf{r}_{N_\uparrow})\chi_\downarrow(\uparrow) & \dots & \Psi_{N_\downarrow}(\mathbf{r}_{N_\uparrow})\chi_\downarrow(\uparrow) \\ \Psi_1(\mathbf{r}_{N_\uparrow+1})\chi_\uparrow(\downarrow) & \dots & \Psi_{N_\uparrow}(\mathbf{r}_{N_\uparrow+1})\chi_\uparrow(\downarrow) & \Psi_1(\mathbf{r}_{N_\uparrow+1})\chi_\downarrow(\downarrow) & \dots & \Psi_{N_\downarrow}(\mathbf{r}_{N_\uparrow+1})\chi_\downarrow(\downarrow) \\ \vdots & & \vdots & \vdots & & \vdots \\ \Psi_1(\mathbf{r}_N)\chi_\uparrow(\downarrow) & \dots & \Psi_{N_\uparrow}(\mathbf{r}_N)\chi_\uparrow(\downarrow) & \Psi_1(\mathbf{r}_N)\chi_\downarrow(\downarrow) & \dots & \Psi_{N_\downarrow}(\mathbf{r}_N)\chi_\downarrow(\downarrow) \end{vmatrix}. \quad (9.13)$$

We observe that the the spin-up particles sometimes occupy spin-down states, which they are not allowed to. Therefore, half of the elements become zero and the determinant can be further expressed as

$$\Psi_{sd}(\mathbf{r}, \sigma) \propto \begin{vmatrix} \Psi_1(\mathbf{r}_1)\chi_\uparrow(\uparrow) & \dots & \Psi_{N_\uparrow}(\mathbf{r}_1)\chi_\uparrow(\uparrow) & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ \Psi_1(\mathbf{r}_{N_\uparrow})\chi_\uparrow(\uparrow) & \dots & \Psi_{N_\uparrow}(\mathbf{r}_{N_\uparrow})\chi_\uparrow(\uparrow) & 0 & \dots & 0 \\ 0 & \dots & 0 & \Psi_{N_\uparrow+1}(\mathbf{r}_{N_\uparrow+1})\chi_\downarrow(\downarrow) & \dots & \Psi_N(\mathbf{r}_{N_\uparrow+1})\chi_\downarrow(\downarrow) \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & \Psi_{N_\uparrow+1}(\mathbf{r}_N)\chi_\downarrow(\downarrow) & \dots & \Psi_N(\mathbf{r}_N)\chi_\downarrow(\downarrow) \end{vmatrix} \quad (9.14)$$

where the Slater matrix now is block diagonal! For a general block diagonal matrix, the determinant is given by the product of the determinant of each block

$$\Psi_{sd}(\mathbf{r}, \sigma) \propto |\hat{D}_\uparrow(\mathbf{r}_\uparrow, \sigma_\uparrow)| \cdot |\hat{D}_\downarrow(\mathbf{r}_\downarrow, \sigma_\downarrow)| \quad (9.15)$$

which can be seen by writing the total matrix as a product over all the block diagonal matrix.  $\hat{D}_\uparrow$  is the matrix containing all spin-up states and  $\hat{D}_\downarrow$  is the matrix containing all spin-down states. Since all elements in the respective matrices contain the same spin function, it can be factorized out,

$$\Psi_{sd}(\mathbf{r}, \sigma) \propto |\hat{D}_\uparrow(\mathbf{r}_\uparrow)| \{ \chi_\uparrow(\sigma_1^\uparrow) \chi_\uparrow(\sigma_2^\uparrow) \dots \chi_\uparrow(\sigma_{N_\uparrow}^\uparrow) \} \cdot |\hat{D}_\downarrow(\mathbf{r}_\downarrow)| \{ \chi_\downarrow(\sigma_{N_\uparrow+1}^\downarrow) \dots \chi_\downarrow(\sigma_{N-1}^\downarrow) \chi_\downarrow(\sigma_N^\downarrow) \} \quad (9.16)$$

and omitted in the future study since the energy is independent of spin. We are then left with the Slater determinant

$$\Psi_{sd}(\mathbf{r}) \propto |\hat{D}_\uparrow(\mathbf{r}_\uparrow)| \cdot |\hat{D}_\downarrow(\mathbf{r}_\downarrow)| \quad (9.17)$$

which is independent of spin, i.e, the matrices now consist of the spatial functions  $\Psi_j(\mathbf{r}_i)$  as the elements.

It is also worth to notice that the size of the spin-up matrix is determined by the number of spin-up particles, and it is similar for the spin-down matrix. This means that we can change the total spin  $S$  by adjusting the relative sizes of the determinants. In the implementation, however, we stick to the magic quantum numbers, where  $N_\uparrow = N_\downarrow$ . Otherwise, we need to add a spin term to the Hamiltonian.

This section was heavily inspired by Daniel Nissenbaum's doctoral dissertation, see appendix I in [50].

### 9.3.2 Factorizing out elements

We have now seen how the Slater determinant can be split up in a spin-up part and a spin-down part. Before we evaluate these determinants, we should try to make the elements of the Slater matrices as simple as possible to save computational time. If all the elements have the same factor, the computations will get much cheaper if the factor is factorized out of the matrix. How this is possible is easiest to see if we express the Slater determinant on a summation form,

$$\Psi_{sd}(\mathbf{r}) \propto \sum_p (-1)^p \hat{P} \Psi_1(\mathbf{r}_1) \Psi_2(\mathbf{r}_2) \dots \Psi_N(\mathbf{r}_N), \quad (9.18)$$

where the  $\hat{P}$  is the permutation operator, permuting coordinates pairwise and the sum runs over all the possible permutations. If all the (spatial) single particle functions  $\Psi_j(\mathbf{r})_i$  can be split in two functions  $f_j(\mathbf{r}_i)$  and  $g(\mathbf{r}_i)$  where the latter is common for all the single particle functions,

$$\Psi_j(\mathbf{r}_i) = f_j(\mathbf{r}_i) g(\mathbf{r}_i) \quad (9.19)$$

the Slater determinant can be rewritten as

$$\begin{aligned} \Psi_{sd}(\mathbf{r}) &\propto \sum_p (-1)^p \hat{P} f_1(\mathbf{r}_1) g(\mathbf{r}_1) f_2(\mathbf{r}_2) g(\mathbf{r}_2) \dots f_N(\mathbf{r}_N) g(\mathbf{r}_N) \\ &= g(\mathbf{r}_1) g(\mathbf{r}_2) \dots g(\mathbf{r}_N) \sum_p (-1)^p \hat{P} f_1(\mathbf{r}_1) f_2(\mathbf{r}_2) \dots f_N(\mathbf{r}_N) \\ &= \prod_{i=1}^N g(\mathbf{r}_i) \begin{vmatrix} f_1(\mathbf{r}_1) & f_2(\mathbf{r}_1) & \dots & f_N(\mathbf{r}_1) \\ f_1(\mathbf{r}_2) & f_2(\mathbf{r}_2) & \dots & f_N(\mathbf{r}_2) \\ \vdots & \vdots & \ddots & \vdots \\ f_1(\mathbf{r}_N) & f_2(\mathbf{r}_N) & \dots & f_N(\mathbf{r}_N) \end{vmatrix} \end{aligned} \quad (9.20)$$

This is very useful when it comes to many common basises. For instance, the Hermite basis is given by

$$\Psi_j(\mathbf{r}_i) \propto H_j(\mathbf{r}_i) \exp\left(-\frac{1}{2}\omega|\mathbf{r}_i|^2\right) \quad (9.21)$$

where  $H_j(\mathbf{r}_i)$  are the Hermite polynomials and the Gaussian part fulfills the requirement of  $g(\mathbf{r}_i)$ . Therefore, we can construct a Slater determinant containing the Hermite polynomials only, treating the Gaussian as an independent element. This is not only preferable from an efficiency point of view, by doing this the variational parameter in the Gaussian is also removed from the determinant, which means that we can implement the determinant without worrying about the variational parameters.

With this in mind, we will first treat the Gaussian element, obtaining its derivative and optimization schemes. Moreover, in section 9.3.4 we will discuss how the determinant can be treated efficiently.

### 9.3.3 Gaussian

When factorizing out the Gaussian part from the Slater determinant, we obtain the element

$$\Psi_{sg}(\mathbf{r}; \alpha) = \prod_{j=1}^N g(\mathbf{r}_j) = \exp\left(-\frac{1}{2}\omega\alpha \sum_{j=1}^N r_j^2\right) = \exp\left(-\frac{1}{2}\omega\alpha \sum_{j=1}^F x_j^2\right), \quad (9.22)$$

with  $N$  number of particles and  $F = NP$  degrees of freedom.  $\omega$  is the oscillator strength and  $\alpha$  is a variational parameter, which for non-interacting atoms is 1. Because of the appearance of  $r_i^2$ , the function can easily be treated both in Cartesian and spherical coordinates, but in this thesis we will focus on the former.

We now use  $j$  as our summation index, and reserve  $i$  for the moved particle and  $k$  as our the differentiating index. When changing a coordinate  $x_i$  from  $x_i^{\text{old}}$  to  $x_i^{\text{new}}$ , the probability ratio can easily be found to be

$$\frac{|\Psi_{sg}(\mathbf{x}_{\text{new}})|^2}{|\Psi_{sg}(\mathbf{x}_{\text{old}})|^2} = \exp\left(\omega\alpha((x_i^{\text{old}})^2 - (x_i^{\text{new}})^2)\right), \quad (9.23)$$

which is pretty cheap to evaluate. The gradient of  $\ln \Psi_{sg}$  with respect to the coordinate  $x_k$  is

$$\nabla_k \ln \Psi_{sg} = -\omega\alpha x_k, \quad (9.24)$$

and the corresponding Laplacian is

$$\nabla^2 \ln \Psi_{sg} = -\omega\alpha F. \quad (9.25)$$

We observe that the factor  $\omega\alpha$  is found in all the expressions above, and can be computed when the parameter  $\alpha$  is updated only. Finally, we will update  $\alpha$  according to

$$\nabla_\alpha \ln \Psi_{sg} = -\frac{1}{2}\omega \sum_{j=1}^F x_j^2. \quad (9.26)$$

Since this wave function element is quite simple, there is no special optimization available that will cause a noticeable speed-up. The implementation looks like

```

double Gaussian :: evaluateRatio ()
{
    return m_probabilityRatio;
}

double Gaussian :: computeGradient(const int k)
{
    return -m_omegalpha * m_positions(k);
}

double Gaussian :: computeLaplacian()
{
    return -m_omegalpha * m_degreesOfFreedom;
}

Eigen :: VectorXd Gaussian :: computeParameterGradient()
{
    m_gradients(0) = -0.5 * m_omega * m_positions.cwiseAbs2().sum();
    return m_gradients;
}

```

**Listing 9.5:** From gaussian.cpp.

where `m_omegalpha` is  $\omega\alpha$ . The probability ratio is calculated using

```

double void Gaussian :: updateProbabilityRatio(int changedCoord)
{
    m_probabilityRatio = exp(m_omegalpha
        * (m_positionsOld(changedCoord) * m_positionsOld(changedCoord)
           - m_positions(changedCoord) * m_positions(changedCoord)));
}

```

**Listing 9.6:** From gaussian.cpp.

We see that matrix-vector operations are used when it is possible, which makes the computations very fast.

### 9.3.4 The determinant

As discussed in section 9.3.1, the Slater determinant can be split in a spin-up part and a spin-down part, and further the common functions can be factorized out as shown in section 9.3.2. This means that the remaining determinant is not the full Slater determinant, and to distinguish it from the real Slater determinant,  $\Psi_{sd}(\mathbf{r})$ , we will name the element  $\Psi_{det}(\mathbf{r})$ . This determinant is of course still split like the Slater determinant,

$$\Psi_{det}(\mathbf{r}) = |\hat{D}_\uparrow(\mathbf{r}_\uparrow)| \cdot |\hat{D}_\downarrow(\mathbf{r}_\downarrow)|, \quad (9.27)$$

where  $r_\uparrow$  are the coordinates of particles with spin up (defined as the first  $N_\uparrow$  coordinates) and  $r_\downarrow$  are the coordinates of particles with spin down (defined as the last  $N_\downarrow$  coordinates).

We can now utilize the logarithmic scale, by using that the logarithm of a product corresponds to summarize the logarithm of each factor,

$$\ln \Psi_{det} = \ln |\hat{D}_\uparrow(\mathbf{r}_\uparrow)| + \ln |\hat{D}_\downarrow(\mathbf{r}_\downarrow)| \quad (9.28)$$

such that we only need to care about one of the determinants when differentiating, dependent on whether the coordinate we differentiate with respect to is among the spin-up or the spin-down coordinates:

$$\nabla_k \ln \Psi_{det} = \begin{cases} \nabla_k \ln |\hat{D}_\uparrow(\mathbf{r}_\uparrow)| & \text{if } k < N_\uparrow \\ \nabla_k \ln |\hat{D}_\downarrow(\mathbf{r}_\downarrow)| & \text{if } k \geq N_\uparrow. \end{cases} \quad (9.29)$$

Before we go further, we will introduce a more general notation which covers both the cases:

$$\hat{D}(\mathbf{r}) \equiv \hat{D}_\sigma(\mathbf{r}_\sigma) \quad (9.30)$$

where  $\sigma$  is the spin projection. When summarizing, the sum is always over all relevant coordinates. Furthermore, we have that

$$\nabla_k \ln |\hat{D}(\mathbf{r})| = \frac{\nabla_k |\hat{D}(\mathbf{r})|}{|\hat{D}(\mathbf{r})|} \quad (9.31)$$

and

$$\nabla_k^2 \ln |\hat{D}(\mathbf{r})| = \frac{\nabla_k^2 |\hat{D}(\mathbf{r})|}{|\hat{D}(\mathbf{r})|} - \left( \frac{\nabla_k |\hat{D}(\mathbf{r})|}{|\hat{D}(\mathbf{r})|} \right)^2 \quad (9.32)$$

At this point, there are (at least) two possible paths to the final expressions. We can keep on using matrix operations and find the expressions of  $\nabla_k |\hat{D}(\mathbf{r})|/|\hat{D}(\mathbf{r})|$  and  $\nabla_k^2 |\hat{D}(\mathbf{r})|/|\hat{D}(\mathbf{r})|$  using Jacobi's formula, or we can switch to element representations of the matrices. We choose the latter, because we believe that is the path of least resistance.

The determinant of an arbitrary matrix  $\hat{A}$  can be expressed by its comatrix  $\hat{C}$  in the following way,

$$|\hat{A}| = \sum_{ij} a_{ij} c_{ji} \quad (9.33)$$

where  $a_{ij}$  are the matrix elements of  $\hat{A}$  and  $c_{ij}$  are the element of the comatrix. Going further, an element  $c_{ij}$  can be expressed in terms of an element from the inverse of  $\hat{A}$ ,  $a_{ij}^{-1}$  [84],

$$c_{ij} = a_{ij}^{-1} |\hat{A}|. \quad (9.34)$$

Relating this to our particular problem, we can express

$$\begin{aligned} \frac{\nabla_k |\hat{D}|}{|\hat{D}|} &= \frac{\nabla_k \sum_{ij} d_{ij} c_{ji}}{\sum_{ij} d_{ij} c_{ji}} = \frac{\sum_j \nabla_k d_{kj} c_{jk}}{\sum_{ij} d_{ij} c_{ji}} \\ &= \frac{\sum_j \nabla_k d_{kj} d_{jk}^{-1} |\hat{D}|}{\sum_{ij} d_{ij} d_{ji}^{-1} |\hat{D}|} = \sum_j \nabla_k d_{kj} d_{jk}^{-1} \end{aligned} \quad (9.35)$$

where  $d_{ij}$  are the elements of  $\hat{D}$  and we have used the fact that the elements  $\nabla_k d_{ij}$  contribute to the sum if and only if  $i = k$ , such that the sum over  $i$  collapses. Moreover, we use that multiplying a matrix with its inverse is identity, i.e.,  $\sum_{ij} d_{ij} d_{ji}^{-1} = 1$  and the determinants cancel. Similarly, we get

$$\frac{\nabla_k^2 |\hat{D}|}{|\hat{D}|} = \sum_j \nabla_k^2 d_{kj} d_{jk}^{-1} \quad (9.36)$$

for the Laplacian. We are then ready to write up the final expressions for the gradient and Laplacian of the logarithm of the Slater determinant,

$$\begin{aligned} \nabla_k \ln |\hat{D}(\mathbf{r})| &= \sum_j d_{jk}^{-1}(\mathbf{r}) \nabla_k \Psi_j(\mathbf{r}_k) \\ \nabla_k^2 \ln |\hat{D}(\mathbf{r})| &= \sum_j d_{jk}^{-1}(\mathbf{r}) \nabla_k^2 \Psi_j(\mathbf{r}_k) - \left( \sum_j d_{jk}^{-1}(\mathbf{r}) \nabla_k \Psi_j(\mathbf{r}_k) \right)^2 \end{aligned} \quad (9.37)$$

where we have used that  $d_{ij} = \Psi_j(\mathbf{r}_i)$  with  $\Psi_j(\mathbf{r}_i)$  as a single particle function found in the Slater determinant, see above.

### 9.3.4.1 Efficient calculation of the determinant

As aforementioned, dealing with the Slater determinant is very computational expensive, mainly because of the requirement of the inverse Slater matrix. However, by exploiting that only one row in the Slater matrix is updated for each step, we can update the inverse recursively. By using the same element representations as above, the ratio between the new and the old determinant can be expressed as

$$R \equiv \frac{|\hat{D}(\mathbf{r}_{\text{new}})|}{|\hat{D}(\mathbf{r}_{\text{old}})|} = \frac{\sum_j d_{ij}(\mathbf{r}_{\text{new}}) c_{ij}(\mathbf{r}_{\text{new}})}{\sum_j d_{ij}(\mathbf{r}_{\text{old}}) c_{ij}(\mathbf{r}_{\text{old}})} = \sum_j d_{ij}(\mathbf{r}_{\text{new}}) d_{ji}^{-1}(\mathbf{r}_{\text{old}}) \quad (9.38)$$

which is very similar to the calculation given in equation (9.35). To calculate the inverse matrix  $\hat{D}^{-1}$  efficiently, we need to calculate

$$S_j = \sum_{l=1}^N d_{il}(\mathbf{r}_{\text{new}}) d_{lj}^{-1}(\mathbf{r}_{\text{old}}) \quad (9.39)$$

for all columns but the one associated with the moved particle,  $i$ . For all columns where  $j \neq i$ , we then find the new elements using

$$d_{kj}^{-1}(\mathbf{r}_{\text{new}}) = d_{kj}^{-1}(\mathbf{r}_{\text{old}}) - \frac{S_j}{R} d_{ki}^{-1}(\mathbf{r}_{\text{old}}) \quad (9.40)$$

while the remaining column,  $i$ , is updated using the simple formula [84]

$$d_{ki}^{-1}(\mathbf{r}_{\text{new}}) = \frac{1}{R} d_{ki}^{-1}(\mathbf{r}_{\text{old}}). \quad (9.41)$$

Those procedures makes the inverting scale as  $\mathcal{O}(N^2)$  instead of  $\mathcal{O}(N^3)$ , which is largely beneficial for large systems.

We assume that we do not have any variational parameter in the determinant, and obtain three expressions of the case when a particle with spin up is moved and three of the case when a particle with spin down is moved.

$$\text{if } k < N_{\uparrow} : \quad | \Psi_{sd}(\mathbf{r}_{\text{new}}) |^2 / | \Psi_{sd}(\mathbf{r}_{\text{old}}) |^2 = |\hat{D}_{\uparrow}(\mathbf{r}_{\uparrow}^{\text{new}})|^2 / |\hat{D}_{\uparrow}(\mathbf{r}_{\uparrow}^{\text{old}})|^2 \quad (9.42)$$

$$\nabla_k \ln |\hat{D}_{\uparrow}(\mathbf{r}_{\uparrow})| = \sum_{j=1}^{N_{\uparrow}} \nabla_k d_{jk}(\mathbf{r}_{\uparrow}) d_{kj}^{-1}(\mathbf{r}_{\uparrow})$$

$$\text{if } k \geq N_{\uparrow} : \quad | \Psi_{sd}(\mathbf{r}_{\text{new}}) |^2 / | \Psi_{sd}(\mathbf{r}_{\text{old}}) |^2 = |\hat{D}_{\downarrow}(\mathbf{r}_{\downarrow}^{\text{new}})|^2 / |\hat{D}_{\downarrow}(\mathbf{r}_{\downarrow}^{\text{old}})|^2 \quad (9.43)$$

$$\nabla_k \ln |\hat{D}_{\downarrow}(\mathbf{r}_{\downarrow})| = \sum_{j=N_{\uparrow}}^F \nabla_k d_{jk}(\mathbf{r}_{\downarrow}) d_{kj}^{-1}(\mathbf{r}_{\downarrow})$$

$$\nabla^2 \ln |\hat{D}(\mathbf{r})| = \sum_{k=1}^F \left[ \sum_{j=1}^F \nabla_k^2 d_{jk}(\mathbf{r}) d_{kj}^{-1}(\mathbf{r}) - \left( \sum_{j=1}^F \nabla_k d_{ik}(\mathbf{r}) d_{ki}^{-1}(\mathbf{r}) \right)^2 \right] \quad (9.44)$$

We have now presented the theory behind finding the ratio between the new and the old probability and the gradients of the determinant (equation (9.42-9.44)), and we have described how we efficiently can find the inverse of the Slater matrix (equation (9.38-9.41)). However, to make things more clear we will outline some selected parts of the Slater determinant implementation.

In the code, we create a Slater matrix, `m_slaterMatrix` where all the elements are stored. This matrix contains both  $\hat{D}_\uparrow$  and  $\hat{D}_\downarrow$ , and has therefore dimensions  $N \times N/2$ . Furthermore, we store the gradient of the elements with respect to all the  $F$  elements in a matrix `m_slaterMatrixDer`, which naturally gets the dimensions  $F \times N/2$ . We also create a matrix `m_slaterMatrixSecDer` to store all the Laplacians of the elements which also has the dimensions  $F \times N/2$ . In all of them, we only need to update a row when a particle is moved, which makes it quite efficient. The `m_slaterMatrixDer` is updated in the following way

```
void SlaterDeterminant::updateSlaterMatrixDerRow(const int row)
{
    int particle = int(row / m_numberOfDimensions);
    int dimension = row % m_numberOfDimensions;
    for (int col = 0; col < m_numberOfParticlesHalf; col++) {
        m_slaterMatrixDer(row, col) = m_basis->basisElementDer(col,
                                                               dimension, m_positions.col(particle));
    }
}
```

**Listing 9.7:** Taken from `slaterdeterminant.cpp`.

where each element is taken from the `basisElementDer` function in the `Basis` class. This function returns just the derivative of the single particle function called for the chosen basis set. Note also that only the coordinates of the moved particle, stored in a column of the `m_positions` matrix, is needed for the update. The update of `m_slaterMatrix` and `m_slaterMatrixSecDer` are very straightforward and similar to the example above, so we will not discuss them further.

Something that might be less intuitive, is how to update the inverse of the Slater matrix. We also store this in a dedicated matrix `m_slaterMatrixInverse`, and we use LU decomposition only to initialize it. Thereafter, we use the formulas above to update the inverse. We implement it as

```
void SlaterDeterminant::updateRatio()
{
    m_ratio = m_slaterMatrix.row(m_particle) *
              ↪ m_slaterMatrixInverse.col(m_particle);
}

void SlaterDeterminant::updateSlaterMatrixInverse(int start, int end)
{
    updateRatio();
    for (int j = start; j < m_particle; j++) {
        double S = m_slaterMatrix.row(m_particle) * m_slaterMatrixInverse.col(j);
        m_slaterMatrixInverse.col(j) -= S * m_slaterMatrixInverse.col(m_particle)
                                      / m_ratio;
    }
    for (int j = m_particle+1; j < end; j++) {
        double S = m_slaterMatrix.row(m_particle) * m_slaterMatrixInverse.col(j);
        m_slaterMatrixInverse.col(j) -= S * m_slaterMatrixInverse.col(m_particle)
                                      / m_ratio;
    }
    m_slaterMatrixInverse.col(m_particle) /= m_ratio;
}
```

**Listing 9.8:** Taken from `slaterdeterminant.cpp`.

where `m_ratio` is a global variable also returned by the function `evaluateRatio` (see section 9.2.3). Note that the loops never affect the  $i$ 'th columns, where particle  $i$  is moved (in the code denoted by

the global variable `m_particle`). The arguments to the function `updateSlaterMatrixInverse` specify which part of the matrix that should be updated, based on whether the moved particle has spin-up or spin-down.

We will end our discussions of the Slater determinant by presenting the implementation of the gradient and the Laplacian of the logarithm of the determinant. These were decided to be stored in the vectors `m_determinantDer` and `m_determinantSecDer` for  $(\nabla_k |\hat{D}(\mathbf{r})|)/|\hat{D}(\mathbf{r})|$  and  $(\nabla_k^2 |\hat{D}(\mathbf{r})|)/|\hat{D}(\mathbf{r})|$  respectively. These vectors are updated using vector operations in the following fashion

```
void SlaterDeterminant::updateSlaterDeterminantDerivatives(int start, int end)
{
    for (int i = start * m_numberOfDimensions; i < end * m_numberOfDimensions;
         i++) {
        int particle = int(i / m_numberOfDimensions);
        m_determinantDer(i) = m_slaterMatrixDer.row(i)
            * m_slaterMatrixInverse.col(particle);
        m_determinantSecDer(i) = m_slaterMatrixSecDer.row(i)
            * m_slaterMatrixInverse.col(particle);
    }
}
```

**Listing 9.9:** Taken from `slaterdeterminant.cpp`.

We avoid a double loop by taking a inner product instead. However, we are left with one loop which can also be avoided using smart matrix operations.

## 9.4 Jastrow factor

The second main part of the trial wave function is the Jastrow factor, which is meant to take care of the electron-electron correlations. The optimization scheme of this element is not as complex as the determinant, and this section will therefore be notably shorter than the previous. We will first discuss the two Jastrow factors given in section 3.2.4: the simple Jastrow and the Padé-Jastrow factor, and then we look at how the distance matrix can be updated efficiently.

### 9.4.1 Simple Jastrow factor

Recall the simple Jastrow factor from (3.12),

$$\Psi_{sj}(\mathbf{r}; \boldsymbol{\beta}) = \exp \left( \sum_{i=1}^N \sum_{j>i}^N \beta_{ij} r_{ij} \right). \quad (9.45)$$

with  $N$  as the number of particles,  $r_{ij}$  as the distance between particle  $i$  and  $j$  and  $\beta_{ij}$  as variational parameters.

This is a quite simple element, but one challenge is that we operate in Cartesian coordinates, while the expressed Jastrow factor obviously is easier to deal with in polar coordinates. Since we need to differentiate this with respect to all degrees of freedom, we need to be attentive not confusing the particle indices with the coordinate indices. Let us reserve  $j$  as the coordinate index and  $j'$  as the index of the corresponding particle. The relationship between  $j$  and  $j'$  is *always*  $j' = j \backslash D$ , where the backslash denotes integer division. The other way around, we have  $j = j' + d$  where  $d$  is the respective dimension of the coordinate  $j$ . With that notation, the probability ratio is given by

$$\frac{|\Psi_{sj}(\mathbf{r}_{\text{new}})|^2}{|\Psi_{sj}(\mathbf{r}_{\text{old}})|^2} = \exp \left( 2 \sum_{j'=1}^N \beta_{i'j'} (r_{i'j'}^{\text{new}} - r_{i'j'}^{\text{old}}) \right) \quad (9.46)$$

where  $i'$  again is the moved particle. The gradient is straight-forward to find, and reads

$$\nabla_k \ln \Psi_{sj} = \sum_{j'=1}^N \beta_{k'j'} \frac{x_k - x_j}{r_{k'j'}} \quad (9.47)$$

where  $j$  is related to the same dimension as  $k$ . Here we use  $x_j$  as a general coordinate, no matter whether it is associated with the x-direction or not. This also applies for the Laplacian,

$$\nabla^2 \ln \Psi_{sj} = \sum_{k=1}^F \sum_{j'=1}^N \frac{\beta_{k'j'}}{r_{k'j'}} \left[ 1 - \left( \frac{x_k - x_j}{r_{k'j'}} \right)^2 \right]. \quad (9.48)$$

Finally, the parameter update is given by

$$\nabla_{\beta_{m'l'}} \ln \Psi_{sj} = r_{m'l'}. \quad (9.49)$$

For this element, the most important thing we can do to keep the computational cost as low as possible is to reveal that only a row and a column of the distance matrix is changed as we change a coordinate. Updating the entire distance matrix means updating  $N^2$  elements, while updating a row and a column means updating  $2N$  elements, which is an essential difference for large systems. This is detailed in section 9.4.3.

We also observe that the factor  $(x_k - x_j)/r_{k'j'}$  is found in both the gradient and the Laplacian, so by storing this matrix we can speed-up the computations. Most naturally, the matrix has dimensions  $F \times F$ , but using that we only are interested in the elements where  $x_k$  and  $x_j$  have the same dimension and that the diagonal is zero, only  $F$  of the elements need to be found. Further, the matrix is obviously anti-symmetric, so we actually only need to calculate  $F/2$  of the elements.

When one particle is moved, only the elements related to the moved particle need to be updated, which is  $2(N - 1)$  elements. Again, we utilize that the matrix is anti-symmetric and get the following efficient update scheme

```
void SimpleJastrow::updatePrincipalDistance(int i)
{
    int i_d = i % m_numberOfDimensions;
    for (int j_p = 0; j_p < i_p; j_p++) {
        int j = i_d + j_p * m_numberOfDimensions;
        m_principalDistance(i, j) = (m_positions(i) - m_positions(j))
            / m_distanceMatrix(i_p, j_p);
        m_principalDistance(j, i) = -m_principalDistance(i, j);
    }
    for (int j_p = i_p + 1; j_p < m_numberOfParticles; j_p++) {
        int j = i_d + j_p * m_numberOfDimensions;
        m_principalDistance(i, j) = (m_positions(i) - m_positions(j))
            / m_distanceMatrix(i_p, j_p);
        m_principalDistance(j, i) = -m_principalDistance(i, j);
    }
}
```

**Listing 9.10:** Taken from simplejastrow.cpp.

where  $i_p$  is the moved particle  $i_d$  is the dimension the particle is moved in and  $i$  thus is the coordinate index. Similarly, the loop goes over the particles  $j_p$  with the associated coordinate  $j$ . Note that we split the loop in two parts to avoid calculating the distance from a particle to itself. This trick is also done in many of the other functions in the simple Jastrow class, and also in the Padé-Jastrow factor class.

### 9.4.2 The Padé-Jastrow factor

The Padé-Jastrow factor is a more complicated Jastrow factor, and was specified in equation (3.13),

$$\Psi_{pj}(\mathbf{r}; \beta) = \exp \left( \sum_{i=1}^N \sum_{j>i}^N \frac{a_{ij} r_{ij}}{1 + \beta r_{ij}} \right). \quad (9.50)$$

where  $\beta$  is a variational parameter and the  $a_{ij}$  is **not** a variational parameter, but rather constants dependent on the spin of particles  $i$  and  $j$ .

Similarly to the simple Jastrow, we also here need to distinguish between particle indices and coordinate indices because of the radial distances  $r_{ij}$ . We do the same trick as presented for the simple Jastrow denoting  $j$  as the coordinate index and  $j'$  as the particle index, and obtain the gradient

$$\nabla_k \ln \Psi_{pj} = \sum_{j' \neq k'=1}^N \frac{a_{k'j'}}{(1 + \beta r_{k'j'})^2} \frac{x_k - x_j}{r_{k'j'}} \quad (9.51)$$

with respect to the coordinate  $x_k$ . By again differentiating this with respect to  $x_k$ , we obtain the Laplacian

$$\nabla^2 \ln \Psi_{pj} = \sum_{k=1}^F \sum_{j' \neq k'=1}^N \frac{a_{k'j'}}{(1 + \beta r_{k'j'})^2} \left[ 1 - \left( 1 + 2 \frac{\beta r_{k'j'}}{1 + \beta r_{k'j'}} \right) \left( \frac{x_k - x_j}{r_{k'j'}} \right)^2 \right] \frac{1}{r_{k'j'}}. \quad (9.52)$$

Similar to the simple Jastrow factor, we again observe the factor  $(x_k - x_j)/r_{k'j'}$  in both the gradient and the Laplacian, which can be stored as a matrix and updated in the same way as described for the simple Jastrow factor. The last expression we need is the one used to update the variational parameter  $\beta$ , which is found to be

$$\nabla_\beta \ln \Psi_{pj} = - \sum_{i'=1}^N \sum_{j'>i'}^N \frac{a_{ij} r_{ij}^2}{(1 + \beta r_{ij})^2}. \quad (9.53)$$

In addition to the factor  $g_{ij} \equiv (x_k - x_j)/r_{k'j'}$ , there are multiple factors that we can store to make the computations cheaper. The factor  $f_{ij} \equiv a_{ij}/(1 + \beta r_{ij})^2$  is found both in the gradient, Laplacian and parameter gradient, and storing it will save a significant amount of computational time. Lastly, the factor  $h_{ij} \equiv r_{ij}/(1 + \beta r_{ij})$  is found in several places and will be stored as well. As a summary, we use

$$f_{ij} = \frac{a_{ij}}{(1 + \beta r_{ij})^2} \quad g_{ij} = \frac{x_i - x_j}{r_{i'j'}} \quad h_{ij} = \frac{r_{ij}}{1 + \beta r_{ij}}. \quad (9.54)$$

and obtain the simplified expressions

$$\begin{aligned} \frac{|\Psi_{pj}(\mathbf{r}_{\text{new}})|^2}{|\Psi_{pj}(\mathbf{r}_{\text{old}})|^2} &= \exp \left( 2 \sum_{j'=1}^N a_{i'j'} (h_{i'j'}^{\text{new}} - h_{i'j'}^{\text{old}}) \right) \\ \nabla_k \ln \Psi_{pj} &= \sum_{j' \neq k'=1}^N f_{k'j'} \cdot g_{kj} \\ \nabla^2 \ln \Psi_{pj} &= \sum_{k=1}^F \sum_{j' \neq k'=1}^N \frac{f_{k'j'}}{r_{k'j'}} \left[ 1 - \left( 1 + 2\beta h_{k'j'} \right) g_{kj}^2 \right] \\ \nabla_\beta \ln \Psi_{pj} &= - \sum_{l'=1}^N \sum_{j>l}^N a_{l'j'} h_{l'j'}^2 = - \sum_{l'=1}^N \sum_{j>l}^N f_{l'j'} r_{l'j'}^2 \end{aligned} \quad (9.55)$$

with marked indices ( $j'$ ) as the particle related ones and the unmarked ( $j$ ) as the coordinate related ones.  $i'$  is the moved particle. We now proceed further to the update of the distance matrix, which is where we can find the remaining optimization possibilities.

### 9.4.3 Updating the distance matrix

The distance matrix, which is used in the Jastrow factors, gives an illustrating example on how we can avoid repeating calculations. The matrix, henceforth named  $M$ , contains the relative distances between all the particles, for three particles given by

$$M = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} = \begin{pmatrix} 0 & r_{12} & r_{13} \\ r_{12} & 0 & r_{23} \\ r_{13} & r_{23} & 0 \end{pmatrix} \quad (9.56)$$

where  $r_{ij}$  means the distance between particles  $i$  and  $j$ . Since  $r_{ij} = r_{ji}$  and  $r_{ii} = 0$ , the matrix becomes symmetric with zeros on the diagonal, which means that we only need to calculate  $N(N - 1)/2$  elements instead of  $N^2$ . Further, we can utilize that only a particle is moved at a time, which means that only a row and a column are changed when a particle is moved. For instance, if particle 1 is moved, the upper row and the left-hand-side column in matrix  $M$  need to be updated. In our program, we have implemented this in the following way

```
double Metropolis::calculateDistanceMatrixElement(const int i, const int j)
{
    double dist = 0;
    int parti = m_numberOfDimensions*i;
    int partj = m_numberOfDimensions*j;
    for(int d=0; d<m_numberOfDimensions; d++) {
        double diff = m_positions(parti+d)-m_positions(partj+d);
        dist += diff*diff;
    }
    return sqrt(dist);
}

void Metropolis::calculateDistanceMatrixCross(const int particle) {
    for(int i=0; i<m_numberOfParticles; i++) {
        m_distanceMatrix(particle, i) = calculateDistanceMatrixElement(particle,
            ↪ i);
        m_distanceMatrix(i, particle) = m_distanceMatrix(particle, i);
    }
}
```

**Listing 9.11:** Taken from `metropolis.cpp`.

where the function `calculateDistanceMatrixElement(i,j)` returns element  $i,j$  of the matrix, which is called from the function `calculateDistanceMatrixCross(particle)`. The latter takes the moved particle index as input, and updates the necessary row and column of the matrix.

For systems of non-interacting particles, the distance matrix is redundant, and should therefore not be calculated. We have solved this by giving all the wave function elements and the Hamiltonians a number which indicated whether they require the distance matrix or not, as mentioned above. If no part of the code needs the distance matrix, it is never calculated.

We also calculate the radial position globally when it is required by any part of the code. The components are stored in a vector named `radialVector`, applying the same optimization ideas as the distance matrix.

## 9.5 Sampling

Also when it comes to the sampling itself, there exist optimization schemes to speed-up the process. Remember that the sampling algorithm often is repeated millions of times for each iteration, so even a small impact can give a massive speed-up. We will initially present the brute force sampling implementation in its entirety, before we move on to the importance sampling implementation. For the latter, we will discuss the optimization possibilities and connect them to the actual implementation.

### 9.5.1 Brute force sampling

The brute force sampling was introduced in section 7.2.1, and is the most basic sampling method implemented. The sampling function `BruteForce::acceptMove`, which returns true if the move is accepted, is implemented as

```
bool BruteForce :: acceptMove()
{
    int i = m_RNG->nextInt(m_degreesOfFreedom);

    m_positionsOld = m_positions;
    m_radialVectorOld = m_radialVector;
    m_distanceMatrixOld = m_distanceMatrix;

    m_positions(i) += (m_RNG->nextDouble() - 0.5) * m_stepLength;
    if (m_calculateDistanceMatrix) {
        Metropolis :: calculateDistanceMatrixCross(int(i / m_numberOfDimensions));
    }
    if (m_calculateRadialVector) {
        Metropolis :: calculateRadialVectorElement(int(i / m_numberOfDimensions));
    }
    m_system->updateAllArrays(m_positions, m_radialVector, m_distanceMatrix, i);

    double p = m_system->evaluateProbabilityRatio();
    if (p < m_RNG->nextDouble()) {
        m_positions = m_positionsOld;
        m_distanceMatrix = m_distanceMatrixOld;
        m_radialVector = m_radialVectorOld;
        m_system->resetAllArrays();
        return false;
    }
    return true;
}
```

**Listing 9.12:** Taken from `bruteforce.cpp`.

where `i` is the changed coordinate which is drawn from the random number generator `m_RNG`. Initially the old positions, radial vector and distance matrix are stored in case the move is rejected, and then a new move is proposed in positive or negative direction. If the radial vector or distance matrix (or both) are needed somewhere in the code, they are updated in this function, using the ideas and implementation presented in section 9.4.3. The they are distributed to the wave function elements using the function `updateAllArrays`.

In the end, the probability ratio is evaluated using the function `evaluateProbabilityRatio` presented in section 9.2.3. If this ratio is larger than a random number between 0 and 1, the move is accepted, and otherwise we set all the arrays back to the old ones (also the ones in the wave function elements).

### 9.5.2 Importance sampling

The importance sampling implementation is very similar to the brute force sampling implementation, and we will therefore not repeat it. However, we need to calculate the quantum force and the ratio between the new and the old Green's function, which can be calculated in clever ways to keep the code efficient.

We have already seen that the quantum force takes the same form as the gradient of the trial wave function,  $\mathbf{F}(\mathbf{r}) = 2(\nabla\Psi_T(\mathbf{r}))/\Psi_T(\mathbf{r})$ , and we can therefore simply reuse the function `computeGradient`, which is a part of the local energy computations from section 9.2.1. We call this from the function `ImportanceSampling::QuantumForce`, which contains the few lines of code

```
double ImportanceSampling::QuantumForce(const int i)
{
    double QF = 0;
    for (auto &j : m_waveFunctionVector) {
        QF += j->computeGradient(i);
    }
    return 2 * QF;
}
```

**Listing 9.13:** Taken from `importancesampling.cpp`.

where the force in dimension  $i$  is returned.

The Green's function was first presented in section 7.2.2, and at first glance it might look computational expensive to evaluate. Fortunately, we only need the ratio between the old and the new function which can be found in a quite simple fashion. Actually, both the diffusion constant  $D$  and the time step  $\Delta t$  cancel in the exponent, and the ratio can be expressed in the elegant form

$$g(\mathbf{r}', \mathbf{r}, \Delta t) \equiv \frac{G(\mathbf{r}', \mathbf{r}, \Delta t)}{G(\mathbf{r}, \mathbf{r}', \Delta t)} = \exp((\mathbf{r}' - \mathbf{r}) \cdot (\mathbf{F}(\mathbf{r}) - \mathbf{F}(\mathbf{r}'))/2) \quad (9.57)$$

where  $\mathbf{r}$  and  $\mathbf{r}'$  differ by one element and so does  $\mathbf{F}(\mathbf{r})$  and  $\mathbf{F}(\mathbf{r}')$ . It can therefore be evaluated

```
double ImportanceSampling::GreenRatio(const int i)
{
    double dQF = m_quantumForceOld(i) - m_quantumForceNew(i);
    return exp(0.5 * dQF * m_dx) + 1;
}
```

**Listing 9.14:** Taken from `importancesampling.cpp`.

where  $dQF$  is the difference between the new and the old force and  $m_dx$  is the distance particle  $i$  is moved. 1 appears from the term where  $m_dx$  is zero, such that we get zero in the exponent.

## 9.6 Update of parameters

The parameter update is a central part of a VMC implementation, and a good VMC implementation requires a good optimization algorithm. Since the optimization functions are called outside the sampling, they are just called a fraction number of times, compared to the function called from the sampling. Therefore, we will not put too much effort in making them efficient, but they should still be thought-through. We will here discuss the gradient descent method with momentum and monotonic decaying step, and the ADAM optimizer.

The `Optimizer` class contains a pure virtual function `updateParameters` which is thus forced to be included in the optimizer subclasses. This function returns the update of the new parameters, and is the function we will discuss in this section.

### 9.6.1 Gradient descent

Gradient descent is a simple optimization algorithm, and so is the implementation. Based on the theory presented in section 5.6.1, the implementation is really straight-forward and reads

```
Eigen::MatrixXd GradientDescent::updateParameters()
{
    m_step += 1;
    double monotonic = 1 / pow(m_step, m_monotonicExp);
    m_v = m_gamma * m_v + m_eta * Optimization::getEnergyGradient() * monotonic;
    return m_v;
}
```

**Listing 9.15:** Taken from `gradientdescent.cpp`.

where `m_v` is the momentum vector and `m_monotonicExp` describes how fast the rate should decrease. Further, `m_gamma` is the momentum parameter defining the relative size of the momentum. The function `Optimization::getEnergyGradient` returns a matrix with the gradients of the energy expectation value with respect to all the parameters, given in equation (7.27).

### 9.6.2 ADAM optimizer

The ADAM optimizer implementation also is very straight-forwardly based on the algorithm given in section 5.6.4. The momentum vectors were actually implemented as matrices to match the dimensions of the parameter matrix. By matrix operations we could also have made the function really efficient, but since that is not the aim here, we decided to keep the loops in order to make the code legible. The implementation looks like:

```
Eigen::MatrixXd ADAM::updateParameters()
{
    m_step += 1;
    m_g = Optimization::getEnergyGradient();
    m_m = m_beta1 * m_m + (1 - m_beta1) * m_g;
    m_v = m_beta2 * m_v + (1 - m_beta2) * m_g.cwiseAbs2();
    m_mHat = m_m / (1 - pow(m_beta1, m_step));
    m_vHat = m_v / (1 - pow(m_beta2, m_step));
    for (int i = 0; i < m_numberOfElements; i++) {
        for (int j = 0; j < m_maxParameters; j++) {
            m_theta(i, j) = m_eta * m_mHat(i, j) / (sqrt(m_vHat(i, j) +
                m_epsilon));
        }
    }
    return m_theta;
}
```

**Listing 9.16:** Taken from `adam.cpp`.

The parameter matrix, named `m_parameters`, can then easily be updated by the code

```
if (m_myRank == 0) {
    m_sampler->computeAverages();
    m_parameters = m_optimization->updateParameters();
}
```

**Listing 9.17:** Taken from `system.cpp`.

where `m_optimization` is the specified optimizer and `m_myrank` is the *rank* of the process. Parallel processing is not discussed yet, but we will describe it briefly in the following section.

## 9.7 Parallel processing

The code was parallelized using MPI to make study of large systems possible. This means that the code can run multiple parallel threads and in that manner utilize the processors. Most notably, this allows us to run on computer clusters which typically reduce the running time with a factor 10-100. We will not explain how MPI works in detail, nor will we detail the implementation of MPI since the commands are distribution over the entire code. The thing we present, is a sketch of the idea behind the parallelization used for our particular code.

Actually, one of the things that makes VMC preferred over other many-body methods, is that the algorithm quite easily can be split into independent parts, which encourages parallelization. The entire sampling can actually be split into as many parallel processes as needed, such that the code can be run on an arbitrary number of CPUs.

We typically distinguish between wall clock time,  $t_{clock}$  and CPU time  $t_{cpu}$  where the former is the time measured by a clock and the latter is the total computation time from all the CPUs. The speed-up will in general not be 100%, i.e.,  $t_{clock} \neq t_{cpu}/N$  with  $N$  as the number of processes, mainly because all of the samplings should have the same burn-in period as if we only run one process, but also because the code that is not part of the sampling cannot be parallelized and needs to be run on a CPU. The process that takes care of this part is the main process with rank 0.

---

**Algorithm 3:** Sketch of the parallelization.

---

```

1 MPI_Init() (Initialize MPI);
2 while not converged do
3    $E_L = 0;$ 
4   gradient= 0;
5   Egradient= 0;
6   for  $i \leftarrow 1$  to  $M$  do
7      $E_L += (\hat{\mathcal{H}}\Psi)/\Psi;$ 
8     gradient+ =  $\nabla_\theta \ln \Psi;$ 
9     Egradient+ =  $(\hat{\mathcal{H}}\Psi)/\Psi * \nabla_\theta \ln \Psi;$ 
10  end
11  if something goes wrong then
12    | MPI_Abort() (Terminate all processes);
13  end
14  MPI_Barrier() (Align processes);
15  MPI_Reduce( $E_L$ , gradients, Egradients) (Collect cumulative values);
16  if myrank==0 then
17    |  $\bar{E}_L = E_L/M;$ 
18    |  $\bar{\text{gradient}} = \text{gradient}/M;$ 
19    |  $\bar{\text{Egradient}} = \text{Egradient}/M;$ 
20    |  $G = 2 * (\bar{\text{Egradient}} - \bar{E}_L * \bar{\text{Egradient}});$ 
21    |  $\theta -= \eta G;$ 
22  end
23  MPI_Bcast( $\theta$ ) (Broadcast parameters);
24 end
25 MPI_Finalize() (Finalize MPI);
Result: Optimal variational parameters  $\theta$ .

```

---

In algorithm 3, we have sketched very roughly how the parallelization goes. We first run the entire sampling individually for all the  $N$  processes, and if something goes wrong, we call the MPI\_Abort function. To align the processes before we collect all the cumulative values, we use the function MPI\_BARRIER and we use MPI\_Reduce for the actual collection. Thereafter, the average energies are calculated *by the main process only*, and in the end the updated parameters are broadcast to all the other processes. Note that this is just a sketch where we avoid the arguments and the actual implementation of the MPI functions. This is of course found in the code.

## 9.8 Electron density

We presented the theory behind the electron density in section 3.3, where we saw that the  $P$ -body density is given by an integral over all probability density functions  $|\Psi(\mathbf{r}_1, \dots, \mathbf{r}_N)|^2$  but  $P$  of them. Usually, we look at the one-body density or the two-body density, leaving out one or two particles from the integration. Further in section 7.4, we gave a brief explanation of how the one-body density can be found using Monte Carlo integration in a VMC scheme. In this section, we will discuss the technique in more detail, and of course give the actual implementation.

In our particular implementation, we decided to divide the space into annuluses<sup>2</sup> of equal width  $d$ , which obviously gives bins of different sizes. For an illustration of the idea behind the Monte-Carlo integration for the one-body density in two dimensions, see figure (9.1). In two dimensions, a bin  $i$  has the area

$$A_i = (2i + 1)\pi d^2 \quad (9.58)$$

and in three dimensions the volume of bin  $i$  is

$$V_i = 4(i(i + 1) + 1/3)\pi d^3. \quad (9.59)$$

The most intuitive way of finding the correct bin of a particle, is to loop through all the bins and check if the particles belongs to the particular bin. However, this is a rather inefficient method of doing it, and it can be done much smarter revealing that a particle of radius  $r$  belong to the bin of index

$$i = r \backslash d + 1 \quad (9.60)$$

where  $\backslash$  indicates integer division.

By defining a vector `m_particlesPerBin` with the length number of bins, we can find the number of particles in each bin by a simple loop over all particles,

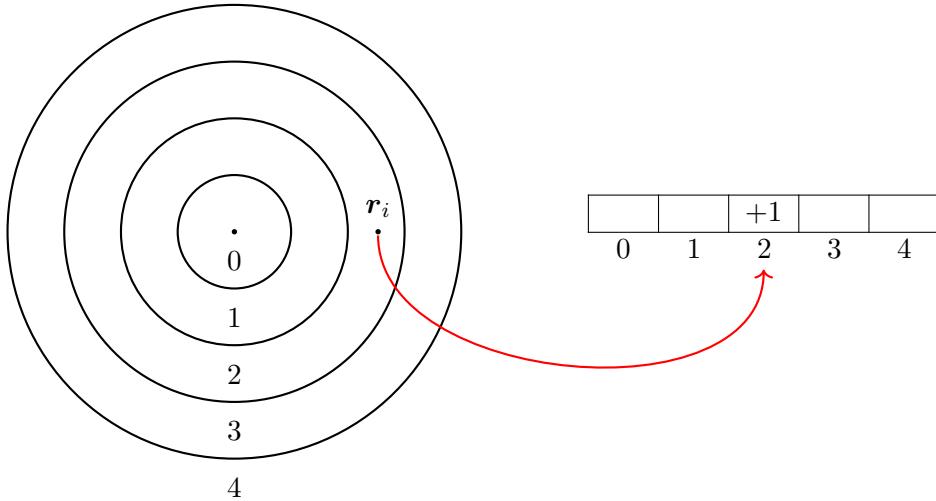
```
void Sampler::computeOneBodyDensity(const Eigen::VectorXd radialVector)
{
    for (int i_p = 0; i_p < m_numberOfParticles; i_p++) {
        int bin = int(radialVector(i_p) / m_radialStep) + 1;
        m_particlesPerBin(bin)++;
    }
}
```

**Listing 9.18:** Taken from `sampler.cpp`.

where `m_radialStep` is the width of each annulus, above denoted by  $d$ . In the end, `m_particlesPerBin` is printed to file, and we do the normalization when this file is read by a script, see `plot_ob_density.py`. For the two-body density, the implementation is very similar, and it will therefore not be detailed here.

---

<sup>2</sup>An annulus is a ring-shaped object with a region bounded by two concentric circles.



**Figure 9.1:** This figure is meant to illustrate how the one-body density is calculated using Monte-Carlo integration. One divides the space into  $n$  bins (here 5), and count the number of particles in each bin throughout the sampling. Afterwards, the bins need to be normalized.

## 9.9 Random number generator

In Monte Carlo integration, we are dependent on random numbers which we get from a random number generator (RNG). The RNG should have to main properties: It should give many independent uncorrelated random numbers and it should be fast. The former depends on the *period* of the RNG, where a long period gives many independent numbers.

In this work, we have used the mersenne twister random number generator, as it has a period of  $2^{19937} - 1$  which is known as the Mersenne prime. This is an incredibly large number, and should be more than sufficient for our purpose. We use built-in package in C++, `std::mt19937`, which also is quite fast.

## Chapter 10

# Implementation: Restricted Boltzmann Machines

In the previous chapter, we described common optimization procedures for a standard variational Monte Carlo (VMC) implementation, and we also presented implementation examples taken from the code. In this section we will do the same, but for the restricted Boltzmann machines (RBM). As we have pointed out before, the same sampling methods and optimization algorithms can be used both for the VMC implementation and the RBM implementation, such that much of the VMC framework is reused. The already described parts of the code will naturally not be described again, and for that reason this chapter will more or less exclusively concern the RBM wave function elements.

The main goal of this work, is to reduce the physical intuition needed when doing quantum computations, and that is the task of the restricted Boltzmann machines. The idea is to use a flexible basis set based on RBMs, which need to be the elements of the Slater matrix, as first seen in section 3.2.2. Further, we proved that the Slater determinant can be split in a spin-up part and a spin-down part in section 9.3.4, such that the spin can be factorized out and avoided. We therefore only need the spatial part of the wave functions, and we will henceforth assume that this spatial part is defined by the marginal distribution of the visible units.

Even though we want to reduce the need of physical intuition about the system, we still need to add some intuition to get reasonable results. For instance, for quantum dot systems, we add the Hermite polynomials to the marginal distribution such that each basis function becomes unique. The RBM basis functions for quantum dots then read

$$\Psi_n(\mathbf{x}) = H_n(\mathbf{x})P(\mathbf{x}) \quad (10.1)$$

where  $\Psi_n(\mathbf{x})$  are the spatial parts of RBM single particle functions,  $H_n(\mathbf{x})$  is the Hermite polynomial of degree  $n$ , possible multi-dimensional and  $P(\mathbf{x})$  is the marginal distribution of the visible nodes. In section 9.3.2, we saw that a Slater determinant containing single particle functions on the form  $\Psi_j(\mathbf{r}_i) = f_j(\mathbf{r}_i)g(\mathbf{r}_i)$  can be simplified by factorizing out the function  $g(\mathbf{r}_i)$ . For that reason, we can treat the marginal distributions  $P(\mathbf{r})$  as separate elements in combination with the determinant containing Hermite polynomials. In the following section, we will describe how these elements can be treated in the code.

## 10.1 Restricted Boltzmann machines

Back in chapter 6, we presented the marginal distribution of the visible units of a Gaussian-binary restricted Boltzmann machine. We will implement the distribution as a wave function, writing

$$\Psi_{rbm}(\mathbf{x}; \mathbf{a}, \mathbf{b}, \mathbf{w}) = \exp\left(-\sum_{i=1}^F \frac{(x_i - a_i)^2}{2\sigma^2}\right) \prod_{j=1}^H \left(1 + \exp\left(b_j + \sum_{i=1}^F \frac{w_{ij}x_i}{\sigma^2}\right)\right), \quad (10.2)$$

where  $\mathbf{x}$  contains all the coordinates,  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{w}$  are variational parameters (weights),  $\sigma$  is the width of the Gaussian distribution,  $F$  is the degrees of freedom and  $H$  is the number of hidden units. The wave function can naturally be split in a Gaussian part and a product, and we will henceforth work with them separately to simplify the calculations. They will also be implemented as separate wave function elements as this will reduce the complexity of the derivatives associated with each element. The first part will henceforth be denoted by RBM-Gaussian, while the last part will be denoted by RBM-Product.

### 10.1.1 RBM-Gaussian

The RBM-Gaussian reads

$$\Psi_{rg}(\mathbf{x}; \mathbf{a}) = \exp\left(-\sum_{i=1}^F \frac{(x_i - a_i)^2}{2\sigma^2}\right) \quad (10.3)$$

and is really similar to the simple Gaussian presented in section 9.3.3. Also the gradient, Laplacian and the gradient with respect to the variational parameters become similar, and we will for that reason just list them up,

$$\begin{aligned} \frac{|\Psi_{rg}(\mathbf{x}_{\text{new}})|^2}{|\Psi_{rg}(\mathbf{x}_{\text{old}})|^2} &= \exp\left(\frac{(x_i^{\text{old}} - a_i)^2 - (x_i^{\text{new}} - a_i)^2}{2\sigma^2}\right) \\ \nabla_k \ln \Psi_{rg} &= -\frac{x_k - a_k}{\sigma^2} \\ \nabla_k^2 \ln \Psi_{rg} &= -\frac{1}{\sigma^2} \\ \nabla_{a_l} \ln \Psi_{rg} &= \frac{x_l - a_l}{\sigma^2}. \end{aligned} \quad (10.4)$$

Further, the frequency of the quantum dots should be inversely proportional with the Gaussian sampling width from the Gaussian-binary RBM,  $\sigma^2$ , such that we can set

$$\omega = \frac{1}{\sigma^2} \quad (10.5)$$

for the RBMs.

An obvious optimization concerning this element, is that we can introduce a vector  $\mathbf{x}\mathbf{a} \equiv \mathbf{x} - \mathbf{a}$ , which we deal with instead of the position vector  $\mathbf{x}$  and the parameter vector  $\mathbf{a}$ . We update the arrays using the pure virtual function `updateArrays`, which looks like

```
void RBMGaussian::updateArrays(const Eigen::VectorXd& positions,
                                const Eigen::VectorXd& radialVector,
                                const Eigen::MatrixXd& distanceMatrix,
                                const int i)
{
    m_positions = positions;
```

```

m_Xa = positions - m_a;
double expDiff = m_XaOld(i) * m_XaOld(i) - m_Xa(i) * m_Xa(i);
m_probabilityRatio = exp(expDiff / (2 * m_sigmaSqr));
}

```

**Listing 10.1:** Taken from `rbmgaussian.cpp`.

We see that the vector `m_Xa` is declared globally, such that it can be used also in the gradients of the element. `i` is again the updated coordinate.

### 10.1.2 RBM-product

The RBM product is the last part of (6.19), and is thus given by

$$\Psi_{rp}(\mathbf{x}; \mathbf{b}, \mathbf{w}) = \prod_{j=1}^H \left[ 1 + \exp \left( b_j + \sum_{i=1}^F \frac{w_{ij}x_i}{\sigma^2} \right) \right]. \quad (10.6)$$

In appendix C, section (C.2), a general Gaussian-binary RBM product in the form of

$$\Psi(\mathbf{x}; \boldsymbol{\theta}) = \prod_{j=1}^H \left[ 1 + \exp \left( f_j(\mathbf{x}; \boldsymbol{\theta}) \right) \right] \quad (10.7)$$

is differentiated, which for this element corresponds to setting  $f_j = b_j + \mathbf{w}_j^T \mathbf{x} / \sigma^2$ . As we further claim, the only expressions that need to be calculated are  $\nabla_k(f_j)$ ,  $\nabla_k^2(f_j)$  and  $\partial_{\theta_i}(f_j)$  for all the coordinates  $k$  and all the parameters  $\theta_i$ . They can easily be found to be

$$\begin{aligned}
\nabla_k(f_j) &= \frac{w_{kj}}{\sigma^2} \\
\nabla_k^2(f_j) &= 0 \\
\partial_{b_l}(f_j) &= \delta_{lj} \\
\partial_{w_{ml}}(f_j) &= \frac{x_m}{\sigma^2} \delta_{lj}
\end{aligned} \quad (10.8)$$

for our specific function.  $\delta_{lj}$  is the Kronecker delta. By reintroducing the sigmoid function and the counterpart

$$n_j(x) = \frac{1}{1 + \exp(-x)} \quad \wedge \quad p_j(x) = n_j(-x) = \frac{1}{1 + \exp(x)} \quad (10.9)$$

we can express the required derivatives in the following fashion

$$\begin{aligned}
\frac{|\Psi_{rp}(\mathbf{x}_{\text{new}})|^2}{|\Psi_{rp}(\mathbf{x}_{\text{old}})|^2} &= \prod_{j=1}^H \frac{p_j(\mathbf{x}_{\text{old}})^2}{p_j(\mathbf{x}_{\text{new}})^2} \\
\nabla_k \ln \Psi_{rp} &= \sum_{j=1}^H \frac{w_{kj}}{\sigma^2} n_j \\
\nabla_k^2 \ln \Psi_{rp} &= \sum_{j=1}^H \frac{w_{kj}^2}{\sigma^4} p_j n_j \\
\nabla_{b_l} \ln \Psi_{rp} &= n_l \\
\nabla_{w_{ml}} \ln \Psi_{rp} &= \frac{x_m n_l}{\sigma^2}.
\end{aligned} \quad (10.10)$$

This is the same result as obtained for the gradient of the log-likelihood function presented in chapter 5. In this element, there are plenty of optimization possibilities. For the RBM-Gaussian, we saw that the distribution width  $\sigma$  was set such that  $\omega = 1/\sigma^2$ , for but this product the value of  $\sigma$  has no impact on the outcome since it is always multiplied with the weights which are adjusted freely. By further revealing that some sums are vector products, we can get a significant speed-up. Firstly, we will define a vector

$$\mathbf{v} = \mathbf{b} + \mathbf{w}^T \mathbf{x} \quad (10.11)$$

which is what we above have called  $\mathbf{f}(\mathbf{x}; \theta)$ . Thereafter, we define the vectors  $\mathbf{n}$  and  $\mathbf{p}$  as described above. These vectors are declared as `m_v`, `m_n` and `m_p` respectively, and are initialized and updated using the function `updateVectors` in the following way

```
void RBMProduct::updateVectors()
{
    m_v = m_b + m_W.transpose() * m_positions;
    Eigen::VectorXd e = m_v.array().exp();
    m_p = (e + Eigen::VectorXd::Ones(m_numberOfHiddenNodes)).cwiseInverse();
    m_n = e.cwiseProduct(m_p);
}
```

**Listing 10.2:** Taken from `rbmproduct.cpp`.

One can see that all the operations are vectorized, which makes the operations quite affordable.

## 10.2 Partly restricted Boltzmann machine

For the partly restricted Boltzmann machine given in equation (6.26), we observe that the only difference from a standard Boltzmann machine is the factor

$$\Psi_{pr} = \exp \left( \sum_{i=1}^F \sum_{j=1}^F x_i c_{ij} x_j \right) \quad (10.12)$$

which we can threat separately. To run a computation with the partly restricted Boltzmann machine, we thus need to add the elements `RBMGaussian`, `RBMProduct` and `PartlyRestricted` in a similar way as in the example 9.1. When differentiating, we end up with the expressions

$$\begin{aligned} \frac{|\Psi_{pr}(\mathbf{x}_{\text{new}})|^2}{|\Psi_{pr}(\mathbf{x}_{\text{old}})|^2} &= \exp \left( 2 \sum_{j=1}^F c_{ij} x_j (x_i^{\text{new}} - x_i^{\text{old}}) \right) \\ \nabla_k \ln \Psi_{pr} &= 2 \sum_{j=1}^F c_{kj} x_j \\ \nabla_k^2 \ln \Psi_{pr} &= 2 c_{kk} \\ \nabla_{c_{ml}} \ln \Psi_{pr} &= x_m x_l \end{aligned} \quad (10.13)$$

where  $x_i$  is the changed coordinate. Also here can we use vectorization to speed-up the computations, most elegantly shown by the `computeParameterGradient`,

```
Eigen::VectorXd PartlyRestricted::computeParameterGradient()
{
    Eigen::MatrixXd out = m_positions * m_positions.transpose();
    m_gradients.head(out.size()) = WaveFunction::flatten(out);
    return m_gradients;
}
```

**Listing 10.3:** Taken from `partlyrestricted.cpp`.

where we use that the parameter gradient  $\nabla_{c_{ml}} \ln \Psi_{pr}$  is given by the outer product between the coordinate vectors.



## **Part IV**

# **Results and Conclusion**



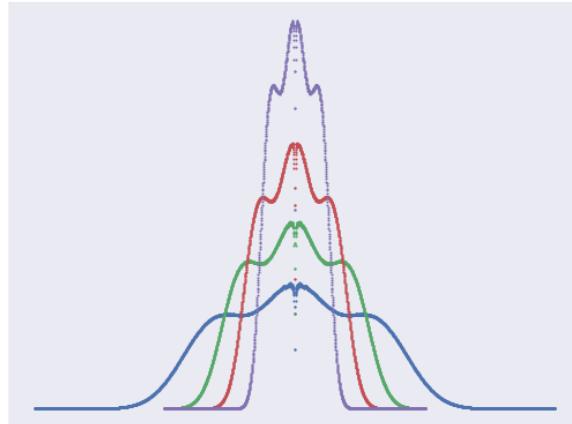
# Chapter 11

## Selected Results

Results! Why, man, I have gotten a lot of results. I know several thousand things that won't work.

---

Thomas A. Edison, [97]



**Figure 11.1:** One-body density plots for a two-dimensional single quantum dot containing 12 electrons, popularly called an artificial Magnesium atom. The four graphs correspond to four different oscillator frequencies, where the weakest oscillator gives the broadest density distribution. It's quite artistic, isn't it?

We are finally ready to discuss the most interesting part of this thesis, namely the results. Since this work, after all, is about physics the physical insight should be our focus, and in this chapter we finally get the opportunity to discuss pure physics. However, we will also compare the restricted Boltzmann machine to standard variational Monte Carlo to see which one that works best and eventually where the Boltzmann machine fails.

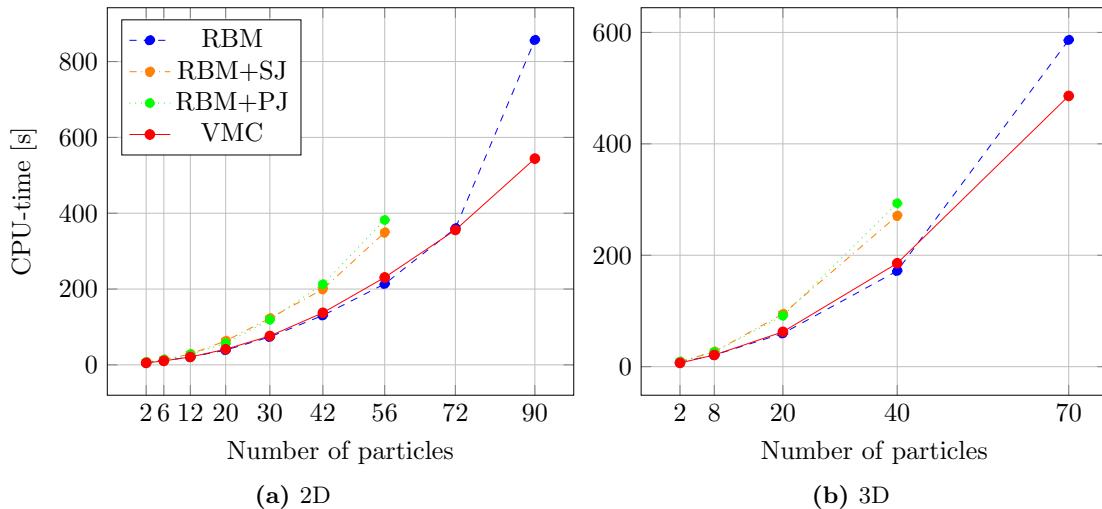
As we can look at various systems, system sizes, system strengths and innumerable hyper-parameter settings by different methods, the number of results we possibly can obtain is uncountable. In total, we have looked at more than 150 different quantum dot systems using four methods, which means that we have generated a large set of results. Just a small subset of

those results will be presented and discussed in this chapter, while a larger collection of results is presented in appendix D. The results presented above were selected since they describe physical properties and compare the various methods in a good way. Typically, we will choose settings that are comparable to what others have obtained, such that we can benchmark the calculations. Our main focus will be on ground-state energy and electron density calculations.

Before we move on to the physical results, we will take a quick look at some more technical results, more precisely the computational cost of various wave function structures and the energy convergence using various optimization tools. For validation purposes, we will present a few selected results on the case without repulsive interaction and compare to analytical results. Thereafter, we study the case with repulsive interaction in a much larger scale, where we first look at single quantum dots and then on double quantum dots.

## 11.1 Computational cost

One of the major problems when simulating quantum mechanical systems is the computational cost, which is due to the large amount of information that needs to be stored in a wave function. Albeit the standard variational Monte Carlo (VMC) method is known to have a high performance-cost ratio, the it is still not cheap. In this section we will find the average cost of VMC and compare the cost to a plain restricted Boltzmann machine (RBM), restricted Boltzmann machine with a simple Jastrow factor (RBM+SJ) and a restricted Boltzmann machine with a Padé-Jastrow factor (RBM+PJ). In figure (11.2) the CPU-time is plotted as a function of the number of particles for two-dimensional (a) and three-dimensional (b) quantum dots. To obtain accurate times, all the simulations were run on the Abel computer cluster with  $M = 2^{20} = 1,048,576$  Monte Carlo cycles per iteration. The time presented is the average time over at least four independent runs with thousands of iterations each. To see the actual numbers, go to appendix D, section D.1.



**Figure 11.2:** CPU-time per iteration for  $M = 2^{20} = 1,048,576$  cycles as a function of number of electrons for two- dimensional (a) and three-dimensional (b) quantum dots. The solid line is associated with standard variational Monte-Carlo (VMC), while the dashed lines are associated with restricted Boltzmann machines without Jastrow factor (RBM), with simple Jastrow factor (RBM+SJ) and with Padé-Jastrow factor (RBM+PJ). Table with numbers can be found in appendix D, section D.1.

Our immediate observation is that the methods are pairwise quite similar, with RBM and VMC as the cheapest methods and RBM+SJ and RBM+PJ are the most expensive ones. This

**Table 11.1:** Optimal constants  $\alpha$  and  $\beta$  for restricted Boltzmann machine (RBM), restricted Boltzmann machine with a simple Jastrow factor (RBM+SJ), restricted Boltzmann machine with Padé-Jastrow factor (RBM+PJ) and standard variational Monte-Carlo sampling (VMC).

	2D		3D	
	$\alpha$	$\beta$	$\alpha$	$\beta$
RBM	1.345	1.269	2.423	1.187
RBM+SJ	2.137	1.189	1.208	1.453
RBM+PJ	2.034	1.206	1.200	1.462
VMC	1.294	1.265	2.263	1.197

is not surprising, as the RBM requires a neural network, VMC requires a Jastrow factor while RBM+SJ and RBM+PJ requires both a neural network and a Jastrow factor.

For two-dimensional dots, the RBM is the cheapest among all the methods, but for larger systems ( $N = 42, 56$  with  $N$  as number of electrons), VMC get cheaper due to an explosion in CPU-time for the RBM. This explosion can be described by our choice of number of hidden nodes,  $H$ , which consequently is set to the number of electrons, i.e.,  $H = N$ , which was found to give the lowest energies when testing on small quantum dots [80]. Since the RBM has  $N \cdot D \cdot (1 + H) + H$  variational parameters with  $D$  as the number of dimensions, the number of variational parameters for a dot containing 90 particles in two dimensions in 16,470! On the other hand, VMC has two variational parameters which obviously makes a difference.

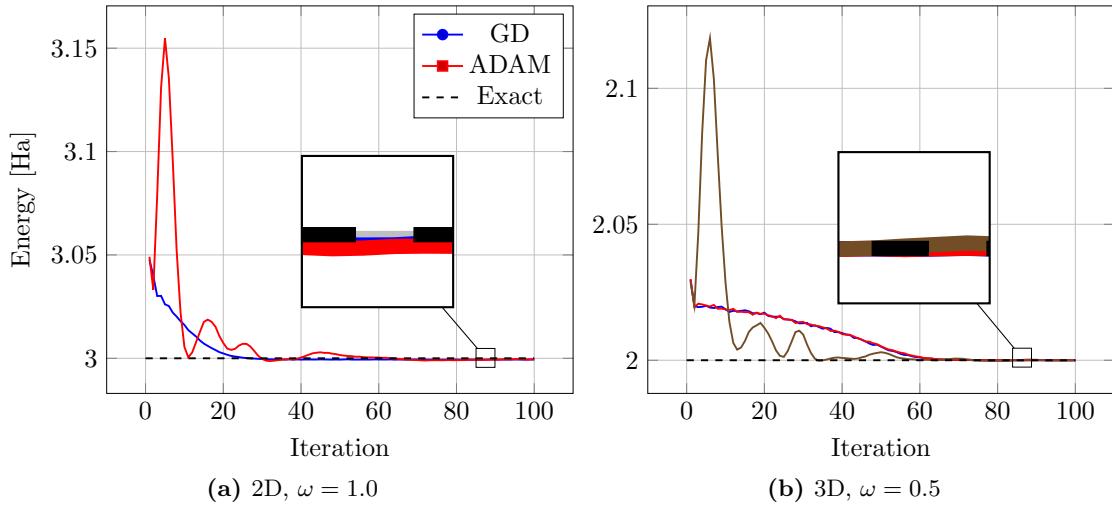
We also see that RBM+PJ is cheaper than RBM+SJ for systems up to  $N = 42$ , but after that the RBM+SJ gets slightly cheaper. This might be surprising, as the simple Jastrow contains  $N^2$  variational parameters, but a possibility is that BLAS is optimized for large matrix-vector operations and is thus fully utilized first when the matrices get large. The exact same behavior is observed for three-dimensional quantum dots, and the discussion above is describing for them as well.

To estimate how fast the cost increases as we add more particles, we do linear regression with a function in form of  $f(x) = \alpha x^\beta$  where  $x$  is the number of particles while  $\alpha$  and  $\beta$  are the unknown parameters to be found. From the limited number of points, we have found the parameters and presented them in table (11.1). Although the RBM was found to be cheaper than VMC, we can see that the estimated exponential factor  $\alpha$  is actually slightly larger. The prefactor  $\beta$  is significantly lower though.

## 11.2 Energy convergence

We want our calculations to converge fast and to be stable, and that is what the optimization tools are responsible for. In figure (11.3), we compare standard gradient descent to stochastic gradient descent and ADAM for quantum dots containing two interacting electrons in a two- and three dimensions. The frequency  $\omega = 1.0$  is used for the two-dimensional case since we know that the exact energy is  $E = 3.0$  for this case [26]. Similarly, we use the frequency  $\omega = 0.5$  for the three-dimensional case since the exact energy is  $E = 2.0$  [30].

The first thing we observe, is that all the three optimization tools manage to converge to the exact energy. The stochastic and non-stochastic gradient descent methods are hard to distinguish, they both converge smoothly. The ADAM optimizer, on the other hand, fluctuates much more. This behavior can be described by the momentum, as discussed in section 5.6.3. It is also important to remember that this is the case when we use standard variational Monte-Carlo wave function. The ADAM optimizer is known to be good at machine learning problems, so we will



**Figure 11.3:** Energy convergence for circular quantum dots where we use the optimization tools gradient descent (GD), stochastic gradient descent with 10 batches (SGD) and the ADAM optimizer. The plot in (a) shows a two-dimensional quantum dot of frequency  $\omega = 1.0$  containing two interacting electrons with exact energy  $E = 3.0$ . The plot in (b) shows a three-dimensional quantum dot of frequency  $\omega = 0.5$  containing two interacting electrons with exact energy  $E = 2.0$ . We use a standard variational Monte-Carlo wave function, the learning rate was set to  $\eta = 0.5$  and the number of Metropolis steps used for each iteration was  $M = 2^{24} = 16,777,216$ . All energies are given in units of  $\hbar$  (Hartree units).

stick to it even though gradient descent seems like a clever choice seen from the figure.

### 11.3 No repulsive interaction

We start with the non-interacting case in order to validate the implemented code. For this case, we know the exact energy and the exact one-body density for both single and double quantum dots, which makes it a good test for both systems. The physical significance is though limited, such systems do not appear in the real world. We will first present the quantum dots, before we move on to double quantum dots.

#### 11.3.1 Quantum dots

Quantum dots will be the system that we will study in highest detail, and therefore most of the validation will be on the quantum dots as well.

##### 11.3.1.1 Ground-state energy

The single quantum dot has analytical ground-state energies given by equation (4.4) and a number of particles given by the magic numbers in equation (4.5). For some selected number of electrons and for frequencies  $\omega = 0.5$  and  $\omega = 1.0$ , the obtained energies are presented in table (11.2).

We observe that both the standard VMC, with  $\alpha = 1$ , and standard RBM, with all parameters set to zero, are able to reproduce the analytical expression. This is as expected, since the exact wave functions are found when the parameters have these particular values.

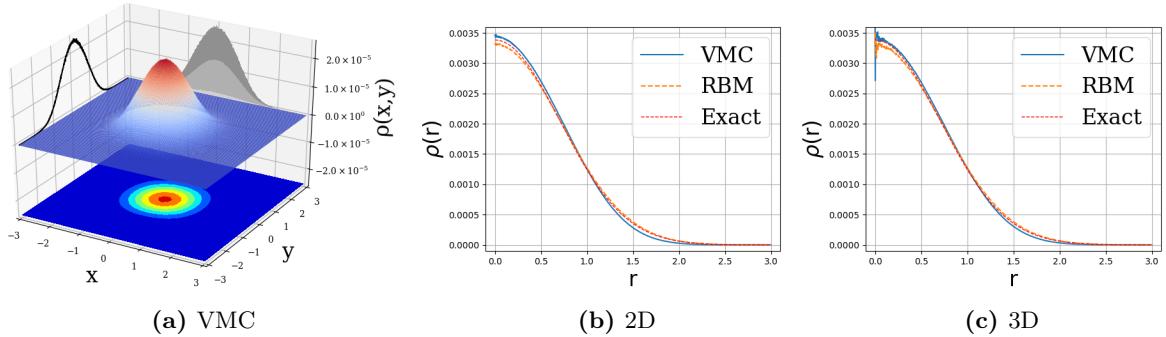
**Table 11.2:** Energy of circular quantum dots of frequency  $\omega = 0.5$  and  $\omega = 1.0$  consisting of  $N$  non-interacting particles. RBM is a single Slater determinant with a plain Boltzmann machine baked in, while VMC is a standard variational Monte-Carlo Slater determinant. Exact values are obtained by  $E = \omega(n + 1/2)$ , and all values are given in units of  $\hbar$ .

	$\omega = 0.5$			$\omega = 1.0$			
	$N$	RBM	VMC	Exact	RBM	VMC	Exact
2D	2	1.0	1.0	1	2.0	2.0	2
	12	14.0	14.0	14	28.0	28.0	28
	30	55.0	55.0	55	110.0	110.0	110
3D	2	1.5	1.5	1.5	3.0	3.0	3
	20	30.0	30.0	30	60.0	60.0	60
	70	157.5	157.5	157.5	315.0	315.0	315

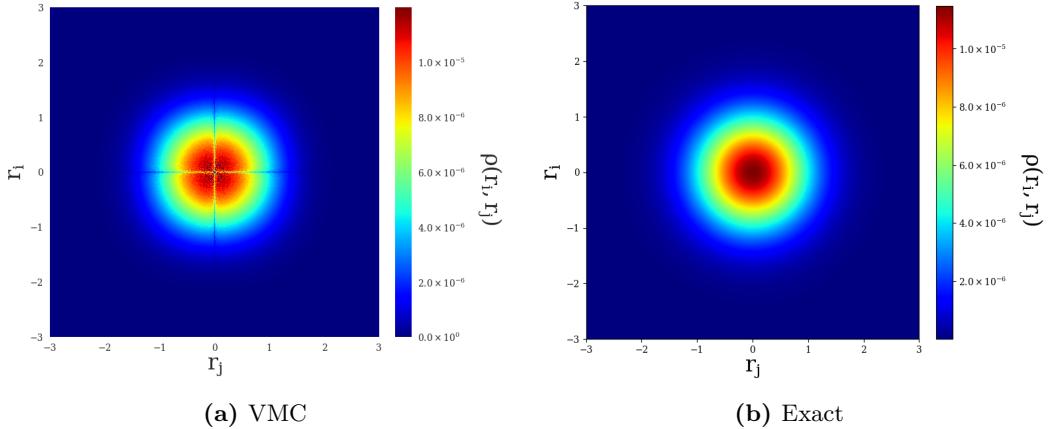
### 11.3.1.2 Electron density

We will also focus on the electron density throughout this chapter, and comparing the obtained densities to the analytical ones is a good indicator on whether the implementation is correct or not. In figure (11.4) the one-body densities are plotted for quantum dots of two non-interacting electrons in two and three dimensions with frequency  $\omega = 1.0$ . In part (a) the simulated distribution using VMC throughout the space is given, while in (b) and (c) the radial density profile for VMC, RBM and the exact profile is given in two and three dimensions respectively. We observe that both VMC and RBM are more or less able to reproduce the exact distribution, which indicated that we have done the implementation correctly. However, the densities get noisy when they approach  $r = 0$ , most notably in three dimensions, which is caused by the way we calculate the one-body densities. In section 9.8, we saw that the innermost bins are also the smallest, which means that particles are less likely to be observed there when the number of Monte Carlo cycles is finite. This artifact is not found in part (a), because that one-body density was computed by dividing the space into equally sized squares. Apart from this noise, the density plot in two and three dimensions are identical, which can be explained mathematically by the fact that the non-interacting electron densities are separable. The analytical one-body densities are found from the definition of one-body density in equation (3.15).

Also the two-body density is interesting to investigate as it might provides information about the interactions that the one-body density cannot give us. The radial two-body density profile is computed in the same way as the radial one-body density profile, and we expect therefore to observe the same artifact where the density drops in the center. In figure (11.5), we plot the obtained two-body density using VMC (a) and the exact density found from equation (3.15) (b). We observe that the simulated two-body density is similar to the exact one, but apparently not as smooth. We also get a cross on the simulated density profile, which occurs since we in practice only simulate one quadrant and add the three other quadrants to make the plot more intuitive. As we have seen before, the electron densities are separable without interaction, which means that the radial two-body density presented here simply is the radial one-body density rotated in space.



**Figure 11.4:** One-body density for two-dimensional (a) and three-dimensional (b) quantum dots with frequency  $\omega = 1.0$  containing two non-interacting electrons. The two methods used are standard variational Monte Carlo (VMC) and a plain restricted Boltzmann machine (RBM), and the exact solution was found from the definition of one-body density found in equation (3.15).



**Figure 11.5:** Two-body density for a two-dimensional quantum dot with frequency  $\omega = 1.0$  containing two non-interacting electrons. We use standard variational Monte Carlo (VMC) for the simulation (a) and compare it to the exact solution was found from the definition of two-body density found in equation (3.15).

### 11.3.2 Double quantum dots

For the double quantum dots, we can also find analytical energies without interaction, as explained in section 4.2. We first find the matrix

$$\hat{h}_{\nu\lambda} = \langle \phi_{\nu}^{\text{HO}}(x) | \hat{\mathcal{H}}_{\text{HO}} | \phi_{\lambda}^{\text{HO}}(x) \rangle + \langle \phi_{\nu}^{\text{HO}}(x) | \hat{\mathcal{H}}_{+} | \phi_{\lambda}^{\text{HO}}(x) \rangle$$

for a satisfying number of harmonic oscillator functions. All the symbols are detailed in section 4.2. The energies are the eigenvalues of the matrix, which were obtained by the script `doublewell_functions.py`. For one particle in one-dimensional potentials, we get the energy  $E = 0.3095$  for  $\omega = 1.0$  and  $b = 2$ , and  $E = 0.1916$  for  $\omega = 0.5$  and  $b = 4$ . This gives the analytical energies presented in table (11.3).

**Table 11.3:** Energy of double quantum dots of two non-interacting electrons. RBM is a single Slater determinant with a plain Boltzmann machine baked in, while VMC is a standard variational Monte-Carlo Slater determinant.

D	$\omega$	b	RBM	VMC	Exact
2D	0.5	4.0	-	-	0.8832
	1.0	2.0	-	-	1.619
3D	0.5	4.0	-	-	1.3832
	1.0	2.0	-	-	2.619

## 11.4 Quantum dots

We now move on to the more interesting case with repulsive interaction, where we no longer have analytical results, apart from a few semi-analytical energies and wave functions for the two- and three-dimensional single quantum dots.

To achieve good results, we apply an adaptive step number, which means that the number of Monte Carlo cycles per iteration is increased for the last iterations. Firstly, this makes the final energy more accurate due to better statistics. Secondly, we get less noisy electron density plots by using this technique. All results below are produced using  $2^{20} = 1,048,576$  number of steps per iteration for the initial iterations. Then the number of steps is increased to  $2^{24} = 16,777,216$  when we have 11 iterations left, and for the very last iteration we use  $2^{28} = 268,435,456$  steps.

Initially we look at standard quantum dots of size up to  $N = 56$  electrons in 2D and  $N = 40$  electrons in 3D and frequencies between  $\omega = 1.0$  and  $\omega = 0.1$ . For those systems, we will compute the ground state energy, the one-body density and the two-body density.

After that, we move on to some special cases where the dots have low frequency ( $\omega = 0.01$ ) and large dots ( $N > 56$ ) to test how far we can go. For those systems, we will typically focus on either the ground-state energy or the one-body density dependent on what we want to investigate. For instance, we will focus on the one-body density for low frequency dots because of the search for Wigner crystals. We also have a thorough discussion of how the energy is distributed between kinetic and potential energy, and compare this to the virial theorem. In section 11.4.7, we will investigate the various obtained wave function and try to reveal similarities and differences.

### 11.4.1 Ground-state energy

The ground-state energy is a natural start as it is what we try to minimize, and it can easily be compared to benchmarks. By exploiting the symmetry of quantum dots of two electrons, M.Taut was able to obtain semi-analytical energies for some specific frequencies  $\omega$ . More specifically, he found the energy to be  $E = 3$  for the frequency  $\omega = 1$  and  $E = 2/3$  for the frequency  $\omega = 1/6$  for the two-dimensional case, and  $E = 2$  for the frequency  $\omega = 1/2$  and  $E = 1/2$  for the frequency  $\omega = 1/10$  for the three-dimensional case [26, 30].

For other references, we need to rely on what researchers have found before us. Since diffusion Monte-Carlo (DMC) is known to give very accurate results, we will mainly compare our results to J. Høgberget's DMC computations, which exist for quantum dots containing maximum 56 electrons in two dimensions and maximum 20 particles in three dimensions [56]. Comparing the energy to the Hartree-Fock limit is also interesting, mainly because the Boltzmann machines need to be better to have any significance as Hartree-Fock is much cheaper. We use A.Mariadason's computations for this for quantum dots up to 20 electrons in two dimensions, and up to 8 particles in three dimensions [79].

Ground state energy computations of two- and three dimensional quantum dots are found in tables (11.4) and (11.5) respectively. They are computed by a restricted Boltzmann machine (RBM), restricted Boltzmann machine with a simple Jastrow factor (RBM+SJ), restricted Boltzmann machine with Padé-Jastrow factor (RBM+PJ) and standard variational Monte-Carlo (VMC). In addition, the Hartree-Fock limit (HF) and diffusion Monte-Carlo (DMC) are present for reference purposes. The exact values are found in the last column with the same name.

We observe that the method where less physical intuition is used, RBM, is the one that gives the highest energies. This is as expected, but considering that no Jastrow factor is added to account for the correlations, the result is not bad. For small quantum dots (less than 8 particles), the RBM energy is lower than the Hartree-Fock limit, and for low frequencies ( $\omega = 0.1$  and  $0.28$ ), the RBM energy is also lower. However, for higher frequencies and larger dots, the Hartree-Fock limit is lower than the obtained energy with the RBMs.

**Table 11.4:** The ground state energy of two-dimensional circular quantum dots of frequency  $\omega$  containing  $N$  electrons. From the left-hand-side column, the methods used restricted Boltzmann machine (RBM), restricted Boltzmann machine with simple Jastrow factor (RBM+SJ), restricted Boltzmann machine with Padé-Jastrow factor (RBM+PJ), the Hartree-Fock limit (HF), standard variational Monte-Carlo with Hartree-Fock basis (VMC+HF), standard variational Monte-Carlo with Hermite basis (VMC), diffusion Monte-Carlo (DMC) and semi-analytical results (Exact). Hartree-Fock results are taken from Ref.[79], DMC results are taken from Ref.[56] and semi-analytical results are taken from Ref.[30]. The energy is given in units of  $\hbar$ , and the numbers in parenthesis are the statistical uncertainties in the last digit.

$N$	$\omega$	RBM	RBM+SJ	RBM+PJ	HF (Ref.[79])	VMC	DMC (Ref.[56])	Exact (Ref.[30])
2	0.1	0.4728(1)	0.44858(6)	0.440975(8)	0.525635	0.44129(1)	0.44079(1)	2/3
	1/6	0.7036(1)	0.67684(7)	0.66715(6)	0.768675	0.66710(1)	-	
	0.28	1.0707(2)	1.03470(7)	1.021668(7)	1.14171	1.02192(1)	1.02164(1)	
	0.5	1.7234(2)	1.67739(9)	1.659637(6)	1.79974	1.65974(1)	1.65977(1)	
	1.0	3.0829(2)	3.0259(1)	2.999587(5)	3.16190	2.99936(1)	3.00000(1)	3.0
6	0.1	3.697(1)	3.6584(4)	3.5700(2)	3.85238	3.5695(1)	3.55385(5)	
	0.28	7.9273(9)	7.7347(4)	7.6203(2)	8.01957	7.6219(1)	7.60019(6)	
	0.5	12.241(1)	11.9659(5)	11.8074(2)	12.2713	11.8104(2)	11.78484(6)	
	1.0	20.716(1)	20.4061(7)	20.1832(2)	20.7192	20.1918(2)	20.15932(8)	
12	0.1	12.703(2)	12.5964(7)	12.3416(4)	12.9247	12.3196(3)	12.26984(8)	
	0.28	26.389(2)	26.083(1)	25.7331(5)	26.5500	25.7049(4)	25.63577(9)	
	0.5	40.440(3)	39.6340(7)	39.2743(6)	40.2161	39.2421(5)	39.1596(1)	
	1.0	67.632(3)	66.1898(8)	65.7911(7)	66.9113	65.7026(4)	65.7001(1)	

20	0.1	30.824(2)	30.567(3)	30.1553(9)	31.1902	30.086(1)
	0.28	63.788(4)	62.786(3)	62.148(1)	63.5390	62.0755(7)
	0.5	96.491(4)	94.920(4)	94.104(1)	95.7328	94.0433(9)
	1.0	159.645(5)	156.816(4)	156.104(1)	158.004	155.8900(4)
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30	0.1	61.829(5)	61.351(4)	60.774(2)	60.585(1)	60.4205(2)
	0.28	126.958(6)	126.067(5)	124.437(2)	124.195(2)	123.9683(2)
	0.5	191.495(7)	188.995(5)	187.488(2)	187.325(3)	187.0426(2)
	1.0	315.364(8)	311.468(7)	308.989(2)	308.576(1)	308.5627(2)
<hr/>						
42	0.1	109.892(6)	109.892(7)	108.753(6)	107.928(2)	107.6389(2)
	0.28	224.462(8)	224.184(9)	222.200(5)	220.224(2)	219.8426(2)
	0.5	337.523(8)	333.582(9)	331.410(3)	331.276(3)	330.6306(2)
	1.0	553.40(1)	549.76(1)	543.746(3)	542.977(2)	542.9428(8)
<hr/>						
56	0.1	-	179.59(1)	-	176.774(3)	175.9553(7)
	0.28	364.85(1)	364.165(9)	-	359.63(1)	358.145(2)
	0.5	547.46(1)	545.74(1)	538.810(7)	538.686(9)	537.353(2)
	1.0	894.12(2)	890.70(2)	881.010(5)	879.514(3)	879.3986(6)

**Table 11.5:** The ground state energy of three-dimensional circular quantum dots of frequency  $\omega$  containing  $N$  electrons. From the left-hand-side column, the methods used restricted Boltzmann machine (RBM), restricted Boltzmann machine with simple Jastrow factor (RBM+SJ), restricted Boltzmann machine with Padé-Jastrow factor (RBM+PJ), the Hartree-Fock limit (HF), standard variational Monte-Carlo with Hermite basis (VMC), diffusion Monte-Carlo (DMC) and semi-analytical results (Exact). Hartree-Fock results are taken from Ref.[79], DMC results are taken from Ref.[56] and semi-analytical results are taken from Ref.[26]. The energy is given in units of  $\hbar$ , and the numbers in parenthesis are the statistical uncertainties in the last digit.

$N$	$\omega$	RBM	RBM+SJ	RBM+PJ	HF (Ref.[79])	VMC	DMC (Ref.[56])	Exact (Ref.[26])
2	0.1	0.5177(1)	0.50214(3)	0.500080(6)	0.529065	0.500083(7)	0.499997(3)	0.5
	0.28	1.2261(1)	1.20475(4)	1.201710(6)	1.23722	1.201752(6)	1.201725(2)	
	0.5	2.0269(1)	2.00371(4)	1.999912(5)	2.03851	1.999977(5)	2.000000(2)	2.0
	1.0	3.7574(1)	3.73543(4)	3.729827(5)	3.77157	3.730030(5)	3.730123(3)	
8	0.1	5.8910(6)	5.7498(4)	5.718(4)	5.86255	5.7126(1)	5.7028(1)	
	0.28	12.650(1)	12.2492(4)	12.2056(2)	12.3987	12.2050(2)	12.1927(1)	
	0.5	19.487(2)	19.0241(4)	18.9747(2)	19.1916	18.9759(1)	18.9611(1)	
	1.0	33.302(1)	32.7159(6)	32.6820(2)	32.9246	32.6863(2)	32.6680(1)	
20	0.1	27.813(2)	27.470(1)	27.3382(8)		27.3144(5)	27.2717(2)	
	0.28	57.700(4)	56.600(1)	56.4477(6)		56.4297(5)	56.3868(2)	
	0.5	87.840(4)	85.893(1)	85.7153(6)		85.7161(5)	85.6555(2)	
	1.0	146.292(4)	143.209(2)	142.9409(6)		142.9560(7)	142.8875(2)	
40	0.1	89.45(8)	89.618(4)	88.596(4)		88.182(1)		
	0.28	182.714(6)	181.877(4)	179.630(1)		179.567(1)		
	0.5	275.262(7)	271.030(4)	269.782(2)		269.746(1)		
	1.0	452.732(8)	442.874(4)	442.630(1)		442.602(2)		

When we add more intuition in form of a simple Jastrow factor, the energy drops significantly. The RBM+SJ is not very far from reproducing the reference and provide, without exception, lower energy than the Hartree-Fock limit. This is impressive as it contains among the simplest Jastrow factors we could attain, with absent of the correct cusp condition.

Furthermore, we added a more complex Jastrow factor in the form of a Padé-Jastrow factor, and we observe that the energy drops further. Actually, the RBM+PJ provides energies completely on par with VMC, at least for small quantum dots. It is also the method that provides the lowest standard error, together with VMC.

We also see that all the methods are able to obtain energies closer to the reference energy in three dimensions compared to two dimensions. This is a well-known concept, and occurs because we have more degrees of freedom. One can imagine that the electrons can move in the space instead of just in the plane, which makes it easier to form a configuration that minimizes the energy.

Efforts was made to produce results with the partly restricted Boltzmann machine and the deep restricted Boltzmann machine, but they did not behave nicely.

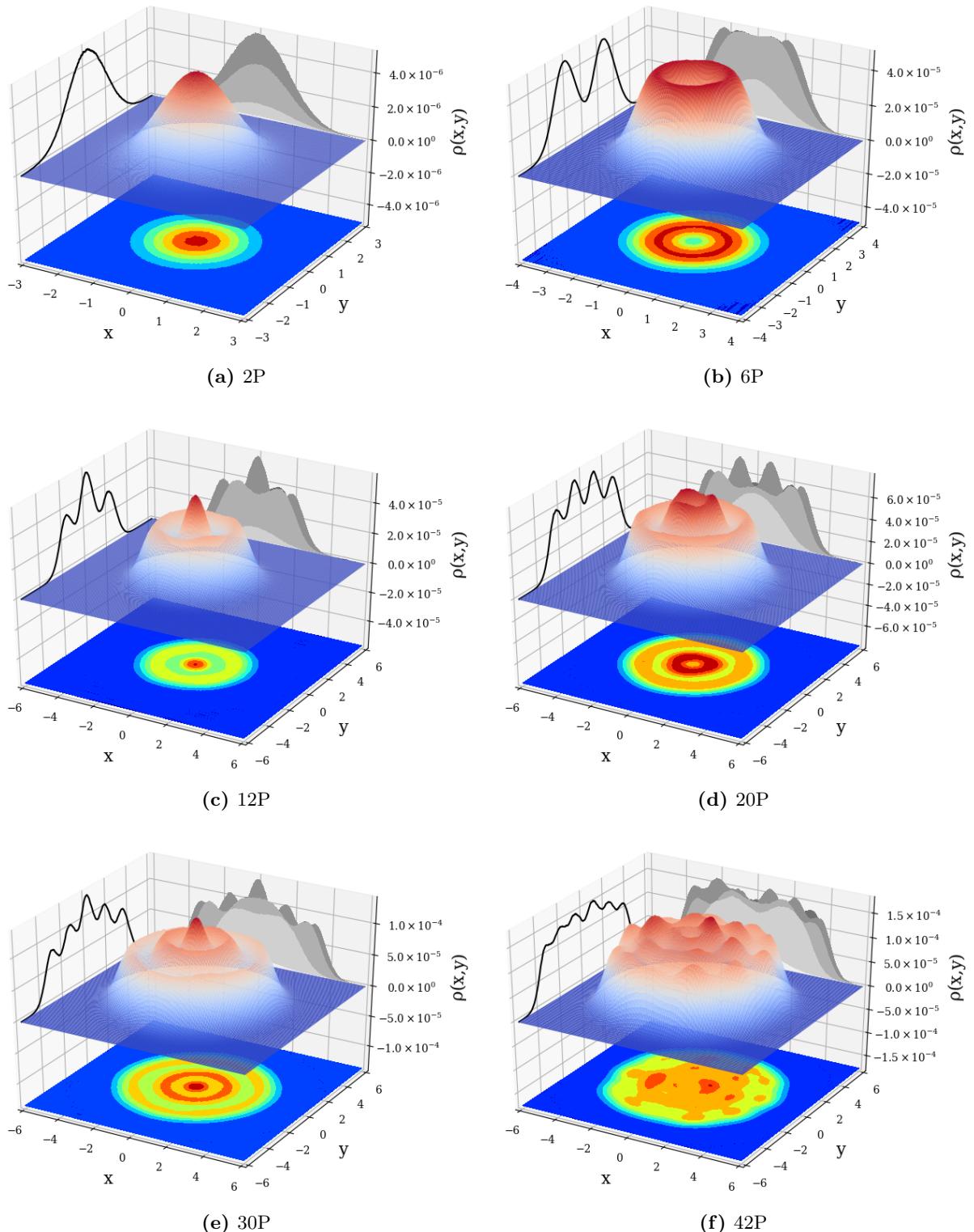
### 11.4.2 One-body density

Another quantity of particular interest is the one-body density. For all the systems presented in tables (11.4) and (11.5) we have also calculated the one-body density. However, since those plots would occupy tens of pages, we will not present all of them here. Instead, we select a few illustrative plots that might me interesting for the reader and give a representative picture of the density. The total collection of one-body density plots can be found in section D.3 in appendix D.

We have computed the one-body density in two different ways: by dividing the space into annuluses and obtaining the radial density profile, as discussed in section 9.8, and by dividing the space into a grid of equally sized bins and obtaining the density profile throughout the space. The former is computational beneficial as the density typically is stored in  $n$  bins compared to  $n^2$  bins for the second method, but the latter is obviously able to store more information. The second method is thereby able to give more informative plots, but when we want to compare the various wave function choices a radial density profile is more convenient. In other words, there are pros and cons associated with both methods, and we will therefore present both to give an as comprehensive description of the results as possible.

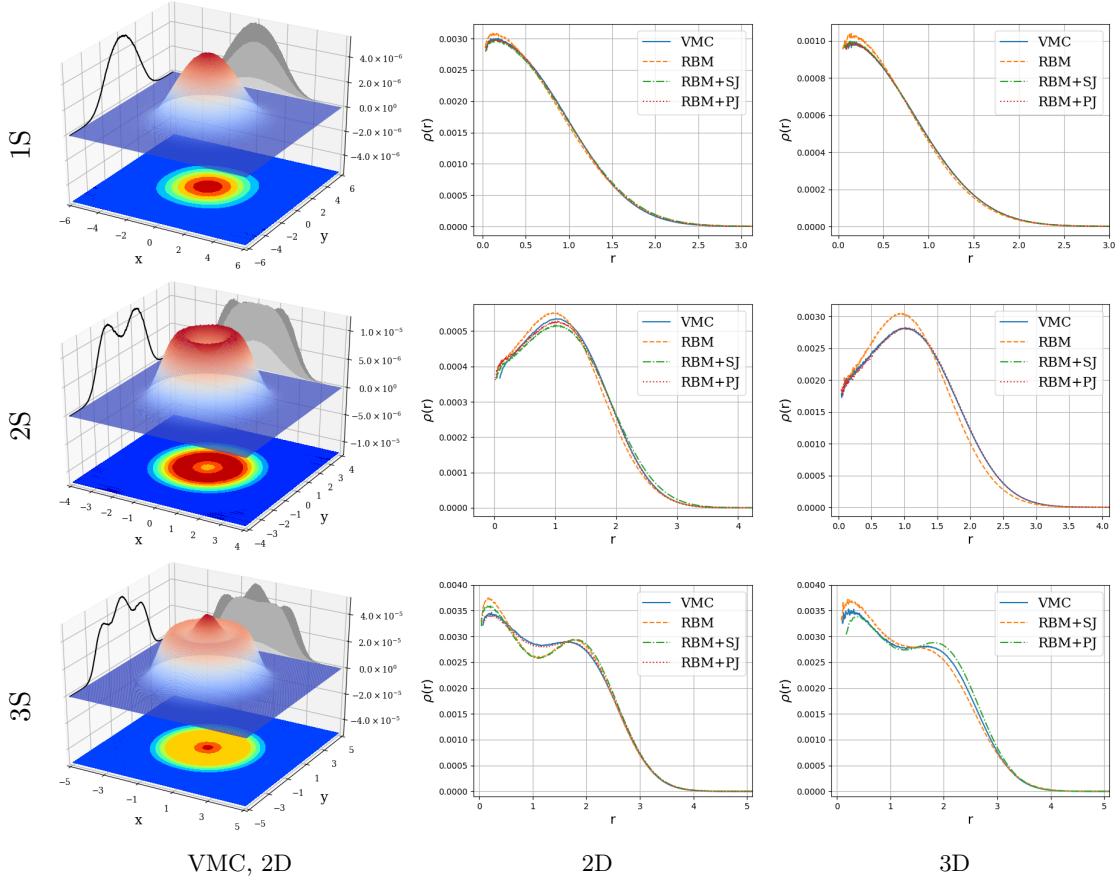
We will start with density profiles of the spaces for two-dimensional quantum dots of frequency  $\omega = 1.0$ . In figure (11.6), dots of sizes  $N = 2, 6, 12, 20, 30$  and  $40$  are presented, produced with a restricted Boltzmann machine. To maximize the amount of information in the plot, we have included a contour plot and a shadow plot in addition to a 3D plot of the density. What we observe is that the density gets a wave form, with more tops as the number of particles increase. Actually, the density plots behave like a water ripple if one sees it from the smallest to the largest dots, as J.Høgberget also points out [56]. The shapes of the densities are more or less consistent with what he found.

In figure (11.7), we stick to frequency  $\omega = 1.0$  for small two- and three-dimensional quantum dots and compare the results obtained by VMC, RBM, RBM+SJ and RBM+PJ. We investigate dots of 2, 6 and 12 electrons in two dimensions and 2, 8 and 20 particles in three dimensions, which corresponds to one shell (1S), two shells (2S) and three shells (3S) respectively. For all the cases, we immediately see that the density plots are identical in two and three dimensions for the same closed shell. After all, this is not so surprising considering the fact that particles in a shell of a quantum dot should be at the same distance from the center no matter if we look at the two-, three- or even one-dimensional case. However, what is most interesting for us is to analyze the performance of the Boltzmann machines. We observe that all the methods give quite similar



**Figure 11.6:** One-body density plots for two-dimensional circular quantum dots with frequency  $\omega = 1.0$ . The plots (a)-(f) present quantum dots containing 2, 6, 12, 20, 30 and 42 electrons respectively, and were all obtained using a plain restricted Boltzmann machine. ADAM optimizer was used, and after convergence the number of Monte-Carlo cycles was  $M = 2^{28} = 268,435,456$ .

radial density profiles, but RBM is always the guy standing out. Apparently, the RBM tends to over-exaggerate the ripples discussed above and gives an even more wavy density plot, which can be explained by the lack of Jastrow factor to take care of correlations. Further, the RBM+SJ is more similar to the VMC and RBM+PJ, which are almost identical. This is consistent with the energy analysis in section 11.4.1, where we found VMC and RBM+PJ to be most accurate, followed by RBM+SJ and RBM. We also observe an effect where the density plots consequently decrease close to the center, but this is an artifact caused by the fact that the number of Monte Carlo cycles is finite.



**Figure 11.7:** One-body density plot for small circular quantum dots with frequency  $\omega = 1.0$ . We look at the three lowest shells 1S, 2S and 3S corresponding to 2, 6 and 12 particles in two dimensions and 2, 8 and 20 particles in three dimensions. In the first column, the density profile for the space are presented for the two-dimensional dots using standard variational Monte Carlo (VMC), and in the second row the corresponding radial density profiles are obtained using VMC, plain restricted Boltzmann machine (RBM), restricted Boltzmann machine with simple Jastrow factor (RBM+SJ) and restricted Boltzmann machine with Padé-Jastrow factor. On the right-hand side, the radial density profiles for the three-dimensional dots are given using the same methods. ADAM optimizer was used, and after convergence the number of Monte-Carlo cycles was  $M = 2^{28} = 268, 435, 456$ .

We now move on to larger dots, containing  $N = 20$  and  $N = 42$  electrons with frequencies  $\omega = 0.1$  and  $\omega = 0.5$  in addition to the above-studied frequency  $\omega = 1.0$  to see if we can observe something interesting. In figure (11.8) we plot the radial density profile to again compare all the methods. What we see, is that the ripples discussed above are conserved for lower frequencies, and they possibly become more significant as the frequency drops. Moreover, both VMC, RBM+PJ

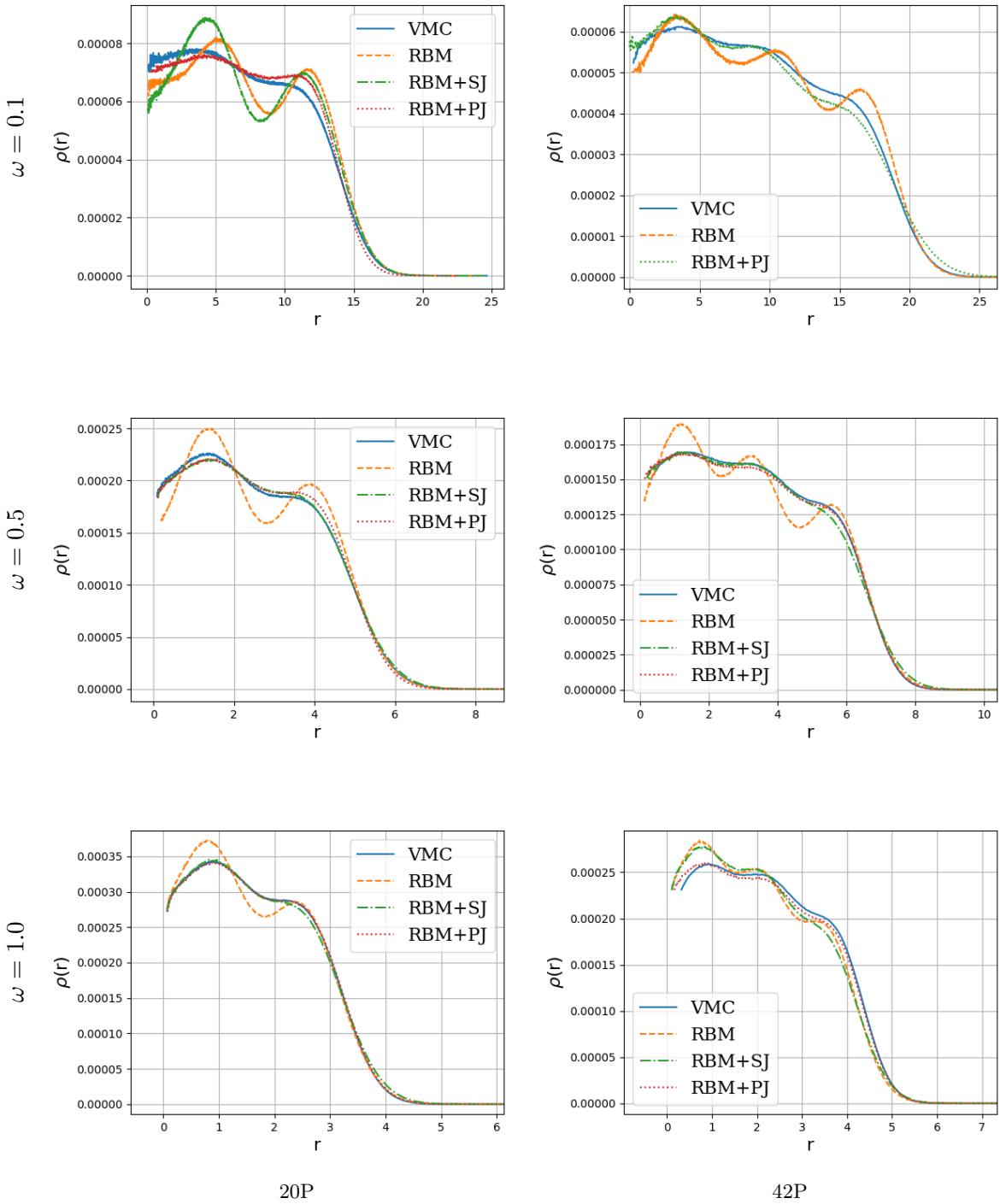
and RBM+SJ appear to give quite similar density plots, but for lower frequencies the difference gets more and more visible. Again RBM+SJ also cuts some of the density edges and RBM over-exaggerate the oscillations. Physically, this means that the methods is capable of finding the most likely places to find the electrons, but fails to determine the actual density there. Exactly why the Boltzmann machine gives this result is hard to say, but in section (11.4.7) we will try to reveal the secrets behind the Boltzmann wave functions.

### 11.4.3 Two-body density

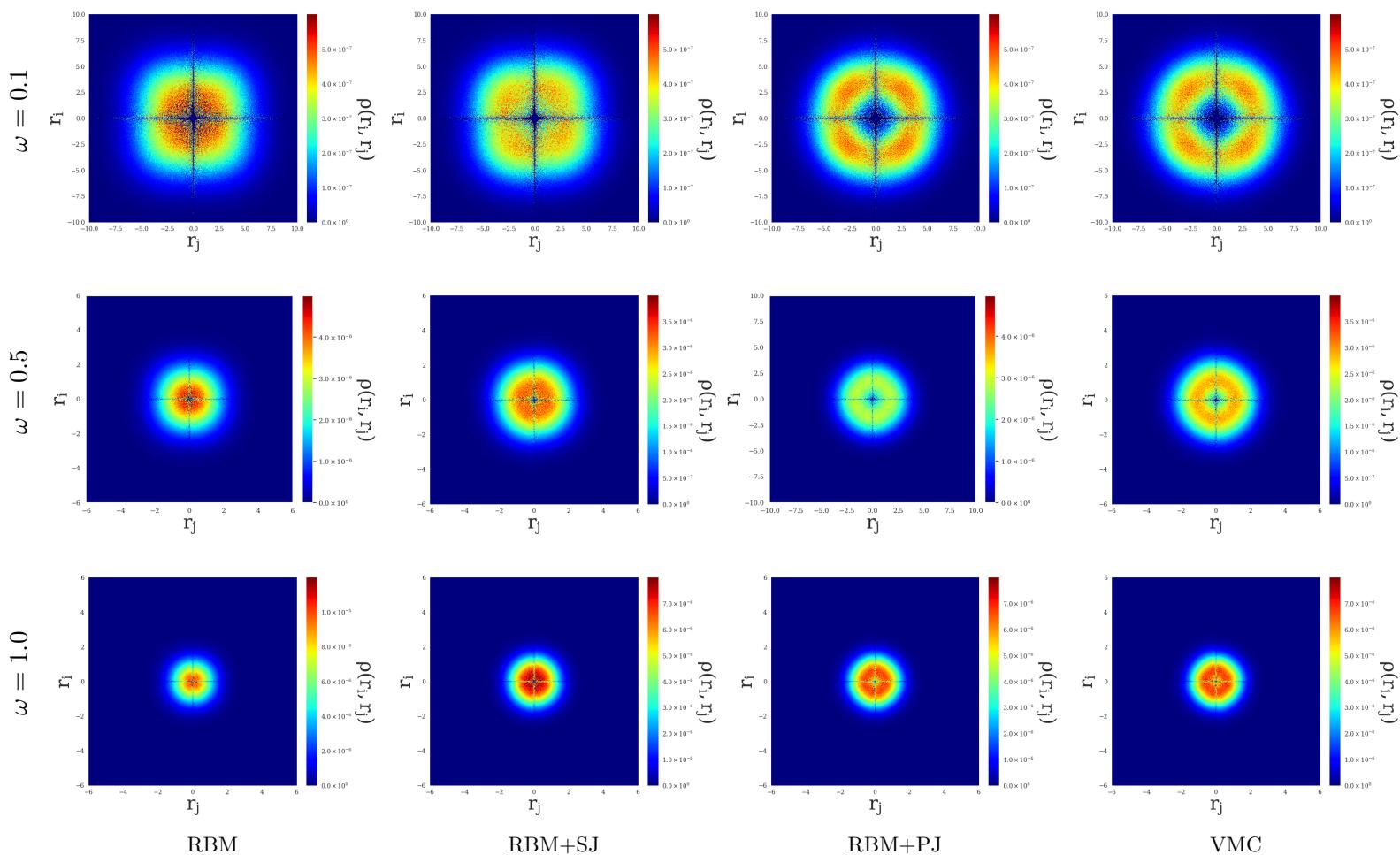
We have on multiple occasions discussed the electron density, and we have also mentioned the two-body density. The two-body density is interesting because it gives the probability of finding two particles at certain relative distances from the center of the dot. Similarly to the one-body density, we can both plot the radial two-body density profile and the actual two-body density distribution throughout the space. However, since the latter results in a higher-dimensional plot, we stick to the radial density profile. Also for this quantity it would be convenient to have references to benchmark our results to, but for some reason it is hard to find papers that discuss the two-body density for quantum dots. Nevertheless, someone needs to be the first and we will use the plots to discuss the physics and compare the various methods in the best possible way. Again we will emphasize that a negative radial distance does not make sense, so all the quadrants are the same and presented in this way to make it more illustrative and intuitive.

Again, we will focus on the two-dimensional case, as the three-dimensional case becomes similar as for the one-body density. In figure (11.9) we plot the two-body densities for a two-dimensional quantum dot at various frequencies containing two electrons. For frequency  $\omega = 1.0$ , the density plots for all the methods give quite similar results, but it seems like the RBM find the electrons closer to the center than the other methods. This can be explained by the absent of a Jastrow factor. Also the RBM+SJ gives a slightly more narrow distribution than the RBM+PJ and VMC, so it is apparent that the Padé-Jastrow factor results in a more repulsive factor than the simple Jastrow factor. We known that the potential, and thus the interactions get more dominating as the frequency is decreased. For VMC and RBM+PJ we can observe this by a circle of higher density at a certain distance from the origin both for  $\omega = 0.5$  and  $\omega = 0.1$ , which means that both particles are less likely to be found in the center of the dot at the same time. If one particle is close to the center, the other particle is probably far from the center. For the RBM+SJ, this behavior can only be observed for  $\omega = 0.1$ , while the RBM is not able to reproduce this phenomenon at all. This shows that the Padé-Jastrow factor is more correct than the simple Jastrow factor, as expected.

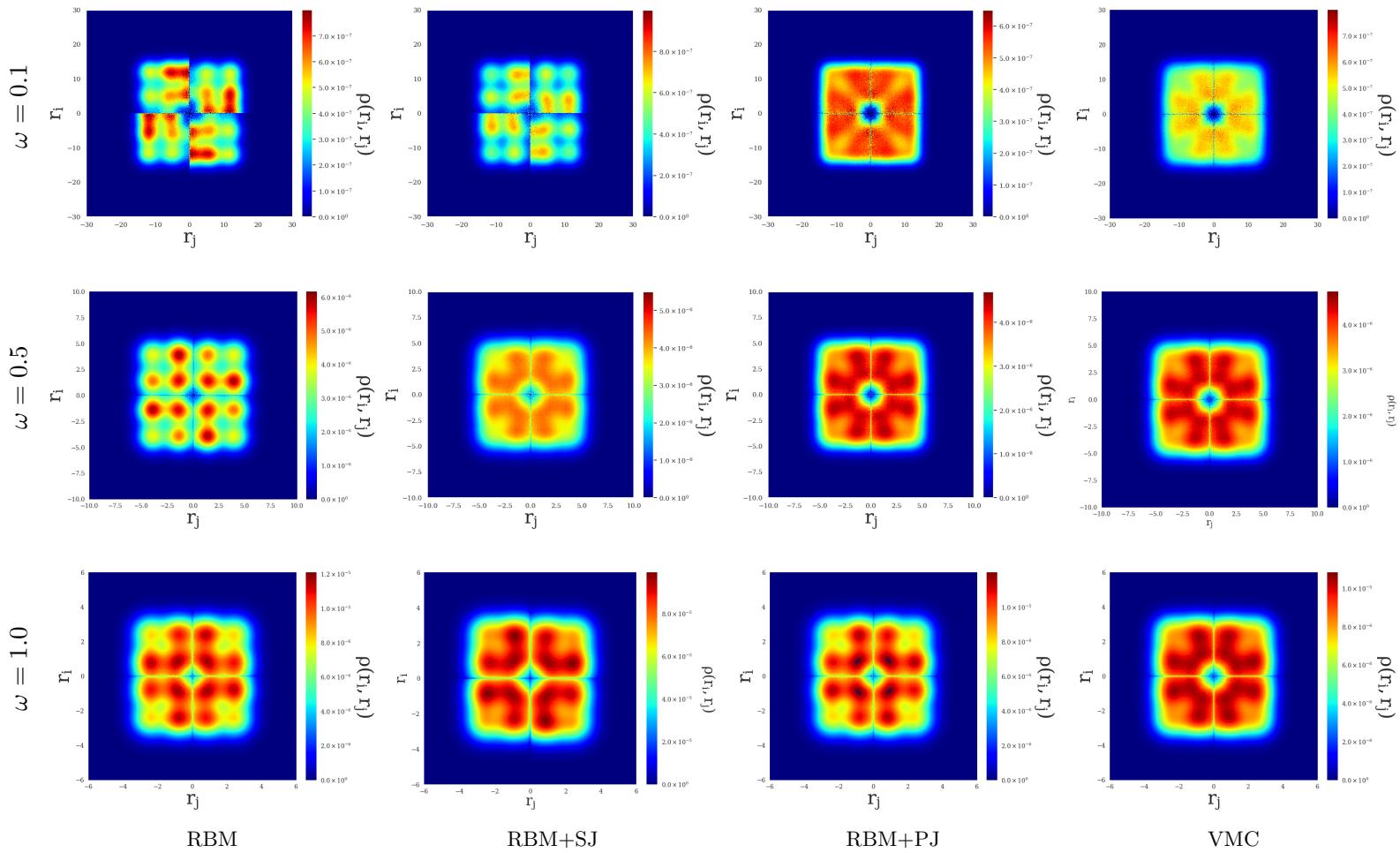
The calculations are repeated for two-dimensional quantum dots of 20 electrons, which are presented in figure (11.10). For  $\omega = 1.0$ , all the methods give approximately the same density distribution where the most likely area is located where an electron is far from the center and the other is close to the center and vice versa. However, the RBM finds the most likely configurations to be more discrete distributed than the other methods, which is even more significant for the frequency  $\omega = 0.5$ . This is the same behavior as we saw for the two-body density, where we argued that it can be explained by the absent of a Jastrow factor. For frequency  $\omega = 0.1$ , the same circle around the origin as we saw for the two-particle case can be observed, but it is not as notable since the scales along the axis are totally different. We also see that the RBM+PJ and VMC give a low probability of finding an electron pair at the same distance from the center at the same time. This can also be recognized from the one-body density plot in (11.8), where we see that the radial density profile is quite flat. We also see that the VMC always gives symmetric two-body density plots around  $r_i = r_j$ , which means that the wave function is symmetric with respect to the radial distance of two particles. On the other hand, the RBM wave function is not necessary symmetric, which results in non-symmetric plots for RBM, RBM+SJ and RBM+PJ.



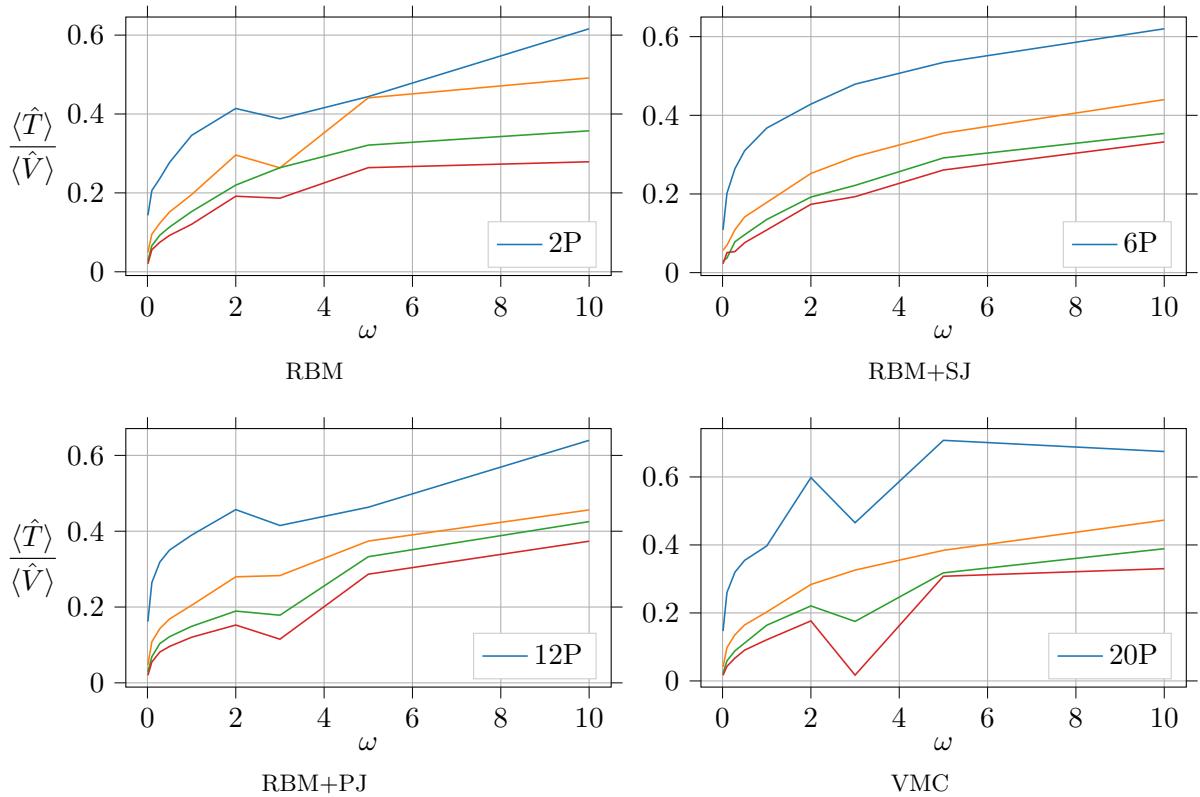
**Figure 11.8:** One-body density plot for two-dimensional circular quantum dots of 20 and 42 interacting electrons for the oscillator frequencies  $\omega = 0.1$ , 0.5 and 1.0. The results were produced by standard variational Monte-Carlo (VMC), plain restricted Boltzmann machine (RBM), restricted Boltzmann machine with simple Jastrow factor (RBM+SJ) and restricted Boltzmann machine with Padé-Jastrow factor (RBM+PJ). ADAM optimizer was used, and after convergence the number of Monte-Carlo cycles was  $M = 2^{28} = 268,435,456$ .



**Figure 11.9:** Two-body density plots for two-dimensional circular quantum dots containing up to two electrons with oscillator frequencies  $\omega = 0.1, 0.5$  and  $1.0$ . The density plots were produced using a plain restricted Boltzmann machine (RBM), restricted Boltzmann machine with simple Jastrow factor (RBM+SJ), restricted Boltzmann machine with Padé-Jastrow factor (RBM+PJ) and standard variational Monte-Carlo (VMC). The ADAM optimizer was used, and after convergence the number of Monte-Carlo cycles was  $M = 2^{28} = 268, 435, 456$ .



**Figure 11.10:** Two-body density plots for two-dimensional circular quantum dots containing up 20 electrons with oscillator frequencies  $\omega = 0.1, 0.5$  and  $1.0$ . The density plots were produced using a plain restricted Boltzmann machine (RBM), restricted Boltzmann machine with simple Jastrow factor (RBM+SJ), restricted Boltzmann machine with Padé-Jastrow factor (RBM+PJ) and standard variational Monte-Carlo (VMC). The ADAM optimizer was used, and after convergence the number of Monte-Carlo cycles was  $M = 2^{28} = 268,435,456$ .



**Figure 11.11:** The kinetic-potential energy ratio  $\langle \hat{T} \rangle / \langle \hat{V} \rangle$  plotted as a function of the oscillator frequency for two-dimensional circular quantum dots. The frequencies  $\omega = 0.01, 0.1, 0.28, 0.5, 1.0, 2.0, 3.0, 5.0, 10.0$  were run. The methods used are plain restricted Boltzmann machine (RBM), restricted Boltzmann machine with simple Jastrow factor (RBM+SJ), restricted Boltzmann machine with Padé-Jastrow factor (RBM+PJ) and standard variational Monte-Carlo (VMC).

#### 11.4.4 Energy distribution

It is also interesting to investigate how the energy is distributed between kinetic energy, external energy and interaction energy for the different methods and various oscillator frequencies. This makes us able to check if the results agree with the virial theorem presented in section 2.6, and it is also interesting to see if the different methods give different energy distribution. In figure (11.11), the ratio between the kinetic and potential energy as a function of oscillator frequency for two-dimensional quantum dots of various sizes is plotted. The numbers are taken from the tables (D.3-D.6) in appendix D.

We see a significant trend where the ratio drops as the frequency decreases. This means that the potential energy dominates the kinetic energy at low frequencies, which is a known phenomenon already mentioned several times throughout this thesis. For all the methods, the ratio is also lower for larger dots, which is because we have more particles and thereby gradually more interaction energy. The various methods also give more or less the same ratios spanning from around 0.65 to 0.01, but some of the plots are more noisy than the others. Most notably, the RBM+SJ provides smooth curves, but this is probably just a coincidence, there is no reason why this method should behave more nicely than the others.

It also looks like the ratio approaches a linear slope for higher frequencies, even though we have too few points to say anything with certainty. If we now recall the virial theorem in section

**Table 11.6:** This table shows how the total energy ( $\langle \hat{H} \rangle$ ) is distributed between kinetic energy ( $\langle \hat{T} \rangle$ ), external potential energy ( $\langle \hat{V}_{\text{ext}} \rangle$ ) and interaction energy ( $\langle \hat{V}_{\text{int}} \rangle$ ) of two-dimensional circular quantum dots at a wide range of frequencies  $\omega$  and 12 interacting electrons. The methods used are standard variational Monte-Carlo (VMC), plain restricted Boltzmann machine (RBM), restricted Boltzmann machine with a simple Jastrow factor (RBM+SJ) and restricted Boltzmann machine with Padé-Jastrow factor (RBM+PJ). The energy is given in units of  $\hbar$ , and the numbers in parenthesis are the statistical uncertainties in the last digit.

	$\omega$	$\langle \hat{H} \rangle$	$\langle \hat{T} \rangle$	$\langle \hat{V}_{\text{ext}} \rangle$	$\langle \hat{V}_{\text{int}} \rangle$
RBM	0.01	6.217(2)	0.1236(4)	2.244(2)	3.849(2)
	2.0	269.086(8)	43.262(8)	95.17(2)	130.65(1)
	10.0	1003.71(4)	218.87(4)	492.87(8)	291.97(3)
RBM+SJ	0.01	6.239(2)	0.1372(6)	2.184(2)	3.919(3)
	2.0	265.66(9)	39.31(8)	95.78(1)	130.57(2)
	10.0	952.71(2)	237.65(4)	392.06(7)	323.00(3)
RBM+PJ	0.01	6.210(1)	0.1208(5)	2.189(2)	3.900(2)
	2.0	262.598(1)	34.758(6)	108.546(9)	119.293(7)
	10.0	947.33(2)	257.67(5)	348.35(6)	341.31(3)
VMC	0.01	6.2097(8)	0.1005(4)	2.270(3)	3.839(3)
	2.0	262.5339(9)	38.402(3)	95.681(7)	128.451(5)
	10.0	944.1(1)	234.4(4)	383.9(3)	325.9(1)

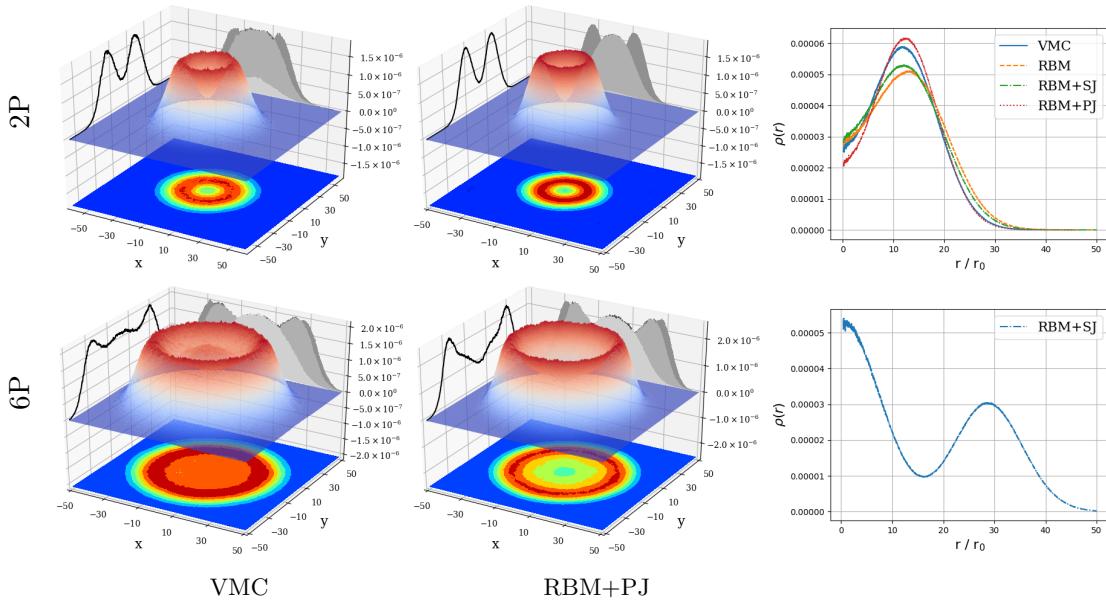
2.6, it states that the kinetic energy is related to the potential energies in a certain way. For the interaction quantum dot system, the virial theorem reads

$$2\langle \hat{T} \rangle = 2\langle \hat{V}_{\text{ext}} \rangle - \langle \hat{V}_{\text{int}} \rangle \quad (11.1)$$

which can be verified from the energy splitting. To do this accurately, we better list up the exact energies, which for a two-dimensional quantum dot with 20 electrons is given in table (11.6) for some selected frequencies. For more frequencies and system sizes, please look at appendix D. By doing the math, we see that the virial theorem is satisfied for high frequencies, but as the frequency decreases the theorem gets more and more off. This is the case for all the methods, and is also observed for other system sizes.

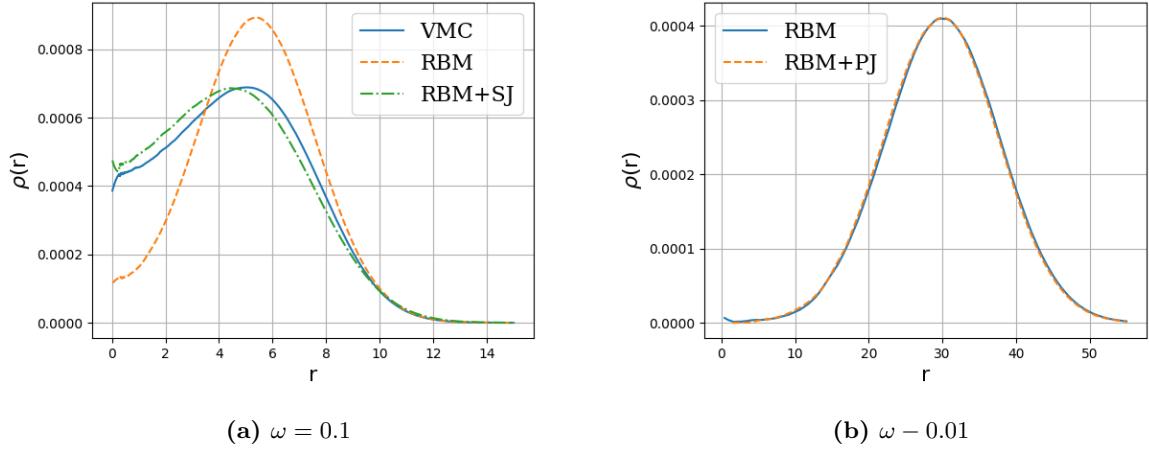
### 11.4.5 Low frequency dots

As we pointed out in section 3.3.1, and as shown in the previous section, the potential energy dominates the kinetic energy at low frequencies. This means that the electrons have a lower average momentum, which makes Wigner crystallization possible. As already discussed in the theory part, this phenomenon can be observed by some distinct peaks in the one-body density plots. In figure (11.12), the one-body density is plotted for low frequency dots containing 2, 6, 12 and 20 particles.



**Figure 11.12:** One-body density of two-dimensional circular quantum dots of frequency  $\omega = 0.01$ .

For two particles, we can see that the one-body density has a peak at a distance from origin, and a local minimum in origin. This means that the probability of finding the two electrons close to each other is low, which is as expected due to the low momentum.



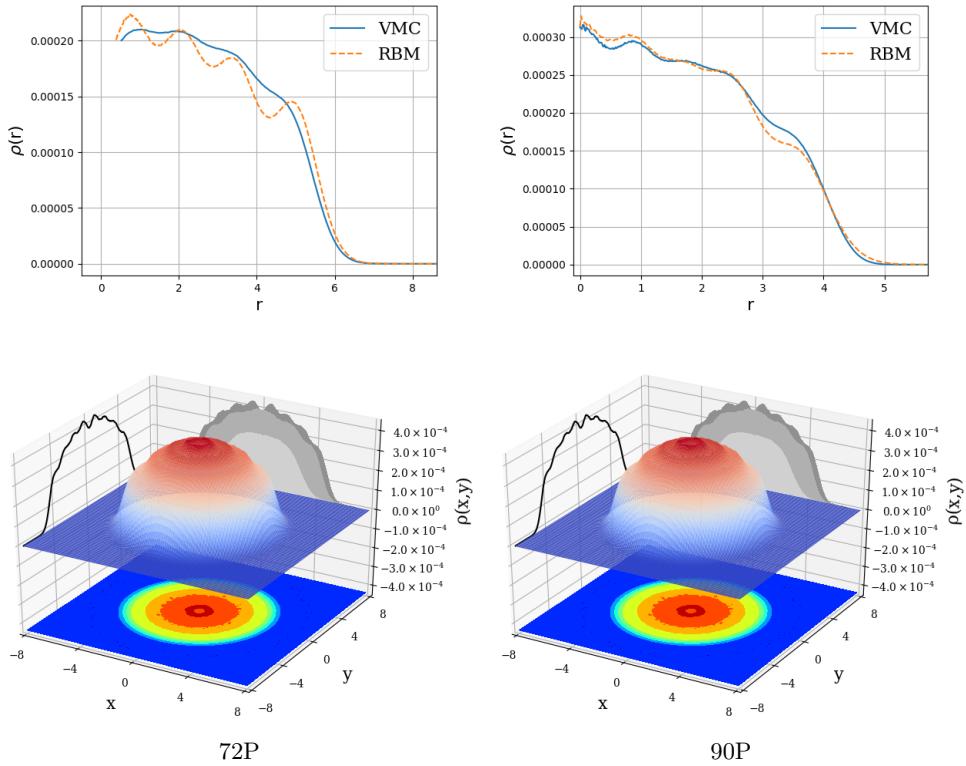
**Figure 11.13:** One-body density of three-dimensional circular quantum dots containing 8 electrons.

#### 11.4.6 Large dots

In order to test the code, we also decided to run for systems with  $N > 56$ . This has no scientific significance, other than testing how far a VMC code can go. One thing is that the computations get extremely expensive as the number of particle increases, but also the standard error is assumed to increase due to.. This means that we cannot just crack up the wall clock time and wait when studying large systems, at some point the standard error gets insufficiently large and we need to further increase the number of Monte Carlo cycles. The learning rate also needs to be decreased as the system sizes increase, which requires more iterations. We look at weakly interacting particles, so we will not get any relativistic effects even when adding many particles.

**Table 11.7:** Energy of large circular quantum dots,  $\omega = 1.0$ . All energies are given in units of  $\hbar$ , and the numbers in parenthesis are the statistical uncertainties in the last digit.

	$N$	RBM	VMC
2D	72	1355.37(2)	1340.520(7)
	90	2194.12(9)	1990.89(2)
3D	70	1129.40(2)	1108.950(4)



**Figure 11.14:** One-body density of large circular quantum dots with frequency  $\omega = 1.0$ .

#### 11.4.7 Wave functions

### 11.5 Double quantum dots

Double quantum dots are systems that are very interesting when it comes to testing the Boltzmann machines. Even though we know the wave functions for the non-interacting case, they are expansions that are computational expensive to work with.

#### 11.5.1 Ground state energy

For the interacting case, we will use Alocias Mariadason's VMC calculations with and without a Hartree-Fock basis as a reference. In addition, Jørgen Høgberget provides a DMC energy for the case with two particles and  $\omega = 1.0$ . All the results are given in table (11.8). What we observe, is that the Boltzmann machine with a Padé-Jastrow factor outperforms a Hermite expansion of 10 functions, which is surprising. The reason might be that the Boltzmann machine consists of a sum over all hidden nodes, which makes it behave similar to an expansion. However, the Boltzmann machine was then again beat by the Hermite expansion with a Hartree-Fock basis.

**Table 11.8:** Double quantum dots. F is the number of functions used in the expansion.

$N$	$\omega$	VMC (F=1)	RBM+PJ	VMC (F>1) (Ref.[79])	VMC+HF (Ref.[79])	DMC (Ref.[56])
2	0.1	0.41982(4)	0.41939(3)			
	0.28	0.9365(1)	0.92618(4)			
	0.5	1.4847(2)	1.44004(4)			
	1.0	2.6331(5)	2.38342(3)	2.42238(4)	2.36618(4)	2.3496(1)
4	0.1					
	0.28					
	0.5					
	1.0	-	-	7.95247(4)	7.90232(4)	-
6	0.1					
	0.28					
	0.5					
	1.0	-	-	16.61419(4)	16.55609(4)	
8	0.1					
	0.28					
	0.5					
	1.0	-	-			

## Chapter 12

# Conclusion and future work

The idea behind the simple Jastrow factor, was that it is not optimized with respect to the interactions and thus represents an attempt of reducing the physical intuition used. It turned out that the simple Jastrow did not work very well compare to the Padé-Jastrow factor and when it more expensive as well, there is actually no point of using the simple Jastrow factor.

See if machine learning is able to describe the three-body interaction, with nuclear physics applications.

Run linear algebra operations on GPU, we only tried CPU.

In-medium

Multi-quantum dot to compare to experiments

If we go back to the goals presented in the introduction, we can see that most of them are met. However, we did not have time or manage to study atoms using Boltzmann machines as it is not obvious to me how to do that.



# Appendix A

## Dirac Formalism

The Dirac formalism, also called braket notation, was suggested by Paul Dirac in a 1939 paper with the purpose of improving the reading ease [12]. The notation unites the integral representation and the matrix representation in an elegant fashion by representing all elements in the *Hilbert-space*.

The Hilbert-space is a complete linear vector space, which allows length, inner products and angles between vectors to be measured. A column vector in the space is denoted by  $|\psi\rangle$ , which is called the *ket*, and by taking the Hermitian conjugate of it we obtain the corresponding row vector,  $(|\psi\rangle)^+ = \langle\psi|$  called the *bra*. As the Hilbert-space is characterized by linearity, it requires that the sum of two element in the space is also an element in the space,

$$|\psi\rangle + |\phi\rangle = |\psi + \phi\rangle, \quad (\text{A.1})$$

and that two elements always *commute*,

$$|\psi\rangle + |\phi\rangle = |\phi\rangle + |\psi\rangle. \quad (\text{A.2})$$

Another important property is that an element of the space multiplied with a complex number is also an element of the space,

$$c|\psi\rangle = |c\psi\rangle \quad \forall c \in \mathbb{C}. \quad (\text{A.3})$$

To fully utilize the notation, orthogonality properties need to be taken into account. Assume that we have a orthogonal basis set

$$\{\psi_1, \psi_2, \dots, \psi_n\}. \quad (\text{A.4})$$

The inner product between two basis elements is then given by

$$\langle\psi_i|\psi_j\rangle = \begin{cases} \neq 0 & \text{if } i = j \\ = 0 & \text{otherwise} \end{cases} \equiv \delta_{ij} \quad (\text{A.5})$$

where we have introduced the Kronecker delta  $\delta_{ij}$ . If we further require that our basis is *orthonormal*, the inner product is 1, and we can prove the *completeness relation*. Assume we want to expand a vector  $|\phi\rangle$  in our orthonormal basis set,

$$|\phi\rangle = \sum_{i=1}^n c_i |\psi_i\rangle. \quad (\text{A.6})$$

By multiplying with one of the basis vectors on the left hand side, the sum collapses, and we are just left with one of the coefficients  $c_j$ ,

$$\langle \psi_j | \phi \rangle = \sum_{i=1}^n c_i \langle \psi_j | \psi_i \rangle = c_j. \quad (\text{A.7})$$

If we now again insert this into the expansion, we obtain

$$|\phi\rangle = \sum_{i=1}^n \langle \psi_i | \phi \rangle |\psi_i\rangle = \left[ \sum_{i=1}^n |\psi_i\rangle \langle \psi_i| \right] |\phi\rangle \quad (\text{A.8})$$

which implies that the outer product

$$\sum_{i=1}^n |\psi_i\rangle \langle \psi_i| = 1. \quad (\text{A.9})$$

We can use these properties to demonstrate how the normalization constant can be obtained without explicitly solving any integrals. Consider two spin-1/2 particles, for example the electron and the proton in the ground state of Hydrogen. There are then four possible states: the *triplet* with  $s = 1$  and the *singlet* with  $s = 0$ . The latter reads [40]

$$|00\rangle = A(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \quad (\text{A.10})$$

which is associated with the bra

$$\langle 00| = (|00\rangle)^+ = A(\langle \uparrow\downarrow| - \langle \downarrow\uparrow|). \quad (\text{A.11})$$

The inner product is then given by

$$\begin{aligned} \langle 00|00\rangle &= A(\langle \uparrow\downarrow| - \langle \downarrow\uparrow|)A(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \\ &= A^2(\langle \uparrow\downarrow|\uparrow\downarrow\rangle - \langle \uparrow\downarrow|\downarrow\uparrow\rangle - \langle \downarrow\uparrow|\uparrow\downarrow\rangle + \langle \downarrow\uparrow|\downarrow\uparrow\rangle) \\ &= A^2(1 - 0 - 0 + 1) \\ &= 2A^2 = 1 \end{aligned} \quad (\text{A.12})$$

where we have assumed that also the states  $|\uparrow\downarrow\rangle$  and  $|\downarrow\uparrow\rangle$  are orthonormal. From this, we can see that the normalization constant  $A$  must be equal to  $1/\sqrt{2}$ .

Further, we also use the Dirac notation as a short-hand notation of the Slater determinant, discussed in section 3.2.2. Instead of writing out the entire determinant, it is common to write it as

$$|\psi_1 \psi_2, \dots, \psi_N\rangle \equiv \frac{1}{\sqrt{N!}} \begin{vmatrix} \psi_1(\mathbf{r}_1, \sigma_1) & \psi_2(\mathbf{r}_1, \sigma_1) & \dots & \psi_n(\mathbf{r}_1, \sigma_1) \\ \psi_1(\mathbf{r}_2, \sigma_2) & \psi_2(\mathbf{r}_2, \sigma_2) & \dots & \psi_N(\mathbf{r}_2, \sigma_2) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_1(\mathbf{r}_N, \sigma_N) & \psi_2(\mathbf{r}_N, \sigma_N) & \dots & \psi_N(\mathbf{r}_N, \sigma_N) \end{vmatrix} \quad (\text{A.13})$$

which is extensively used in for instance second quantization.

## Appendix B

# Natural units

In the everyday life, we usually stick to the standard SI units when measuring or expressing distances, energies, weights, time and so on. A standard unit system is important because it is common to people around the world, so that people can communicate with each other conveniently across borders with a reduced risk of misconceptions. Another point is that people gradually develop an intuition about a set of units when they are frequently used, such that they immediately observe if a number makes sense or not when it is expressed in the preferred units. As an European, I always measure length in meters so when people say they are 1.80m tall I can easily imagine their height. On the other hand, when Americans tell their height, I do not have an intuition for what 5.0 feet is.

However, in science the SI units are often not the preferred ones, especially when things get very large or very small. For instance, measuring cosmological distances in meters is very unpractical, as the distance to the Sun is  $\sim 1.5 \cdot 10^{11}$ m and the distance to our closest neighbor galaxy Andromeda is  $\sim 2.4 \cdot 10^{22}$ m! Instead, we use units like the astronomical unit [a.u.] (should not be confused with atomic units) and light years.

For small scales, the situation is similar. For instance, the most probable distance between the nucleus and the electron is the Bohr radius, which is  $a_0 \approx 5.3 \cdot 10^{-11}$ m, which again is very unpractical to work with. Instead, we define so-called natural units where  $a_0 = 1$ , and scale the other quantities after that. We will in this chapter present how the quantum dot and atomic Hamiltonian can be scaled in natural units. We will stick to Hartree atomic units, as it gives elegant expressions of the Hamiltonians.

### B.1 Quantum dots

For circular quantum dots, the one-dimensional Hamiltonian in SI units reads

$$\hat{\mathcal{H}} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{1}{2} m \omega^2 x^2 \quad (\text{B.1})$$

with  $\hbar$  as the reduced Planck's constant,  $m$  as the electron mass and  $\omega$  as the oscillator frequency. The corresponding wave functions read

$$\phi_n(x) = \frac{1}{\sqrt{2^n n!}} \cdot \left( \frac{m\omega}{\pi\hbar} \right)^{1/4} \exp \left( -\frac{m\omega}{2\hbar} x^2 \right) H_n \left( \sqrt{\frac{m\omega}{\hbar}} x \right). \quad (\text{B.2})$$

We want to get rid of  $\hbar$  and  $m$  in equation (B.1) to make it dimensionless, which can be accomplished by scaling  $\hat{\mathcal{H}}' = \hat{\mathcal{H}}/\hbar$ , such that the Hamiltonian reduces to

$$\hat{\mathcal{H}}' = -\frac{\hbar}{2m} \frac{\partial^2}{\partial x^2} + \frac{1}{2} \frac{m\omega^2}{\hbar} x^2. \quad (\text{B.3})$$

Moreover, we observe that the fraction  $\hbar/m$  comes in both terms, which can be avoided by introducing a characteristic length  $x' = x/\sqrt{\hbar/m}$ . The final Hamiltonian is

$$\hat{\mathcal{H}} = \frac{1}{2} \frac{\partial^2}{\partial x^2} + \frac{1}{2} \omega^2 x^2 \quad (\text{B.4})$$

which corresponds to setting  $\hbar = m = 1$ . In natural units, one often sets  $\omega = 1$  as well by scaling  $\hat{\mathcal{H}}' = \hat{\mathcal{H}}/\hbar\omega$ , but since we want to keep the  $\omega$ -dependency, we do it slightly different. This means that the exact wave functions for the one-particle one-dimensional case goes as

$$\phi_n(x) \propto \exp\left(-\frac{\omega}{2}x^2\right) H_n(\sqrt{\omega}x) \quad (\text{B.5})$$

where we omit the normalization factor with the Metropolis algorithm in mind.

## B.2 Atoms

The atomic Hamiltonian for an electron in subshell  $l$  affected by a nucleus with atomic number  $Z$  reads

$$\hat{\mathcal{H}} = -\frac{\hbar^2}{2m_e} \nabla^2 - \frac{1}{4\pi\epsilon_0} \frac{Ze^2}{r} + \frac{\hbar^2 l(l+1)}{2m_e r^2}. \quad (\text{B.6})$$

in SI units where  $e$  is the elementary charge and  $k_e = 1/4\pi\epsilon_0$  is Coulomb's constant. Again we want to get rid of the reduced Planck's constant  $\hbar$  and the electron mass  $m_e$  in order to make the Hamiltonian dimensionless, which we do by multiplying all terms by  $(4\pi\epsilon_0)^2 \hbar^2 / (m_e e^4 Z^2)$ ,

$$\hat{\mathcal{H}} \cdot \frac{(4\pi\epsilon_0)^2 \hbar^2}{m_e e^4 Z^2} = -\frac{(4\pi\epsilon_0)^2 \hbar^4}{2m_e^2 e^4 Z^2} \nabla^2 + \frac{4\pi\epsilon_0 \hbar^2}{m_e e^2 Z r} - \frac{(4\pi\epsilon_0)^2 \hbar^4 l(l+1)}{m_e^2 e^4 Z^2} \frac{l(l+1)}{r^2}. \quad (\text{B.7})$$

This might look very chaotic, but by exploiting that the Bohr radius,

$$a_0 \equiv \frac{4\pi\epsilon_0 \hbar^2}{m_e e^2 Z}, \quad (\text{B.8})$$

is found in all the right hand side terms, we can simplify the Hamiltonian to be

$$\hat{\mathcal{H}} \cdot \frac{(4\pi\epsilon_0)^2 \hbar^2}{m_e e^4 Z^2} = -\frac{a_0^2}{2} \nabla^2 + a_0 \frac{Z}{r} - a_0^2 \frac{l(l+1)}{2r^2}. \quad (\text{B.9})$$

We obtain the dimensionless Hamiltonian in Atomic units by scaling  $r' = r/a_0$  and  $\hat{\mathcal{H}}' = \hat{\mathcal{H}}/(m_e e^4 Z^2 / (4\pi\epsilon_0)^2 \hbar^2)$ , getting

$$\hat{\mathcal{H}} = -\frac{1}{2} \nabla^2 - \frac{Z}{r} + \frac{l(l+1)}{2r^2}, \quad (\text{B.10})$$

which again corresponds to setting  $\hbar = m_e = k_e = e = 1$ .

## Appendix C

# Calculations of a general Gaussian-binary RBM wave function

In this appendix, we start from a general Gaussian-binary restricted Boltzmann machine and set up the system energy and joint probability distribution.

The most basic Gaussian-binary restricted Boltzmann machine is the one in the form of

$$\begin{aligned} E(\mathbf{x}, \mathbf{h}) &= \sum_{i=1}^F \frac{(x_i - a_i)^2}{2\sigma_i^2} - \sum_{j=1}^H b_j h_j - \sum_{i,j=1}^{F,H} \frac{x_i w_{ij} h_j}{\sigma_i^2} \\ &= \sum_{i=1}^F \frac{(x_i - a_i)^2}{2\sigma_i^2} - \sum_{j=1}^H h_j \left( b_j + \sum_{i=1}^F \frac{x_i w_{ij}}{\sigma_i^2} \right) \end{aligned} \quad (\text{C.1})$$

as discussed in chapter 5. If we now denote the expression in the last parenthesis by  $f_j(\mathbf{x}; \boldsymbol{\theta})$  where  $\boldsymbol{\theta}$  includes all the parameters, we end up with the expression of the general Gaussian-binary restricted Boltzmann machine,

$$E(\mathbf{x}, \mathbf{h}) = \sum_{i=1}^F \frac{(x_i - a_i)^2}{2\sigma_i^2} - \sum_{j=1}^H h_j f_j(\mathbf{x}; \boldsymbol{\theta}) \quad (\text{C.2})$$

where  $f_j(\mathbf{x}; \boldsymbol{\theta})$  in principle can be any function of  $\mathbf{x}$  and  $\boldsymbol{\theta}$ . From this expression, we obtain the joint probability distribution

$$\begin{aligned} P(\mathbf{x}, \mathbf{h}) &= \frac{1}{Z} \exp(-\beta E(\mathbf{x}, \mathbf{h})) \\ &\propto \exp \left( \sum_{i=1}^F \frac{(x_i - a_i)^2}{2\sigma_i^2} \right) \exp \left( \sum_{j=1}^H \left( h_j f_j(\mathbf{x}; \boldsymbol{\theta}) \right) \right) \\ &\propto \exp \left( \sum_{i=1}^F \frac{(x_i - a_i)^2}{2\sigma_i^2} \right) \prod_{j=1}^H \exp \left( h_j f_j(\mathbf{x}; \boldsymbol{\theta}) \right) \end{aligned} \quad (\text{C.3})$$

where we set  $\beta = 1/k_B T = 1$  and we ignore the partition function  $Z$ . Our main interest is the marginal distribution, which we will derive in detail.

### C.1 Marginal probability

In chapter 5, we presented the marginal distribution of the visible nodes as

$$P(\mathbf{x}) = \sum_{\mathbf{h}} P(\mathbf{x}, \mathbf{h}) \quad (\text{C.4})$$

which for our function is

$$\begin{aligned}
P(\mathbf{x}) &\propto \sum_{\{\mathbf{h}\}} \exp \left( \sum_{i=1}^F \frac{(x_i - a_i)^2}{2\sigma^2} \right) \prod_{j=1}^H \exp \left( h_j f_j(\mathbf{x}; \boldsymbol{\theta}) \right) \\
&= \sum_{h_1=0}^1 \sum_{h_2=0}^1 \dots \sum_{h_H=0}^1 \exp \left( \sum_{i=1}^F \frac{(x_i - a_i)^2}{2\sigma^2} \right) \exp(h_1 f_1) \exp(h_2 f_2) \dots \exp(h_H f_H) \\
&= \exp \left( \sum_{i=1}^F \frac{(x_i - a_i)^2}{2\sigma^2} \right) \sum_{h_1=0}^1 \sum_{h_2=0}^1 \dots \sum_{h_H=0}^1 \exp(h_1 f_1) \exp(h_2 f_2) \dots \exp(h_H f_H) \\
&= \exp \left( \sum_{i=1}^F \frac{(x_i - a_i)^2}{2\sigma^2} \right) \sum_{h_1=0}^1 \exp(h_1 f_1) \sum_{h_2=0}^1 \exp(h_2 f_2) \dots \sum_{h_H=0}^1 \exp(h_H f_H) \\
&= \exp \left( \sum_{i=1}^F \frac{(x_i - a_i)^2}{2\sigma^2} \right) \prod_{j=1}^H \sum_{h_j=0}^1 \exp(h_j f_j) \\
&= \exp \left( \sum_{i=1}^F \frac{(x_i - a_i)^2}{2\sigma^2} \right) \prod_{j=1}^H \left[ 1 + \exp(f_j(\mathbf{x}; \boldsymbol{\theta})) \right]
\end{aligned} \tag{C.5}$$

where we utilize that only an element in the product contains a certain hidden node  $h_j$ .

## C.2 Gradient of wave function and log-likelihood function

Defining the wave function as the marginal probability, we have seen that wave function of a general Gaussian-binary restricted Boltzmann machine has the form

$$\psi(\mathbf{x}; \mathbf{a}, \boldsymbol{\theta}) = \exp \left( - \sum_{i=1}^F \frac{(x_i - a_i)^2}{2\sigma^2} \right) \prod_{j=1}^H \left[ 1 + \exp(f_j(\mathbf{x}; \boldsymbol{\theta})) \right] \tag{C.6}$$

where  $f_j(\mathbf{x}; \boldsymbol{\theta})$  is an arbitrary function of the coordinates  $\mathbf{x}$  and the weights  $\boldsymbol{\theta}$ . The Gaussian part is straight-forward to differentiate, so we will keep our attention on the product,

$$\psi_p(\mathbf{x}; \boldsymbol{\theta}) = \prod_{j=1}^H \left[ 1 + \exp(f_j(\mathbf{x}; \boldsymbol{\theta})) \right]. \tag{C.7}$$

Henceforth, we will omit the variable  $\mathbf{x}$  and  $\boldsymbol{\theta}$ . By introducing the functions

$$p_j \equiv \frac{1}{1 + \exp(+f_j)} \quad \wedge \quad n_j \equiv \frac{1}{1 + \exp(-f_j)}, \tag{C.8}$$

where the last one is the sigmoid function, we find the gradient and Laplacian of  $\ln \psi_p$  to be

$$\nabla_k \ln \psi_p = \sum_{j=1}^H n_j \nabla_k (f_j) \tag{C.9}$$

and

$$\nabla_k^2 \ln \psi_p = \sum_{j=1}^H n_j [\nabla_k^2 (f_j) + p_j (\nabla_k (f_j))^2] \tag{C.10}$$

respectively. The parameter  $\theta_i$  can be updated according to the following rule

$$\frac{\partial}{\partial \theta_i} \ln \psi_p = \sum_{j=1}^H n_j \frac{\partial}{\partial \theta_i} (f_j). \quad (\text{C.11})$$

and the ratio between wave functions can be found by

$$\frac{\psi_p^{\text{new}}}{\psi_p^{\text{old}}} = \prod_{j=1}^H \frac{p_j^{\text{old}}}{p_j^{\text{new}}}. \quad (\text{C.12})$$

As a conclusion, what we actually need to calculate to find respective expressions for each wave function is  $\nabla_k(f_j)$ ,  $\nabla_k^2(f_j)$  and  $\partial_{\theta_i}(f_j)$ . This applies for a general Gaussian-binary restricted Boltzmann machine, also for a deep Boltzmann machine if all the units are Gaussian-binary.



## Appendix D

# Collection of Results

This appendix is meant as a more or less complete collection of the results obtained in this thesis, with purpose of presenting a complementary picture of the observations. This also allows us to include the (subjective) most important results in the chapter on selected results, 11, and avoid it to be too overwhelming. However, in this appendix the results will simply be listed and not commented and discussed.

First the computational time will be listed, which are just the information plotted in figure (11.2). Thereafter we list all the obtained energies, included the distribution between interaction and kinetic energy. We also present a complete set of one-body and two-body densities.

### D.1 CPU time

The CPU-time was calculated for all the systems we have been looking at, i.e., two-dimensional quantum dots containing up to 90 electrons and three-dimensional quantum dots containing up to 70 particles. Only closed-shell dots were considered, and the times are obviously independent of the frequency. To find correct CPU times per iteration, all systems were run with  $M = 2^{20} = 1,048,576$  cycles per iteration on the computer cluster Abel, every time on 8 nodes á 16 cores. We performed at least four independent runs for each system, where the average time over thousands of iterations was calculated automatically by the program. The results can be found in the tables (D.1) and (D.2) for two- and three dimensional quantum dots respectively.

**Table D.1:** The CPU time (in seconds) for each iteration when simulating two-dimensional circular quantum dots of 2-90 electrons. The time was clocked for  $M = 2^{20} = 1,048,576$  Monte Carlo cycles, and to get accurate times we took the average over at least four independent runs with thousands of iterations.

	2P	6P	12P	20P	30P	42P	56P	72P	90P
RBM	6.05	11.25	20.53	38.99	73.72	130.49	213.47	360.22	856.84
RBM+SJ	7.12	14.07	28.42	63.27	122.93	199.60	349.22	-	-
RBM+PJ	7.26	13.50	27.68	57.09	119.17	212.53	382.13	-	-
VMC	5.11	10.51	20.85	41.20	76.26	137.39	230.63	355.81	544.03

**Table D.2:** The CPU time (in seconds) for each iteration when simulating three-dimensional circular quantum dots of 2-70 electrons. The time was clocked for  $M = 2^{20} = 1,048,576$  Monte Carlo cycles, and to get accurate times we took the average over at least four independent runs with thousands of iterations.

	2P	8P	20P	40P	70P
RBM	7.69	20.92	59.67	171.84	586.39
RBM+SJ	8.95	26.86	94.64	270.92	-
RBM+PJ	8.87	26.36	91.40	293.25	-
VMC	6.70	20.99	62.54	185.65	486.02

## D.2 Ground-state energy

In this section, we present the ground-state energy included the distribution between kinetic energy, external potential energy and interaction energy for quantum dots containing up to 20 particles in two and three dimensions and with frequencies spanning from  $\omega = 0.01$  to  $\omega = 10$ . These results are interesting for the validation of the virial theorem, described in section 2.6, but they also makes it possible to compare how the energy spread in two and three dimensions for an equal number of particles.

We have created respective tables for each of our four methods VMC, RBM, RBM+SJ and RBM+PJ in two and three dimensions, resulting in eight tables in total. We first present VMC, RBM, RBM+SJ and RBM+PJ for two dimensions in that order, and then repeat the same order for three dimensions.

### D.2.1 Two dimensions

**Table D.3:** This table shows how the total energy ( $\langle \hat{H} \rangle$ ) is distributed between kinetic energy ( $\langle \hat{T} \rangle$ ), external potential energy ( $\langle \hat{V}_{\text{ext}} \rangle$ ) and interaction energy ( $\langle \hat{V}_{\text{int}} \rangle$ ) of two-dimensional circular quantum dots for a wide range of frequencies  $\omega$ . A standard variational Monte-Carlo wave function is used. The energy is given in units of  $\hbar$ , and the numbers in parenthesis are the statistical uncertainties in the last digit.

	$\omega$	$\langle \hat{H} \rangle$	$\langle \hat{T} \rangle$	$\langle \hat{V}_{\text{ext}} \rangle$	$\langle \hat{V}_{\text{int}} \rangle$
2	0.01	0.074070(8)	0.00947(3)	0.02732(5)	0.03728(4)
	0.1	0.44129(1)	0.09117(9)	0.1789(1)	0.17119(9)
	0.28	1.02192(1)	0.2477(1)	0.4256(2)	0.3487(1)
	0.5	1.65974(1)	0.4346(2)	0.7057(2)	0.5195(2)
	1.0	2.99936(1)	0.8523(3)	1.3149(3)	0.8321(2)
	2.0	5.497567(8)	2.0574(3)	1.9931(3)	1.4471(3)
	3.0	7.881906(9)	2.5028(5)	3.6911(6)	1.6880(3)
	5.0	12.504211(8)	5.1819(6)	4.7651(6)	2.5572(5)
	10.0	23.65035(1)	9.529(1)	10.538(1)	3.583(1)
	-	-	-	-	-
6	0.01	0.6982(1)	0.02735(7)	0.2427(3)	0.4281(3)
	0.1	3.5695(1)	0.3201(3)	1.2934(6)	1.9560(5)
	0.28	7.6219(1)	0.9105(4)	2.8821(9)	3.8292(7)
	0.5	11.8104(2)	1.6710(7)	4.535(1)	5.6045(9)
	1.0	20.1918(2)	3.405(1)	8.046(1)	8.741(1)
	2.0	35.0734(3)	7.751(2)	13.846(3)	13.476(2)
	3.0	48.8728(4)	12.016(2)	19.682(4)	17.175(2)
	5.0	74.9356(5)	20.796(4)	31.043(6)	23.097(3)
	10.0	136.1522(7)	43.712(9)	58.20(1)	34.240(5)
	-	-	-	-	-
12	0.01	2.4972(3)	0.05506(2)	0.858(1)	1.584(1)
	0.1	12.3196(3)	0.6885(6)	4.393(2)	7.238(2)
	0.28	25.7049(4)	2.090(1)	9.355(2)	14.260(2)
	0.5	39.2421(5)	3.939(2)	14.564(3)	20.739(3)
	1.0	65.7026(4)	9.246(2)	23.079(3)	33.378(3)
	2.0	111.8377(3)	19.678(2)	41.349(3)	50.811(2)
	3.0	154.206(1)	29.701(6)	60.74(1)	63.763(7)
	5.0	232.818(2)	56.157(9)	88.18(1)	88.478(9)
	10.0	415.056(4)	112.99(2)	173.91(3)	128.15(1)
	-	-	-	-	-
20	0.01	6.2097(8)	0.1005(4)	2.270(3)	3.839(3)
	0.1	30.086(1)	1.243(1)	10.587(4)	18.257(4)
	0.28	62.0755(7)	3.902(2)	22.228(5)	35.946(4)
	0.5	94.0433(9)	7.823(3)	33.938(6)	52.282(5)
	1.0	155.8900(4)	17.921(2)	54.076(3)	83.893(3)
	2.0	262.5339(9)	38.402(3)	95.681(7)	128.451(5)
	3.0	358.927(1)	61.017(5)	133.99(1)	163.924(7)
	5.0	542.680(7)	127.77(2)	177.89(2)	237.12(2)
	10.0	944.1(1)	234.4(4)	383.9(3)	325.9(1)
	-	-	-	-	-

**Table D.4:** This table shows how the total energy ( $\langle \hat{H} \rangle$ ) is distributed between kinetic energy ( $\langle \hat{T} \rangle$ ), external potential energy ( $\langle \hat{V}_{\text{ext}} \rangle$ ) and interaction energy ( $\langle \hat{V}_{\text{int}} \rangle$ ) of two-dimensional circular quantum dots for a wide range of frequencies  $\omega$ . A plain restricted Boltzmann machine wave function is used. The energy is given in units of  $\hbar$ , and the numbers in parenthesis are the statistical uncertainties in the last digit.

	$\omega$	$\langle \hat{H} \rangle$	$\langle \hat{T} \rangle$	$\langle \hat{V}_{\text{ext}} \rangle$	$\langle \hat{V}_{\text{int}} \rangle$
2	0.01	0.078643(5)	0.009835(3)	0.031930(8)	0.03688(1)
	0.1	0.4743(1)	0.08102(8)	0.2082(2)	0.1851(2)
	0.28	1.0707(2)	0.2047(1)	0.4678(3)	0.3983(3)
	0.5	1.7234(2)	0.3739(2)	0.7611(3)	0.5884(3)
	1.0	3.0805(2)	0.7915(2)	1.3673(4)	0.9217(3)
	2.0	5.5936(3)	1.6377(4)	2.5507(5)	1.4051(4)
	3.0	7.9968(4)	2.2346(5)	4.0201(7)	1.7422(4)
	5.0	12.6070(4)	3.8768(7)	6.4292(9)	2.3010(4)
	10.0	23.7748(7)	9.063(3)	11.132(4)	3.580(1)
	<hr/>				
6	0.01	0.7072(5)	0.033(2)	0.2660(4)	0.4080(6)
	0.1	3.7337(5)	0.3251(3)	1.4070(9)	2.002(1)
	0.28	7.9273(9)	0.8684(6)	3.009(1)	4.050(2)
	0.5	12.241(1)	1.611(1)	4.709(2)	5.921(2)
	1.0	20.716(1)	3.391(1)	7.914(3)	9.411(2)
	2.0	36.383(5)	8.311(7)	13.705(8)	14.367(6)
	3.0	49.415(1)	10.309(3)	21.456(4)	17.649(2)
	5.0	76.801(6)	23.50(1)	27.33(1)	25.967(7)
	10.0	137.338(4)	45.25(1)	55.75(1)	36.336(6)
	<hr/>				
12	0.01	2.5106(8)	0.0682(2)	0.893(1)	1.549(1)
	0.1	12.703(2)	0.790(2)	4.681(3)	7.232(3)
	0.28	26.564(3)	2.254(2)	9.635(3)	14.675(4)
	0.5	40.442(3)	4.116(2)	14.868(4)	21.458(4)
	1.0	67.614(3)	8.953(3)	25.207(6)	33.455(5)
	2.0	115.214(5)	20.760(6)	43.69(1)	50.764(7)
	3.0	158.145(6)	33.020(8)	59.72(1)	65.407(9)
	5.0	239.527(8)	58.22(1)	93.92(2)	87.39(1)
	10.0	435.36(2)	114.61(2)	200.13(4)	120.62(1)
	<hr/>				
20	0.01	6.217(2)	0.1236(4)	2.244(2)	3.849(2)
	0.1	32.308(5)	1.708(2)	7.680(4)	22.919(7)
	0.28	63.788(4)	4.443(3)	22.707(6)	36.638(7)
	0.5	96.491(4)	8.144(3)	34.953(8)	53.394(8)
	1.0	159.645(5)	17.12(5)	58.74(5)	83.397(9)
	2.0	269.086(8)	43.262(8)	95.17(2)	130.65(1)
	3.0	362.52(1)	57.005(9)	148.08(2)	157.43(1)
	5.0	551.21(2)	115.12(2)	219.12(4)	216.97(2)
	10.0	1003.71(4)	218.87(4)	492.87(8)	291.97(3)
	<hr/>				

**Table D.5:** This table shows how the total energy ( $\langle \hat{H} \rangle$ ) is distributed between kinetic energy ( $\langle \hat{T} \rangle$ ), external potential energy ( $\langle \hat{V}_{\text{ext}} \rangle$ ) and interaction energy ( $\langle \hat{V}_{\text{int}} \rangle$ ) of two-dimensional circular quantum dots for a wide range of frequencies  $\omega$ . A restricted Boltzmann machine wave function with a simple Jastrow factor is used. The energy is given in units of  $\hbar$ , and the numbers in parenthesis are the statistical uncertainties in the last digit.

	$\omega$	$\langle \hat{H} \rangle$	$\langle \hat{T} \rangle$	$\langle \hat{V}_{\text{ext}} \rangle$	$\langle \hat{V}_{\text{int}} \rangle$
2	0.01	0.075267(3)	0.00738(2)	0.03071(8)	0.03718(8)
	0.1	0.44858(6)	0.07539(8)	0.1990(2)	0.1742(1)
	0.28	1.03470(7)	0.2163(1)	0.4547(2)	0.3637(2)
	0.5	1.67636(8)	0.3967(2)	0.7366(3)	0.5430(2)
	1.0	3.0213(1)	0.8120(2)	1.3623(4)	0.8469(3)
	2.0	5.5254(1)	1.6572(3)	2.5447(5)	1.3234(3)
	3.0	7.9103(2)	2.5627(7)	3.664(1)	1.6840(5)
	5.0	12.5322(2)	4.369(1)	5.890(2)	2.2838(7)
	10.0	23.6773(3)	9.062(3)	11.208(3)	3.408(1)
	-	-	-	-	-
6	0.01	0.7006(3)	0.0376(2)	0.2467(4)	0.4163(4)
	0.1	3.6458(4)	0.2376(3)	1.3652(9)	2.043(1)
	0.28	7.7347(4)	0.7603(4)	3.000(1)	3.974(1)
	0.5	11.9392(5)	1.4801(5)	4.678(1)	5.781(1)
	1.0	-	-	-	-
	2.0	35.2446(8)	7.098(2)	14.587(3)	13.560(2)
	3.0	49.050(1)	11.164(3)	20.6469(5)	17.240(3)
	5.0	75.116(1)	19.661(5)	32.283(7)	23.172(3)
	10.0	136.331(2)	41.65(1)	60.53(1)	34.152(5)
	-	-	-	-	-
12	0.01	2.4950(5)	0.07(2)	0.845(4)	1.58(2)
	0.1	12.5964(7)	0.4520(4)	4.644(1)	7.500(1)
	0.28	26.340(2)	1.917(2)	10.006(4)	14.417(4)
	0.5	39.6340(7)	3.4852(5)	14.948(2)	21.201(2)
	1.0	66.1898(8)	7.8777(8)	25.822(2)	32.490(2)
	2.0	112.502(2)	18.118(4)	44.118(4)	50.201(6)
	3.0	154.521(3)	28.050(5)	63.84(1)	62.634(6)
	5.0	234.110(6)	52.86(1)	94.50(2)	86.75(1)
	10.0	415.384(7)	108.57(2)	181.90(3)	124.92(1)
	-	-	-	-	-
20	0.01	6.239(2)	0.1372(6)	2.184(2)	3.919(3)
	0.1	30.624(3)	1.487(2)	10.893(5)	18.243(5)
	0.28	62.786(3)	3.190(2)	22.782(7)	36.814(6)
	0.5	94.755(3)	6.709(2)	34.845(7)	53.200(6)
	1.0	156.816(4)	15.340(3)	59.931(9)	81.545(7)
	2.0	265.66(9)	39.31(8)	95.78(1)	130.57(2)
	3.0	360.630(6)	58.36(1)	141.54(2)	160.72(2)
	5.0	543.06(1)	112.38(2)	210.52(4)	220.15(2)
	10.0	952.71(2)	237.65(4)	392.06(7)	323.00(3)
	-	-	-	-	-

**Table D.6:** This table shows how the total energy ( $\langle \hat{H} \rangle$ ) is distributed between kinetic energy ( $\langle \hat{T} \rangle$ ), external potential energy ( $\langle \hat{V}_{\text{ext}} \rangle$ ) and interaction energy ( $\langle \hat{V}_{\text{int}} \rangle$ ) of two-dimensional circular quantum dots for a wide range of frequencies  $\omega$ . A restricted Boltzmann machine with Padé-Jastrow wave function is used. The energy is given in units of  $\hbar$ , and the numbers in parenthesis are the statistical uncertainties in the last digit.

	$\omega$	$\langle \hat{H} \rangle$	$\langle \hat{T} \rangle$	$\langle \hat{V}_{\text{ext}} \rangle$	$\langle \hat{V}_{\text{int}} \rangle$
2	0.01	0.074107(8)	0.01031(3)	0.02703(4)	0.03677(3)
	0.1	0.440975(8)	0.09223(9)	0.1757(1)	0.17304(9)
	0.28	1.021668(7)	0.2468(1)	0.4258(2)	0.3490(1)
	0.5	1.659637(6)	0.4305(2)	0.7112(2)	0.5179(2)
	1.0	2.999587(5)	0.8440(3)	1.3418(3)	0.8238(2)
	2.0	5.49475(1)	1.7234(4)	2.4657(4)	1.3057(3)
	3.0	7.87961(1)	2.3144(5)	3.9349(6)	1.6413(3)
	5.0	12.49832(1)	3.9569(7)	6.3068(8)	2.2347(4)
	10.0	23.65275(7)	9.228(3)	11.059(3)	3.366(1)
	-	-	-	-	-
6	0.01	0.6932(5)	0.031(2)	0.260(2)	0.401(1)
	0.1	3.5700(2)	0.3494(3)	1.2805(9)	1.9401(8)
	0.28	7.6203(2)	0.9519(6)	2.82(1)	3.84(1)
	0.5	11.8074(2)	1.7018(7)	4.513(1)	5.5927(9)
	1.0	20.1832(1)	3.428(1)	8.068(1)	8.687(1)
	2.0	35.0872(3)	7.670(2)	14.139(3)	13.279(2)
	3.0	48.9157(8)	10.789(5)	20.383(5)	17.743(2)
	5.0	74.9545(5)	20.402(5)	31.744(7)	22.809(3)
	10.0	136.1738(8)	42.66(1)	59.71(1)	33.799(5)
	-	-	-	-	-
12	0.01	2.5019(4)	0.0699(2)	0.893(1)	1.539(1)
	0.1	12.361(1)	0.797(1)	4.394(3)	7.169(3)
	0.28	25.7461(6)	2.415(1)	9.050(2)	14.281(2)
	0.5	39.2661(6)	4.262(2)	14.277(2)	20.728(2)
	1.0	65.7911(5)	8.537(3)	25.197(4)	32.067(3)
	2.0	111.9426(5)	17.817(3)	46.532(4)	47.593(3)
	3.0	154.1234(7)	23.34(7)	70.39(7)	60.397(4)
	5.0	233.633(4)	58.33(1)	84.76(1)	90.537(9)
	10.0	415.943(9)	124.13(2)	157.92(3)	133.89(1)
	-	-	-	-	-
20	0.01	6.210(1)	0.1208(5)	2.189(2)	3.900(2)
	0.1	30.156(1)	1.574(1)	10.473(3)	18.109(3)
	0.28	62.210(1)	4.657(2)	21.227(4)	36.106(4)
	0.5	94.127(1)	8.249(3)	33.543(5)	52.335(4)
	1.0	156.099(1)	16.768(6)	58.513(8)	80.818(6)
	2.0	262.598(1)	34.758(6)	108.546(9)	119.293(7)
	3.0	359.220(2)	37.1(7)	171.5(7)	150.646(8)
	5.0	539.13(1)	120.09(2)	188.45(2)	230.58(1)
	10.0	947.33(2)	257.67(5)	348.35(6)	341.31(3)
	-	-	-	-	-

### D.2.2 Three dimensions

**Table D.7:** This table shows how the total energy ( $\langle \hat{H} \rangle$ ) is distributed between kinetic energy ( $\langle \hat{T} \rangle$ ), external potential energy ( $\langle \hat{V}_{\text{ext}} \rangle$ ) and interaction energy ( $\langle \hat{V}_{\text{int}} \rangle$ ) of three-dimensional circular quantum dots for a wide range of frequencies  $\omega$ . A standard variational Monte-Carlo wave function is used. The energy is given in units of  $\hbar$ , and the numbers in parenthesis are the statistical uncertainties in the last digit.

	$\omega$	$\langle \hat{H} \rangle$	$\langle \hat{T} \rangle$	$\langle \hat{V}_{\text{ext}} \rangle$	$\langle \hat{V}_{\text{int}} \rangle$
2	0.01	0.079284(6)	0.01221(4)	0.039757(6)	0.036319(4)
	0.1	0.500083(7)	0.1263(1)	0.2082(2)	0.1656(1)
	0.28	1.201752(6)	0.3606(2)	0.5272(3)	0.3140(1)
	0.5	1.999977(5)	0.6517(3)	0.9032(3)	0.4451(1)
	1.0	3.730030(5)	1.3105(4)	1.7551(5)	0.6644(2)
	2.0	7.065911(7)	3.2766(4)	2.6932(5)	1.0961(2)
	3.0	10.31717(1)	3.8365(7)	5.2770(8)	1.2037(2)
	5.0	16.713925(4)	8.1523(8)	6.7797(9)	1.7819(2)
	10.0	32.449053(8)	14.586(2)	15.470(2)	2.3933(2)
	<hr/>				
8	0.01	1.12283(7)	0.04384(7)	0.3832(2)	0.6958(2)
	0.1	5.7126(1)	0.4930(4)	2.085(1)	3.1342(9)
	0.28	12.2050(2)	1.5332(7)	4.630(2)	6.041(1)
	0.5	18.96747(8)	3.2098(4)	6.7892(8)	8.9647(6)
	1.0	32.6863(2)	6.244(2)	13.378(3)	13.064(1)
	2.0	57.41(2)	15.35(4)	21.47(3)	20.59(2)
	3.0	80.6193(3)	22.281(5)	33.286(7)	25.052(3)
	5.0	124.9024(4)	38.713(8)	52.89(1)	33.300(4)
	10.0	230.1668(7)	81.03(2)	100.56(2)	48.573(6)
	<hr/>				
20	0.01	5.6428(3)	0.1621(4)	1.923(2)	3.558(2)
	0.1	27.3152(5)	1.247(1)	9.392(3)	16.676(3)
	0.28	56.4386(5)	3.991(2)	20.125(5)	32.322(4)
	0.5	85.7197(6)	7.868(2)	31.383(6)	46.469(5)
	1.0	142.9561(7)	17.29(2)	54.45(3)	71.218(6)
	2.0	242.0320(6)	42.246(3)	86.317(9)	113.469(6)
	3.0	332.6976(6)	67.976(5)	119.95(1)	144.772(7)
	5.0	509.45(1)	137.93(2)	163.28(3)	208.24(2)
	10.0	902.58(2)	288.99(4)	310.87(5)	302.72(3)
	<hr/>				

**Table D.8:** This table shows how the total energy ( $\langle \hat{H} \rangle$ ) is distributed between kinetic energy ( $\langle \hat{T} \rangle$ ), external potential energy ( $\langle \hat{V}_{\text{ext}} \rangle$ ) and interaction energy ( $\langle \hat{V}_{\text{int}} \rangle$ ) of three-dimensional circular quantum dots for a wide range of frequencies  $\omega$ . A plain restricted Boltzmann machine wave function is used. The energy is given in units of  $\hbar$ , and the numbers in parenthesis are the statistical uncertainties in the last digit.

	$\omega$	$\langle \hat{H} \rangle$	$\langle \hat{T} \rangle$	$\langle \hat{V}_{\text{ext}} \rangle$	$\langle \hat{V}_{\text{int}} \rangle$
2	0.01	0.85193(5)	0.014853(4)	0.03141(9)	0.03893(1)
	0.1	0.5177(1)	0.1249(1)	0.2065(2)	0.1863(2)
	0.28	1.2261(1)	0.35(2)	0.53(2)	0.3488(6)
	0.5	2.0269(1)	0.6595(3)	0.8778(4)	0.4896(2)
	1.0	3.7574(1)	1.3224(5)	1.7215(5)	0.7136(2)
	2.0	7.0870(1)	2.7338(6)	3.3183(7)	1.0350(2)
	3.0	10.2981(2)	3.7507(8)	5.2896(9)	1.2578(2)
	5.0	16.7018(1)	6.211(1)	8.890(1)	1.6012(2)
	10.0	32.2186(2)	7.879(3)	22.306(3)	2.0343(2)
	<hr/>				
8	0.01	1.1350(1)	0.0626(2)	0.3951(7)	0.6774(7)
	0.1	5.8910(6)	0.6480(6)	2.075(2)	3.168(2)
	0.28	12.650(1)	1.931(1)	4.641(2)	6.078(2)
	0.5	19.680(2)	3.601(2)	7.289(4)	8.786(3)
	1.0	33.305(1)	7.032(2)	12.267(3)	14.006(2)
	2.0	58.889(3)	16.976(4)	19.717(5)	22.195(3)
	3.0	81.648(3)	23.987(6)	31.157(8)	26.504(4)
	5.0	126.03(9)	42.68(6)	47.58(5)	35.77(2)
	10.0	231.410(6)	85.62(2)	95.00(2)	50.789(7)
	<hr/>				
20	0.01	5.6448(4)	0.1624(4)	1.955(2)	3.527(2)
	0.1	27.9277(5)	1.7925(5)	9.676(2)	16.459(2)
	0.28	57.822(1)	5.240(1)	20.384(4)	32.198(3)
	0.5	87.798(5)	9.635(5)	32.12(1)	46.047(9)
	1.0	147.03(1)	1(4)	79(5)	66(1)
	2.0	250.159(7)	49.76(1)	83.69(2)	116.71(1)
	3.0	335.440(5)	64.59(1)	130.27(2)	140.578(9)
	5.0	524.94(2)	121.39(3)	225.61(5)	177.94(2)
	10.0	1005.24(6)	225.76(5)	552.3(1)	227.20(2)
	<hr/>				

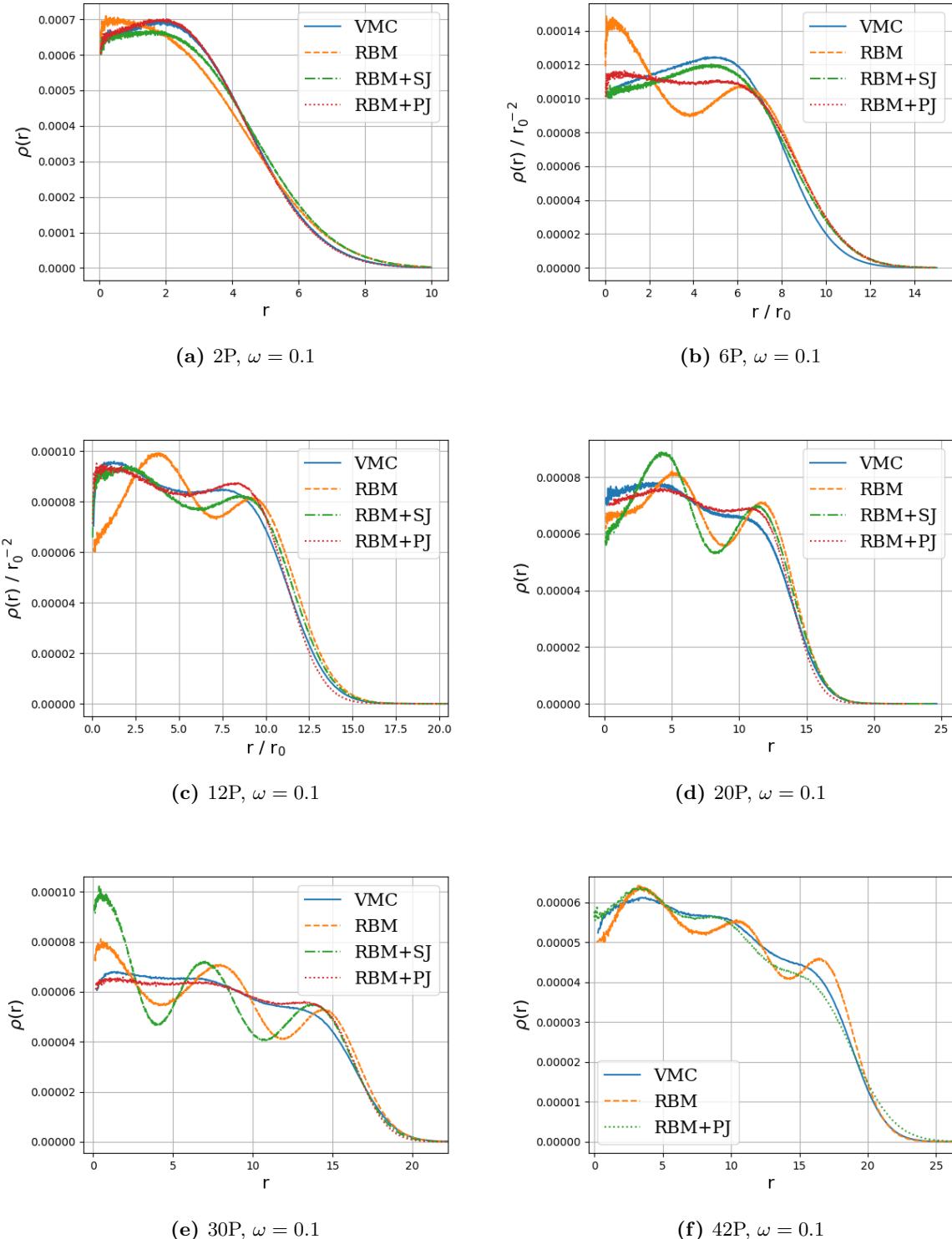
**Table D.9:** This table shows how the total energy ( $\langle \hat{H} \rangle$ ) is distributed between kinetic energy ( $\langle \hat{T} \rangle$ ), external potential energy ( $\langle \hat{V}_{\text{ext}} \rangle$ ) and interaction energy ( $\langle \hat{V}_{\text{int}} \rangle$ ) of three-dimensional circular quantum dots for a wide range of frequencies  $\omega$ . A restricted Boltzmann machine with a simple Jastrow factor is used. The energy is given in units of  $\hbar$ , and the numbers in parenthesis are the statistical uncertainties in the last digit.

	$\omega$	$\langle \hat{H} \rangle$	$\langle \hat{T} \rangle$	$\langle \hat{V}_{\text{ext}} \rangle$	$\langle \hat{V}_{\text{int}} \rangle$
2	0.01	0.07994(2)	0.01069(3)	0.03190(8)	0.03735(8)
	0.1	0.50214(3)	0.1178(1)	0.2177(2)	0.1666(1)
	0.28	1.20475(4)	0.3497(2)	0.5326(3)	0.3225(1)
	0.5	2.00371(4)	0.6340(3)	0.9201(4)	0.4496(2)
	1.0	3.73543(4)	1.2801(4)	1.7871(5)	0.6683(2)
	2.0	7.06343(7)	2.7117(7)	3.3574(9)	0.9944(2)
	3.0	10.32289(5)	3.5281(8)	5.6147(9)	1.1801(2)
	5.0	16.7155(1)	7.035(2)	8.035(2)	1.6462(4)
	10.0	32.6045(9)	14.568(4)	15.613(5)	2.4238(6)
	-	-	-	-	-
8	0.01	1.1371(5)	0.02(1)	0.388(6)	0.73(2)
	0.1	5.7498(4)	0.4107(3)	2.113(1)	3.226(1)
	0.28	12.2492(4)	1.3909(6)	4.756(2)	6.101(1)
	0.5	19.0241(4)	2.7417(9)	7.579(2)	8.704(2)
	1.0	32.7159(6)	6.137(1)	13.440(3)	13.139(2)
	2.0	57.4473(8)	13.451(3)	24.361(5)	19.636(2)
	3.0	80.6370(9)	21.039(5)	34.888(8)	24.710(3)
	5.0	124.955(1)	37.126(9)	54.81(1)	33.020(5)
	10.0	230.149(2)	76.75(2)	105.83(2)	47.560(6)
	-	-	-	-	-
20	0.01	5.6448(4)	0.1624(4)	1.955(2)	3.527(2)
	0.1	27.470(1)	0.9593(9)	9.711(6)	16.800(5)
	0.28	56.600(1)	3.515(1)	20.616(7)	32.469(6)
	0.5	85.893(1)	7.212(2)	31.722(8)	46.958(7)
	1.0	143.209(2)	16.531(7)	54.86(1)	71.819(7)
	2.0	242.195(2)	37.591(8)	96.36(2)	108.24(1)
	3.0	333.07(6)	53.2(5)	138.0(5)	141.835(9)
	5.0	507.35(1)	119.91(2)	196.89(4)	190.55(2)
	10.0	903.79(2)	253.59(5)	372.83(7)	277.37(3)
	-	-	-	-	-

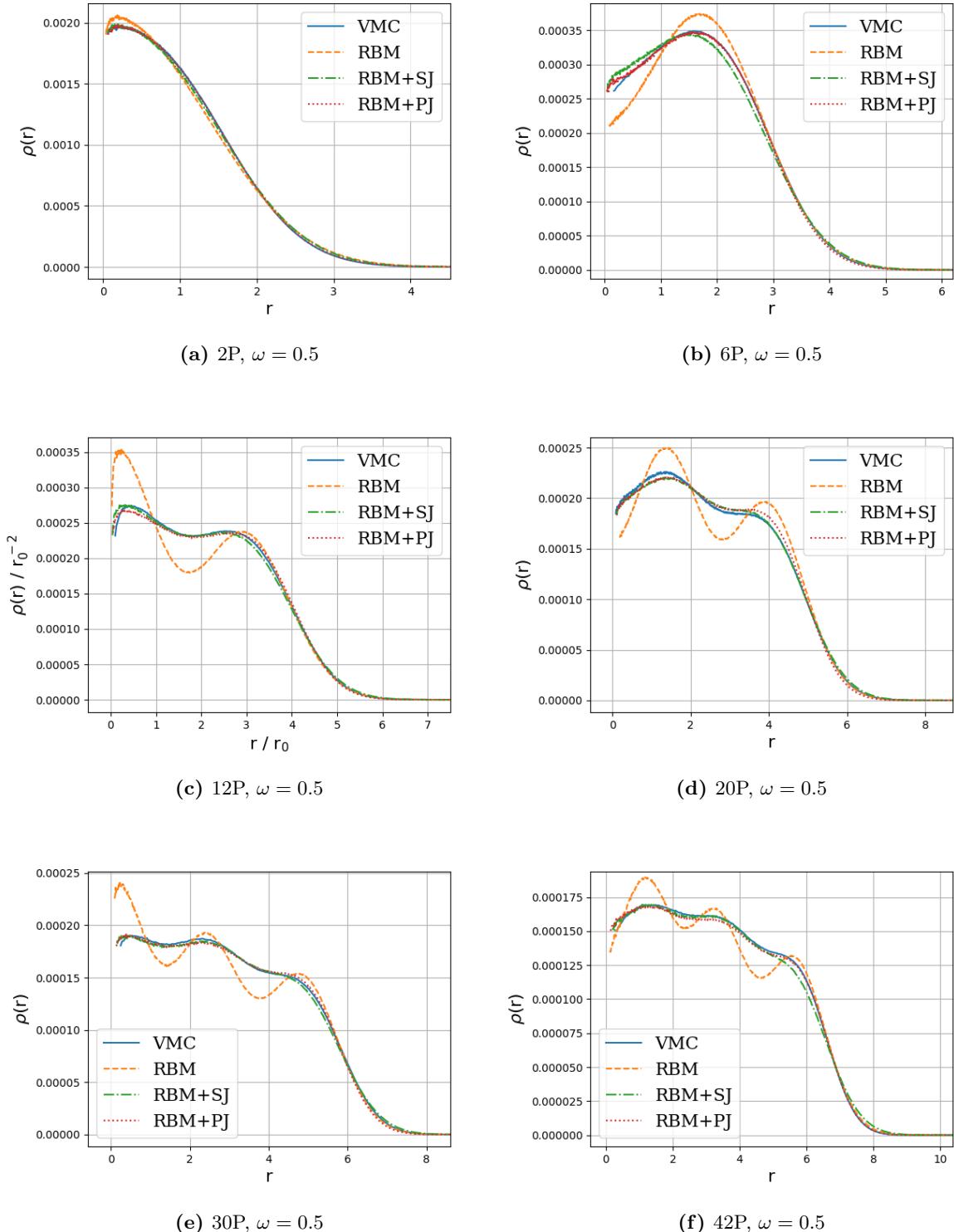
**Table D.10:** This table shows how the total energy ( $\langle \hat{H} \rangle$ ) is distributed between kinetic energy ( $\langle \hat{T} \rangle$ ), external potential energy ( $\langle \hat{V}_{\text{ext}} \rangle$ ) and interaction energy ( $\langle \hat{V}_{\text{int}} \rangle$ ) of three-dimensional circular quantum dots for a wide range of frequencies  $\omega$ . A restricted Boltzmann machine with Padé-Jastrow wave function is used. The energy is given in units of  $\hbar$ , and the numbers in parenthesis are the statistical uncertainties in the last digit.

$\omega$	$\langle \hat{H} \rangle$	$\langle \hat{T} \rangle$	$\langle \hat{V}_{\text{ext}} \rangle$	$\langle \hat{V}_{\text{int}} \rangle$
2	0.01	0.079312(6)	0.01283(4)	0.02987(6)
	0.1	0.500080(6)	0.1271(1)	0.2085(2)
	0.28	1.201710(6)	0.3624(2)	0.5253(3)
	0.5	1.999912(5)	0.6515(3)	0.9040(3)
	1.0	3.729827(5)	1.2995(4)	1.7688(5)
	2.0	7.05785(1)	2.7017(6)	3.3705(6)
	3.0	10.31271(4)	3.5410(8)	5.5989(9)
	5.0	16.7170(1)	7.012(2)	8.072(2)
	10.0	32.44255(9)	8.054(3)	22.384(3)
	-	-	-	-
8	0.01	1.1346(1)	0.0624(2)	0.3910(7)
	0.1	5.8562(9)	0.6134(7)	2.088(2)
	0.28	12.2056(2)	1.5665(7)	4.605(1)
	0.5	18.9747(2)	2.972(1)	7.344(2)
	1.0	32.6820(2)	6.266(2)	13.390(3)
	2.0	57.4148(3)	13.744(3)	24.205(5)
	3.0	80.6280(3)	18.35(2)	38.64(2)
	5.0	-	-	-
	10.0	-	-	-
	-	-	-	-
20	0.01	5.6328(3)	0.1621(4)	1.923(2)
	0.1	27.3382(8)	1.336(3)	9.408(4)
	0.28	56.4477(6)	4.157(2)	20.124(4)
	0.5	85.7153(6)	8.028(2)	31.333(6)
	1.0	142.9409(6)	17.603(3)	54.592(7)
	2.0	242.1168(8)	38.487(5)	96.23(1)
	3.0	333.027(1)	49.3(3)	152.1(3)
	5.0	506.58(1)	130.79(3)	131.618(7)
	10.0	897.68(2)	273.53(5)	203.45(2)
	-	-	-	-

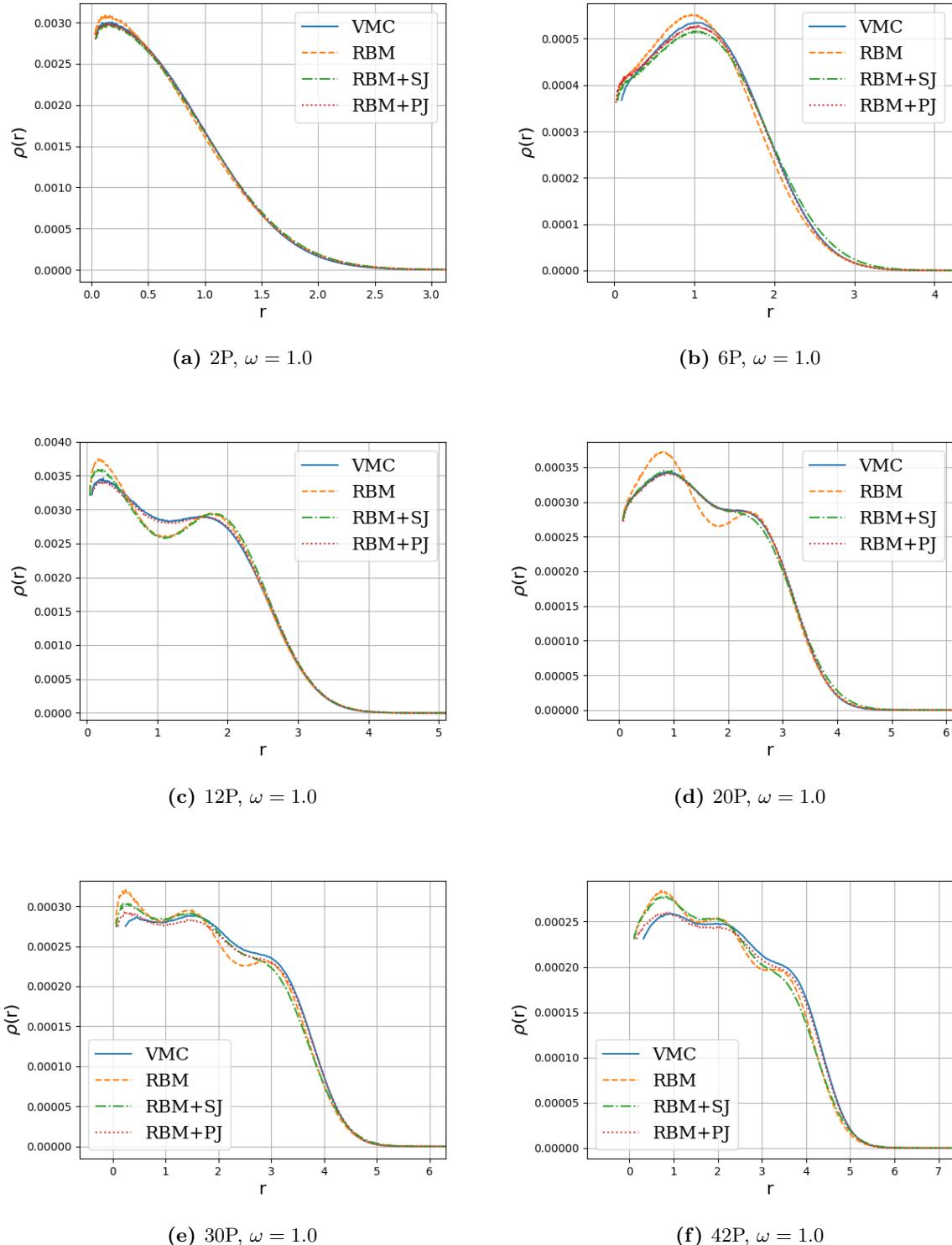
### D.3 One-body density plots



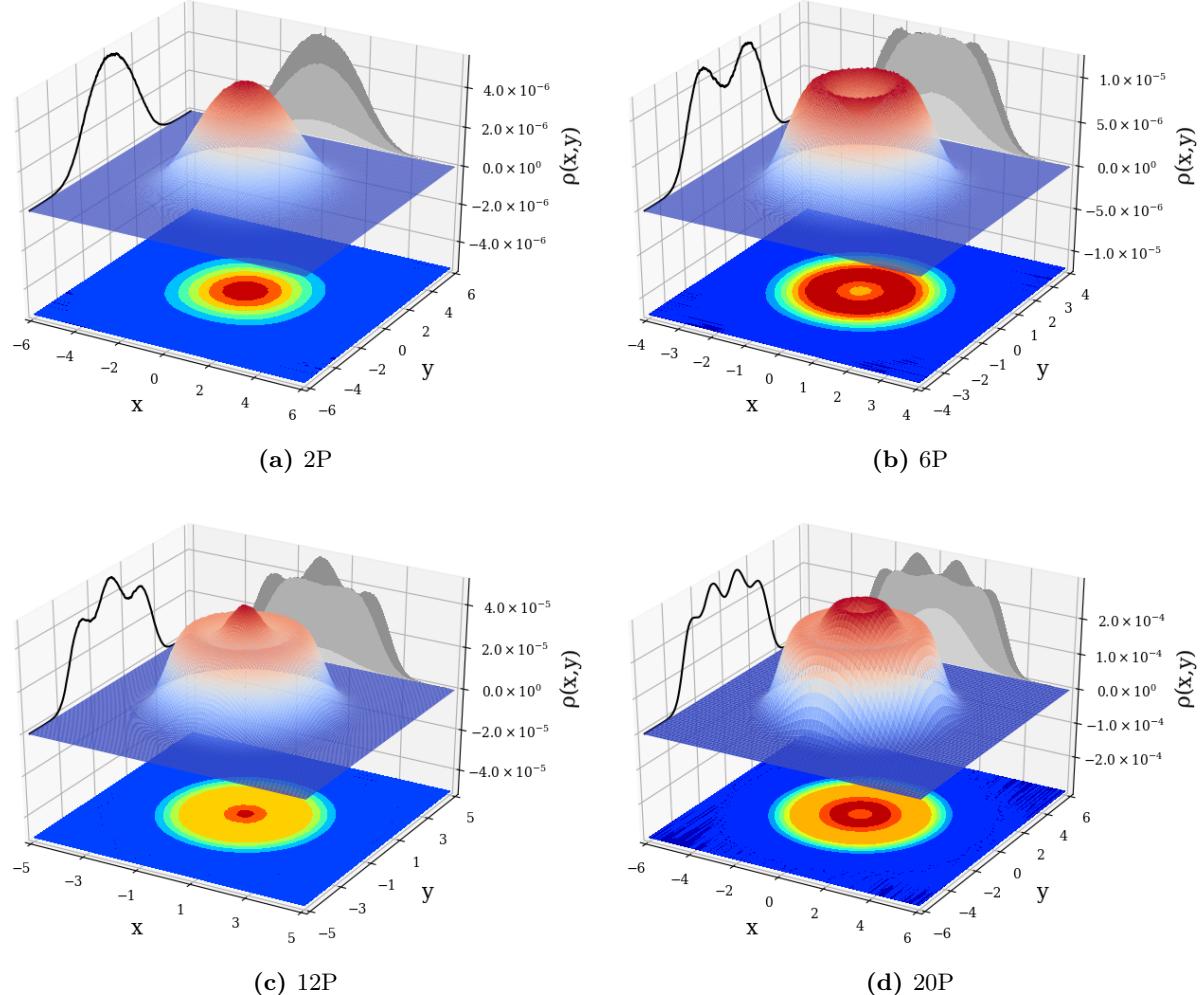
**Figure D.1:** One-body density plots for two-dimensional circular quantum dots of 2, 6, 12, 20, 30 and 42 interacting electrons for the oscillator frequency  $\omega = 0.1$ . The results were produced by standard variational Monte-Carlo (VMC), plain restricted Boltzmann machine (RBM), restricted Boltzmann machine with simple Jastrow factor (RBM+SJ) and restricted Boltzmann machine with Padé-Jastrow factor (RBM+PJ). ADAM optimizer was used, and after convergence the number of Monte-Carlo cycles was  $M = 2^{28} = 268, 435, 456$ .



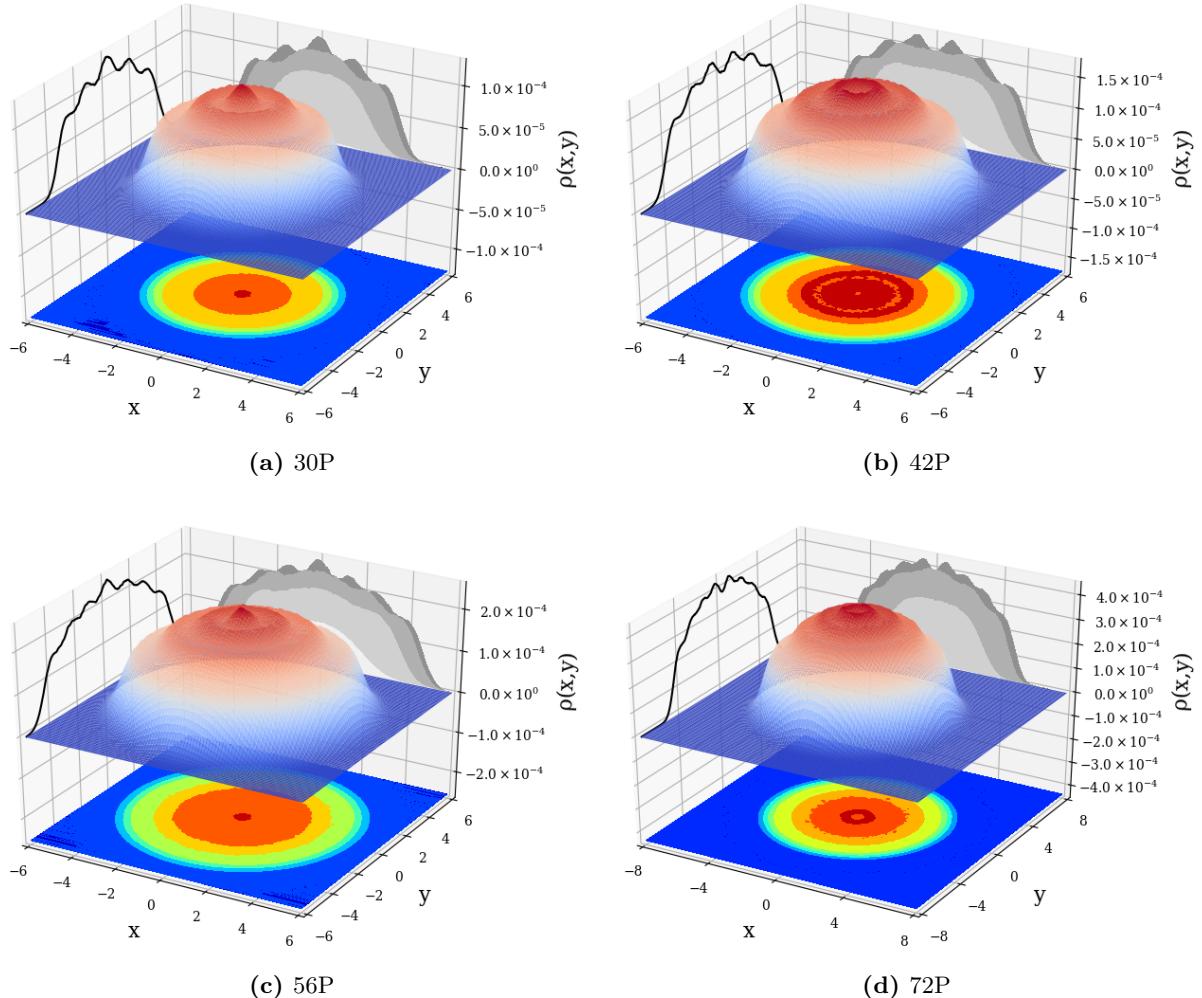
**Figure D.2:** One-body density plots for two-dimensional circular quantum dots of 2, 6, 12, 20, 30 and 42 interacting electrons for the oscillator frequency  $\omega = 0.5$ . The results were produced by standard variational Monte-Carlo (VMC), plain restricted Boltzmann machine (RBM), restricted Boltzmann machine with simple Jastrow factor (RBM+SJ) and restricted Boltzmann machine with Padé-Jastrow factor (RBM+PJ). ADAM optimizer was used, and after convergence the number of Monte-Carlo cycles was  $M = 2^{28} = 268,435,456$ .



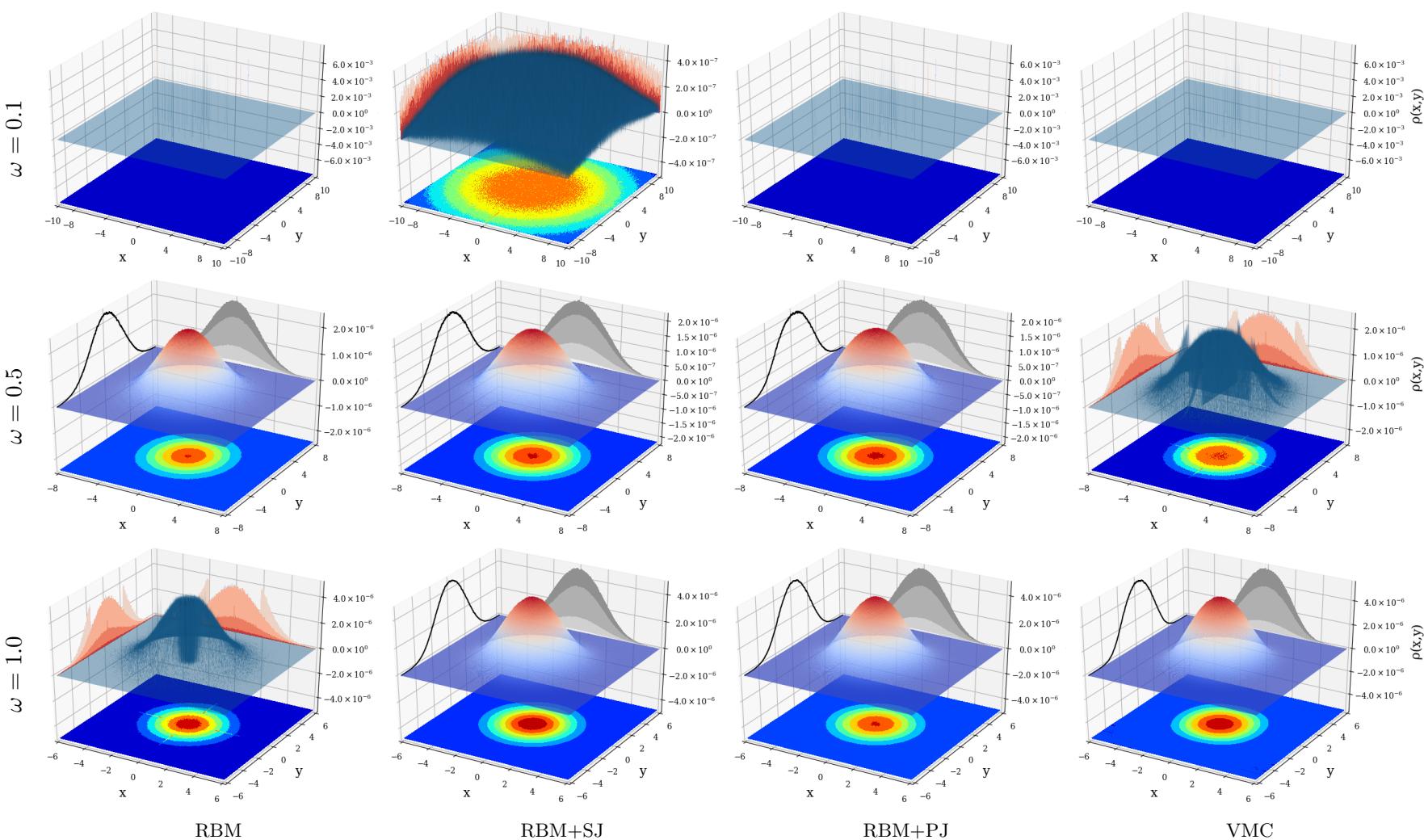
**Figure D.3:** One-body density plots for two-dimensional circular quantum dots of 2, 6, 12, 20, 30 and 42 interacting electrons for the oscillator frequency  $\omega = 1.0$ . The results were produced by standard variational Monte-Carlo (VMC), plain restricted Boltzmann machine (RBM), restricted Boltzmann machine with simple Jastrow factor (RBM+SJ) and restricted Boltzmann machine with Padé-Jastrow factor (RBM+PJ). ADAM optimizer was used, and after convergence the number of Monte-Carlo cycles was  $M = 2^{28} = 268, 435, 456$ .



**Figure D.4:** One-body density plots for two-dimensional circular quantum dots with frequency  $\omega = 1.0$ . The plots (a)-(f) present quantum dots containing 2, 6, 12, 20, 30 and 42 electrons respectively, and were all obtained using a plain restricted Boltzmann machine. ADAM optimizer was used, and after convergence the number of Monte-Carlo cycles was  $M = 2^{28} = 268,435,456$ .



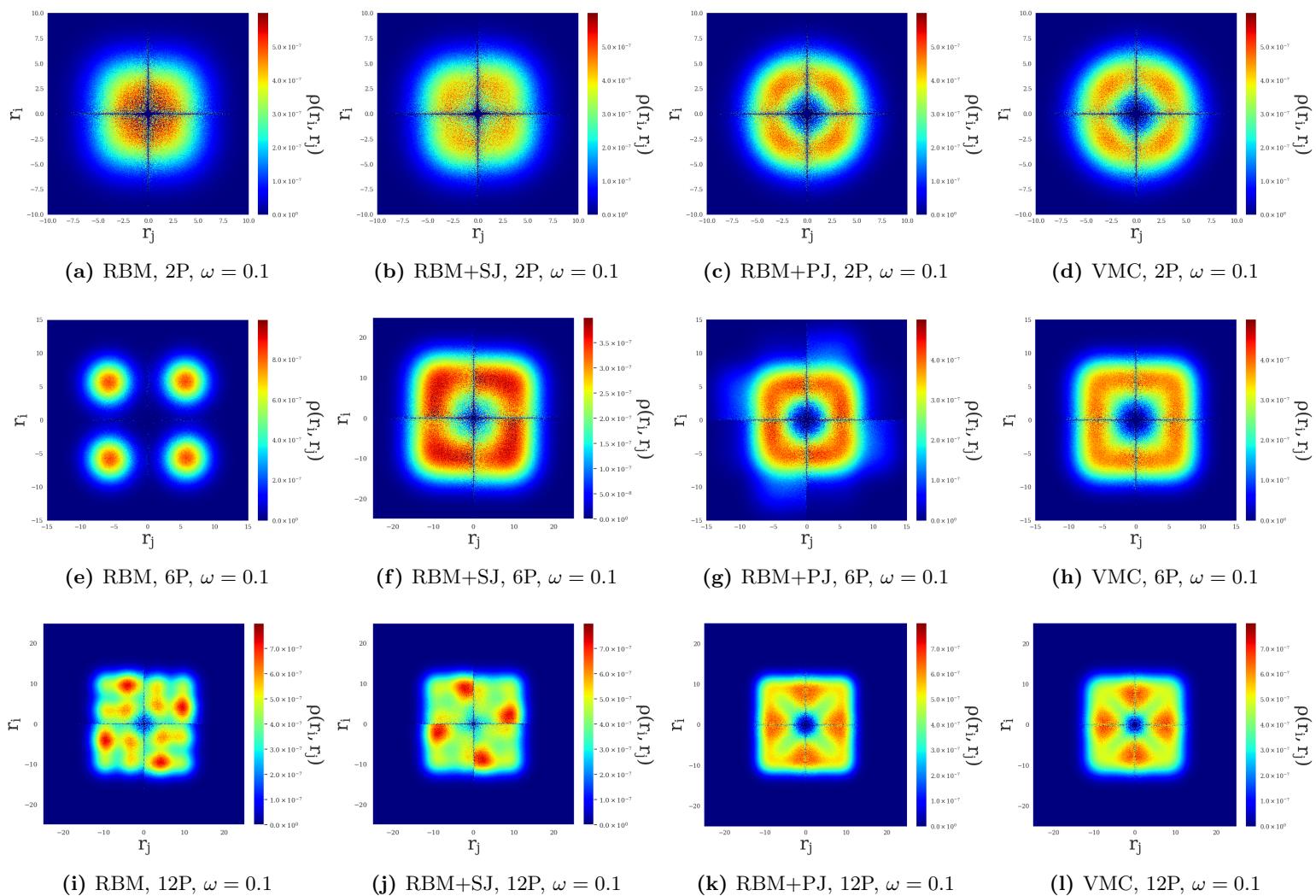
**Figure D.5:** One-body density plots for two-dimensional circular quantum dots with frequency  $\omega = 1.0$ . The plots (a)-(f) present quantum dots containing 2, 6, 12, 20, 30 and 42 electrons respectively, and were all obtained using a plain restricted Boltzmann machine. ADAM optimizer was used, and after convergence the number of Monte-Carlo cycles was  $M = 2^{28} = 268,435,456$ .

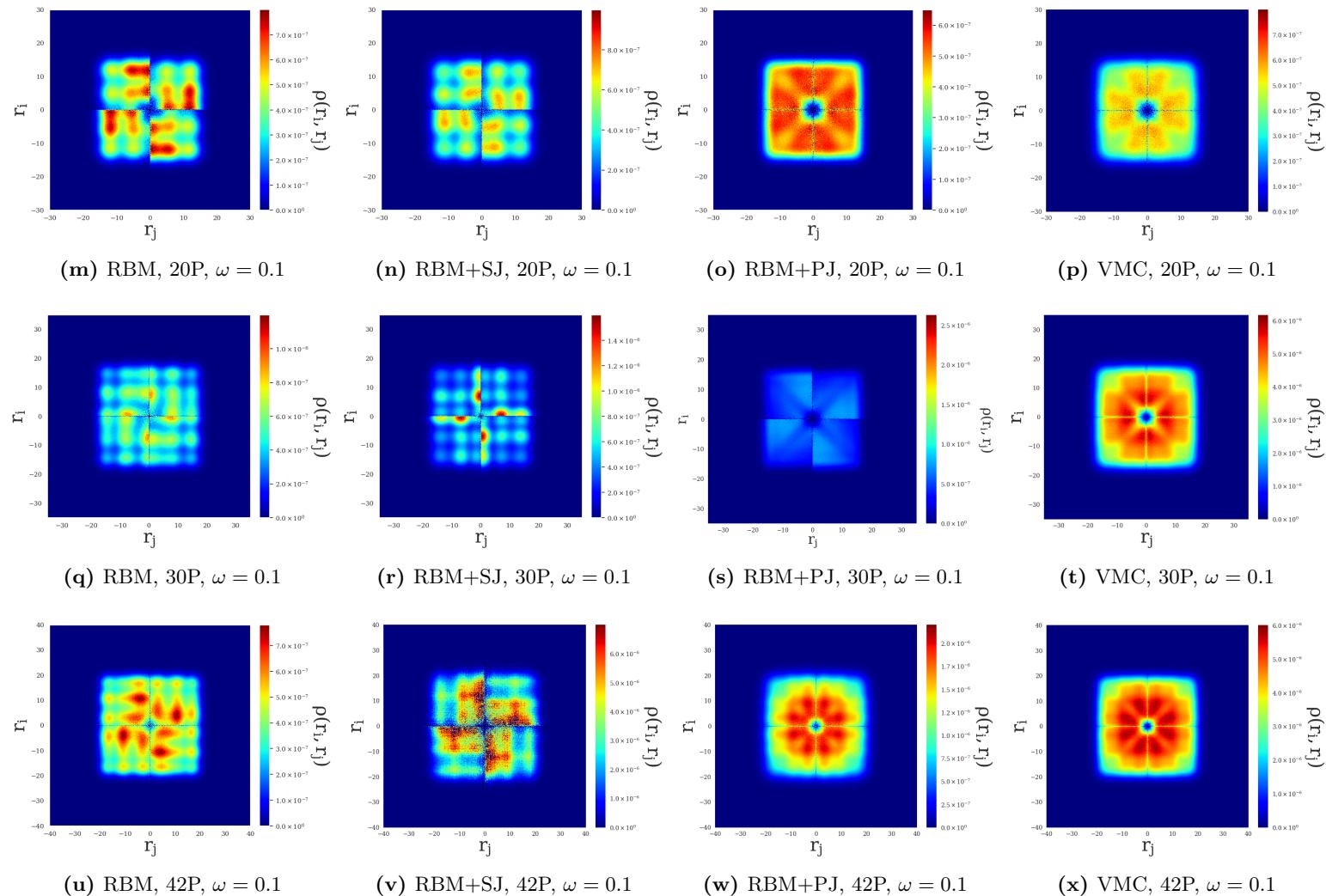


**Figure D.6:** One-body density plots for two-dimensional circular quantum dots containing 2 electrons with oscillator frequencies  $\omega = 0.1$ , 0.5 and 1.0. The density plots were produced using a plain restricted Boltzmann machine (RBM), restricted Boltzmann machine with simple Jastrow factor (RBM+SJ), restricted Boltzmann machine with Padé-Jastrow factor (RBM+PJ) and standard variational Monte-Carlo (VMC). The ADAM optimizer was used, and after convergence the number of Monte-Carlo cycles was  $M = 2^{28} = 268,435,456$ .

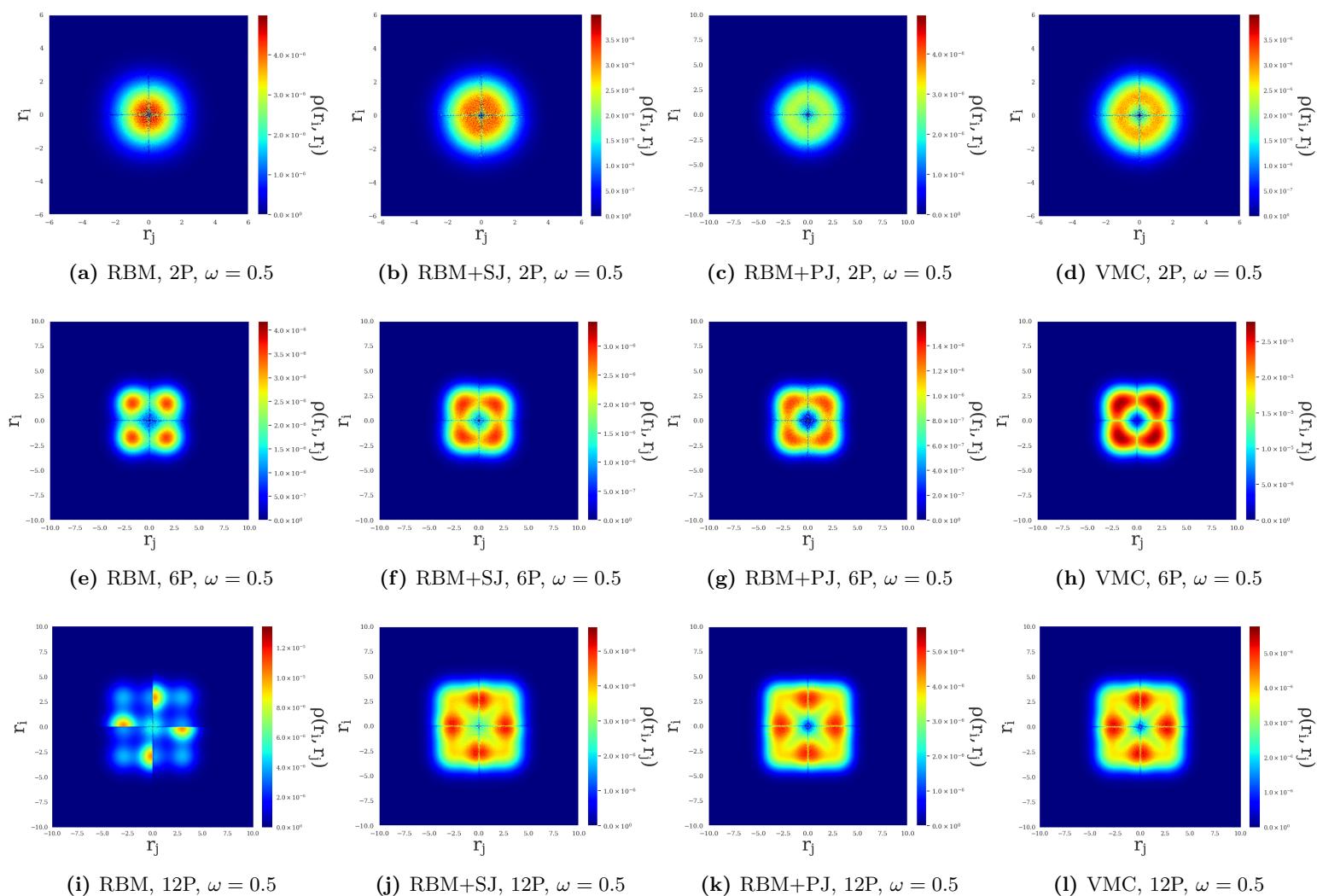
## D.4 Two-body density plots

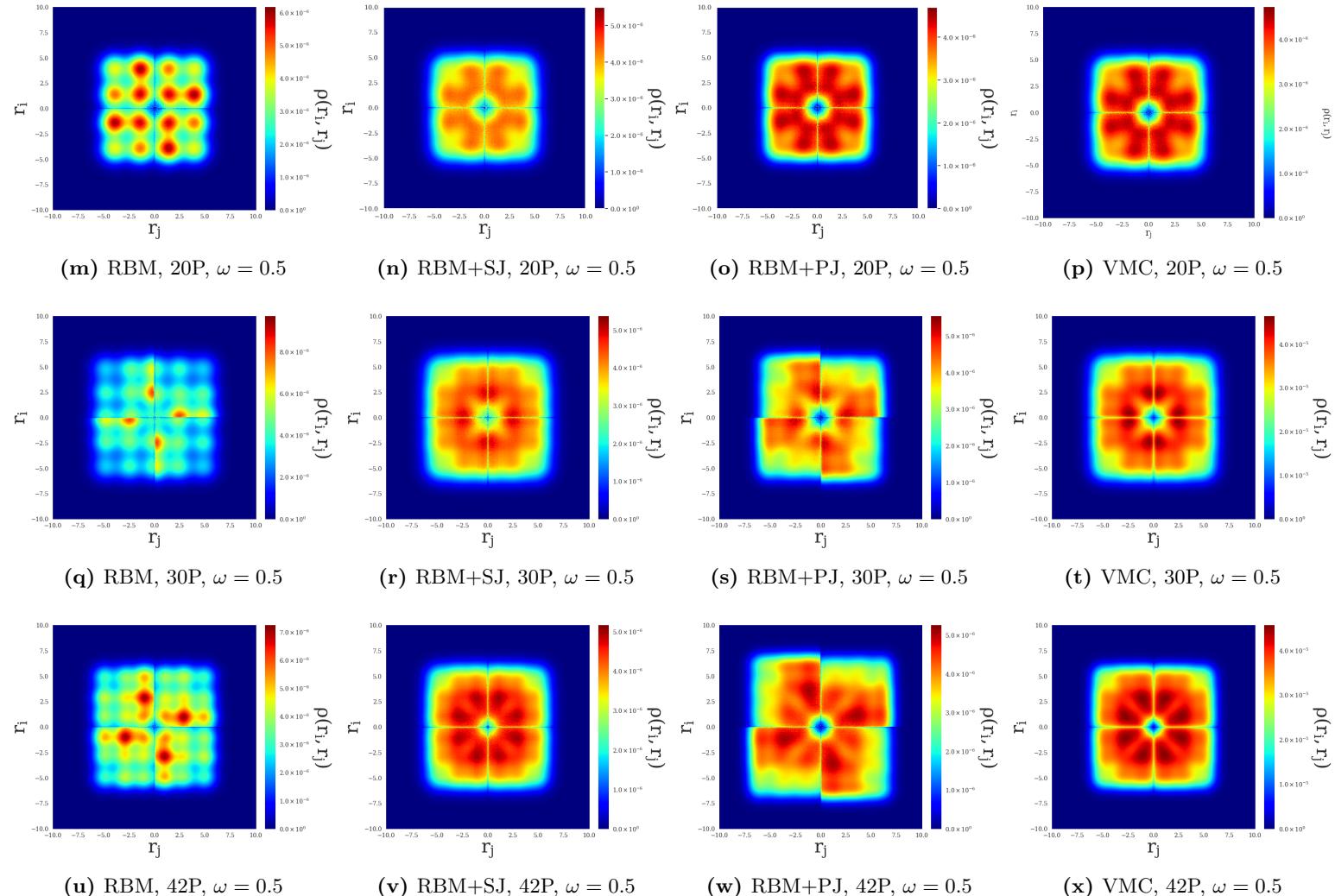
Collection of all two-body densities



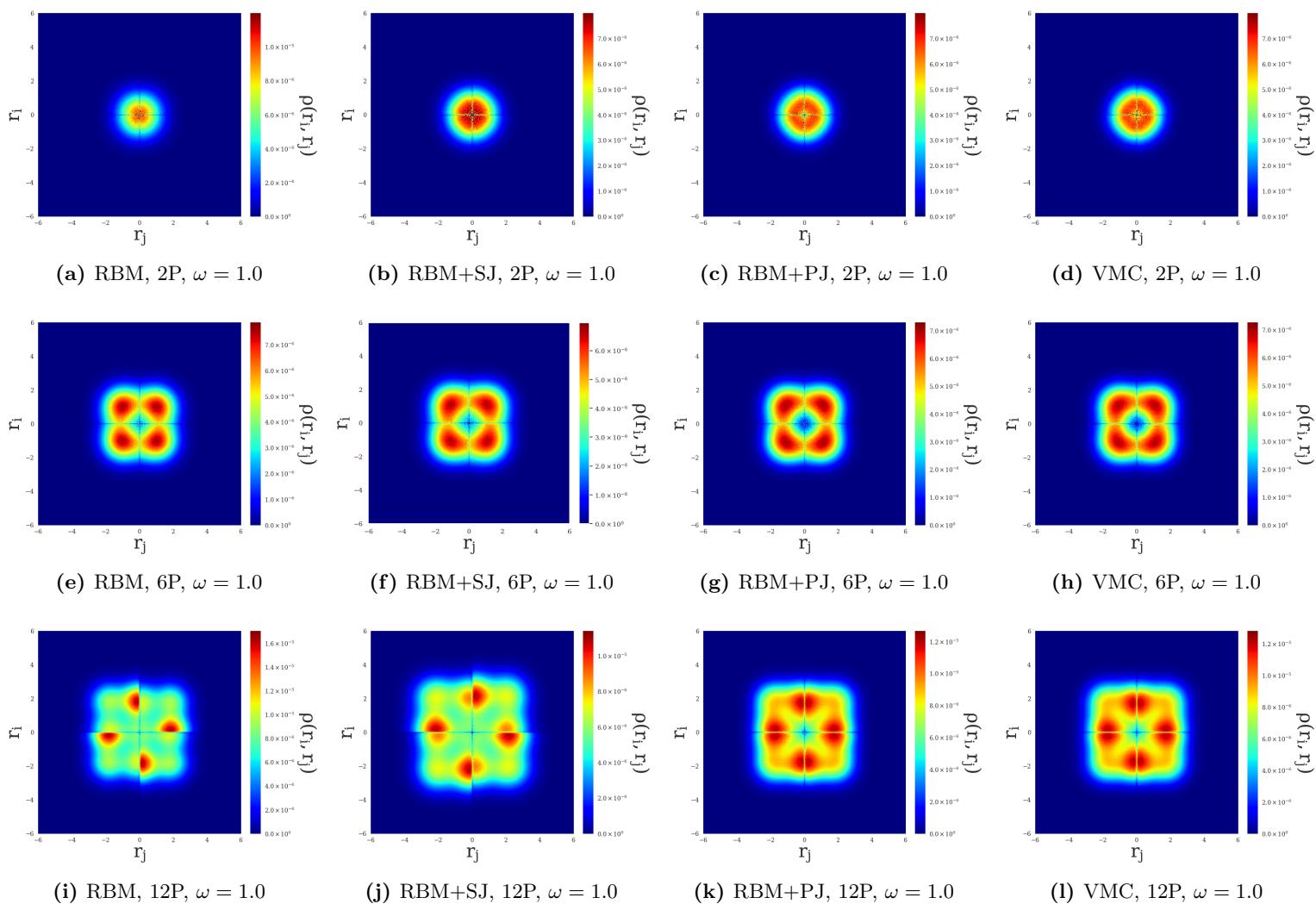


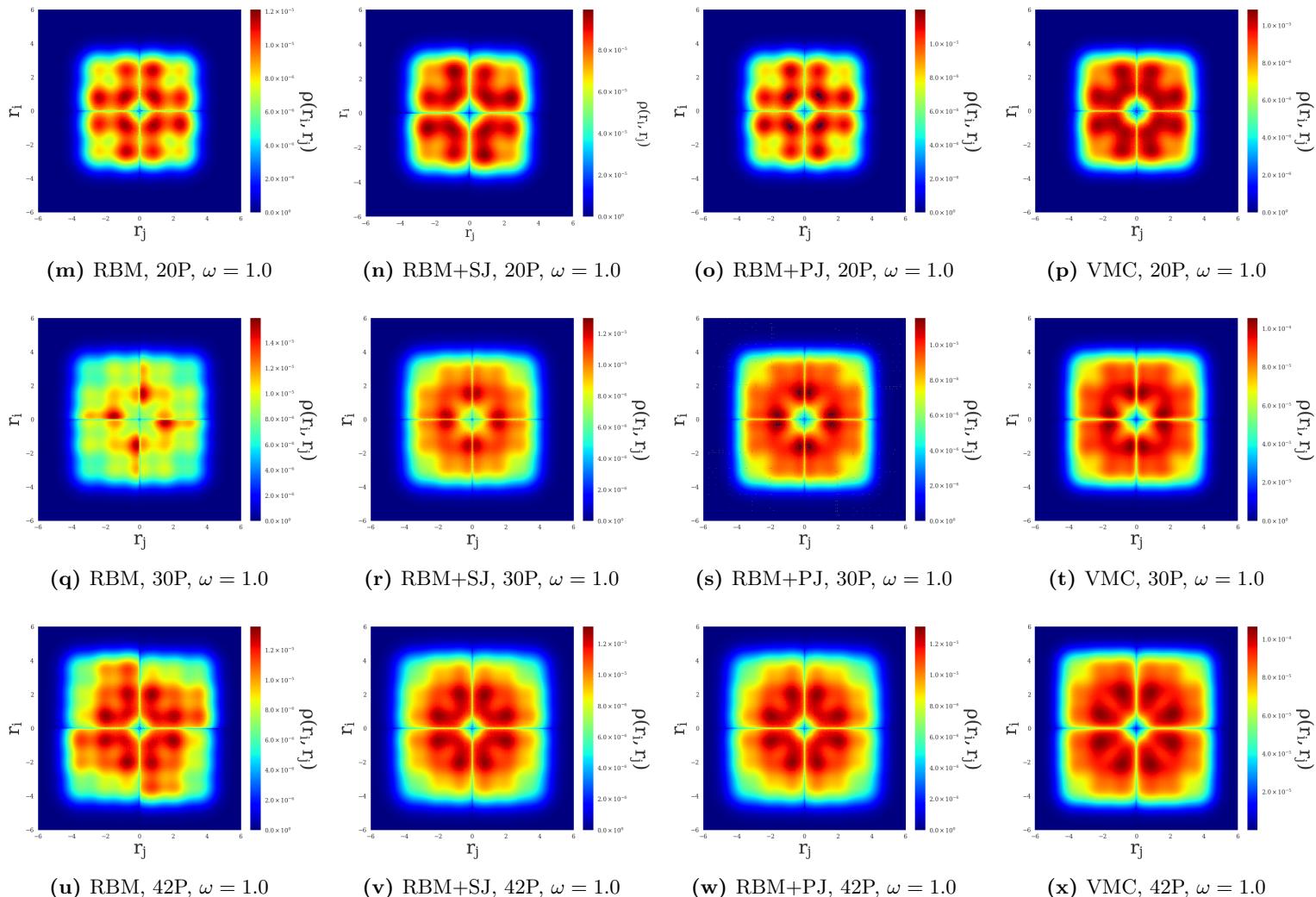
**Figure D.7:** Two-body densities for two-dimensional circular quantum dots containing up to 42 electrons with oscillator frequency  $\omega = 0.1$ . The density plots were produced using a plain restricted Boltzmann machine (RBM), restricted Boltzmann machine with simple Jastrow factor (RBM+SJ), restricted Boltzmann machine with Padé-Jastrow factor (RBM+PJ) and standard variational Monte-Carlo (VMC). The ADAM optimizer was used, and after convergence the number of Monte-Carlo cycles was  $M = 2^{28} = 268,435,456$ .





**Figure D.8:** Two-body densities for two-dimensional circular quantum dots containing up to 42 electrons with oscillator frequency  $\omega = 0.5$ . The density plots were produced using a plain restricted Boltzmann machine (RBM), restricted Boltzmann machine with simple Jastrow factor (RBM+SJ), restricted Boltzmann machine with Padé-Jastrow factor (RBM+PJ) and standard variational Monte-Carlo (VMC). The ADAM optimizer was used, and after convergence the number of Monte-Carlo cycles was  $M = 2^{28} = 268,435,456$ .





**Figure D.9:** Two-body densities for two-dimensional circular quantum dots containing up to 42 electrons with oscillator frequency  $\omega = 1.0$ . The density plots were produced using a plain restricted Boltzmann machine (RBM), restricted Boltzmann machine with simple Jastrow factor (RBM+SJ), restricted Boltzmann machine with Padé-Jastrow factor (RBM+PJ) and standard variational Monte-Carlo (VMC). The ADAM optimizer was used, and after convergence the number of Monte-Carlo cycles was  $M = 2^{28} = 268,435,456$ .



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