

Solving Many-body Quantum Problem using Machine Learning

by

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Abstract

Abstract should be written here

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Secondly, my annoying cohabitants have forced me to spend more time at university. I'm so glad you behave the way you do, if you were more cozy I would certainly spend more time at home and this thesis would not be the same.

Acknowledgements should be written here

-One of my mottos used to be that everything has a reason, which I used to point out whenever people were talking about things that apparently could not be described immediately. It was first when I learned about quantum mechanics that I understood I was wrong, and that is one of the reasons why quantum mechanics caught me so hardly.

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List of abbreviations

Letters	Meaning
RBM	- Restricted Boltzmann Machine
MC	- Monte Carlo
VMC	- Variational Monte Carlo
DMC	- Diffusion Monte Carlo
ML	- Machine Learning
WF	- Wave Function
SPF	- Single Particle Function

Table 1: List of symbols used with explanation.

Source Code

The source code is given in [`https://github.com/evenmn`](https://github.com/evenmn)

1 Introduction

Write introduction here

- Introduce the wavefunction - Mention the uncertainty principle and also quantum entanglement to catch the readers interest

1.1 Many-Body problem

1.2 Machine learning

1.3 Goals and milestones

2 Quantum Many-Body Physics

Here I might present basic quantum mechanics briefly.

2.1 Elementary Quantum Theory

2.1.1 The Schrödinger Equation

Quantum quantities can be found by solving eigenvalue equations with the wavefunction as eigenfunction.

$$\hat{H}\Psi_n(\mathbf{x}) = \epsilon_n \Psi_n(\mathbf{x}) \quad (1)$$

2.1.2 Born-Oppenheimer approximation

The Born-Oppenheimer approximation is the assumption that we can split the Hamiltonian in a one-body part and a two-body part, and calculate the energies separately.

$$\hat{H} = \sum_{i=1}^P \left(-\frac{1}{2} \nabla_i^2 + \frac{1}{2} \omega^2 r_i^2 \right) + \sum_{i < j} \frac{1}{r_{ij}} \quad (2)$$

In atomic units.

2.1.3 Postulates of Quantum Mechanics

We will first list the postulates of quantum mechanics, and then discuss their importance

- The state of a quantum mechanical system is completely specified by its wavefunction
- To every observable in classical mechanics there corresponds a linear, Hermitian operator in quantum mechanics.
-

2.1.4 The Variational Principle

2.1.5 Symmetry and anti-symmetry

Assume we have a permutation operator \hat{P} which switches two coordinates in the wave function,

$$\hat{P}\Psi_n(\mathbf{x}_1, \dots, \mathbf{x}_i, \dots, \mathbf{x}_j, \dots, \mathbf{x}_M) = p\Psi_n(\mathbf{x}_1, \dots, \mathbf{x}_j, \dots, \mathbf{x}_i, \dots, \mathbf{x}_M), \quad (3)$$

where p is just a factor which comes from the transformation. If we again apply the \hat{P} operator, we should switch the same coordinates back, and we expect to end up with the initial wave function. For that reason, $p = \pm 1$.¹

The particles that have an antisymmetric (AS) wavefunction under exchange of two coordinates are called fermions, named after Enrico Fermi, and have half integer spin. On the other hand, the particles that have a symmetric (S) wavefunction under exchange of two coordinates are called bosons, named after Satyendra Nath Bose, and have integer spin.

It turns out that because of their antisymmetric wavefunction, two identical fermions cannot be found at the same position at the same time, known as the Pauli principle. This causes some difficulties when dealing with multiple fermions, because we always need to ensure that the total wavefunction becomes zero if two identical particles happen to be at the same position. To do this, we introduce a Slater determinant as described in the next chapter. In this particular project, we are going to focus on fermions, just because they are more difficult to handle, but much of the theory and implementation applies for bosons as well.

Read <https://manybodyphysics.github.io/FYS4480/doc/pub/secondquant/html/secondquant.bs.html>

2.2 Wavefunction properties

2.2.1 Slater determinant

For a system of more particles we can define a total wavefunction, which is a composition of all the single particle wavefunctions (SPF) and contains all the information about the system. The way we compile the SPFs needs to be based on Pauli's exclusion, which states that two identical fermions cannot possibly be in the same state at the same time. If this happens, we set the total wavefunction to zero, which is done by defining the wavefunction as a determinant.

Consider a system of two identical fermions with SPFs ϕ_1 and ϕ_2 at positions \mathbf{r}_1 and \mathbf{r}_2 respectively. The way we define the wavefunction of the system is then

$$\Psi_T = \begin{vmatrix} \phi_1(\mathbf{r}_1) & \phi_2(\mathbf{r}_1) \\ \phi_1(\mathbf{r}_2) & \phi_2(\mathbf{r}_2) \end{vmatrix} = \phi_1(\mathbf{r}_1)\phi_2(\mathbf{r}_2) - \phi_2(\mathbf{r}_1)\phi_1(\mathbf{r}_2), \quad (4)$$

¹This was true until 1976, when J.M. Leinaas and J. Myrheim discovered the anyon, <https://www.uio.no/studier/emner/matnat/fys/FYS4130/v14/documents/kompendium.pdf>.

which is set to zero if the particles are at the same position. This is called a Slater determinant, and yields the same no matter how big the system is.

Notice that we denote the wavefunction with the ' T ', which indicates that it is a trial wavefunction. We do this because the spin part is avoided with ψ as the radial parts only, thus this wavefunction is not the true wavefunction. We will look closer at how we can factorize out the spin part later. The spin part is assumed to not affect the energies. For bosons the total wavefunction is defined similarly, but with no negative signs since the Pauli principle does not apply for fermions.

A general Slater determinant for a system of N particles takes the form

$$\Psi(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N) = \begin{vmatrix} \psi_1(\mathbf{r}_1) & \psi_2(\mathbf{r}_1) & \dots & \psi_N(\mathbf{r}_1) \\ \psi_1(\mathbf{r}_2) & \psi_2(\mathbf{r}_2) & \dots & \psi_N(\mathbf{r}_2) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_1(\mathbf{r}_N) & \psi_2(\mathbf{r}_N) & \dots & \psi_N(\mathbf{r}_N) \end{vmatrix} \quad (5)$$

where the ψ 's are the true single particle wavefunctions, which are the tensor products

$$\psi = \phi \otimes \xi \quad (6)$$

with ξ as the spin part.

2.2.2 Electron system

For our purpose we will study fermions with spin $\sigma = \pm 1/2$ only, which can be seen as an electron gas. In this particular case, the SPFs can be arranged in spin-up and spin-down parts, such that the Slater determinant can be simplified to

$$\Psi(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N) = \begin{vmatrix} \phi_1(\mathbf{r}_1)\xi_{\uparrow} & \phi_1(\mathbf{r}_1)\xi_{\downarrow} & \dots & \phi_{N/2}(\mathbf{r}_1)\xi_{\downarrow} \\ \phi_1(\mathbf{r}_2)\xi_{\uparrow} & \phi_1(\mathbf{r}_2)\xi_{\downarrow} & \dots & \phi_{N/2}(\mathbf{r}_2)\xi_{\downarrow} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_1(\mathbf{r}_N)\xi_{\uparrow} & \phi_1(\mathbf{r}_N)\xi_{\downarrow} & \dots & \phi_{N/2}(\mathbf{r}_N)\xi_{\downarrow} \end{vmatrix}. \quad (7)$$

This is called the wavefunction ansatz, because assumptions are raised, like two particles with opposite spins are found to be at the same position all the time, i.e, an equal number of fermions have spin up as spin down, are applied. Further the...

SHOULD END UP WITH SPLITTED DETERMINANTS HERE

For a detailed walkthrough, see appendix I in REF(The Stochastic Gradient Approximation: an application to Li nanoclusters, Daniel Nissenbaum).

2.2.3 Basis set

To go further, we need to define a basis set, $\phi_n(\mathbf{r})$ which should be chosen carefully based on the system.

2.2.4 Jastrow factor

Often we add a Jastrow factor to improve the flexibility of the wave function. The Slater determinant itself is usually not able to capture interaction properties, so by multiplying with another function, we might get a better estimate.

2.2.5 CUSP condition

Until now, we have been fiddling with the wavefunction in multiple ways. Are we really allowed to fix it in whichever way we want? The answer is yes, as long as it satisfies the CUSP condition.

The CUSP condition states that

2.3 Systems and basis sets

Specific systems need specific basis sets

2.3.1 Quantum dots

Quantum dots are very small particles, and contain fermions or bosons hold together by an external potential. Since these particles have discrete electronic states like an atom, they are often called artificial atoms.

In this thesis we will study electrons trapped in harmonic oscillators, and we will use the Hermite functions as our basis set. The one dimensional Hermite functions read

$$f_n(x) = H_n(x) \exp(-x^2/2) \quad (8)$$

and are known to be the exact SPFs in a harmonic oscillator.

2.3.2 Atomic and molecular systems

2.3.3 Electron gas

2.3.4 Helium gas

He³ and He⁴

3 Methods

3.1 Configurations Interaction

3.2 Hartree-Fock

3.3 Variational Monte Carlo

Based on the variational principle

3.4 Coupled Cluster

4 Machine Learning

Parameter estimation is a large part of science, because we are often not able to measure things directly. For instance, when Millikan in ... performed his famous electron charge experiment or when we estimate Sun's mass.

4.1 Statistics

4.1.1 Bayesian statistics

4.2 Supervised Learning

4.3 Unsupervised Learning

4.3.1 Boltzmann Machines

5 Implementation

5.1 Foundation

5.2 Structure

Add structure chart

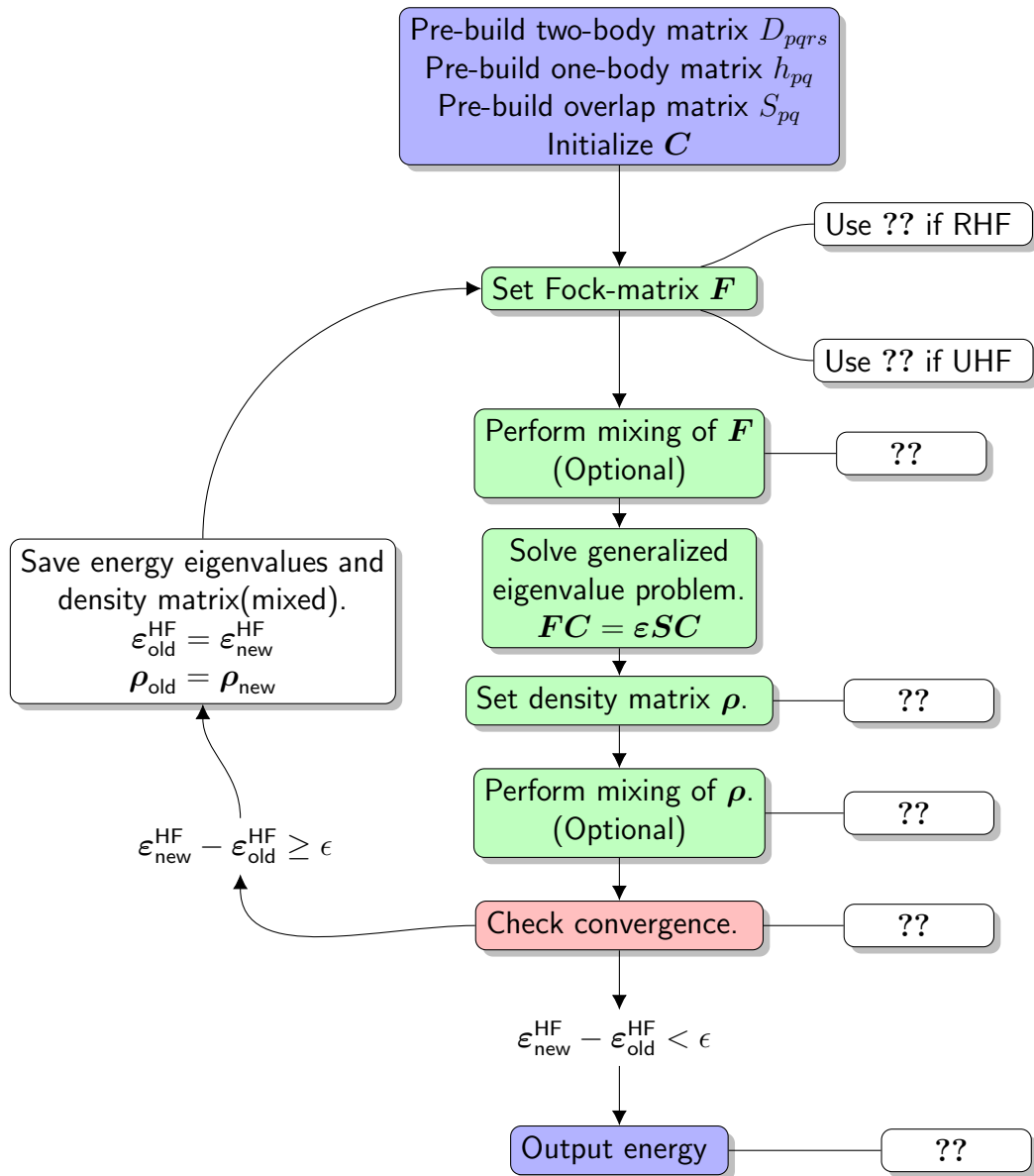


Figure 1: Hartree-Fock Algorithm.

5.3 Optimization algorithms

5.3.1 Stochastic Gradient Descent

5.3.2 ADAM

6 Results

7 Conclusion and future work