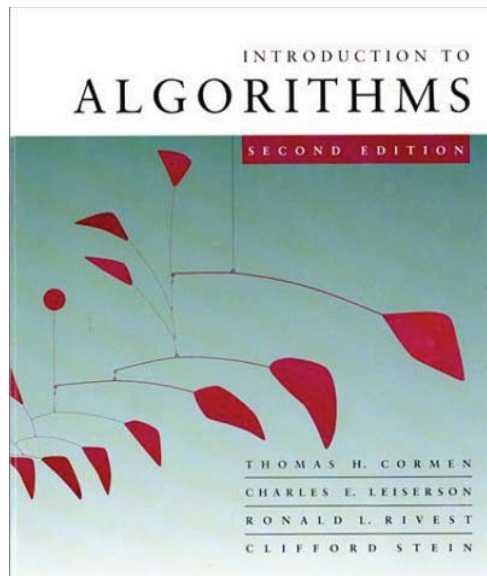


Design and Analysis of Algorithms

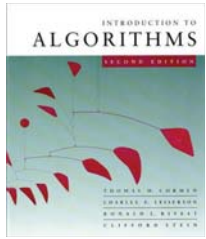
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LECTURE 13

Network Flow

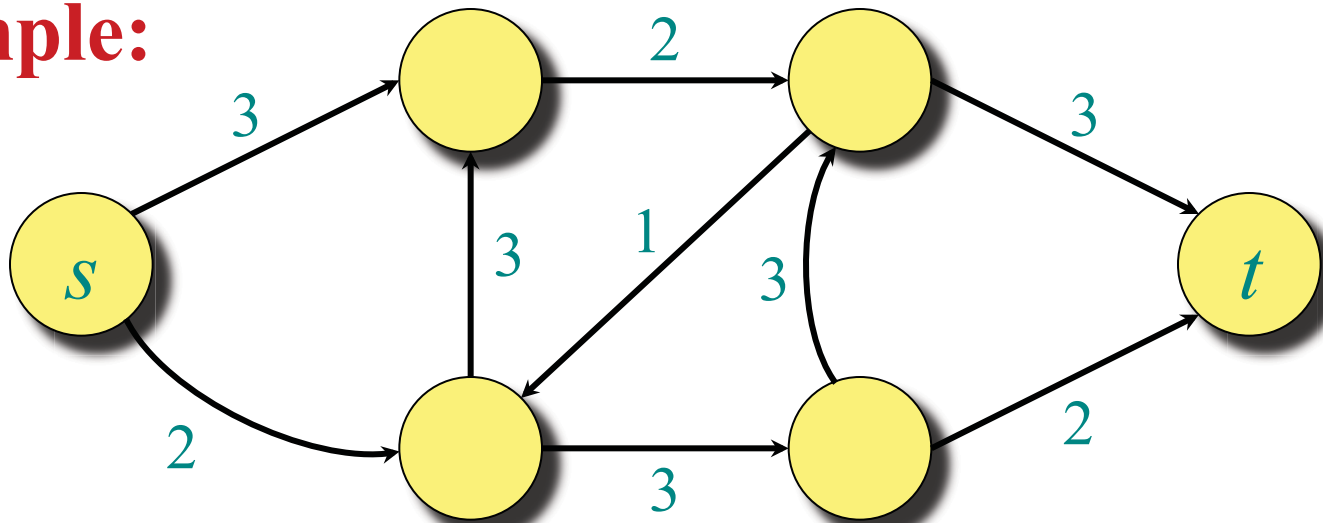
- Flow networks
- Maximum-flow problem
- Cuts
- Residual networks
- Augmenting paths
- Max-flow min-cut theorem
- Ford Fulkerson algorithm

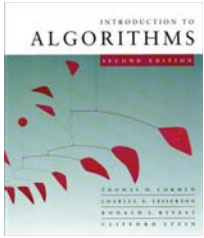


Flow networks

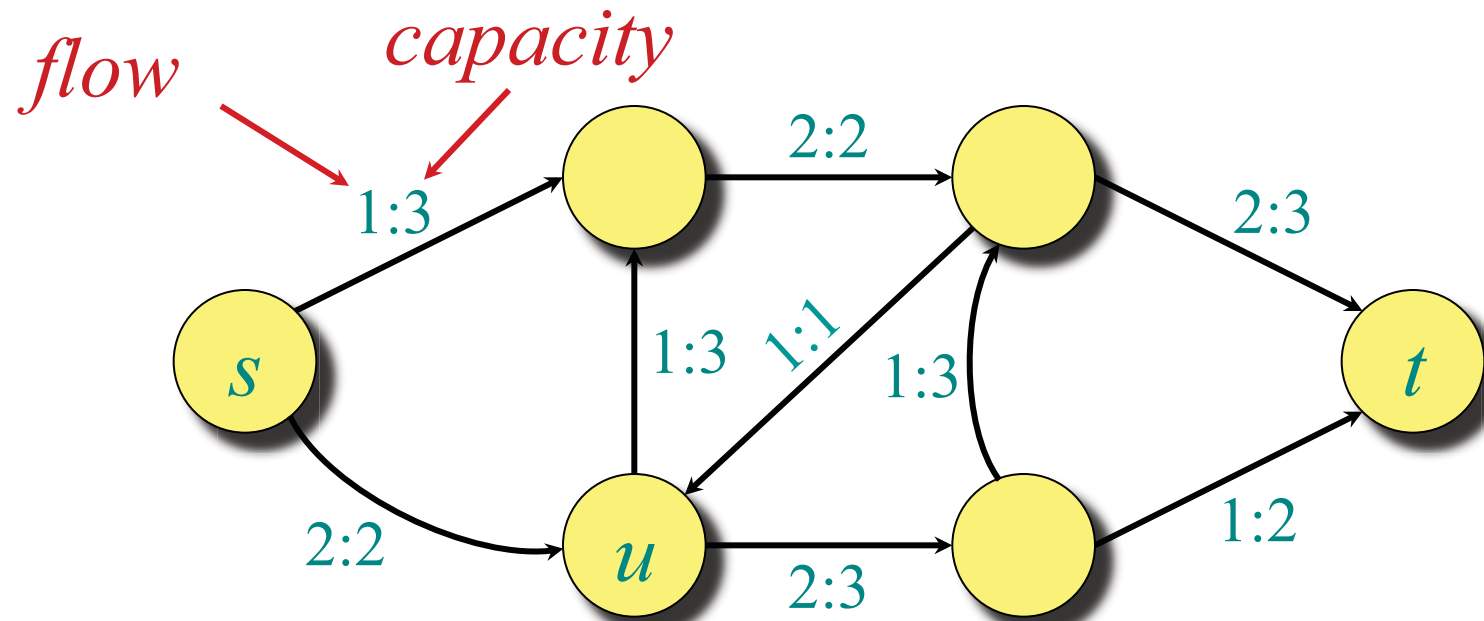
Definition. A *flow network* is a directed graph $G = (V, E)$ with two distinguished vertices: a *source* s and a *sink* t . Each edge $(u, v) \in E$ has a nonnegative *capacity* $c(u, v)$. If $(u, v) \notin E$, then $c(u, v) = 0$.

Example:





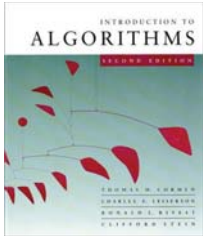
A flow on a network



Flow conservation (like Kirchhoff's current law):

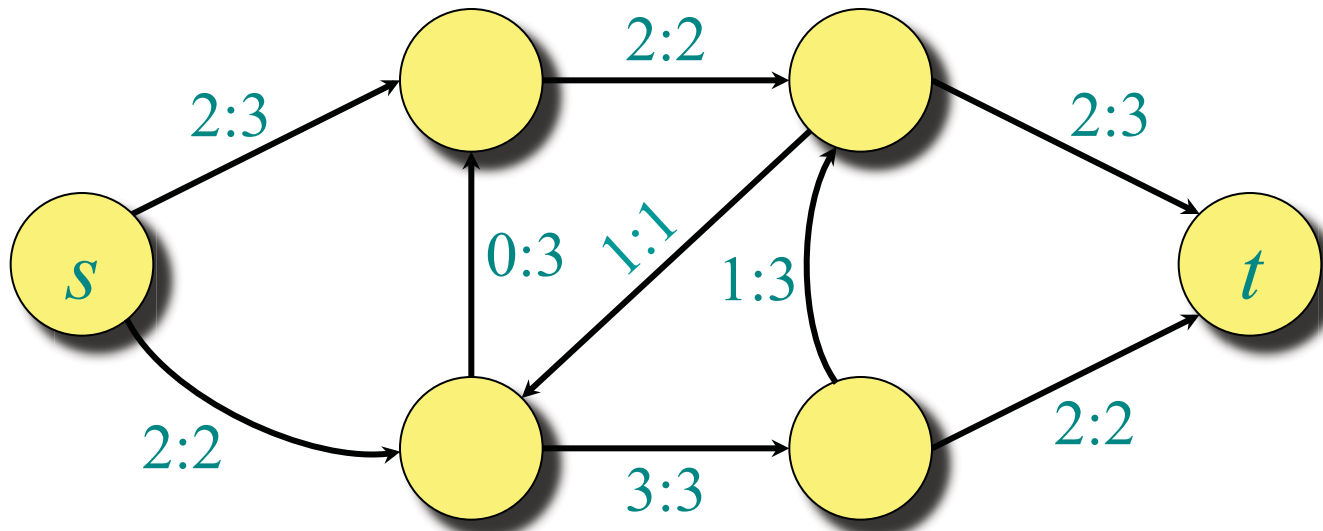
- Flow into u is $2 + 1 = 3$.
- Flow out of u is $1 + 2 = 3$.

INTUITION: View flow as a *rate*, not a *quantity*.

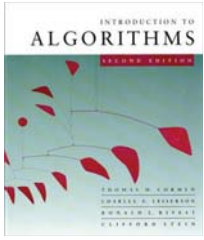


The maximum-flow problem

Maximum-flow problem: Given a flow network G , find a flow of maximum value on G .



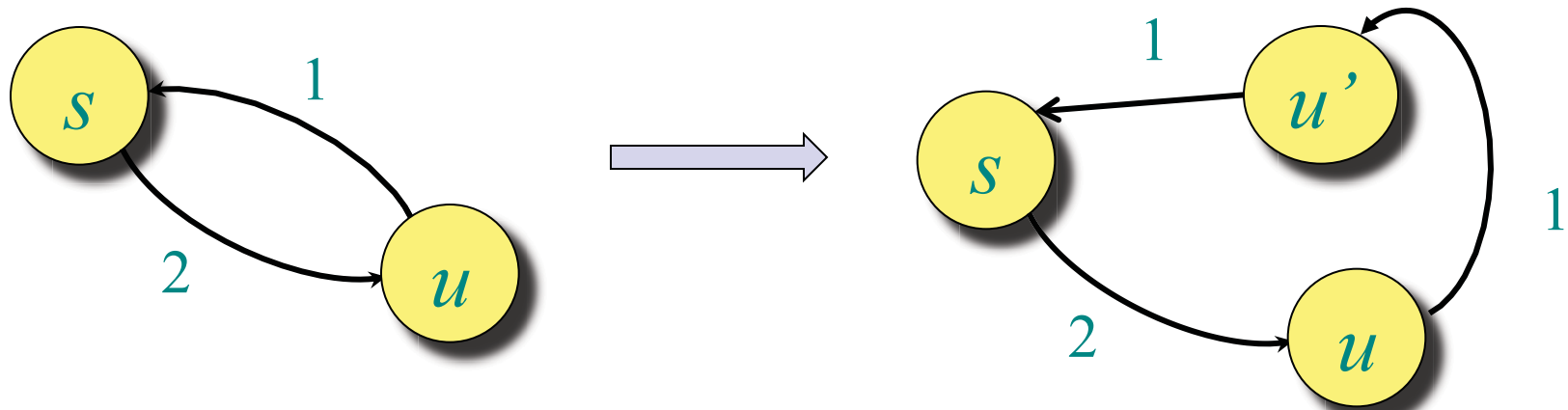
The value of the maximum flow is 4.

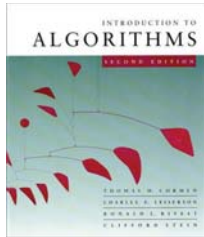


Flow network Assumptions

Assumption. If edge $(u, v) \in E$ exists, then $(v, u) \notin E$.

Assumption. No self-loop edges (u, u) exist





Net Flow

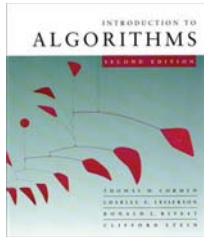
Definition. A *(net) flow* on G is a function $f : V \times V \rightarrow \mathbb{R}$ satisfying the following:

- **Capacity constraint:** For all $u, v \in V$,
 $f(u, v) \leq c(u, v)$.
- **Flow conservation:** For all $u \in V - \{s, t\}$,

$$\sum_{v \in V} f(u, v) = 0.$$

- **Skew symmetry:** For all $u, v \in V$,
 $f(u, v) = -f(v, u)$.

Note: CLRS defines positive flows and net flows; these are equivalent for our flow networks obeying our assumptions.



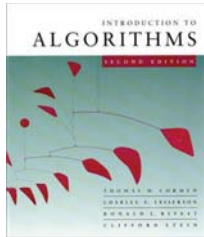
Notation

Definition. The *value* of a flow f , denoted by $|f|$, is given by

$$\begin{aligned}|f| &= \sum_{v \in V} f(s, v) \\ &= f(s, V).\end{aligned}$$

Implicit summation notation: A set used in an arithmetic formula represents a sum over the elements of the set.

- **Example** — flow conservation:
 $f(u, V) = 0$ for all $u \in V - \{s, t\}$.



Simple properties of flow

Lemma.

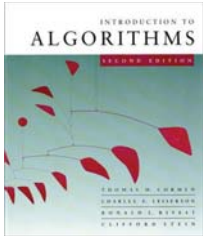
- $f(X, X) = 0$,
- $f(X, Y) = -f(Y, X)$,
- $f(X \cup Y, Z) = f(X, Z) + f(Y, Z)$ if $X \cap Y = \emptyset$. □

Theorem. $|f| = f(V, t)$.

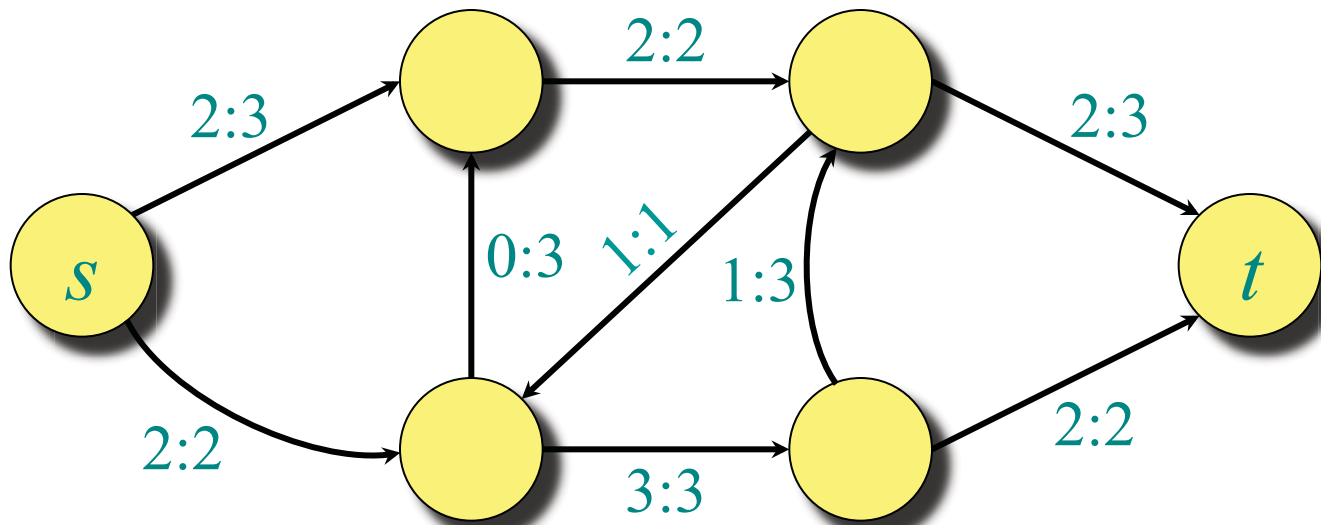
Proof.

$$\begin{aligned} |f| &= f(s, V) \\ &= f(V, V) - f(V-s, V) && \text{Omit braces.} \\ &= f(V, V-s) \\ &= f(V, t) + f(V, V-s-t) \\ &= f(V, t). \quad \text{□} \end{aligned}$$



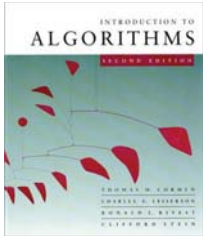


Flow into the sink



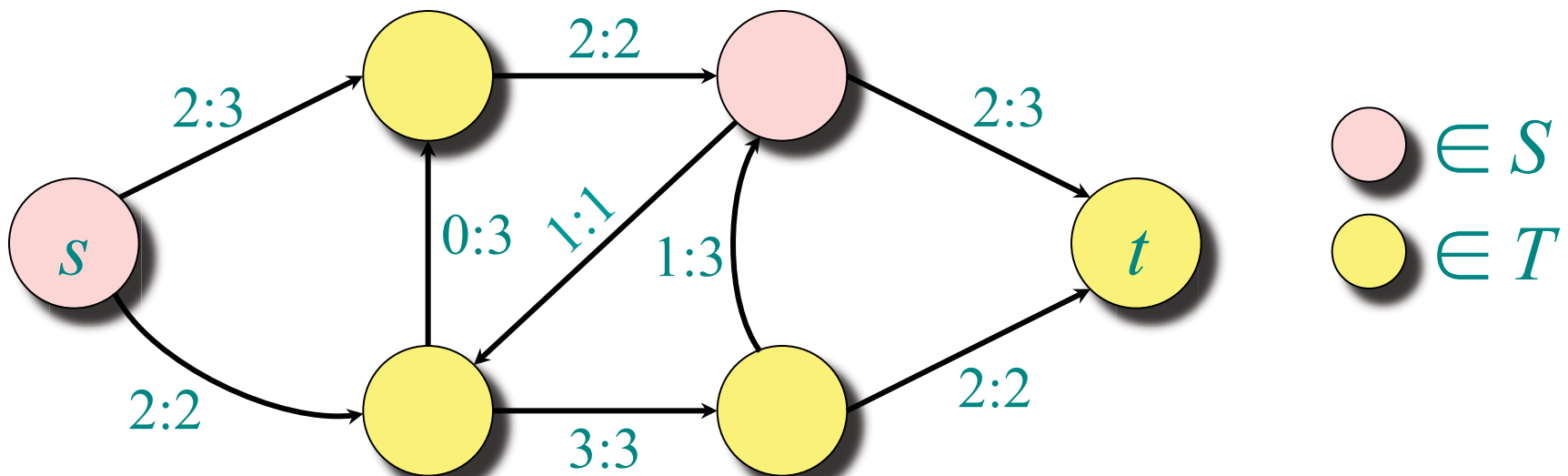
$$|f| = f(s, V) = 4$$

$$f(V, t) = 4$$

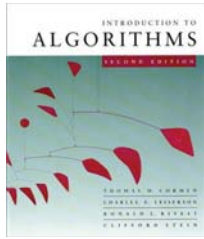


Cuts

Definition. A *cut* (S, T) of a flow network $G = (V, E)$ is a partition of V such that $s \in S$ and $t \in T$. If f is a flow on G , then the *flow across the cut* is $f(S, T)$.



$$f(S, T) = (2 + 2) + (-2 + 1 - 1 + 2) = 4$$

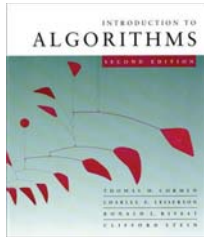


Another characterization of flow value

Lemma. For any flow f and any cut (S, T) , we have $|f| = f(S, T)$.

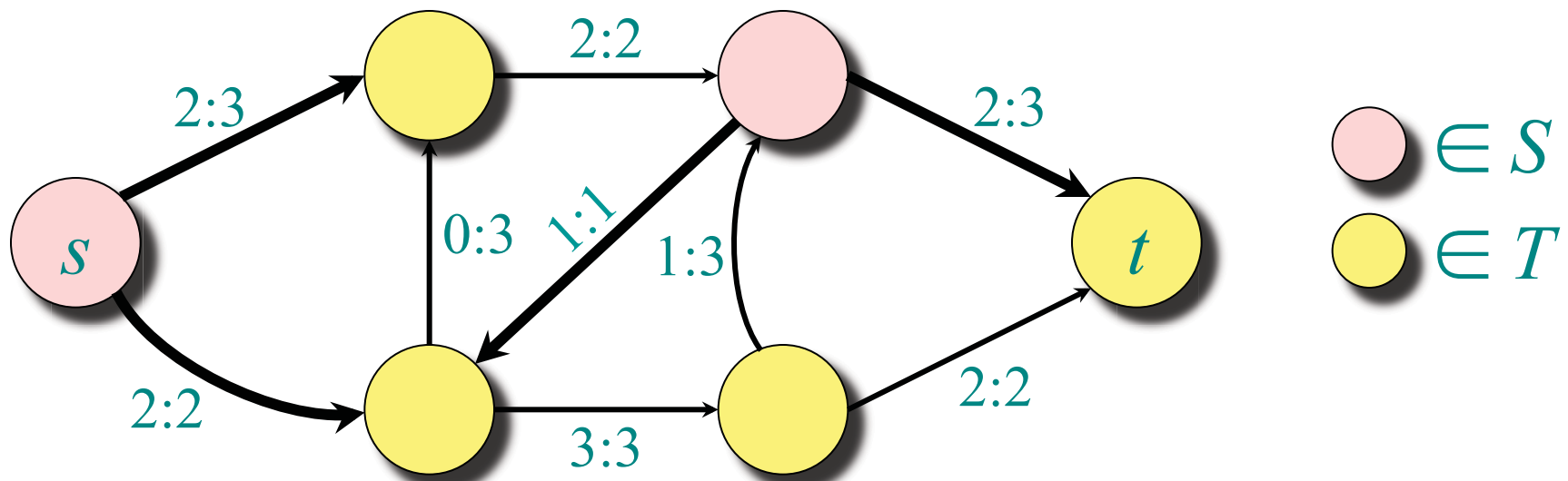
Proof.

$$\begin{aligned} f(S, T) &= f(S, V) - f(S, S) \\ &= f(S, V) \\ &= f(s, V) + f(S-s, V) \\ &= f(s, V) \\ &= |f|. \quad \square \end{aligned}$$

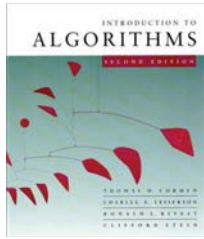


Capacity of a cut

Definition. The *capacity of a cut* (S, T) is $c(S, T)$.



$$c(S, T) = (3 + 2) + (1 + 3) = 9$$



Upper bound on the maximum flow value

Theorem. The value of any flow is bounded above by the capacity of any cut.

Proof.

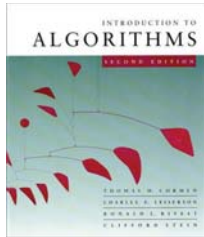
$$|f| = f(S, T)$$

$$= \sum_{u \in S} \sum_{v \in T} f(u, v)$$

$$\leq \sum_{u \in S} \sum_{v \in T} c(u, v)$$

$$= c(S, T) .$$





Residual network

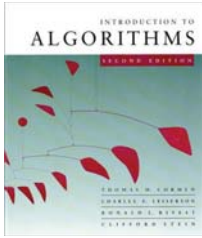
Definition. Let f be a flow on $G = (V, E)$. The *residual network* $G_f(V, E_f)$ is the graph with strictly positive *residual capacities*

$$c_f(u, v) = c(u, v) - f(u, v) > 0.$$

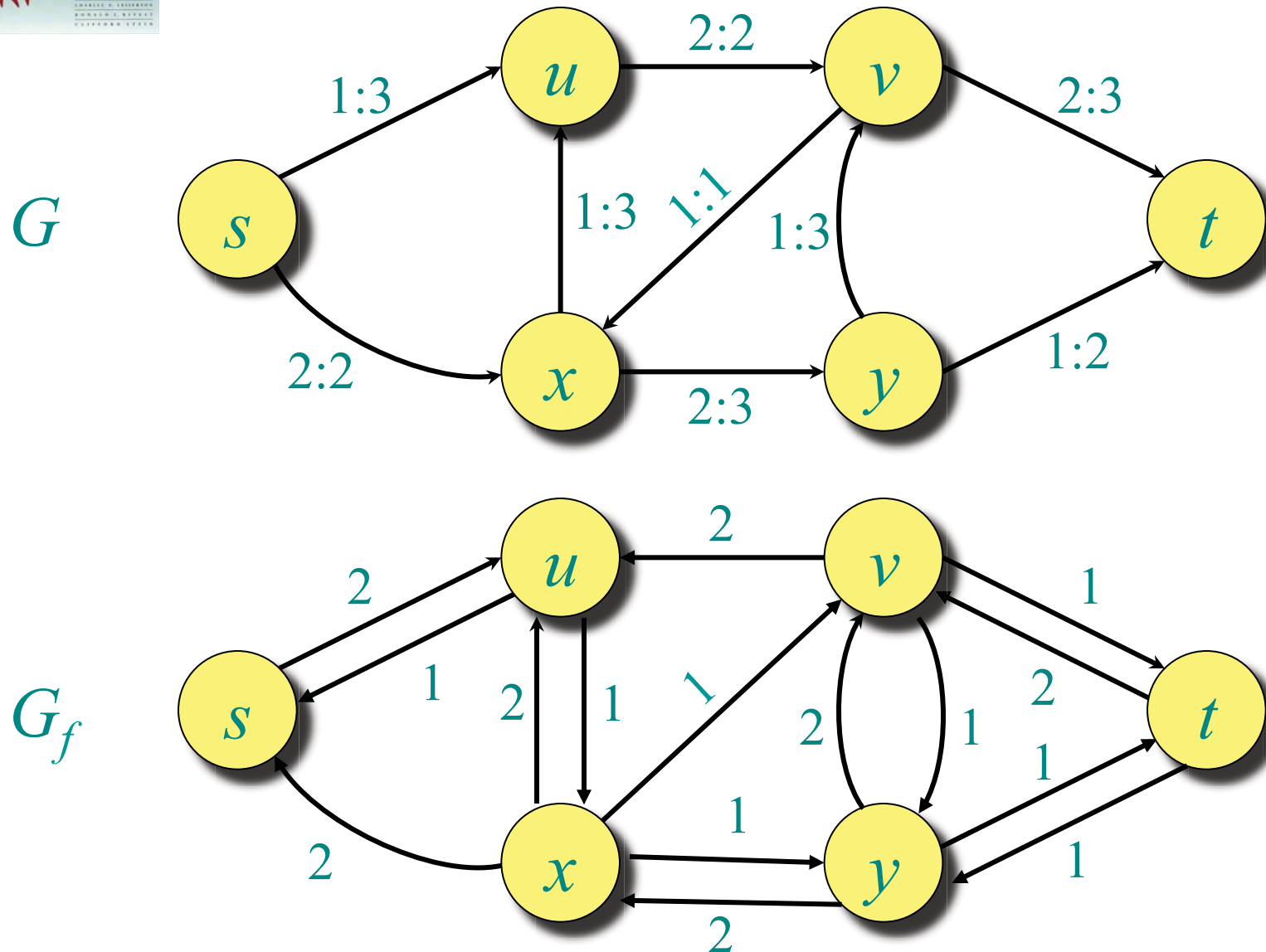
Edges in E_f admit more flow.

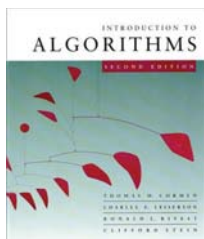
If $(v, u) \notin E$, $c(v, u) = 0$, but $f(v, u) = -f(u, v)$.

$$|E_f| \leq 2 |E|.$$



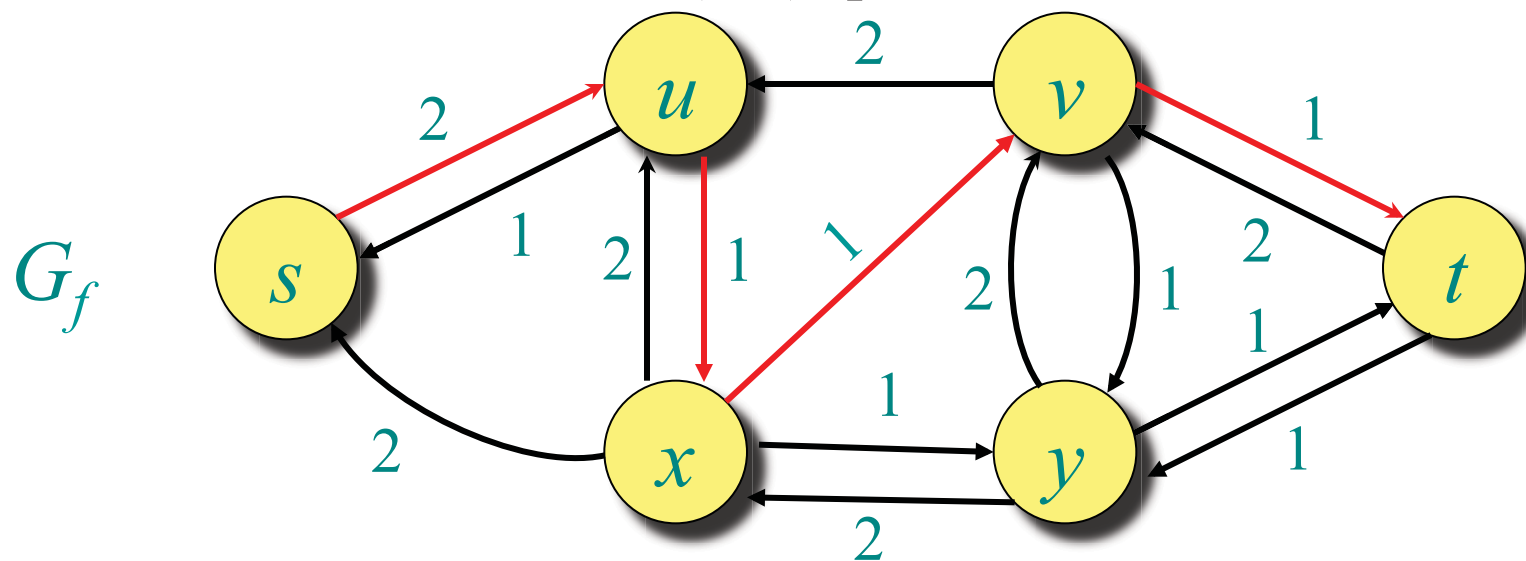
Flow and Residual Network



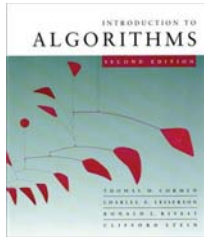


Augmenting paths

Definition. Any path from s to t in G_f is an *augmenting path* in G with respect to f . The flow value can be increased along an augmenting path p by $c_f(p) = \min_{(u,v) \in p} \{c_f(u,v)\}$.

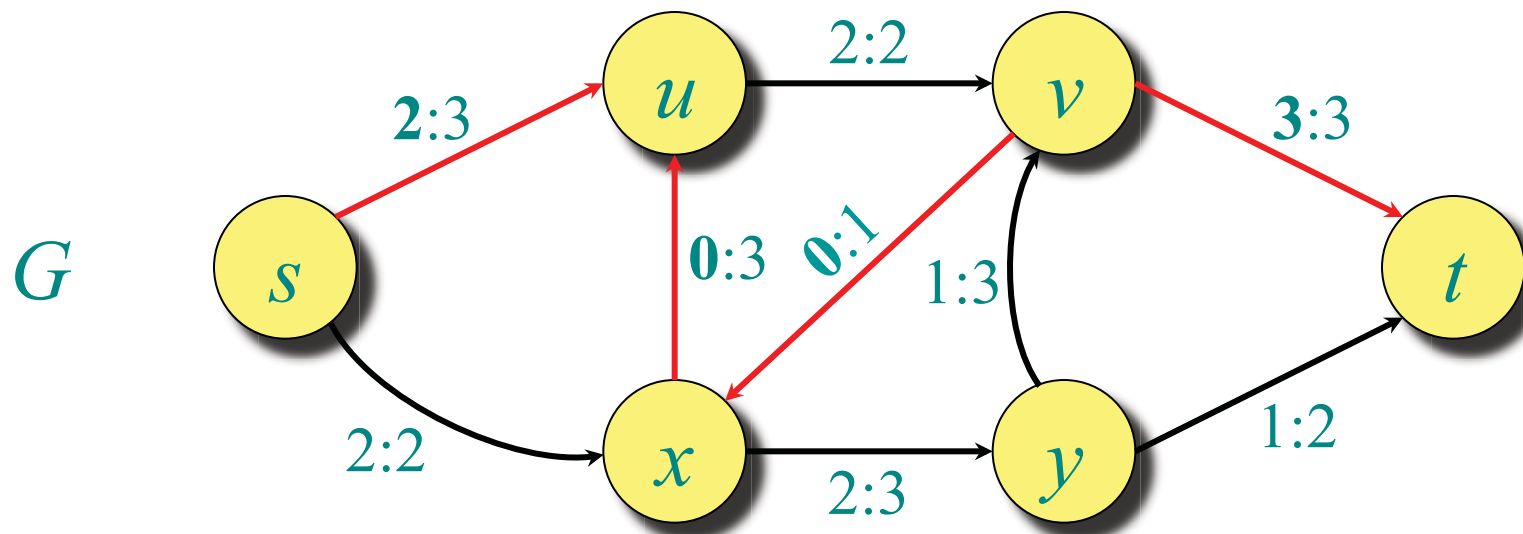


$$p = \{s, u, x, v, t\}, \quad c_f(p) = 1$$



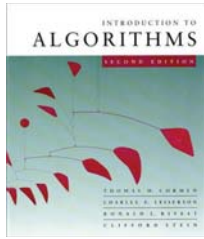
Augmented Flow Network

$$p = \{s, u, x, v, t\}, \quad c_f(p) = 1$$



The value of the maximum flow is 4.

Note: Some flows on edges *decreased*.

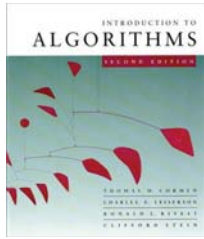


Max-flow, min-cut theorem

Theorem. The following are equivalent:

1. $|f| = c(S, T)$ for some cut (S, T) .
2. f is a maximum flow.
3. f admits no augmenting paths.

Proof. Next time!



Ford-Fulkerson max-flow algorithm

Algorithm:

$f[u, v] \leftarrow 0$ for all $u, v \in V$

while an augmenting path p in G wrt f exists

do augment f by $c_f(p)$

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Spring 2015

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