

Intelligent Systems for Pattern Recognition

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0.1 Introduction

Prof.s: Davide Bacciu and Antonio Carta

Objectives Train ML specialists capable of: designing novel learning models, developing pattern recognition applications using ML, developing intelligent agents using **Reinforcement Learning**.

We're referring to images and signals, but not limited to that: practical applications.

Focusing on challenging and complex data: **machine vision** (noisy, hard to interpret, semantically rich...) and **structured data** (relational information: sequences, trees, graphs...)

Natural Language Processing will be used as an example, but will not be the focus of this course.

Methodology-Oriented Outcomes Gain in-depth knowledge of advanced machine learning models, understanding the underlying theory. This gives the ability to read and understand and discuss research works in the field.

Application-Oriented Outcomes Learn to address modern pattern recognition applications, gain knowledge of ML, PR and RL libraries and be able to develop an application using ML and RL models.

Prerequisites Knowledge of ML fundamentals, mathematical tools for ML and Python.

0.2 Pattern Recognition

Automated recognition of meaningful patterns in noisy data.

Origins

Viola-Jones Algorithm Framework for face recognition. Sum pixel in white area and subtract those in the black portion. The VJ algorithm positions the masks on the image and combines the responses (training set of $\approx 5k$ images with hand-aligned filters)

An historical view

1. Identification of distinguishing features of the object/entity (**feature detection**)
2. Extraction of features for the defining attributes (**feature extraction**)
3. Comparison with known patterns (**matching**)

Basically, lots of time spent hand-engineering the best data features.

A modern view Data is thrown into a neural network. A single stage process with a data crushing-and-munching neural network spitting out prediction, which encapsulates the three historical steps. But the time is now spent in fine-tuning the neural network.

The deep learning Lego Creating applications by putting together various combinations of CNN and LSTM modules.

0.2.1 Signals

Signals are time series: a sequence of measurements in time. Examples of sources are: medicine, finance, geology, IoT, biometrics...

Formalization A time series x is a sequence of measurements in time t

$$x = x_0, \dots, x_N$$

where x_t or $x(t)$ is the measurement at time t .

Observation can be at **irregular** time intervals.

We assume **weakly stationary** (or second-order stationary) data

$$\begin{aligned} \forall t \quad E[x_t] &= \mu \\ \forall t \quad \text{Cov}(x_{t+\tau}, x_t) &= \gamma_\tau \text{ with } \gamma \text{ depending only on the lag } \tau \end{aligned}$$

Goals

Description

Analysis: identify and describe dependencies in data

Prediction: forecast next values given information up to t

Control: adjust parameters of the generative process to make the time series fit a target

Key Methods

Time domain analysis: assesses how a signal changes over time (correlation, convolution, auto-regressive models)

Spectral domain analysis: assesses the distribution of the signal over a range of frequencies (Fourier analysis, wavelets)

Time Domain Analysis

Mean

$$\hat{\mu} = \frac{1}{N} \sum_{t=1}^N x_t$$

Can be used to subtract mean from values and "standardize" the two series.

Autocovariance For lag $-N \leq \tau \leq N$

$$\hat{\gamma}_x(\tau) = \frac{1}{N} \sum_{t=1}^{N-|\tau|} (x_{t+|\tau|} - \hat{\mu})(x_t - \hat{\mu})$$

Autocorrelation The correlation of a signal with itself.

$$\hat{\rho}_x(\tau) = \frac{\hat{\gamma}_x(\tau)}{\hat{\gamma}_x(0)}$$

We can compute this with every possible τ , finding the max/min which gives the τ where the autocorrelation is max/min, which means the lag where the signal starts repeating itself. The lags near zero typically dominates, so we want the maximum lag reasonably far from 0.

Autocorrelation plot It's a revealing view on time series statistics.

Cross-Correlation A measure of similarity of x_1 and x_2 as a function of a time lag τ

$$\phi_{x_1 x_2}(\tau) = \sum_{t=\max\{0,\tau\}}^{\min\{(T_1-1+\tau),(T_2-1)\}} x_1(t-\tau) \cdot x_2(t)$$

Normalized cross-correlation Returns an amplitude independent value

$$\bar{\phi}_{x_1 x_2}(\tau) = \frac{\phi_{x_1 x_2}}{\sqrt{\sum_{t=0}^{T_1-1} (x_1(t))^2 \cdot \sum_{t=0}^{T_2-1} (x_2(t))^2}} \in [-1, +1]$$

With $\bar{\phi}_{x_1 x_2}(\tau) = +1$ mean that the two time series have the exact same shape if aligned at time τ . Nearing -1 we get the maximum anticorrelation, same shape but opposite sign. Near 0 we get that the two signals are completely linearly uncorrelated.

Note that we measure **linear correlation**.

Cross correlation looks like the convolution

$$(f * g)[n] = \sum_{t=-M}^M f(n-t)g(t)$$

but we have a flipped sign ($n-t$ instead of $t-\tau$).

Cross-correlation is not symmetric, whereas convolution is ($f * g = g * f$).

Autoregressive Process A timeseries autoregressive process (AR) of order K is the linear system

$$x_t = \sum_{k=1}^K \alpha_k x_{t-k} + \epsilon_t$$

Autoregressive means x_t regresses on itself

$\alpha_k \Rightarrow$ linear coefficients $|\alpha| < 1$

$\epsilon_t \Rightarrow$ sequence of independent and identically distributed values with mean 0 and fixed variance.

We look backward K steps, so limited memory.

ARMA Autoregressive with Moving Average process

$$x_t = \sum_{k=1}^K \alpha_k x_{t-k} + \sum_{q=1}^Q \beta_q \epsilon_{t-q} + \epsilon_t$$

With ϵ_t Random white noise (again)

The current time series values is the result of a regression on its past values plus a term that depends on a combination of stochastically uncorrelated information

Estimating Autoregressive Models Need to estimate: the values of the linear coefficients α_t and β_t and the order of the autoregressor K and Q

Estimation of the α, β is performed with the Levinson-Durbin Recursion (`levinson(x, K)` in matlab, and included in several Python modules).

The order is often estimated with a Bayesian model selection criterion, choosing the largest K and Q possible. E.g.: BIC, AIC...

The set of autoregressive parameters $\alpha_{i,1}, \dots, \alpha_{i,K}$ fitted to a specific time series x_i is used to confront it with other time series. Same thing for β so we can use α for both sets.

Comparing time series by AR

timeseries clustering: $d(x_1, x_2) = \|\alpha_1 - \alpha_2\|_M^2$

novelty/anomaly detection: $\text{TestErr}(x_t, \hat{x}_t) < \xi$ with \hat{x}_t being the AR predicted value.

Spectral Domain Analysis

Analyze the time series in the frequency domain. Key idea: decomposing the time series into a linear combination of sines and cosines with random and uncorrelated coefficients. So a **regression on sinusoids** with Fourier analysis.

Fourier Transform Discrete Fourier Transform (DFT): transform a time series from the time domain to the frequency domain. Can be easily inverted back to the time domain.

Useful to handle periodicity in the time series: seasonal trends, cyclic processes...

Representing functions We know that, given an orthonormal system for E we can use linear combinations of the basis $\{e_1, \dots, e_k\}$ to represent any function $f \in E$

$$\sum_{k=1}^{\infty} \langle f, e_k \rangle e_k$$

Given the orthonormal system

$$\left\{ \frac{1}{\sqrt{2}}, \sin(x), \cos(x), \sin(2x), \cos(2x), \dots \right\}$$

then the linear combination above becomes the Fourier series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos(kx) + b_k \sin(kx))$$

Representing function in Complex space Using $\cos(kx) - i \sin(kx) = e^{-ikx}$ with $i = \sqrt{-1}$ we can rewrite the Fourier series as

$$\sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

on the orthonormal system

$$\{1, e^{ix}, e^{-ix}, e^{2ix}, e^{-2ix}, \dots\}$$

Representing Discrete Time Series Consider x of length N and $x_n \in R$. Using the exponential formulation, the orthonormal system is finite, from e_0 to e_{N-1} each $\in C^N$

The n -th component of the k -th vector is

$$[e_k]n = e^{\frac{-2\pi ink}{2}}$$

Discrete Fourier Transform Given a time series $x = x_0, \dots, x_{N-1}$ its DFT is the sequence

$$\text{Spectral domain } X_k = \sum_{n=1}^{N-1} x_n e^{\frac{-2\pi ink}{N}} \quad \text{Time domain}$$

And can be inverted

$$x_k = \frac{1}{N} \sum_{k=1}^{N-1} X_k e^{\frac{2\pi ink}{N}}$$

Basic Spectral Quantities in SFT

$$\text{Amplitude } A_k = |X_k| = \sqrt{Re^2(X_k) + Im^2(X_k)}$$

Power $P_k = \frac{|X_k|^2}{N}$, more used in reality and under some conditions this is a reasonable estimate of the power spectral density

DFT in Action We use the DFT elements X_1, \dots, X_K as representation of the signal to train the predictor/classifier. This representation can reveal patterns that are not clear in the time domain.

0.2.2 Image Processing

Bidimensional series. Basically same approach to signals.

Descriptors

An image is a matrix of pixel intensities or color values (RGB). There are other representations, not interesting for this course. CIE-LUV often used in image processing due to perceptual linearity (image difference is more coherent)

Machine Vision Applications For example region of interest, or object classification.

Even pixel-level tasks, for example image segmentation (regions of the image) or semantic segmentation (classifying regions of the image).

Up one level of abstraction: automated image captioning, requiring identifying objects, generating sentences and ranking those sentences.

Key Questions

How to represent visual information? It has to be:

Informative, carrying all the information

Invariant to photometric (different illuminations) and geometric transformation (position in the picture, rotation...)

Efficient for indexing and querying

How to identify informative parts?

Whole image is generally not a good idea

Must lead to good representations

Image Histograms One of the first answer. Describes the distribution of some visual information on the whole image: colors, edges, corners... depending on the goals.

Color Histograms, one of the earliest image descriptors.

Count the number of pixels of a given color (normalize!). We need to discretize and group the RGB colors.

Any information concerning shapes and position is lost. Two images with a random permutation of the same pixels produce the same color histograms.

Images can be compared, indexed and classified based on their color histogram representation.

Can be computed with OpenCV in Python.

Describing Local Image Properties We need something less global, on a local level. Capturing information on image regions, extract **multiple local descriptors**: different location, different scale...

Several approaches, typically performing convolution between a filter and the image region. Using filters sensitive to specific features we can extract many kind of information.

Localized Descriptors

Intensity Vector The simplest form of localized descriptor: a vector $n \cdot m$ of the pixels of a single patch of the image with dimensions n, m . The vector can be normalized to make it invariant to intensity variations.

But rotating gives a different vector. A more robust representation is an histogram of this vector.

Distribution-Based Descriptors Represent local patches by histograms describing properties of the pixels in the patch. The simplest is an histogram of intensity values, but it's not invariant enough even if normalized.

We want a descriptor invariant to illumination (normalization), scale (captured at multiple scale) and geometric transformations (rotation invariant). We want locality, histogram based and invariant to geometric transformation.

SIFT Scale Invariant Feature Transform

1. Center the image patch on a pixel x, y of the image I
2. Represent image at scale σ (controls how close to look at the image)

Convolve the image with a Gaussian filter with standard variation σ , basically computing average of pixels with the coefficient taken from a Gaussian distribution. With a smooth Gaussian, we artificially smooth the object, and vice versa. We can compute different versions of the image.

$$L_\sigma(x, y) = G(x, y, \sigma) * I(x, y)$$

$$G(x, y, \sigma) = \exp\left(-\frac{x^2 + y^2}{2\sigma^2}\right)$$

3. Compute the **gradient of intensity** in the patch, extracting magnitude m and orientation Θ using finite differences.

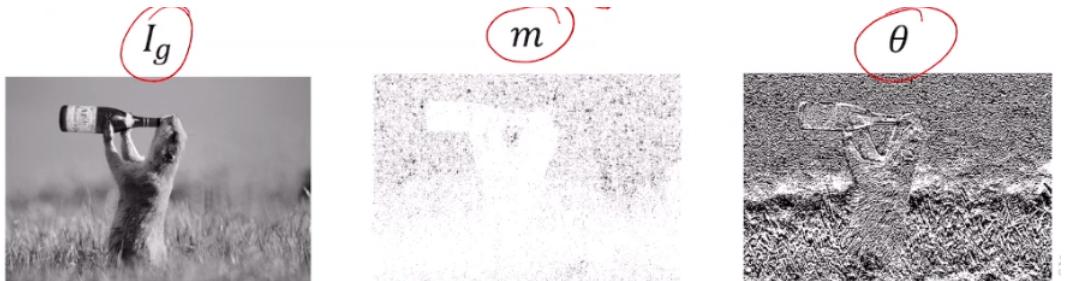
Gaussian Filter of an Image

```
Iscale = imgaussfilt (I, sigma);
```

$\sigma = 5$



$\sigma = 0.05$



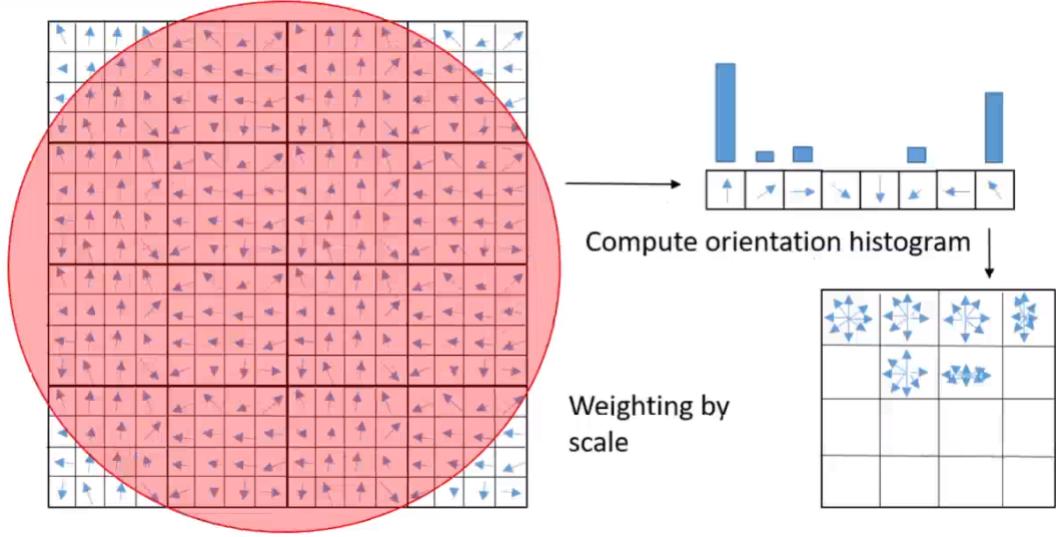
4. Create gradient histogram

4×4 gradient window

Histogram of 4×4 per window on 8 orientation bins

Gaussian weighting on center keypoint (width = 1.5σ)

$4 \times 4 \times 8 = 128$ descriptor size



Detectors

Visual Feature Detector Properties

Repeatability Detect the same feature in different image portions and different images, under different conditions (color, luminance...). So with respect to translation, photometric changes, rotation, scaling and affine transformations (non-isotropic changes, for example the relative position of the camera)...

Edge Detection We need to find interesting points, talking about fundamental elements, basic components. One possible example are the edges of the image.

Reasoning in changes of intensity: edges are those points where the intensity changes.



Typically using an edge detector filter on each pixel and turning pixels white or black by thresholding

Edges and Gradients The image gradient (graylevel) is

$$\nabla I = \left[\frac{\partial I}{\partial x}, \frac{\partial I}{\partial y} \right]$$

which is basically two images, gradient in both x and y directions. Edge are pixel regions where intensity gradient changes abruptly. The return of finite difference methods:

$$G_x = \frac{\partial I}{\partial x} \simeq I(x+1, y) - I(x-1, y)$$

$$G_y = \frac{\partial I}{\partial y} \simeq I(x, y+1) - I(x, y-1)$$

Edge detectors build on this idea combining with some smoothing: average on multiple pixels.

Prewitt operators

$$G_x = \begin{bmatrix} +1 & 0 & -1 \\ +1 & 0 & -1 \\ +1 & 0 & -1 \end{bmatrix} \quad G_y = \begin{bmatrix} +1 & +1 & +1 \\ 0 & 0 & 0 \\ -1 & -1 & -1 \end{bmatrix}$$



Sobel Operator

$$G_x = \begin{bmatrix} +1 & 0 & -1 \\ +2 & 0 & -2 \\ +1 & 0 & -1 \end{bmatrix} \quad G_y = \begin{bmatrix} +1 & +2 & +1 \\ 0 & 0 & 0 \\ -1 & -2 & -1 \end{bmatrix}$$

Often with a constant $c \simeq \frac{1}{8}$ for scaling.



Blob Detection Pixel regions with little gradient variability.

$g_\sigma(x, y)$ has maximum response when centered on a circle of radius $\sqrt{2}\sigma$, with σ being the scale of the gaussian.
Laplace of Gaussian (LoG):

$$\nabla^2 g_\sigma(x, y) = \frac{\partial^2 g_\sigma}{\partial x^2} + \frac{\partial^2 g_\sigma}{\partial y^2}$$

Typically using a scale normalized response

$$\nabla_{norm}^2 g_\sigma(x, y) = \sigma^2 \left(\frac{\partial^2 g_\sigma}{\partial x^2} + \frac{\partial^2 g_\sigma}{\partial y^2} \right)$$

1. Convolve image with a LoG filter at different scales $\sigma = k\sigma_0$ by varying k with a starting σ_0
2. Find maxima of squared LoG responses:

Find maxima on space-scale: focus on a scale and find maxima

Find maxima between scales: do the same for all the scales and pick the maxima

Threshold

The LoG can be approximated by the Difference of Gaussians (DoG) for efficiency, so to reuse part of the computations.

$$g_{k\sigma_0}(x, y) - g_{\sigma_0}(x, y) \simeq (k - 1)\sigma_0^2 \nabla^2 g_{(k-1)\sigma_0}$$

SIFT uses LoG.

Affine Detectors Laplacian-based detectors are invariant to scale thanks to the maximization in scale-space. Still not invariant to affine-transformation.



MSER Maximally Stable Extremal Regions

Extract covariant regions (blobs) that are stable connected components of intensity sets of the image. Interesting areas stay the same at different thresholds: stable with respect to variations in luminance, not scale dependent and doesn't assume circular regions. The key idea is to **take the blobs (extremal regions) which are nearly the same through a wide range of intensity thresholds**.

Blobs are generated (locally) by binarizing the image over a large number of thresholds:

Invariance to affine transformation of image intensities

Stability (they are stable on multiple thresholds)

Multi-scale (connected components are identified by intensity stability not by scale)

Sensitive to local lightning effects, shadows...

Intuitions on MSER

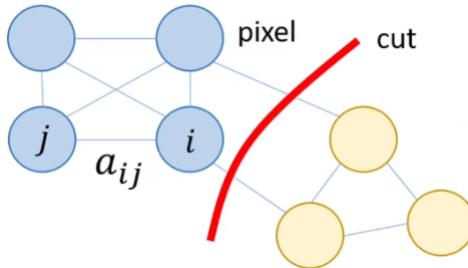
Generate frames from the image by thresholding it on all graylevels.

Capture those regions that from a small seed of pixel grow to a stably connected region. Stability is assessed by looking at derivatives of region masks in time (most stable \Rightarrow minima of connected region variation).

Image Segmentation The process of partitioning an image into a set of homogeneous pixels, hoping to match objects or their subparts.

A naive approach: straighten the image in a $N \cdot M$ vector and use it as a dataset for K-means.

Ncut Normalized cuts



With each node being a pixel: an image is a graph. a_{ij} is the affinity between pixels at a certain scale σ . A cut of G is the set of edges such whose removal makes G a disconnected graph. Breaking the graph into pieces by cutting edges of low affinity.

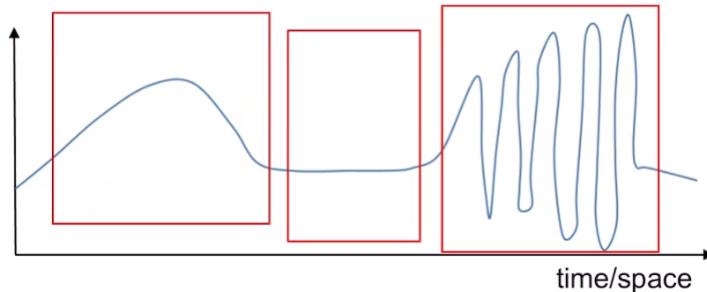
The normalized cut problem is NP-hard, approximate solution as an eigenvalue problem. But the eigenvalue decomposition it's really intractable with big images. We need to reduce the number of pixels. We can use **superpixels**: clustering the pixels with K-means (perhaps with different K) and using the clusters as nodes for segmentation algorithms (Ncut, Markov Random Fields...). We can do multiscale superpixeling and segmenting at different scales, different policies...

Conclusion

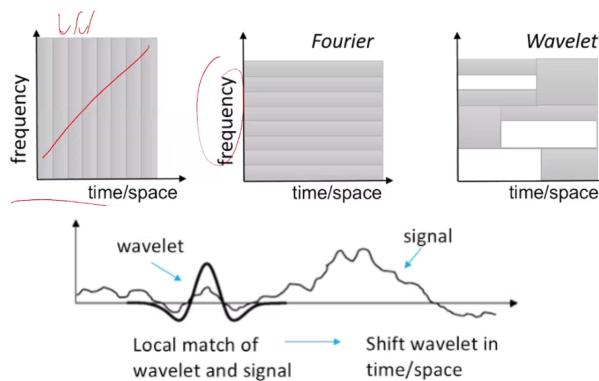
Image processing is a lot about convolutions: linear masks to perform gradient operations, gaussian functions to apply scale changes (zooming in and out). Computational efficiency is a driving factor: convolution in Fourier domain, superpixel, lightweight feature detectors...

0.2.3 Wavelets

Limitations of DFT Sometimes we might need localized frequencies rather than global frequency analysis.



We slice the signal in "time slots" in time analysis and "frequency slots" in frequency analysis. In wavelet analysis you do both.



1. Scale and shift original signal
2. Compare signal to a wavelet
3. Compute coefficient of similarity

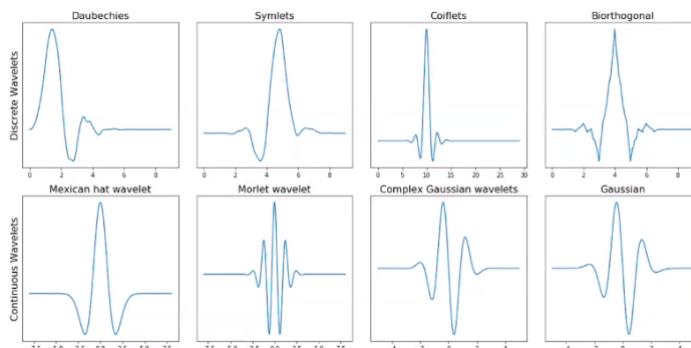
Split the signal with an orthonormal basis generate by translation and dilation of a mother wavelet

$$\sum_t x(t) \phi_{j,k}(t)$$

Terms k, j regulate scaling and shifting of the wavelet

$$\phi_{t,k}(x) = 2^{\frac{k}{2}} \phi\left(\frac{t - j2^k}{2^k}\right)$$

With many different options for the mother wavelet ϕ



Scaling and dilation is akin to a sort of frequency: high scale mean stretched wavelet with slowly changing coarse feature and low frequency, while low scale compressed wavelet with rapidly changing details and high frequency.

DWT Discrete Wavelet Transform: uses a finite set of scales and shifts rather than "any possible value" as in the continuous wavelet transform.

0.3 Generative and Graphical Models

Generative referring to the probability we learn: if we know the distribution probability of data we can generate new data.

Graphical referring to graphical formalisms that describe in a synthetic way the structures we'll see.

Generative Learning ML models that represent knowledge inferred from data under the form of probabilities:

Probabilities can be sampled: new data can be generated

Supervised, unsupervised, weakly supervised tasks

More easily incorporate prior knowledge on data and tasks

Interpretable knowledge (how data is generated)

The majority of modern tasks comprises large number of variables

Modeling the joint distribution of all variables can become impractical

Exponential size of the parameter space

Computationally impractical to train and predict

Representation Graphical models are a compact way to represent exponentially large probability distributions. Encode conditional independence assumptions, and different classes of graph structures imply different assumptions/-capabilities.

Inference How to query (predict with) a graphical model? Probability of unknown X given observations d , $P(X|d)$, the **most likely hypothesis** (parameters) X .

Learning Find the right model parameters.

Representation A graph whose nodes are random variables and edges represent probabilistic relationships between the variables.

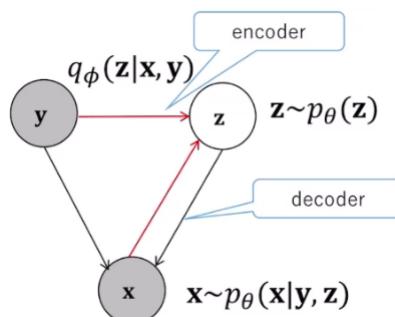
Different classes of graphs:

Directed edges express **causal relationships**

Undirected edges express **soft constraints**, values cannot change independently

Dynamic models, graphs subject to structure changes to reflect dynamic processes. For example RNNs: recurrent neural networks are unfolded using weight sharing, producing a dynamic model.

In Deep Learning Bayesian learning necessary to understand Variational Deep Learning.



Generate new knowledge Complex data can be generated if the model is powerful enough to capture its distribution.

0.3.1 Probability Refresher

todo

Inference

Bayesian: consider all hypothesis weighted by their probabilities

$$P(X | d) = \sum_i P(X | h_i)P(h_i | d)$$

MAP (Maximum a-Posteriori): infer X from $P(X | h_{MAP})$ where h_{MAP} is the maximum a-posteriori hypothesis given d

$$h_{MAP} = \arg \max_{h \in H} P(h | d) = \arg \max_{h \in H} P(d | h)P(h)$$

ML assuming uniform prioris $P(h_i) = P(h_j)$ yields the maximum likelihood (ML) estimate $P(X | h_{ML})$

$$h_{ML} = \arg \max_{h \in H} P(d | h)$$

Any probability can be obtained from the Joint Probability Distribution $P(X_1, \dots, X_n)$ by marginalization but at an exponential cost (e.g. 2^{n-1} for a marginal distribution from binary RV)

0.3.2 Graphical Models

Compact graphical representation for exponentially large joint distributions: simplifies marginalization and inference algorithms, allowing to **incorporate prior knowledge** concerning causal relationships and associations between random variables.

Directed graphical models (Bayesian Networks)

Undirected graphical models (Markov Random Fields)

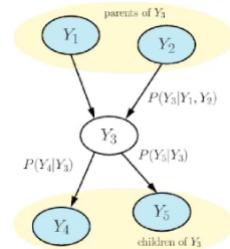
Bayesian Networks

Directed Acyclic Graphs (DAG) $G = (V, E)$

Nodes $v \in V$ represent random variables

Shaded \Rightarrow observed, empty (like Y_3) \Rightarrow unobserved

Edges $e \in E$ describe the conditional independence relationships

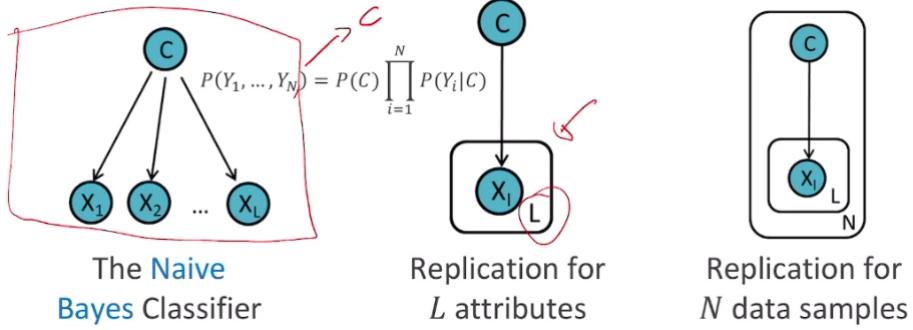


Conditional Probability Tables CPTs are local to each node and describe the probability distribution **given its parents**.

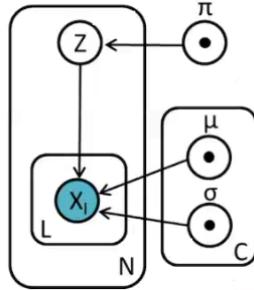
$$P(Y_1, \dots, Y_n) = \prod_{i=1}^N P(Y_i | \text{Parents}(Y_i))$$

Plate notation If the same causal relationship is replicated for a number of variables, we can compactly represent it with plate notation.

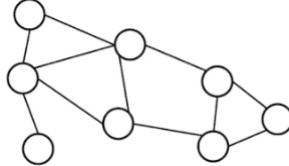
$$P(Y_1, \dots, Y_N, C) = P(C) \prod_{i=1}^N P(Y_i | C)$$



Full-Plate Notation Boxes denote replication for a number of times (denoted by the letter in the corner). Shaded nodes are observed variables, empty nodes are unobserved latent variables. Black dots (optional) identify model parameters.



Markov Random Fields



Undirected graph $G = (V, E)$ (a.k.a. Markov Networks). Also with shaded/empty nodes to denote observed/unobserved variables.

Edges $e \in E$ represent bidirectional dependencies between variables (constraints).

Often arranged in a structure that is coherent with the data/constraint we want to model.

Often used in image processing to impose spatial constraints (e.g. smoothness)

0.3.3 Conditional Independence and Causality

Can we reason on the structure of the graph to infer direct/indirect relationships between random variables?

Local Markov Property Each node (random variable) is conditionally independent of all its non-descendants given a joint state of its parents.

$$Y_v \perp Y_{V \setminus \text{Children}(v)} \text{ given } Y_{\text{Parent}(v)} \quad \forall v \in V$$

There are substructures in the Bayesian networks with which we can build everything.

Markov Blanket A Markov blanket $Mb(A)$ of a node A is the minimal set of vertices that isolates/shields the node from the rest of the Bayesian network. If I know the variables in $Mb(A)$ then I know everything I need to know about A



Taking only the parents it's not sufficient, we need also the children and the co-parents (nodes that are parents of one of my children). So it contains parents, children and children's parents.

$$P(A | Mb(A), Z) = P(A | Mb(A)) \quad \forall Z \notin Mb(A)$$

Joint Probability Factorization An application of the chain rule and local Markov property.

1. Pick a topological ordering of the nodes
2. Apply chain rule following the order

Sampling of a Bayesian Network A BN describes a generative process for observations.

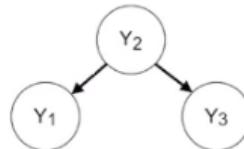
1. Pick a topological ordering of the nodes
2. Generate data by sampling from the local condition probabilities following this order

Generate i th sample for each variable, example $s_i \simeq P(S)$, $h_i \simeq P(H | S = s_i)$

0.3.4 Fundamental Bayesian Network Structures

Three fundamental substructures that determine the conditional independence relationships in a Bayesian network.

Tail to Tail Common cause

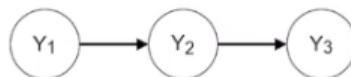


$$P(Y_1, Y_3 | Y_2) = P(Y_1 | Y_2)P(Y_3 | Y_2)$$

If Y_2 is unobserved, then Y_1, Y_3 are marginally dependent $Y_1 \not\perp Y_3$

If Y_2 is observed, Y_1, Y_3 become conditionally independent $Y_1 \perp Y_3 | Y_2$ (the path between Y_1, Y_3 is blocked by the observed (shaded) Y_2)

Head to Tail Causal Effect



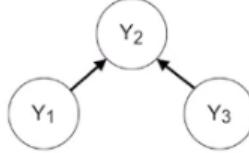
$$P(Y_1, Y_3 | Y_2) = P(Y_1)P(Y_2 | Y_1)P(Y_3 | Y_2) = P(Y_1 | Y_2)P(Y_3 | Y_2)$$

Same behavior as before!

If Y_2 is unobserved, then Y_1, Y_3 are marginally dependent $Y_1 \not\perp Y_3$

If Y_2 is observed, Y_1, Y_3 become conditionally independent $Y_1 \perp Y_3 | Y_2$ (Y_2 again blocks the path)

Heat to Head Common effect



$$P(Y_1, Y_2, Y_3) = P(Y_1)P(Y_3)P(Y_2 | Y_1, Y_3)$$

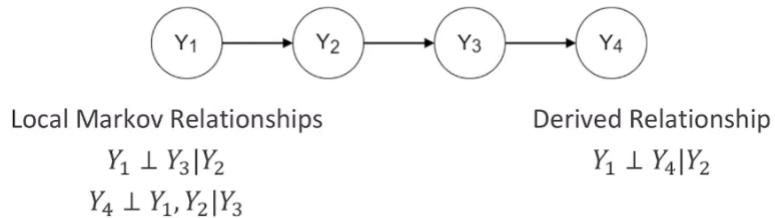
If Y_2 is unobserved, then Y_1, Y_3 are marginally independent $Y_1 \perp Y_3$

If Y_2 is observed, then Y_1, Y_3 are conditionally dependent $Y_1 \not\perp Y_3 | Y_2$

If any Y_2 descendants is observed it unlocks the path.

Derived Conditional Independence Relationships A Bayesian network represent the local relationship encoded by the 3 basic structures plus the derived relationships.

Given the same distribution I can have two different Bayesian Networks, which implies the same factorization.



d-separation Let $r = Y_1 \leftrightarrow \dots \leftrightarrow Y_2$ be an undirected path between Y_1, Y_2 , r is *d*-separated by Z if there exist at least one node $Y_c \in Z$ for which path r is blocked. With Z being the set of variable for which we're assessing this separation.

In other words, this holds if at least one of the following holds:

r contains an head-to-tail structure $Y_i \rightarrow Y_c \rightarrow Y_j$ (or $Y_i \leftarrow Y_c \leftarrow Y_j$) and $Y_c \in Z$

r contains a tail-to-tail $Y_i \leftarrow Y_c \rightarrow Y_j$ and $Y_c \in Z$

r contains head-to-head $Y_i \rightarrow Y_c \leftarrow Y_j$ and neither Y_c nor its descendants are in Z

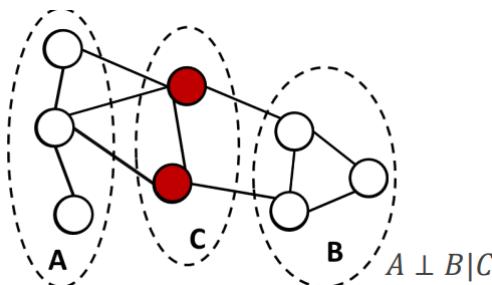
Two nodes Y_i, Y_j in a Bayesian Network G are *d*-separated by $Z \subset V \Leftrightarrow$ all undirected paths between Y_i, Y_j are *d*-separated by Z (denoted by $Dsep_G(Y_i, Y_j | Z)$)

Markov Blanket The Markov Blanket $Mb(Y)$ is the minimal set of nodes which *d*-separates a node Y from all other nodes (i.e. makes Y conditionally independent of all other nodes in the Bayesian Network)

$$Mb(Y) = \{\text{Parents}(Y), \text{Children}(Y), \text{Parents}(\text{Children}(Y))\}$$

Are Directed Models Enough? Bayesian Networks are used to model asymmetric dependencies. But Directed Models cannot express all conditional dependence relationships: expressing some precludes the expressions of others. What if we want to model symmetric dependencies: bidirectional effects, spatial dependencies... we need **undirected approaches**. Directed models cannot represent some bidirectional dependencies in the distributions.

0.3.5 Markov Random Fields



What is the undirected equivalent of d -separation in directed models? It's based on node separation: the two nodes in the middle separate the two lateral parts.

Node subsets $A, B \subset V$ are conditionally independent given $C \subset V \setminus \{A, B\}$ if all paths between nodes in A and B pass through at least one of the nodes in C .

The Markov Blanket of a node includes all and only its neighbors.

Joint Probability Factorization What is the undirected equivalent? We seek a product of functions defined over a set of nodes associated with some local properties of the graph. Markov blanket tells that nodes that are not neighbors are conditionally independent given the remainder of the nodes.

$$P(X_v, X_i | X_{V \setminus \{v, i\}}) = P(X_v | X_{V \setminus \{v, i\}})P(X_i | X_{V \setminus \{v, i\}})$$

Factorization should be chosen in a way that nodes X_v and X_i are not in the same factor: we use a well-known graph structure that includes only nodes that are pairwise connected.

Clique Subset of nodes C in graph G such that G contains an edge between all pair of nodes in C . It's maximal if you cannot add more nodes.

Maximal Clique Factorization Define $X = X_1, \dots, X_n$ as the random variables associated to the N nodes of the undirected graph G

$$P(X) = \frac{1}{Z} \prod_C \psi(X_C)$$

X_C are the random variables in the maximal clique C , $\psi(X_C)$ is the **potential function** over the maximal clique C and Z is the partition function ensuring normalization.

$$Z = \sum_X \prod_C \psi(X_C)$$

The partition function Z is the computational bottleneck of undirected modes: $O(K^N)$ for N discrete random variables with K distinct values.

Potential Functions Potential functions $\psi(X_C)$ are not probabilities, they express which configuration of the local variables are preferred. For example $\psi(X_1, X_2) = \begin{cases} 1 & \text{if } X_1 = X_2 \\ 4 & \text{if } X_2 = 2X_1 \\ 0 & \text{otherwise} \end{cases}$: you can hand-engineer feature functions.

If we restrict to strictly positive potential functions, the Hammersley-Clifford theorem provides guarantees on the distribution that can be represented by the clique factorization.

Boltzmann Distribution A convenient and widely used strictly positive representation of the potential function is

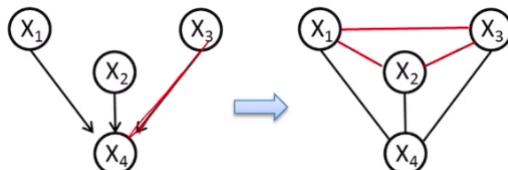
$$\psi(X_C) = e^{-E(X_C)}$$

where $E(X_C)$ is called **energy function**.

From Directed to Undirected

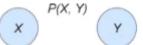
Straightforward when is linear.

Requires some work with v-structures, e.g. **moralization** (a.k.a. marrying of the parents).



0.3.6 Learning Causation from Data

Learning with Bayesian Network

		Structure	
		Fixed Structure	Fixed Variables
			
Data	Complete	Naive Bayes Calculate Frequencies (ML)	Discover dependencies from the data Structure Search Independence tests
	Incomplete	Latent variables EM Algorithm (ML) MCMC, VBEM (Bayesian)	Difficult Problem Structural EM
		Parameter Learning	
		Structure Learning	

Structure Learning Problem Observations are given for a set of fixed random variables, and network structure is not specified:

Determine which arcs exist in the network (causal relationships)

Compute Bayesian network parameters (conditional probability tables)

Determining causal relationships between variables entails deciding on arc presence and directing edges.

Structure Finding Approaches

Search and Score

A model selection approach, a search in the space of the graphs.

Search the space $\text{Graph}(Y)$ of graphs G_k that can be built on the random variables $Y = Y_1, \dots, Y_N$, scoring each structure by $S(G_k)$ and returning the highest scoring graph G^* . So two fundamental aspects: the scoring function and the search strategy.

Scoring function: two fundamental properties:

Consistency: same score for graphs in the same equivalence class

Decomposability: can be locally computed

Two approaches:

Information theoretic: based on data likelihood plus some model-complexity penalization terms

Bayesian: score the structures using a graph posterior (likelihood plus proper prior choice)

Search strategy:

Finding maximal scoring structures is NP complete

Constrain search strategy: starting from a candidate structure we modify iteratively by local operations (edge/node addition/deletion). Each operation has a cost, so a cost optimization problem.

Constrain search space can be

Known node order: can reduce the search space to the parents of each node (Markov Blankets)

Search in the space of structure equivalence classes

Search in the space of node ordering

Constraint Based

Tests of conditional independence $I(X_i, X_j | Z)$, constraining the network. Based on measures of association between two variables X_i and X_j given their neighbor nodes Z .

Testing strategy:

Choice of the testing order is fundamental in avoiding a super-exponential complexity.

Level-wise testing: tests $I(X_i, X_j | Z)$ are performed in order of increasing size of the conditioning set Z starting from $Z = \emptyset$ (PC algorithm)

Node-wise testing: tests are performed on a single edge at the time, exhausting independence checks on all conditioning variables (TPDA algorithm)

The nodes entering Z are chosen in the neighborhood of X_i, X_j

Hybrid

Model selection of constrained structures. Multi-stage algorithm combining previous approaches: independence tests to find a good sub-optimal skeleton as starting point, then search and score refining the skeleton.

Max-Min Hill Climbing (MMHC) model: optimized constraint-based approach to reconstruct the skeleton, using the candidate parents in the skeleton to run a search and score approach.

0.3.7 Hidden Markov Models

Sequence A sequence y is a collection of observations y_t where t represent the position of the element according to a complete order (e.g. time)

$$y_1 \rightarrow \dots \rightarrow y_{t-1} \rightarrow y_t \rightarrow \dots \rightarrow y_T \\ P(y_t | y_{t-1})$$

Also head-to-tail: observation at time t is independent from $t = 1, \dots, t-1$: **first-order Markov assumption**.

Reference population is a set of independent and identically distributed sequences y^1, \dots, y^N

Difference sequences generally have different lengths T^1, \dots, T^N

Markov Chain First-Order Markov Chain is a directed graphical model for sequences such that element x_t only depends on the $t-1$ previous nodes.

We have $\mathbf{X} = x_1, \dots, x_T$ that can be represented as

$$x_1 \rightarrow \dots \rightarrow x_{t-1} \rightarrow x_t \rightarrow \dots \rightarrow x_T$$

So we can write

$$P(\mathbf{X}) = P(x_1, \dots, x_T) = P(x_1) \cdot \prod_{i=2}^T P(x_i | x_{i-1})$$

because $P(x_i | x_{i-1})$ is the same whenever the t .

$P(x_1)$ is the **prior distribution** (x_1 has nothing "before" it) and $P(x_i | x_{i-1})$ is the **transition distribution**.

If I assume $x_t \in \{a, \dots, z\}$, so of 25 elements, this gives $P(x_1) = P(x_1 = \text{letter})$ so $P(x_1)$ is a vector with each position being the probability of x_1 being that letter. Summing the vector elements gives 1, because it's a distribution of probabilities.

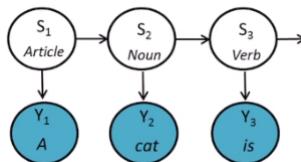
$P(x_i | x_{i-1})$ is a 25×25 matrix: in position n, b is $P(x_i = n | x_{i-1} = b)$. The elements in a single column will give 1, because conditional probability gives a family of distribution: for each assignment I have a distribution.

The general form is the L th order Markov chain, when x_i depends on L predecessors.

Observed Markov Chains We can use the Markov chain to model the relationships between observed elements in a sequence. The problem is that we can do that only pairwise: computational issue (very large matrices) and e.g. only co-occurrence of 2 words so unapplicable to natural language.

So we need to abstract from symbols to category: not relationship between words, but relationships between the general concepts represented by those words. The categories are not observable: Markov chain over non-observable elements.

Hidden Markov Models HMM infer categories: stochastic process where transition dynamics is disentangles from observations generated by the process.



S_i are **hidden states**, finite $i = 1, \dots, C$.

We need **clustering algorithms**: clustering symbols into a finite set of non-observable elements.

Multinomial state transition

$$A_{ij} = P(S_t = i | S_{t-1} = j)$$

Prior probability (**stationary assumption**)

$$\pi_i = P(S_1 = i)$$

Emission distribution (the "down arrow") \downarrow
 S_t
 Y_t

$$b_i(y_t) = P(Y_t = y_t | S_t = i)$$

HMM Joint Probability Factorization Discrete state HMMs are parameterized by the finite number of hidden states C and $\Theta = (\pi, A, B)$:

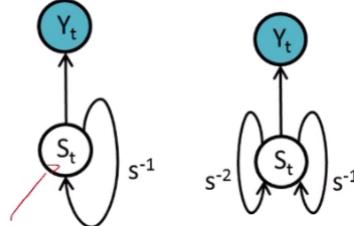
π prior distribution

A state transition

B emission distribution (or its parameters)

$$\begin{aligned} P(Y = y) &= \sum_s P(Y = y, S = s) = \\ &= \sum_{s_1, \dots, s_T} \left(P(S_1 = s_1) P(Y_1 = y_1 | S_1 = s_1) \prod_{t=2}^T P(S_t = s_t | S_{t-1} = s_{t-1}) P(Y_t = y_t | S_t = s_t) \right) \end{aligned}$$

HMMs as Recursive Models A graphical framework describing how contextual information is recursively encoded by both probabilistic and neural models.



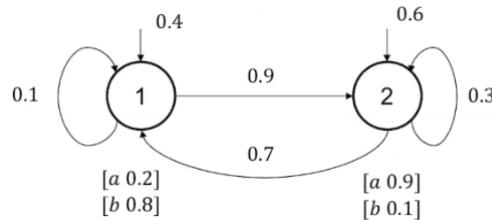
Indicates that the hidden state S_t at time t is dependent on context information from

the previous timestep s^{-1} , first-order

the previous two timesteps s^{-1}, s^{-2} , second-order

and so on.

HMMs as Automata Can also be generalized to transducers.



0.3.8 Notable Inference Problems

Smoothing Given a model Θ and an observed sequence y , determine the distribution of the hidden state at time t : $P(S_t | Y = y, \Theta)$
 Forward Backward algorithm.

Learning Given a dataset of N sequences $D = \{y^1, \dots, y^N\}$ and the number of hidden states C , find the parameters π, A, B that maximize the probability model $\Theta = \{\pi, A, B\}$ having generated the sequences in D

Optimal State Assignment Given a model Θ and an observed sequence y , find an optimal state assignment $s = s_1^*, \dots, s_T^*$ for the hidden Markov chain.
 Viterbi algorithm.

Forward-Backward Algorithm

Smoothing: how do we determine $P(S_t = i | \hat{y})$? We will compute $P(S_t = i, \hat{y})$, it's proportional (just divide by $P(\hat{y})$).
 I know Θ , the model (its parameters). So I know $P(S_1) = \pi, P(S_t | S_{t-1} = A)$ and $P(y_t | S_t) = B$: I need to express the quantity I want in terms of $\Theta = \{\pi, A, B\}$.
 \hat{y} are all the observations for each timestep $\Rightarrow P(S_t = i, y_1, \dots, y_{t-1}, y_t, y_{t+1}, \dots, y_T)$

$$\begin{array}{ccccccccccccc} S_1 & \rightarrow & \dots & \rightarrow & S_{t-1} & \rightarrow & S_t & \rightarrow & S_{t+1} & \rightarrow & \dots & \rightarrow & S_T \\ \downarrow & & \dots & & \downarrow & & \downarrow & & \downarrow & & \dots & & \downarrow \\ y_1 & \rightarrow & \dots & \rightarrow & y_{t-1} & \rightarrow & y_t & \rightarrow & y_{t+1} & \rightarrow & \dots & \rightarrow & y_T \end{array}$$

We are at time t , so everything after that is the future ($y_{t+1:T}$), and everything up to t included is the past ($y_{1:t}$).

$$P(S_t = i, y_1, \dots, y_{t-1}, y_t, y_{t+1}, \dots, y_T) = P(y_{t+1:T} | S_t = i, y_{1:t})P(S_t = i, y_{1:t})$$

If I observe S_t we block the path $y_{1:t}$, so $P(y_{t+1:T} | S_t = i, y_{1:t}) = P(y_{t+1:T} | S_t = i)$

$$P(S_t = i, y_1, \dots, y_{t-1}, y_t, y_{t+1}, \dots, y_T) = P(y_{t+1:T} | S_t = i)P(S_t = i, y_{1:t})$$

I can derive two "messages"

Past message $\alpha_t(i) = P(S_t = i, y_{1:t})$ (**forward recursion**)

Future message $\beta_t(i) = P(y_{t+1:T} | S_t = i)$ (**backward recursion**)

$$P(S_t, y_{1:t}) = \sum_{j=1}^c P(S_t, S_{t-1} = j, y_{1:t}) = \sum_{j=1}^c P(y_t | S_t, S_{t-1} = j, y_{1:t-1})P(S_t, S_{t-1}, y_{1:t-1})$$

But we can get rid of $S_{t-1} = j$ and $y_{1:t-1}$ leaving us with $P(y_t | S_t)$ which is just the emission.

The second factor can be rewritten as $P(S_t | S_{t-1} = j, y_{1:t-1})P(S_{t-1} = j | y_{1:t-1})$ and observing S_{t-1} allows us to get rid of $y_{1:t-1}$, giving us the transition distribution $P(S_t | S_{t-1} = j)$ and $\alpha_{t-1}(j)$

$$\alpha_t(i) = P(S_t = i, y_{1:t}) = \sum_{j=1}^c P(y_t | S_t = i)P(S_t = i | S_{t-1} = j)\alpha_{t-1}(j)$$

$$\alpha_1(j) = P(y_1 | S_1 = j)P(S_1 = j)$$

This just by reasoning with conditional independence.

Same thing can be done for

$$P(y_{t+1:T} | S_t = i) = \sum_j P(y_{t+1:T}, S_{t+1} = j | S_t = i) = \sum_j P(y_{t+2:T} | S_t, S_{t+1}, y_{t+1})P(S_{t+1}, y_{t+1} | S_t = i)$$

Same as before, I can exclude S_t, y_{t+1} because we observe S_{t+1} so that factor is $\beta_{t+1}(j)$.

The second factor is rewritten as $P(S_{t+1} | S_t, y_{t+1})P(y_{t+1} | S_{t+1})$ which is the transition distribution (we can exclude y_{t+1}) times the emission distribution.

$$\beta_t(i) = \sum_j P(y_{t+1} | S_{t+1} = j)P(S_{t+1} = j | S_t = i)\beta_{t+1}(j)$$

$$\beta_T = 1$$

Sum-Product Message Passing The Forward-Backward algorithm is an example of a sum-product message passing algorithm.

A forward recursion computing a generic message μ_α , backward recursion computing a generic message μ_β

A general approach to efficiently perform exact inference in graphical models, with $\alpha_t \equiv \mu_\alpha(X_n)$ and $\beta_t \equiv \mu_\beta(X_n)$

$$\mu_\alpha(X_n) = \sum_{X_{n+1}} \psi(X_n, X_{n+1}) \mu_\beta(X_{n+1})$$

Learning in HMM

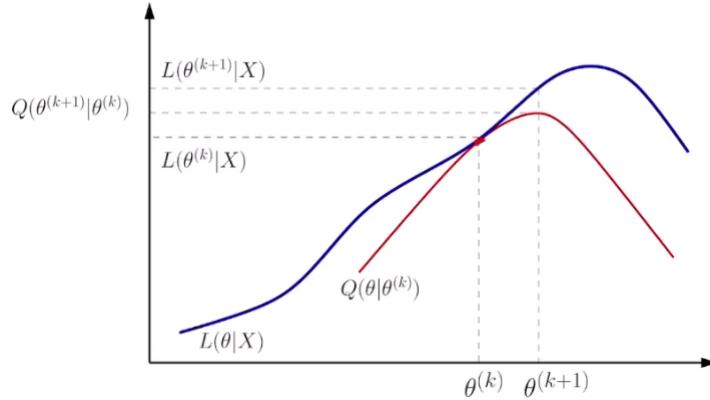
Learning parameters $\Theta = (\pi, A, B)$ by **maximum (log) likelihood**

$$L(\Theta) = \log \prod_{n=1}^N P(Y^n | \Theta) = \log \prod_{n=1}^N \left(\sum_{S_1^n, \dots, S_{T_n}^n} P(S_1^n) P(Y_1^n | S_1^n) \prod_{t=2}^T P(S_t^n | S_{t-1}^n) P(Y_t^n | S_t^n) \right)$$

Maximizing the joint likelihood of the sequences given the parameters considering them independent and identically distributed. We have to deal with the unobserved S_t^n and the nasty sum in the log.

Expectation-Maximization of the **complete likelihood** $L_c(\Theta)$, optimizing a slightly different problem obtaining a not-reducing similar result. It's completed with indicator variables $z_{ti}^n = \begin{cases} 1 & \text{if } n\text{th chain is in state } i \text{ at time } t \\ 0 & \text{otherwise} \end{cases}$ about the assignments S_i^n

Expectation-Maximization Gives the red line: touching in the estimating point and not greater in the other points.



It's a matter of picking the right $Q(\Theta | \Theta^k)$

Introduce indicator variables in $L(\Theta)$ together with model parameters $\Theta = (\pi, A, B)$

$$\begin{aligned} L_C(\Theta) &= \log P(X, Z | \Theta) = \\ &= \log \prod_{n=1}^N \left(\prod_{i=1}^C (P(S_1 = i) P(Y_1^n | S_1 = i))^{z_{1i}^n} \prod_{t=2}^T \prod_{i,j=1}^C P(S_t = i | S_{t-1} = j)^{z_{ti}^n z_{(t-1)j}^n} P(Y_t^n | S_t = i)^{a_{ti}^n} \right) = \\ &= \sum_{n=1}^N \left(\sum_{i=1}^C z_{1i}^n \log \pi_i + \sum_{t=2}^T \sum_{i,j=1}^C z_{ti}^n z_{(t-1)j}^n \log A_{ij} + \sum_{t=1}^T \sum_{i=1}^C z_{ti}^n \log b_i(y_t^n) \right) \end{aligned}$$

We built it assuming to know z , but we don't know it. The *expectation* part is in this: we don't know z_{ti}^n , but you can optimize the function in expectation $E[L_C(\Theta)]$

Expectation-Maximization Algorithm It's a 2-step iterative algorithm for the maximization of complete likelihood $L_C(\Theta)$ with respect to the model parameters Θ

E-step: given the current estimate of the model parameters Θ^t , compute

$$Q^{t+1}(\Theta | \Theta^t) = E_{Z|X,\Theta^t}[\log P(X, Z | \Theta)]$$

So compute the expectation of the complete log likelihood with respect to indicator variables z_{ti}^n assuming estimated parameters $\Theta^t = (\pi^t, A^t, B^t)$ fixed at time t .

Expectation with respect to a discrete random variable is

$$E_z[Z] = \sum_z z \cdot P(Z = z)$$

To compute the conditional expectation $Q^{t+1}(\Theta | \Theta^t)$ for the complete HMM log likelihood we need to estimate

$$E_{Z|Y,\Theta^k}[z_{ti}] = P(S_t = i | y)$$

$$E_{Z|Y,\Theta^k}[z_{ti}z_{(t-1)j}] = P(S_t = i, S_{t-1} = j | y)$$

And we know how to compute the posteriors thanks to the forward-backward algorithm:

$$\begin{aligned} y_t(i) &= P(S_t = i | Y) = \frac{\alpha_t(i)\beta_t(i)}{\sum_{j=1}^C \alpha_t(j)\beta_t(j)} \\ y_{t,t-1}(i,j) &= P(S_t = i, S_{t-1} = j | Y) = \frac{\alpha_{t-1}(j)A_{ij}b_i(y_t)\beta_t(i)}{\sum_{m,l=1}^C \alpha_{t-1}(m)A_{lm}b_j(y_t)\beta_t(l)} \end{aligned}$$

M-step: find the new estimate of the model parameters

$$\Theta^{t+1} = \arg \max_{\Theta} Q^{t+1}(\Theta | \Theta^t)$$

Optimization problem, using the posteriors computed at the E-step. As usual with

$$\frac{\partial Q^{t+1}(\Theta | \Theta^t)}{\partial \Theta}$$

where $\Theta = (\pi, A, B)$ are now variables. The parameters can be distributions, so we need to preserve sum-to-one constraints (Lagrange Multipliers).

State distributions are

$$\begin{aligned} A_{ij} &= \frac{\sum_{n=1}^N \sum_{t=2}^{T^n} \gamma_{t,t-1}^n(i,j)}{\sum_{n=1}^N \sum_{t=2}^{T^n} \gamma_{t-1}^n(j)} \\ \pi_i &= \frac{\sum_{n=1}^N \gamma_1^n(i)}{N} \end{aligned}$$

and the emission distribution, multinomial, is

$$B_{ki} = \frac{\sum_{n=1}^N \sum_{t=1}^{T^n} \gamma_t^n(i) \delta(y_t = k)}{\sum_{n=1}^N \sum_{t=1}^{T^n} \gamma_t^n(i)}$$

With appropriate Lagrange multiplier is multinomial.

Usefulness of HMMs

Regime Detection: for example, you can only observe the volatility and you can model it according to a HMM that can capture it. For example with 2 states a model can be too simple, you can add hidden state (for example a 5-state HMM).

The hidden states are **clustering the observations**.

Decoding Problem Find the optimal state assignment $s = s_1^*, \dots, s_T^*$ for an observed test sequence y given a trained HMM. No unique interpretation of the problem.

Can be done identifying the single hidden states s_t that maximize the posterior

$$s_t^* = \arg \max_{i=1, \dots, C} P(S_t = i | Y)$$

or find the most likely **joint hidden state assignment**

$$s^* = \arg \max_s P(Y, S = s)$$

Viterbi Algorithm Efficient dynamic programming algorithm based on a backward-forward recursion, example of max-product message passing algorithm. When exchanging, instead of \sum we maximize the \prod .

$$\max_{\hat{s}=s_1, \dots, s_T} P(\hat{y}, \hat{s}) = \max_{\hat{s}} \prod_{t=1}^T P(y_t | s_t) P(s_t | s_{t-1})$$

because is emission and prior as always. Let's focus on a simplified problem: first try to find the state that maximize s_T . Let's focus on T

$$\max_{s_T} \prod_{t=1}^T P(y_t | s_t) P(s_t | s_{t-1}) =$$

we can exclude a lot

$$= \prod_{t=1}^{T-1} P(y_t | s_t) P(s_t | s_{t-1}) \cdot \max_{s_T} P(y_T | s_T) P(s_T | s_{T-1})$$

which is a unique term, let's call $\epsilon(s_{T-1}) = \max_{s_T} P(y_T | s_T) P(s_T | s_{T-1})$, and s_{T-1} has c possible values, the number of hidden states.

So $\epsilon(s_{T-1})$ it's a vector of c positions, and in position j we have $\epsilon(s_{T-1} = j)$

$$\prod_{t=1}^{T-1} P(y_t | s_t) P(s_t | s_{t-1}) \epsilon(s_{T-1})$$

Let's try solving

$$\max_{s_{T-1}} \prod_{t=1}^{T-1} P(y_t | s_t) P(s_t | s_{t-1}) \epsilon(s_{T-1})$$

we would do the same procedure. So we can iteratively start from the last item and use the information iteratively to compute the previous one.

In general we compute

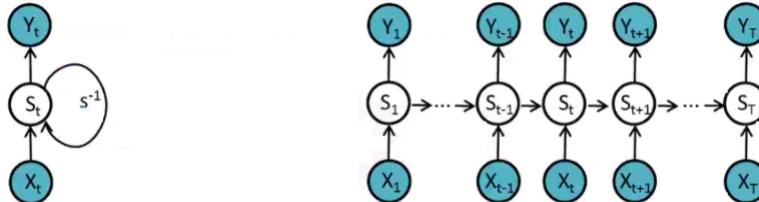
$$\epsilon(s_{t-1}) = \max_{s_t} P(y_t | s_t) P(s_t | s_{t-1}) \epsilon(s_t)$$

So s_t is received by s_{t-1} to compute the new $\epsilon(s_{t-1})$ which is in turn passed to s_{t-2} and so on, ending at the root. In practice we never choose the state, only computing the maximum. At the root, I have no predecessor states and can solve the maximization problem

$$s_1^* = \arg \max_{s_1} P(y_1 | s_1) P(s_1)$$

From state s_1 we pick up s_1^* and send it to s_2 , which will use this information and pick the correct state that maximize the ϵ .

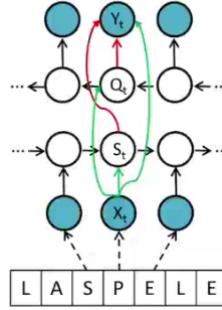
0.3.9 Input-output Hidden Markov Models



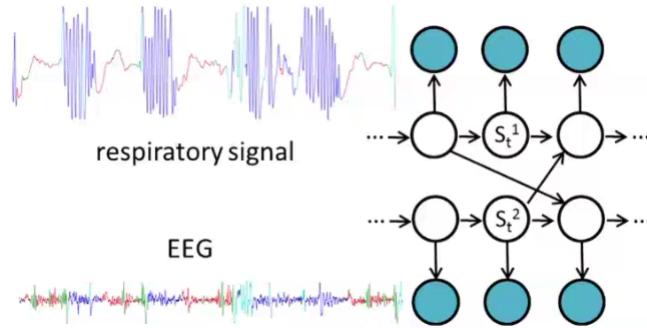
Translates an input sequence in an output sequence (**transduction**). State transition and emission depend on input observations (**input-driven**).

Recursive model highlights analogy with RNNs.

Bidirectional Input-Driven Models Removes the causality assumption that current observation doesn't depend on the future and homogeneity assumption that a state transition doesn't depend on the position in the sequence.



Coupled HMMs Describing interacting processes whose observation follow different dynamics while the underlying generative processes are interlaced.



Dynamic Bayesian Networks HMMs are a specific and the simplest case of a class of directed models that represent dynamic processes and data with changing connectivity template. Other examples are: Hierarchical HMMs and structure changing information.

DBNs are graphical models whose structure changes to represent evolution across time and/or between different samples.

0.3.10 Markov Random Fields

MFRs are undirected graphs $G = (V, E)$. Nodes $v \in V$ are random variables X_v , edges $e \in E$ are bi-directional dependencies between variables.

Likelihood Factorization Define $\mathbf{X} = X_1, \dots, X_N$ as the random variables associated to the N nodes in the undirected graph G

$$P(\mathbf{X}) = \frac{1}{Z} \prod_C \psi_C(X_C)$$

X_C are the random variables associated to the maximal clique C , $\psi_C(X_C)$ is the **potential function** for clique C . With Z normalization term used to transform to probability, from a **partition function**:

$$Z = \sum_{\mathbf{X}} \prod_C \psi_C(X_C)$$

As already stated, potential functions are not probabilities, they express which configurations of the local variables are preferred. A conveniently and widely used strictly positive representation of the potential function is the **Boltzmann Distribution**:

$$\psi_C(X_C) = \exp(-E(X_C))$$

with $E(X_C)$ called **energy function**.

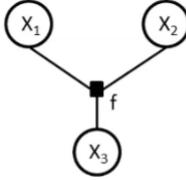
In general we will assume to work with Markov Random Fields where the partition functions factorize as

$$\psi_C(X_C) = \exp \left(- \sum_{k=1}^K \theta_{Ck} f_{Ck}(X_C) \right)$$

K defines the number of feature functions that we use, so the cardinality of a dictionary of feature functions f_{Ck}
 $\theta_{Ck} \in R$ are parameters

Undirected graphical models do not express the factorization of potentials into feature functions, you cannot express f graphically \Rightarrow **factor graphs**.

Factor Graphs Random Variables are still circular nodes, factors f_{Ck} are denoted with square nodes and edges connect a factor to the random variable.



$$\psi(X_1, X_2, X_3) = f(X_1, X_2, X_3)$$

Sum-Product Inference A powerful class of exact inference algorithms. Use factor graph representation to provide a unique algorithm for directed/undirected models. So factor graph are a "unifying language" to represent both models, directed and undirected.

Inference is **feasible for chain and tree structures**. We restructure the graph to obtain a tree-like structure to perform message passing (junction tree algorithm) and then approximated inference (variational, sampling). Even better: we constrain the MRF to obtain tractable classes of undirected models.

Restricting to Conditional Probabilities In ML a part of the random variables can be assumed to be always observable (input data).

X_k are observable inputs in the factor k

Y_k are hidden random variables

$f_k(X_k, Y_k)$ is the factor feature function

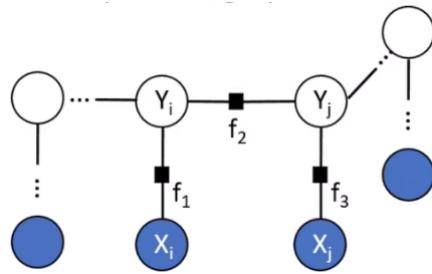
Instead of $P_\theta(x, y)$ we compute $P_\theta(y|x) \cdot P_\theta(x)$. Under this assumption we can directly model the conditional distribution

$$P(Y | X) = \frac{1}{Z(X)} \prod_k \exp(\theta_k f_k(X_k, Y_k))$$

We note that Z depends on X

$$Z(X) = \sum_y \prod_k \exp(\theta_k f_k(X_k, Y_k = y))$$

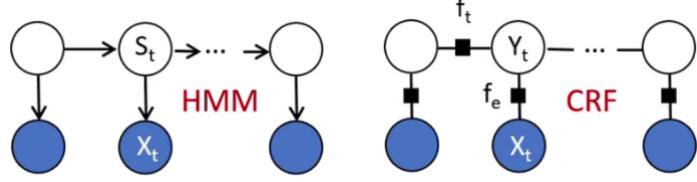
Conditional Random Field CRF are constrained MRF models representing input-conditional distributions.



Feature Functions Represent coupling or constraints between random variables, and are often very simple such as linear functions.

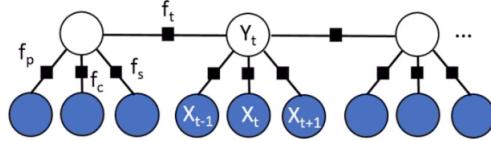
Discriminative Learning in Graphical Models X is always observable input while Y **can** be unobserved. Let's consider a single Y and multiple X s. We can observe the Y_n corresponding to X_n for some n , and we can use this information to fit θ in $P(Y | X, \theta)$

CRF for Sequences Undirected and discriminative equivalent of an HMM.



Meaning that $f_t(Y_{t-1}, Y_t)$ and it looks like $P(S_t | S_{t-1})$ of the HMM. $f_e(X_t, Y_t)$ looks like the emission $P(X_t | S_t)$, but **I can place as many feature functions f_t I want between the same variables** while I can't place more transition probabilities. The other difference is the main essences.

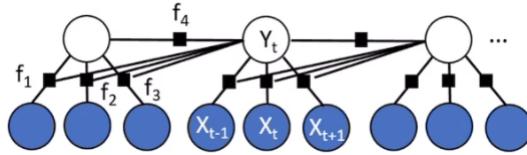
Generalization of HMM CRF are much more powerful



Each hidden state may depend on the previous, with f_t , but also on the emissions for the previous, current and next symbols. This cannot be easily implemented in HMMs. Kind of time stationarity.

$$P(Y | X, \Theta) = \frac{1}{Z(X)} \prod_t \exp (\Theta_p f_p(X_{t-1}, Y_t) + \Theta_c f_c(X_t, f_t) + \Theta_s f_s(X_{t+1}, Y_t) + \Theta_t f_t(Y_{t-1}, Y_t))$$

The cliques to consider are simply $(Y_t, X_{t-1}), (Y_t, X_t), (Y_t, X_{t+1}), (Y_t, Y_{t-1})$ for each t .
We can also model explicitly input influence on transition.



The general Linear Conditional Random Fields likelihood is

$$P(Y | X, \Theta) = \frac{1}{Z(X)} \prod_t \exp (\Theta_k f_k(Y_t, Y_{t-1}, X_t))$$

Uses indicator variables in f_k definition to include or disregard the influence of specific random variables: e.g. $I_{Y_t=i}, I_{X_t=o}$

Posterior Inference in LCRF Is there an equivalent of the smoothing problem? Yes: $P(Y_t, Y_{t-1} | X)$ Solved by exact forward backward inference. Sum-product message passing (alpha-beta recursion) on the LCRF factor graph.

$$P(Y_t, Y_{t-1} | X) \simeq \alpha_{t-1}(Y_{t-1})\psi_t(Y_t, Y_{t-1}, X_t)\beta_t(Y_t)$$

$$\text{Clique weighting } \psi_t(Y_t, Y_{t-1}, X_t) = \exp(\Theta_c f_c(X_t, Y_t) + \Theta_t f_t(Y_{t-1}, Y_t))$$

$$\text{Forward message } \alpha_t(i) = \sum_j \psi_t(i, j, X_t) \alpha_{t-1}(j)$$

$$\text{Backward message } \beta_t(i) = \sum_i \psi_{t+1}(i, j, X_{t+1}) \beta_{t+1}(j)$$

Also Viterbi can be used, because we can do Max-Product. The expensive part is the computation of the exponential summation in $Z(X)$. The forward-backward algorithm computes it efficiently as normalization term of $P(Y_t, Y_{t-1} | X)$. More articulated posteriors interact with $Z(X)$, which is a summation over everything that's not observable, difficult when there's a lot of unobservable variables. Exact inference in CRF other than chain-like is likely to be computationally impractical. We need to approximate: Markov Chain Monte Carlo (sample y rather than estimate $P(y)$) or Variational Belief Propagation (reduce to message passing on trees).

Training LCRF Maximum (conditional) log-likelihood, for training

$$\max_{\Theta} L(\Theta) = \max_{\Theta} \sum_{n=1}^N \log P(y^n | x^n, \Theta)$$

We can substitute the LCRF conditional formulation because $P(y^n | x^n, \Theta) = \frac{1}{Z(X)} \exp(\sum \Theta_k f_k)$

$$L(\Theta) = \sum_n \sum_t \sum_k \Theta_k f_k(Y_t^n, Y_{t-1}^n, X_t^n) - \sum_n \log Z(X^n) \left(-\sum_k \frac{\Theta_k^2}{2\sigma^2} \right)$$

With the last being a regularization term based on $\|\Theta\|^2$.

To get proper marginalization $Z(X^n) = \sum_{y_t, y_{t-1}} \sum_t \exp(\sum_k \Theta_k f_k(y_t, y_{t-1}, x_t^n))$

With $\frac{\partial L(\Theta)}{\partial \Theta_k}$ we can maximize it because typically $L(\Theta)$ cannot be maximized in closed form.

$$\frac{\partial L(\Theta)}{\partial \Theta_k} = \sum_{n,k} f_k(Y_t^n, Y_{t-1}^n, X_t^n) - \sum_{n,t} \sum_{y,y'} f_k(y, y', X_t^n) P(y, y' | X^n) - \frac{\Theta_k}{\sigma^2}$$

We have sum of expectations: the first term is $E[f_k]$ when Y is not random, with samples drawn from a finite dataset (**empirical distribution**), and the right term is $E[f_k]$ using the posterior so the expectation of the feature function under the **model distribution**.

$$\frac{\partial L(\Theta)}{\partial \Theta_k} = E[f_k(y, y', x_t^n)] - E_{P(Y | X, \Theta)}[f_k(y, y', x_t^n)]$$

We need to match those two expectations, meaning that when the gradient is zero these are equal.

There's a regularization term $\sum_k \frac{\Theta_k^2}{2\sigma^2}$ on $\|\Theta\|^2$, a posteriori regularization on the gaussian $P(\Theta)$ with $\mu = 0$ mean and $\sigma^2 I$ variance.

In practice, then, Θ can be learned with stochastic gradient descent or variants.

Applications Linear CRF have various applications: POS-tagging, semantic role identification, bioinformatics. Feature functions have the form $f_k(X_k, Y_k) = I_{y_k=\hat{y}_k} q(X_c)$, with f_k non-zero only for a specific output configuration \hat{y}_k , and then depending only on X_k (the features not shared by the classes). $q(X_c)$ is the observation functions: word begins with capital, ends with -ing...

In computer vision they can be used to define bi-dimensional lattice on images, bg/fg segmentation and to impose constraints.

0.3.11 Bayesian Learning and Variational Inference

Introducing basic concepts of variational learning useful for both generative models and deep learning.

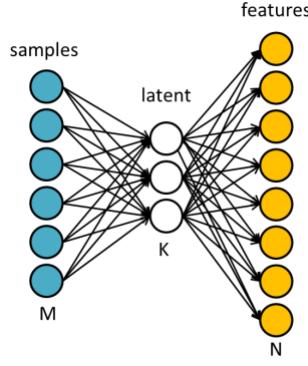
Latent Variables Unobserved random variables that define a hidden generative process of observed data. They explain complex relation between many observable variables. An example: hidden states in HMM/CRF.

Latent variables likelihood

$$P(x) = \int_z \prod_{i=1}^N P(x_i | z) P(z) dz$$

with a graph like $Z \rightarrow X$ so of nodes $\{Z, X\}$ and a directed arc (Z, X) .

Latent Spaces Spaces where high-dimensional data can be represented. Each of the M samples is made of N features represented with $k << N$ dimensions.



The assumption is that latent variables conditional and marginal distribution are more tractable than the joint distribution $P(X)$.

Tractability Hidden variables can make intractable the posteriors. We need stuff that simplify those posteriors: Bayesian learning introduces priors which introduce integrals in the posteriors computation which are not always analytically or computationally tractable.

Kullback-Leiber Divergence

An information theoretic measure of closeness of two distributions p and q

$$KL(q\|p) = E_q \left[\log \frac{q(z)}{q(z|x)} \right] = \langle \log q(z) \rangle_q - \langle \log p(z|x) \rangle_q$$

Tells how the distribution q differs from p . It's a divergence so it is asymmetric, $KL(q\|p) \neq KL(p\|q)$

if q and p high, then good

if q is high and p is low, then it's unhappy

if q is low we don't care (due to the expectation)

The expectation E_q means taking all possible assignments z weighted according to the probability $q(z)$.

Jensen Inequality Property of linear operators on convex/concave functions

$$\lambda f(x) + (1 - \lambda)f(x) \geq f(\lambda x + (1 - \lambda)x)$$

The curve is longer than the line connecting the two points. With concave we have \leq . Applied to probability

$$f(E[X]) \geq E[f(X)]$$

$$\log(E[X]) \geq E[\log(X)]$$

The log-likelihood for a model with a single hidden variable Z and parameters θ with a single sample assumed for simplicity is the following

$$\log P(x|\Theta) = \log \int_z P(x,z|\Theta) dz = \log \int_z \frac{Q(z|\phi)}{Q(z|\phi)} P(x,z|\Theta) dz$$

Q is a distribution, used over z with parameters ϕ and $Q(z|\phi) \neq 0$. That is the definition of expectation, we have $\int_z Q(z|\phi) \cdot \frac{1}{Q(z|\phi)} P(x,z|\Theta) dz$ which is $\sum_z q \cdot g(z)$. Using Jensen we have

$$\log P(x|\Theta) = \log E_Q \left[\frac{P(x,z)}{Q(z)} \right] \geq E_Q \left[\log \frac{P(x,z)}{Q(z)} \right] = E_Q[\log P(x,z)] - E_Q[\log Q(z)] = L(x, \Theta, \phi)$$

So we are lower bounding $\log P(x|\Theta)$ with the expected joint distribution minus the entropy. So we have a lower bound on something I want to maximize, so we can maximize the lower bound. Maximizing this term entails that

we're supported by the data (using the expectation of joint distribution $E_Q[\log P(x, z)]$).

How good is this lower bound? Meaning $\log P(x | \Theta) - L(x, \Theta, \phi) = ?$ We introduce $Q(z)$ by marginalization

$$\int_z Q(z) \log P(x) - \int_z Q(z) \log \frac{P(x, z)}{Q(z)} = \int_z Q(z) \log \frac{P(x)Q(z)}{P(x, z)} = E_Q \left[\log \frac{Q(z)}{P(z | x)} \right] = KL(Q(z | \phi) \| P(z | x, \Theta))$$

Because $\frac{P(x)}{P(x, z)} = \frac{1}{P(z | x)}$. So it's an optimization problem of finding Θ and ϕ that maximize $L(x, \Theta, \phi)$ reduced to a minimization problem with KL.

We can assume the existence of a probability $Q(z | \phi)$ which allows to bound the likelihood $P(x | \Theta)$ from below using $L(x, \Theta, \phi)$.

$L(x, \Theta, \phi)$ is called **variational bound** or **ELBO** (evidence lower bound). The optimal bound is obtained from $KL(Q(z | \phi) \| P(z | x, \Theta)) = 0$ choosing $Q(z | \phi) = P(z | x, \Theta)$. So the problem now becomes maximizing ELBO

$$\max_{\Theta, \phi} \sum_{n=1}^N L(x_n, \Theta, \phi)$$

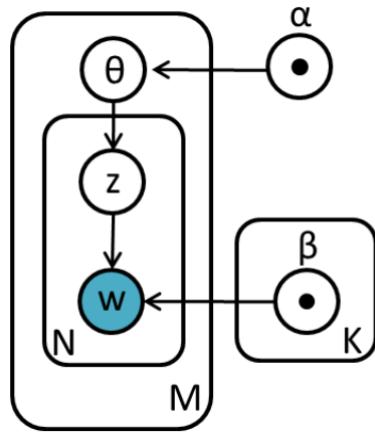
where Θ are the model parameters and ϕ is used in $Q(z | \phi)$.

If $P(z | x, \Theta)$ is tractable then we use it's $Q(z | \phi)$ (optimal ELBO). Otherwise we choose $Q(z | \phi)$ as a tractable family of distributions: find ϕ that minimize $KL(Q(z | \phi) \| P(z | x, \Theta))$ or find ϕ that maximize $L(, \phi)$.

Example Bag of Words representations are classical examples of multinomial data. A BOW dataset X is the $N \times M$ document matrix, with N number of vocabulary items w_j and M is the number of documents d_i and $x_{ij} = n(w_j, d_i)$ the number of occurrences of w_j in d_i .

Often M is very very large ($\simeq 30k$ elements). So we want a smaller representation. Mixture of topics: a topic identifies a pattern in the co-occurrence of multinomial items w_j within the documents. Mixture, so we associate an interpretation (topic) to each item in a document, whose interpretation is then a mixture of the items' topics. We use Latent Variables.

Latent Dirichlet Allocation LDA models a document as a mixture of topics z . We assign one topic z to each item w with probability $P(w | z, \beta)$ and pick a topic for the whole document with probability $P(z | \Theta)$



Each document has its personal topic proportion Θ sampled from a distribution. Θ defines a multinomial distribution, but it is a random variable as well. α is the prior.

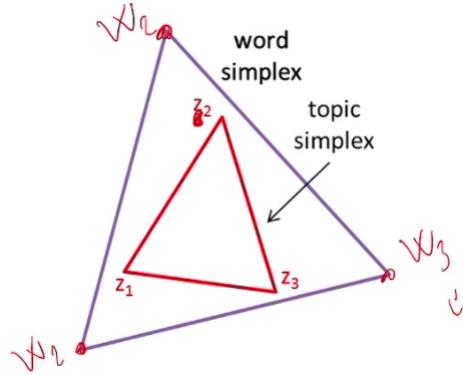
Dirichlet distribution Conjugate prior to multinomial distribution.

If the likelihood is multinomial with a Dirichlet prior, then posterior is Dirichlet.

$$P(\Theta | \alpha) = \frac{\Gamma(\sum_{k=1}^K \alpha_k)}{\prod_{k=1}^K \Gamma(\alpha_k)} \prod_{k=1}^K \Theta_k^{\alpha_k - 1}$$

Γ is the generalization of the factorial. With a big α we make the topics almost equiprobable, while with lower α we get substantially different proportions (less and less topics active). So big α means that every document can express each topic, while low α is almost deterministic. Usually, $\alpha = 1$ or similar but smaller.

LDA finds a set of K projection functions on the K -dimensional topic simplex.



LDA Generative Process For each of the M documents, we choose Θ with $\text{Dirichlet}(\alpha)$ and for each of the N items, we choose a topic z with $\text{Multinomial}(\Theta)$ and pick an item w_j with multinomial probability $P(w_j | z, \beta)$. We get a multinomial topic-item parameter matrix $[\beta]_{k \times V}$

$$\beta_{kj} = P(w_j = 1 | z_k = 1) \text{ or } P(w_j = 1 | z = k)$$

$$P(\Theta, z, w | \alpha, \beta) = P(\Theta | \alpha) \prod_{j=1}^N P(z_j | \Theta) P(w_j | z_j, \beta)$$

It's a completed likelihood with the conditional independence assumption.

Learning Marginal distribution of a document $d = w$

$$P(w | \alpha, \beta) = \int \sum_z P(\Theta, z, w | \alpha, \beta) d\Theta = \int P(\Theta | \alpha) \prod_{j=1}^N \sum_{z_j=1}^k P(z_j | \Theta) P(w_j | z_j, \beta) d\Theta$$

given w_1, \dots, w_M find α, β that maximize.

Key problem is inferring latent variables posterior:

$$P(\Theta, z | w, \alpha, \beta) = \frac{P(\Theta, z, w | \alpha, \beta)}{P(w | \alpha, \beta)}$$

but the denominator is intractable because of couplings between β and Θ under exponentiation in the summation over topics. So we don't use the posterior, we pick a function Q that helps in solving the problem (variational inference).

Variational Inference We write $Q(z|\phi)$ function that is sufficiently similar to the posterior but tractable. It should be such that β and Θ are no longer coupled, fitting ϕ so that is close to the posterior according to KL.

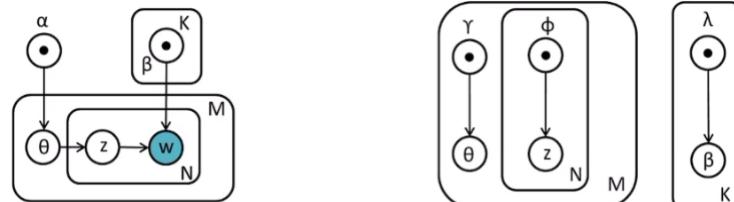
Fast convergence but it's an approximation.

The key idea is to assume that $Q(z|\phi)$ is tractable: **mean-field assumption**.

$$Q(z | \phi) = Q(z_1, \dots, z_K | \phi) = \prod_{k=1}^K Q(z_k | \phi_k)$$

Can be generalized by factorizing on groups of latent variables. Does not contain the true posterior because hidden variables are dependent.

We optimize ELBO using $Q(z|\phi)$ using the factorized distribution. Simple **coordinate ascent inference**: iteratively optimize each variational distribution holding the others fixed, so when learning we use the model on the right, breakdown of the independence.



Given $\Phi = \{\gamma, \phi, \lambda\}$ as **variational approximation parameters**

$$Q(\theta, z, \beta | \Phi) = Q(\theta | \gamma) \prod_{n=1}^N Q(z_n | \phi_n) \prod_{k=1}^K Q(\beta_k | \lambda_k)$$

Variational Expectation-Maximization Find the Φ, Ψ that maximize the ELBO

$$L(w, \Phi, \Psi) = E_Q[\log P(\Theta, z, w | \Psi)] - E_Q[\log Q(\Theta, z, \Psi | \Phi)]$$

by alternate maximization:

Fix Ψ and update variational parameters Ψ^* (**E-Step**)

Fix $\Phi = \Phi^*$ and update the model parameters Ψ^* (**M-Step**)

Repeat until little improvement on the likelihood

Unlike EM, variational EM has no guarantee to reach a local maximizer of L .

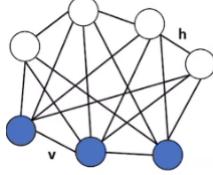
0.3.12 Boltzmann Machines

Examples of Markov Random Fields:

Visible random variables $v \in \{0, 1\}$

Latent random variables $h \in \{0, 1\}$

$s = [vh]$ (concatenation)



Linear energy function

$$E(s) = -\frac{1}{2} \sum_{ij} M_{ij} s_i s_j - \sum_j b_j s_j = -\frac{1}{2} s^T M s - b^T s$$

with symmetric and no self-recurrent connectivity. Model parameters $\Theta = \{M, b\}$ encode the interactions between the variables (observable and not).

Boltzmann machines are a type of Recurrent Neural Networks. They can be interpreted as stochastic neural network: the state of a unit at a given timestep is sampled from a given probability distribution and the network learns a probability distribution $P(V)$ from the training patterns. The network includes both visible v and hidden h units, and the activity is a sample from posterior probability given the inputs (visible data).

Stochastic Binary Neurons Binary output s_j at any time t . Typically discrete time model with time into small Δt intervals. At each time interval $t + 1 = t + \Delta t$ the neuron can emit a spike with probability $p_j^{(t)}$

$$s_j^{(t)} = \begin{cases} 1 & \text{with probability } p_j^{(t)} \\ 0 & \text{with probability } 1 - p_j^{(t)} \end{cases}$$

The key is in the definition of the spiking probability (local potential x_j)

$$p_j^{(t)} \simeq \sigma(x_j^{(t)})$$

The Boltzmann machine has N neurons with binary activations s_j , a weight matrix $M = [M_{ij}]_{i,j} \in \{1, \dots, N\}$ and a bias vector $b = [b_j]_j \in \{1, \dots, N\}$

Local neuron potential x_j is the usual

$$x_j^{(t+1)} = \sum_{i=1}^N M_{ij} s_j^{(t)} + b_j$$

This assuming full connectivity with weight 0 if you want to ignore a neuron in the potential. A chosen neuron fires with spiking probability which is a sigmoid

$$p_j^{(t+1)} = P(s_j^{(t+1)} = 1 | s^{(t)}) = \sigma(x_j^{(t+1)}) = \frac{1}{1 + e^{-x_j^{(t+1)}}}$$

Clearly has Markovian dynamics, $P(s^{t+1} | s^t)$

How does the model state (activation of all neurons) evolve in time?

Parallel Dynamics Assuming we can compute each activation in parallel every Δt , updating each random variable in parallel.

$$P(s^{t+1} | s^t) = \prod_{j=1}^N P(s_j^{t+1} | s^t) = T(s_j^{t+1} | s^t)$$

Yielding a Markov process for state update

$$P(s^{t+1} = s') = \sum_s T(s' | s) P(s^t = s)$$

Glauber Dynamics One neuron at random is chosen for update at each step. No fixed-point guarantees for s , but it has a stationary distribution for the network at equilibrium state when its connectivity is symmetric.

Boltzmann-Gibbs Distribution Undirected connectivity enforces detailed balance condition

$$P(s)T(s' | s) = P(s')T(s | s')$$

Ensures reversible transitions guaranteeing existence of equilibrium distribution (Boltzmann-Gibbs)

$$P_\infty(s) = \frac{e^{-E(s)}}{Z}$$

where $E(s)$ is the energy function and $Z = \sum_s e^{-E(s)}$ is the partition function.

Learning Boltzmann machines can be trained so that the equilibrium distribution tends towards any arbitrary distribution across binary vectors given samples from that distribution. Basically, you can represent any distribution of binary variables.

Couple of simplifications:

bias b is just another row in the weight matrix M

consider only visible random variables, meaning $s = v$

We use probabilistic learning techniques to fit the parameters, i.e. maximizing the log-likelihood

$$L(M) = \frac{1}{L} \sum_{l=1}^L \log P(v_l | M)$$

given the P visible training patterns v_l , the set of all the visible units (so it's a joint distribution v_{l1}, \dots, v_{ln}). We need a way to write the likelihood $P(v_l | M)$, and we can use the Boltzmann-Gibbs. So we can optimize it solving a maximum likelihood problem.

$$\begin{aligned} \log P(v_l | M) &= \log \frac{e^{-E(v)}}{Z} = -E(v) - \log Z \\ \frac{\partial L}{\partial M_{ij}} &\Rightarrow \frac{\partial(-E(v) - \log Z)}{\partial M_{ij}} \Leftrightarrow \end{aligned}$$

Given that $E(v) = -\frac{1}{2}v^T M^T v = -\sum_{ij} M_{ij} v_i v_j$ we get that every M_{ij} will be constant except for the M_{ij} we are differentiating leaving with $v_i v_j$. As for $\log Z$, differentiating the potential function leaves us with $\sum_v P(v | M) v_i v_j$ from $\partial \log \partial \exp \cdot \partial E$

$$\Leftrightarrow v_i v_j - \sum_v P(v | M) v_i v_j = 0$$

The second term is $\sum_v P(v | M) v_i v_j = E_{v_i v_j \in P(v | M)} [v_i v_j] = \langle v_i v_j \rangle_M$ the expectation of the coactivation of two units v_i, v_j when the values of those two units are taken from $P(v | M)$ the distribution learned by the model M .

So for a single l :

$$v_i^l v_j^l - \langle v_i v_j \rangle_M$$

not introducing l in the second term because it's marginalized, all the possible values in all possible configurations, it's an expectation and the data is already included in the formulation.

So $\forall v^l \in L$

$$\frac{\partial L}{\partial M_{ij}} = \frac{1}{L} \sum_{l=1}^L (v_i^l v_j^l) - \langle v_i v_j \rangle_M =$$

We have the clamped expectation under the empirical distribution $\frac{1}{L} \sum_{l=1}^L (v_i^l v_j^l) = E_{v_i^l v_j^l \in L} [v_i^l v_j^l] = \langle v_i v_j \rangle_c$

$$= \langle v_i v_j \rangle_c - \langle v_i v_j \rangle_M = \Delta M_{ij}$$

So we're focusing on the neurons i, j and the link between them M_{ij} , and we increase the connection when both are 1 so when $\langle v_i v_j \rangle_c$ is larger, so Hebbian learning. The second term $\langle v_i v_j \rangle_M$, if they are in disagreement then they are flipped then it's zero, if instead they are coactive (both on or off) then it's close to -1 . It's anti-Hebbian, has the purpose of nearing the expectation of the model towards the reality.

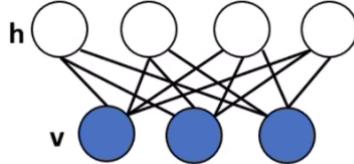
When $\langle v_i v_j \rangle_c - \langle v_i v_j \rangle_M = 0 - 1$, we get that the model thinks that they are correlated while they are different, so the model stops believing that they are correlated.

With hidden variables doesn't change much. We have $s = [hv]$ The wake hebbian term $\sum_h s_i s_j P(h | v)$ and the dream anti-hebbian term $\sum_s s_i s_j P(s)$

$$\frac{\partial P(v | M)}{\partial M_{ij}} = \sum_h s_i s_j P(h | v) - \sum_s s_i s_j P(s) = \langle s_i s_j \rangle_c - \langle s_i s_j \rangle_M = \Delta M_{ij}$$

Again intractable. So we restrict.

Restricted Boltzmann Machines RBM are special Boltzman machines: bipartite graphs and connections only between hidden and visible units, not with "themselves".



This becomes tractable because we can compute the activation of each hidden in parallel and then compute the visible. The energy function is a specialization that highlights the bipartition in hidden and visible units

$$E(v, h) = -v^T M h - b^T v - c^T h$$

Hidden units are conditionally independent given visible units and vice versa

$$P(h_j | v) = \sigma \left(\sum_i M_{ij} v_i + c_j \right)$$

$$P(c_i | v) = \sigma \left(\sum_j M_{ij} h_j + b_i \right)$$

Training is again based on the likelihood maximization

$$\frac{\partial L}{\partial M_{ij}} = \langle v_i h_j \rangle_c - \langle v_j h_i \rangle = \Delta M_{ij}$$

Again, we have data – model which are both expectations that need to be estimated. The first has the sum on just h , the second is a full summation over both v and h .

With a Gibbs sampling approach, for the wake/data term

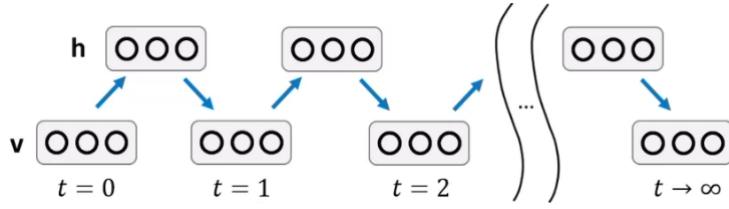
clamp data on v

sample v_i, h_j for all pairs of connected units

Repeat for all elements of dataset

as to stay as much as possible to the empirical data, and for the model/dream term

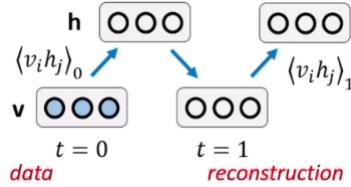
computing a correlation between ideally v^∞, h^∞ , but of course can't wait until the infinity sample and we cut at some sample k so v^k, h^k .



We start with a training vector on the visible units, alternating between updating all the hidden units in parallel and updating all the visible units in parallel

$$\frac{\partial L}{\partial M_{ij}} = \langle v_i h_j \rangle_0 - \langle v_j h_i \rangle_\infty = \text{data} - \text{model}$$

Contrastive-Divergence Learning Because Gibbs sampling converges really slowly: clamp a training vector v^l on visible units. CD-1.



Clamping a training vector v^l on visible units, update all hidden units in parallel, update all the visible units in parallel to get a **reconstruction** and update the hidden units again.

$$\begin{array}{c} \langle v_i h_j \rangle_0 \\ \text{data} \end{array} - \begin{array}{c} \langle v_i h_j \rangle_1 \\ \text{reconstruction} \end{array}$$

Learns a very crude approximation of the gradient of log-likelihood, not even following it closely. Why use it? Because in practice it works well.

0.3.13 Wrap Up

We've seen three families of models

Bayesian Networks: unsupervised data understanding, interpretability but weak on supervised performance. Directed.

Markov Random Fields: undirected version of BN, powerful when knowledge and constraints can be expressed with dictionaries. CRF: supervised way to be generative.
But computationally heavy.

Dynamic Models: topology unfolds on data structure, structured data processing but complex causal relationships.

Tractability Generative models to be used when:

Need interpretability

Need to incorporate prior knowledge

Unsupervised learning or learning with partially observable supervision

Need reusable/portable knowledge

To be avoided when:

Having tight computational constraints

Dealing with raw, noisy, low-level data

Variational inference and sampling are efficient ways to learn approximations to intractable distribution. Neural networks can be used as variational functions or to implement sampling processes.

0.4 Sampling Methods

Probability Recap

Sampling Drawing a set of realisations $X = \{x_1, \dots, x_L\}$ of a random variable x with a distribution $p(x)$. The set contains L samples.

If we have a sampling set we can use it to approximate $p(x)$

$$p(x) \simeq \frac{1}{L} \sum_{l=1}^L I[X_l = i]$$

with $I[c] = 1 \Leftrightarrow c$ is true.

We can also approximate the expectation of a function f

$$E_{p(x)}[f(x)] \simeq \frac{1}{L} \sum_{l=1}^L f(x_l)$$

We need sampling when $p(x)$ is intractable. For example, in Bayesian models the parameters are random variables but the posteriors are often intractable so we can use sampling to obtain the model parameters.

Sampling Procedure as Distributions A sampler S is a procedure that generates a sample set X from a generic distribution $p(x)$. Since X contains realizations of random variables, also X has a probability distribution. We denote with $\hat{p}_S(X)$ the probability to obtain the sample set X from the sampler S .

A **sampler** and its properties **are fully defined by its distribution** $\hat{p}_S(X)$. In general

$$\hat{p}(X) \neq p(x)$$

$p(x)$ is the distribution we would like to sample from, usually intractable

$\hat{p}(X)$ is the distribution over the samples set and defined by the sampling procedure

Let us consider the sampling approximation of the expectation

$$E_{p(x)}[f(x)] \simeq \frac{1}{L} \sum_{l=1}^L f(x_l) = \hat{f}_X$$

Since \hat{f}_X estimates a value, we could ask:

Is \hat{f}_X an unbiased estimator?

How much is the variance of the approximation?

An unbiased estimator $\hat{\Theta}$ of the unknown $\Theta \Rightarrow$ the approximation is exact on average. Meaning: let $\hat{p}(X)$ the distribution over all possible realizations of the sampling set X , then \hat{f}_X is unbiased estimator if

$$E_{\hat{p}(X)}[\hat{f}_X] = E_{p(x)}[f(x)]$$

This is true provided that $\hat{p}(x_l) = p(x_l)$

A sampler with this property is called **valid** because it draws samples from the desired distribution.

The variance of $\hat{f}(X)$ tells us how much we can rely on the approximation computed using the sampling set X . Let

$$\Delta \hat{f}_X = \hat{f}_X - E_{\hat{p}(X)}[\hat{f}_X]$$

then the variance of $\hat{f}(X)$ is given by

$$E_{\hat{p}(X)}[(\Delta \hat{f}_X)^2]$$

we would like low variance meaning that $\hat{f}(X)$ is quite always close to the average, i.e. to $E_{p(x)}[f(x)]$.

So if the sampler has same marginals and draws sample independently we obtain

$$E_{\hat{p}(X)}[(\Delta \hat{f}_X)^2] = \frac{1}{L} \text{Var}_{p(x)}[f(x)]$$

The variance reduces linearly with respect to the number of samples provided that $\text{Var}_{p(x)}[f(x)]$ is finite.

We've shown that we need sampling to approximate expectations and to do inference in Bayesian models. Properties of the sampling procedure depends on $\hat{p}(X)$: valid sampler and low approximation variance.

0.4.1 Univariate Sampling

Easy: random number generator R which produces a value uniformly at random in $[0, 1]$ and $p(x)$. We use $p(x)$ to divide $[0, 1]$ in bins and sample accordingly.

0.4.2 Multivariate Sampling

In the multivariate case $p(x)$ represents a joint distribution over a set of discrete variables $\{s_1, \dots, s_n\}$ each with C states. Hence each sample $x_i \in X$ contains n values.

We build a univariate distribution $p(S)$ where S is a discrete variables with C^n states (all possible combinations) and we can sample from $p(S)$ using the univariate schema.

S	s_1	s_2	s_3	s_4	s_5	$p(S)$
1	1	1	1	1	1	$p(1, 1, 1, 1, 1)$
2	1	1	1	1	2	$p(1, 1, 1, 1, 2)$
3	1	1	1	1	3	$p(1, 1, 1, 1, 3)$
\vdots						
C^n	C	C	C	C	C	$p(C, C, C, C, C)$

The problem is the number of possible states C^n , so computationally unfeasible.

Using the chain rule we can rewrite the joint distribution as a chain of conditional distributions. Then we sample in order:

$$\hat{s}_1 \simeq p(s_1)$$

$$\hat{s}_2 \simeq p(s_2 | \hat{s}_1)$$

...

$$\hat{s}_n \simeq p(s_n | \hat{s}_1, \dots, \hat{s}_{n-1})$$

Each is univariate, so easy, but unfortunately computing the distribution $p(s_i | s_{j < i})$ often requires summation over an exponential number of states.

This approach is called **Ancestral Sampling**. There are cases where it cannot be used.

Example We can apply directly if a distribution $p(s_1, \dots, s_n)$ is already presented as a belief network:



The BN ancestral order tells us the sampling order

$$\{s_1, s_2, s_4\} \prec \{s_3\} \prec \{s_6\} \prec \{s_5\}$$

$$\text{Sample } \hat{s}_1 \simeq p(s_1)$$

$$\text{Sample } \hat{s}_4 \simeq p(s_4)$$

$$\text{Sample } \hat{s}_2 \simeq p(s_2)$$

$$\text{Sample } \hat{s}_3 \simeq p(s_3)$$

$$\text{Sample } \hat{s}_6 \simeq p(s_6)$$

$$\text{Sample } \hat{s}_5 \simeq p(s_5)$$

We obtain a single sample $x^l = \hat{s}_1, \hat{s}_2, \hat{s}_3, \hat{s}_4, \hat{s}_5, \hat{s}_6$.

Suppose a subset s_ϵ of variables that are visible, with $s = s_\epsilon \cup s_{\setminus \epsilon}$ we want to sample from

$$P(s_{\setminus \epsilon} | s_\epsilon) = \frac{p(s_{\setminus \epsilon}, s_\epsilon)}{p(s_\epsilon)}$$

Can we still use Ancestral Sampling?

Clamping variables changes the structure of the bayesian network, and computing the new structure is as complex as running exact inference.

We could run AS on the original structure and discarding any samples which do not match the evidence.

Sampling under evidence is very important, in probabilistic models the inference is based on the posterior which is exactly that. So we need an efficient method of sampling under evidence.

Gibbs Sampling Procedure The idea is to start from a sample $x^1 = \{s_1^1, \dots, s_n^1\}$ and **update only one variable at a time**. Dealing with evidence is easy, we just do not select the visible variables.

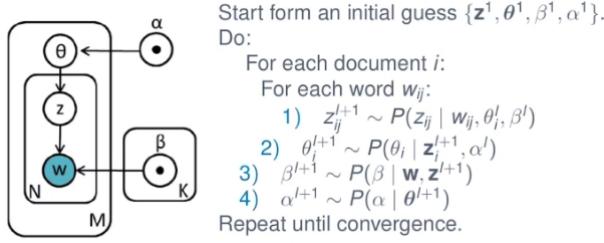
During the $(l+1)$ th iteration:

Select a variable s_j

we sample the new value according to

$$s_j^{l+1} \simeq p(s_j | s_{\setminus j}) = \frac{1}{Z} p(s_j | \text{Parents}(s_j)) \prod_{k \in \text{Children}(s_j)} P(s_k | \text{Parents}(s_k))$$

Which depends only on the Markov blanket of s_j and $s_{\setminus j}$ is clamped to $\{s_1^l, \dots, s_{j-1}^l, s_{j+1}^l, \dots, s_n^l\}$



$\hat{p}(X)$ of Gibbs sampling Considering a set of variables $X = \{x^1, \dots, x^L\}$ obtained with a Gibbs sampler: each sample x^{l+1} is obtained from the previous sample x^l . We can define a probability $q(x^{l+1} | x^l)$, the probability to obtain x^{l+1} given x^l :

$$q(x^{l+1} | x^l) = \sum_{j=1}^N P(\text{Variable } j \text{ is selected}) P(\text{New value } s_j^l \text{ given } x^l)$$

With $P(\text{New value } s_j^l \text{ given } x^l)$ from equation 4 and $P(\text{Variable } j \text{ is selected})$ up to you.

With this we can compute $\hat{p}(X)$

$$\hat{p}(X) = \prod_{l=1}^L q(x^{l+1} | x^l)$$

Markov Chain Monte Carlo Sampling Framework Gibbs sampling is a specialization of the Markov Chain Monte Carlo sampling framework. The idea is to build a Markov Chain whose stationary distribution is $p(x)$. Let $q(x^{l+1} | x^l)$ be the Markov Chain state-transition distribution, if the Markov Chain is

Irreducible, meaning it's possible to reach any state from anywhere

Aperiodic, meaning at each time-step we can be anywhere

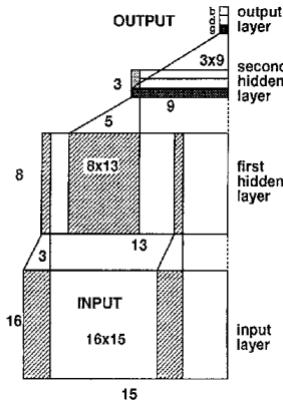
then it has a unique stationary distribution. So we define $q(x^{l+1} | x^l)$ such that the Markov Chain converges to $p(x)$.

0.5 Convolutional Neural Networks

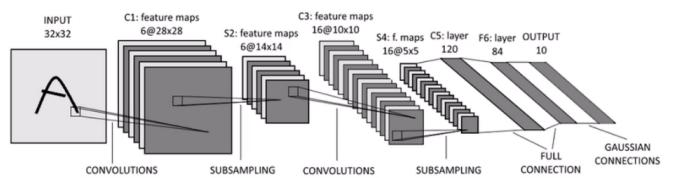
Deep Learning Effectively is learning the representation of data.

CNNs The principle is complex neurons that take the outputs of simple neurons and assemble it into more complex structures.

For sequences:

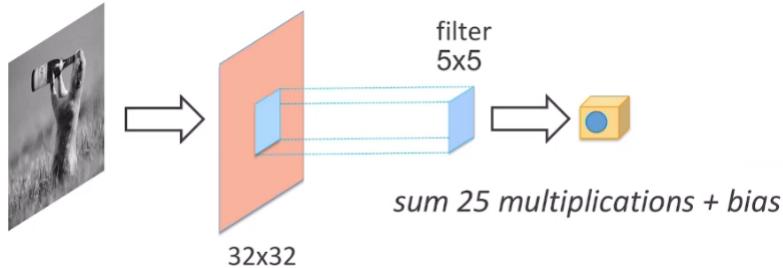


For images:

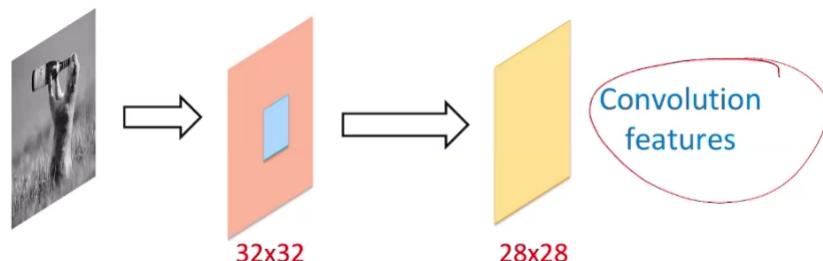
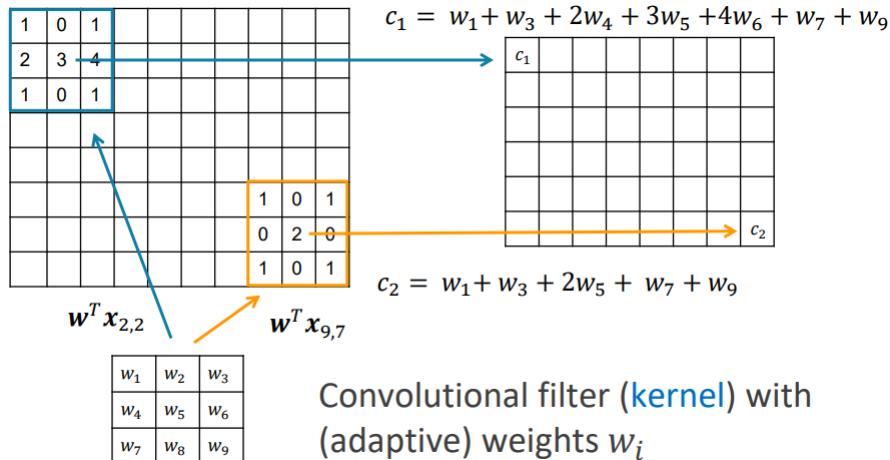


Apply a bank of 16 convolution kernels to sequences (window of 15 elements) trained by backpropagation with parameter sharing.

Dense Vector Multiplication E.g. take the 32x32 image and reshape into a vector of 3072. Input-sized weight vector for each hidden neuron W of e.g. 100 neurons: 100×3072 weights, a huge amount of parameters. That's why we want convolutional filters.



A set of weights that are applied to a 5×5 region of the image. It's a **kernel** that outputs into the **feature map**. No longer a 1:1 correspondence between a pixel and a parameter.



With a number of slices equal to the number of image channels. Basically a filter per channel color, but they are convolved together still giving a 28×28 feature map even with 3 color channels for example. The **convolution map stays bidimensional**.

Stride Basic convolution slides the filter on the image one pixel at the time (stride = 1). The slide value is an hyperparameter. The stride reduces the number of multiplication, sumbsampling the image. With a stride of 2 for example we would end up with a 14×14 map.

Activation Map Size What is the size of the image after applying a filter with a given stride S and size $K \times K$, on a image $W \times H$?

For example, a 3×3 filter with stride 2, $K = 3, S = 2$ on a 7×7 image: output image is 3×3 .

$$W' = \frac{W - K}{S} + 1$$

$$H' = \frac{H - K}{S} + 1$$

Zero Padding Add columns and rows of zeros to the border of image to handle border pixels, padding of P rows/columns.

$$W' = \frac{W - K + 2P}{S} + 1$$

Zero padding is also used to retain the original size of the image

$$P = \frac{K - 1}{2}$$

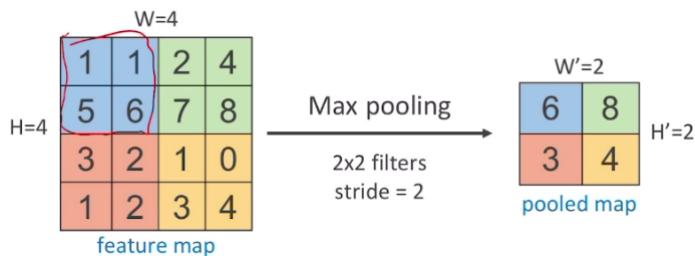
Feature Map Transformation Convolution is a linear operator, applying an element-wise nonlinearity we obtain a transformed feature map.

$$\max(0, w^T x_{ij} + b)$$

The above is ReLU, used because of the simplicity of computing the gradient.

Pooling Operates on the feature map to make the representation: smaller (subsampling) and robust to some transformations.

Very simple:

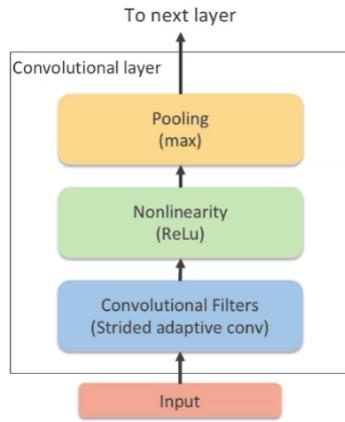


Max pooling is the one used more frequently, but there's more: average pooling, L2-norm pooling, even random pooling.

It's uncommon to use zero padding with pooling.

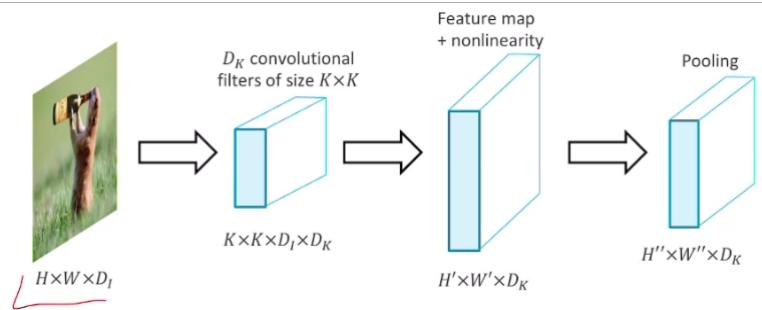
$$W' = \frac{W - K}{S} + 1$$

Convolutional Architecture



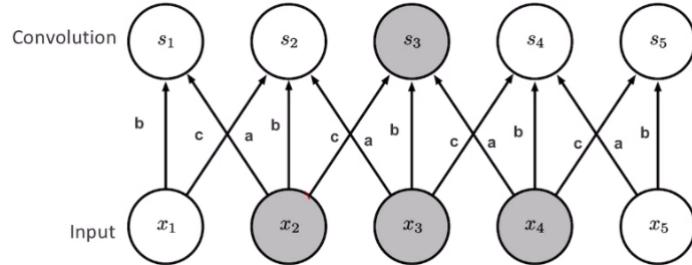
Usually use a certain number of convolutional layers ending in a small final image feeded into some fully connected layers.

Filter Banks



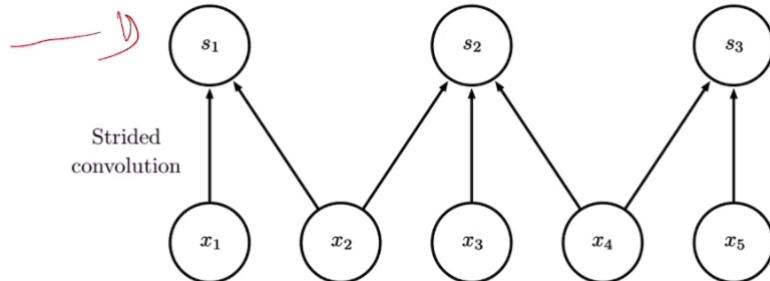
The number of parameters in this model is $K \times K \times D_I \times D_K$ because pooling is non parametric and feature maps have no parameters, so the parameters are just in the filters.

CNNs as Sparse Neural Networks Let's use a 1D input sequence to ease graphics.

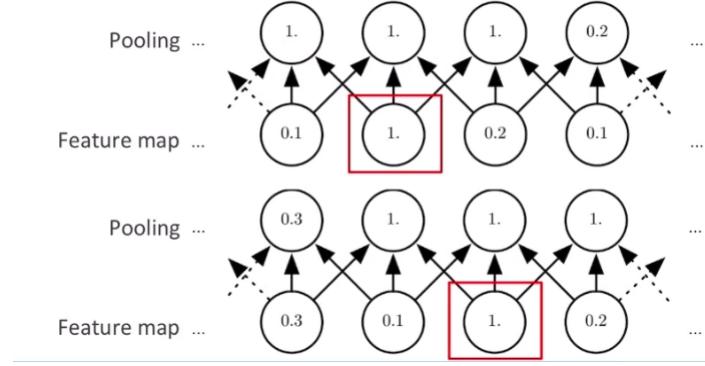


Convolution amounts to sparse connectivity (reduce parameters) with parameter sharing (enforces variance). s_i is a convolutional neuron with a kernel size 3×1 .

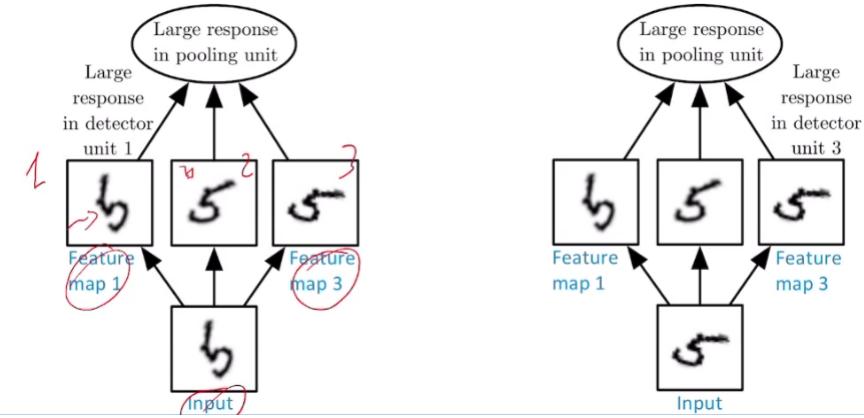
Strided Convolution A neuron after every stride, making connectivity sparser.



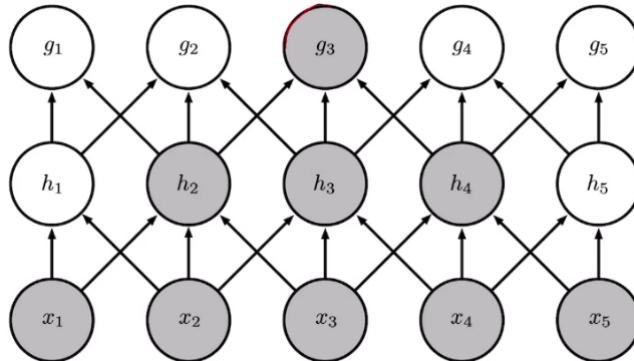
Pooling Just another neuron.



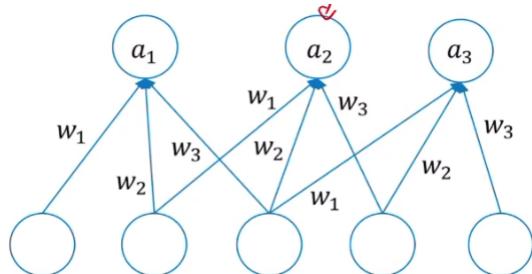
Cross-Channel Pooling and Spatial Invariance



Hierarchical Feature Organization

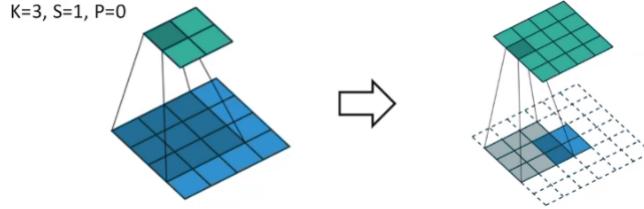


CNN Training Training by variants of the standard backpropagation that account for the fact that connections share weights (convolution parameters).



The gradient Δw_i is obtained by summing the contributions from all the connections sharing that weight.
We can write convolution as dense multiplication with shared weights.

Deconvolution



We can obtain the transposed convolution using the same logic of the forward convolution. If you had no padding in the forward convolution you need to use much padding when doing the transposed convolution.

ReLU ReLU helps counteract gradient vanish: sigmoid first derivative vanishes as we increase or decrease z , ReLU first derivative is 1 when unit is active and 0 otherwise. ReLU second derivative is 0 (no second order effects). Easy to compute and favors sparsity.

VGGNet Standardized convolutional layer:

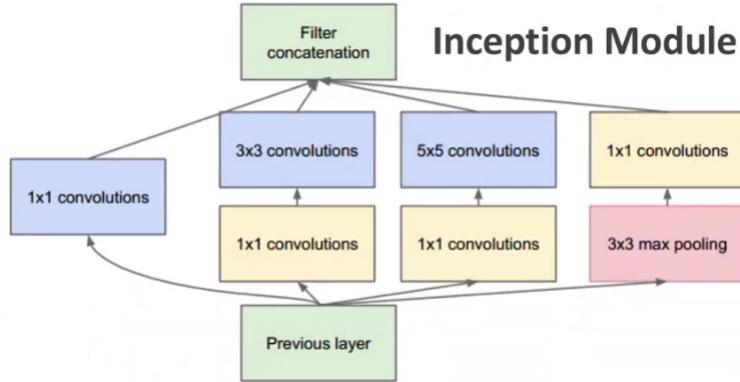
3×3 convolutions of stride 2

2×2 convolutions of stride 2

16 convolutional layers with 3 fully connected layers

Not very good because very limited, need to push for diversity.

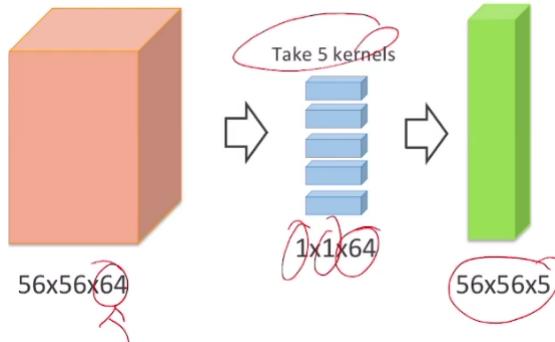
GoogLeNet With the Inception module, based on the idea of diversifying the convolution.



But many parameters: number of channels times number of convolutional layers times the dimensions. If not careful the application of this module explodes the number of parameters.

Kernels of different size to capture details at varied scale, aggregated before sending to next layer.

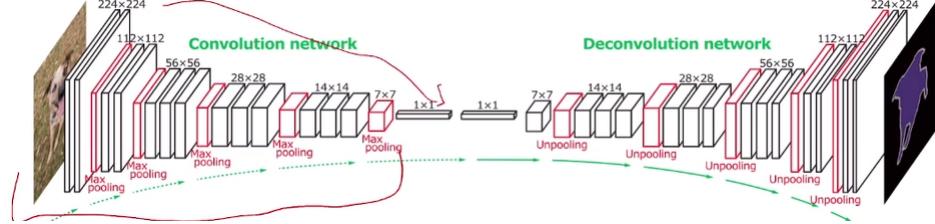
1×1 convolutions are helpful.



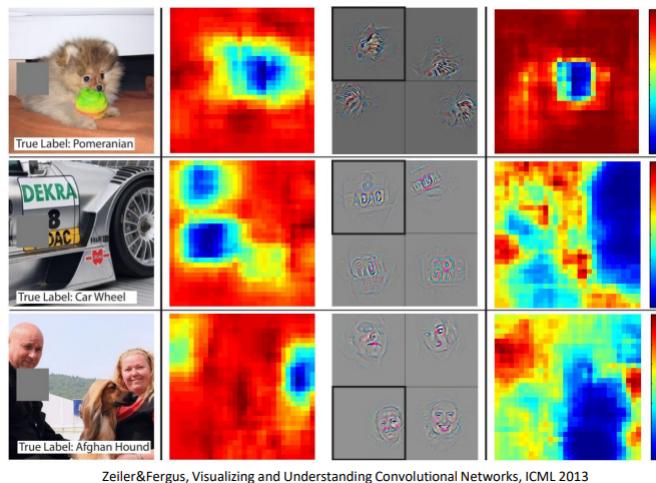
Only 5M parameters, followed by v2, v3 and v4 which added more filter factorization and introduced heavy use of Batch normalization.

Batch Normalization Very deep neural networks are subject to internal covariate shift. Distribution of inputs to a layer N may vary (shift) with different minibatches due to adjustments at layer $N - 1$. Layer N can get confused by this, so (simplifying) we normalize for mean and variance in each minibatch.

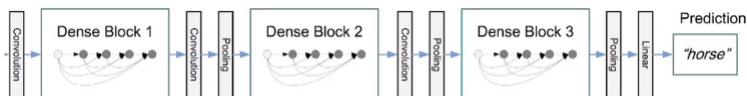
Deconvolution Network Attach a DeConvNet to a target layer, pluggin an input and forward porpagation activations until layers, then backpropagate on the DeConvNet and see what parts of the reconstructed image are affected.



Occlusions Measure what happens to feature maps and object classification if we occlude part of the image. Slide a grey mask across the image and project back the response of the best filters using deconvolution.

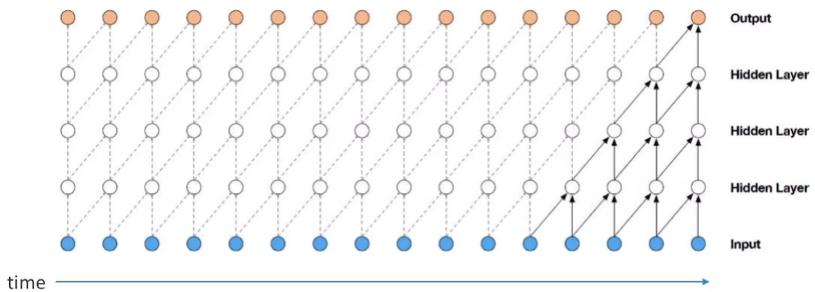


Dense CNN



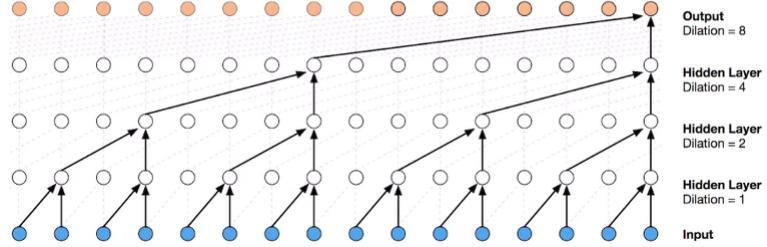
Gradient flows well in bypass connections, and each layer in the dense blocks has access to all the information from previous layers.

Causal Convolutions Basically meaning asymmetrical, to prevent a convolution to see into the future.



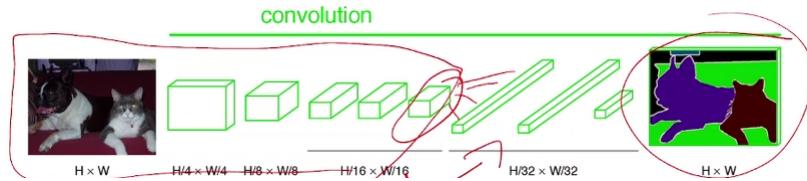
Problem is the context size grows slow with depth. So Dilated Convokutions

$$(I * K)(i, j) = \sum_m \sum_n I(i - lm, i - ln)K(m, n)$$

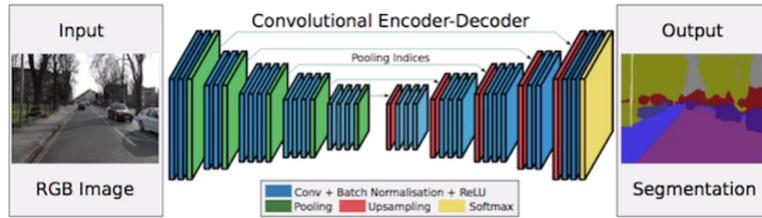


Basically convoluting every now and then. Similar to striding but size is preserved.

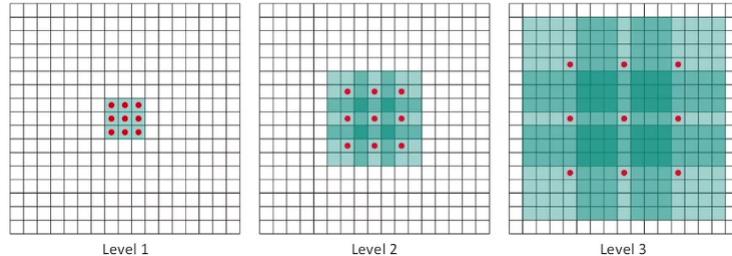
Fully Convolutional Networks Due to not maintaining the original dimension we cannot perform semantic segmentation, so we need **Fully Convolutional Networks**



Deconvolutional Architecture Basically you decompress the segmentation.



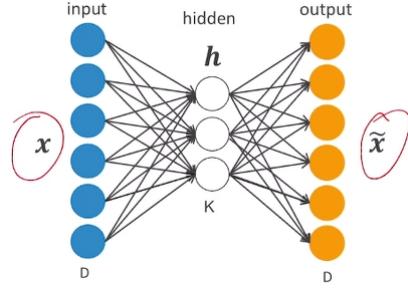
With Dilated Convolution out of the temporal domain we can perform semantic segmentation efficiently. For example with 3×3 convolutions with no pooling at each level:



0.6 Autoencoders

The first and the latest deep learning model.

Basic Autoencoder Train a model to reconstruct the input, passing through some form of **information bottleneck**.



h is known as **latent space projection**, like with probabilistic models.

The architecture is composed of the encoder $f(x \rightarrow h)$, and the decoder $g(h \rightarrow \tilde{x})$. In principle $K \ll D$, the bottleneck. Or you can penalize the model by making h sparsely active.

Neural Autoencoders Generally we want to train nonlinear AEs with possibly $K > D$ that do not learn trivial identity. Regularized AE:

Sparse Autoencoder Add a term to the cost function to penalize h (want the number of active units to be small)

$$J_{SAE}(\Theta) = \sum_{x \in S} (L(x, \tilde{x}) + \lambda \Omega(h))$$

Typically norm 1

$$\Omega(h) = \Omega(f(x)) = \sum_j |h_j(x)|$$

Probabilistic interpretation: training with regularization is MAP inference

$$\max \log P(h, x) = \max(\log P(x | h) + \log P(h))$$

MAP is formed by the likelihood plus the prior $P(h)$

$$P(h) = \frac{\lambda}{2} \exp\left(-\frac{\lambda}{2}|h|_1\right) \Leftrightarrow \log P(h) = \lambda|h|_1 = \Omega(h)$$

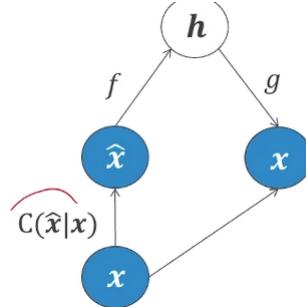
Denoising Autoencoders Train the AE to minimize the function

$$L(x, g(f(\hat{x})))$$

where \hat{x} is a version of the original input corrupted by some noise process $C(\hat{x} | x)$, so the usual $\hat{x} = x + \epsilon$ with ϵ from a gaussian of mean 0 and variance 1.

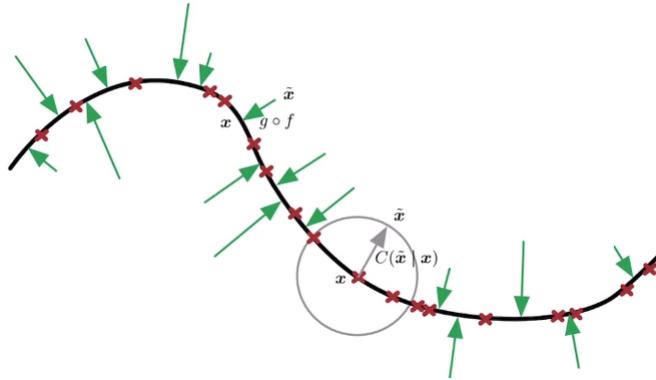
Key intuition: learned representations should be robust to partial destruction of the input.

Probabilistic interpretation:



Learns the denoising distribution $P(x | \hat{x})$ by minimizing $-\log P_d(x | h = f(\hat{x}))$.

Manifold Learning: learning a vector field (green arrows) approximating the gradient of the unknown data generating distribution.



$$C(\hat{x} | x) = N(\hat{x} | x, \sigma^2)$$

$$g(h) - x \propto \frac{\partial \log p(x)}{\partial x}$$

Remembering that h is the encoding $h(x)$ depending on x . So basically compares the reconstruction with the original data.

The Manifold Assumption: assume data lies on a lower dimensional non-linear manifold since variables in data are typically dependent. Regularized AR can afford to represent only variations that are needed to reconstruct training examples. AR mapping is sensitive only to changes in manifold direction.

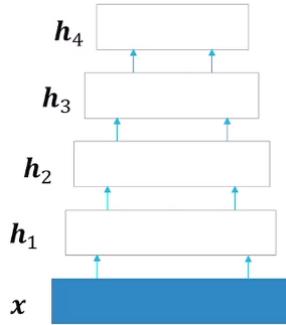
Contractive Autoencoder Penalize encoding function for input sensitivity.

$$J_{CAE}(\Theta) = \sum_{x \in S} (L(x, \tilde{x}) + \lambda \Omega(h))$$

$$\Omega(h) = \Omega(f(x)) = \left\| \frac{\partial f(x)}{\partial x} \right\|_F^2$$

You can as well penalize on higher order derivatives.

0.6.1 Basic Autoencoders

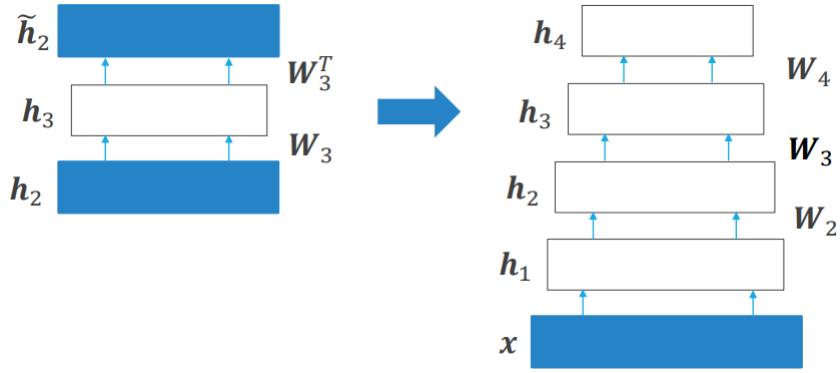


Unsupervised training

Hierarchical autoencoder

Extracts a representation of inputs that facilitates: data visualization, exploration, indexing... also facilitates the realization of a supervised task: adding another layer on top we can perform supervised learning using the representation learned by the autoencoder.

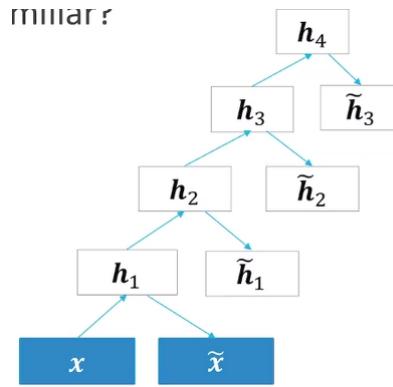
Unsupervised Layerwise Pretraining Incremental unsupervised construction of the deep Autoencoders.



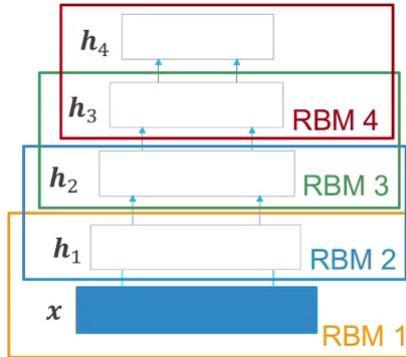
Train h_1 without \tilde{x} getting all the encoding vectors. Then train a new layer h_2 reconstructing h_1 in output ad so on, obtaining the desired structure.

At the end, fine tune the whole autoencoder to optimize input reconstruction. You can use backpropagation, but it remains a supervised task.

If we rearrange the graph we obtain a stack of restricted Boltzmann machine:



This is called a **Deep Belief Network**: a stack of pairwise Restricted Boltzmann Machines.



A DBN it's **not recurrent**, is a deep autoencoder but not a deep RBM.

Training of a Deep Boltzmann Machine requires attention because of the recurrent interactions from higher layers to the bottom.

$$P(h_j^1 | x, h^2) = \sigma \left(\sum_j W_{ij} x_i + \sum_m W_{jm}^2 h_m^2 \right)$$

$$P(x_i | h^1) = \sigma \left(\sum_j W_{ij}^1 h_j^1 \right)$$

To train this, first pretrain the first layers, meaning fitting this model:

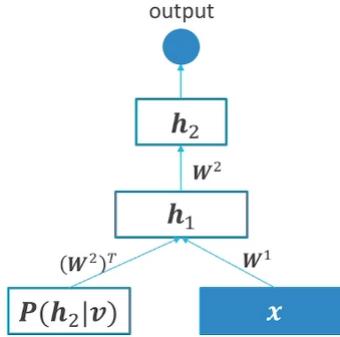
$$P(x | \Theta) = \sum_{h^1} P(h^1 | W^1) P(x | h^1, W^1)$$

Then pretrain the second layer changes h^1 prior by

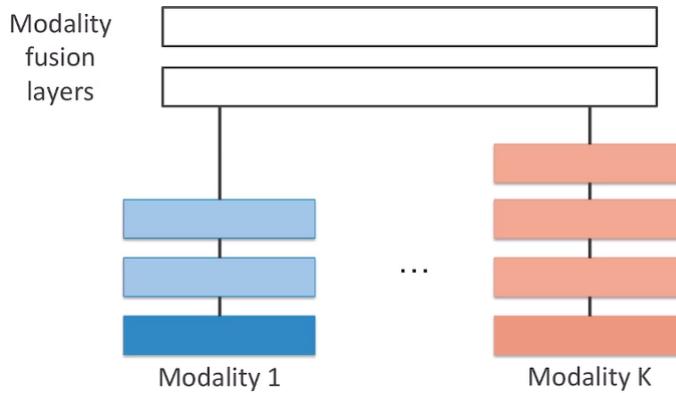
$$P(h^1 | W^2) = \sum_{h^2} P(h^1, h^2 | W^2)$$

A trick, averaging the two models of h^1 can be approximated by taking half contribution from W^1 and half from W^2 . Using full both would double count x contributions as h^2 depends on x .

Discriminative Fine Tuning The pretrained DBM matrices can be used to initialize a deep autoencoder. Add input from h^2 to the first hidden layer, add output layer and fine tune the RBM matrices by backpropagation.



Multimodal DBM For example layer fusion.



0.7 Gated Recurrent Networks

Main motivation: difficulty in learning long-term dependencies. Also gradient issues (exploding/vanishing gradients), solved with constant error propagation and adaptive learning.

Sequences are variable sized data describing sequentially dependent information.

RNN Design

Inductive Bias/Expressiveness: network structure influences the sequential data processing.

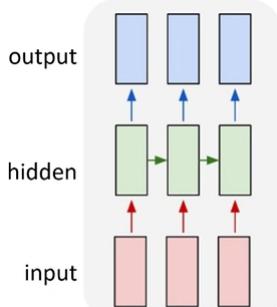
Training: the network should be easy to train.

Depends on architecture, initialization and learning algorithm.

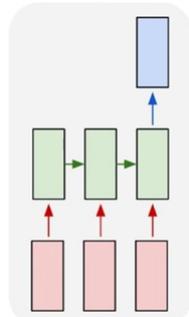
Computational Efficiency: the network should be efficient.

Supervised Recurrent Tasks

Element-to-Element: an output for each element of the input. For example a program where we classify at each timestep.

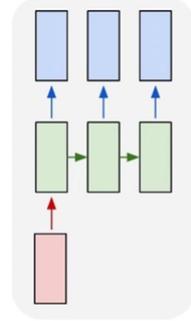


Sequence-to-Element

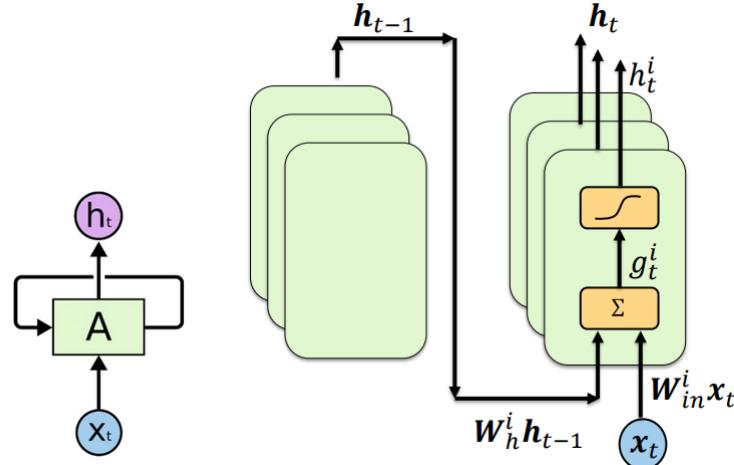
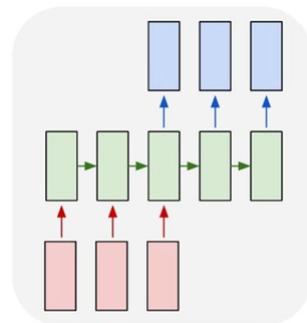


Vanilla RNN

Item-to-Sequence: for example given the genre produce a song, generative models in general



Sequence-to-Sequence: for example machine translation



Given the input, there's an update function A that uses the hidden state h_t as input to the next one.

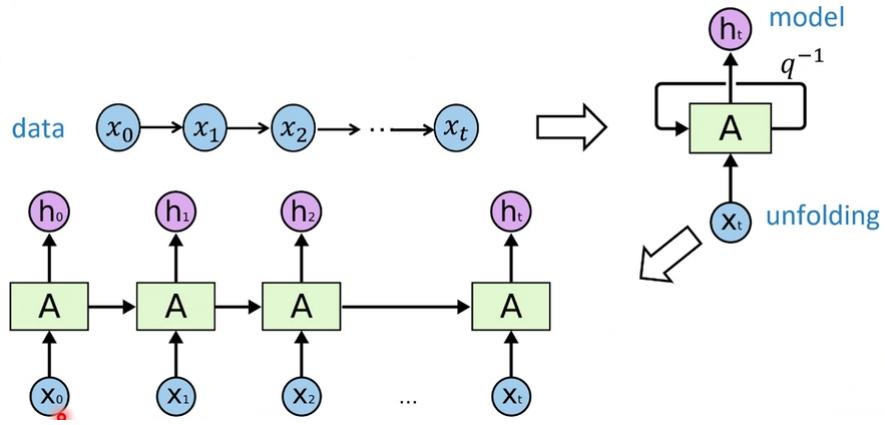
$$y_t = f(W_{out}h_t + b_{out})$$

$$h_t = \tanh(g_t)$$

$$g_t(h_{t-1}, x_t) = W_h h_{t-1} + W_{in} x_t + b_h$$

Unfolding RNN

Forward pass: should be familiar with the unfolding/unrolling on the data.



To be successful, the hidden state h_t of the RNN should be able to summarize the information on the history of the input signal up to time t . In practice, learning long-term dependencies is really difficult, as time grows there's little residual information of the input inside of the memory. When the time gap between the observation and the state grows there is little residual information of the input inside of the memory.

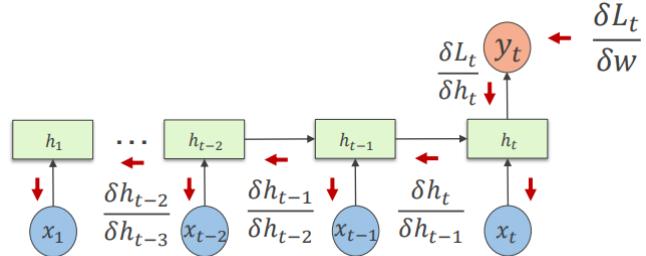
Exploding/Vanishing Gradients Gradients propagated over many stages tend to:

(Often) Vanish \Rightarrow no learning

(Rarely) Explode \Rightarrow instability and oscillations

Weights are shared between timesteps.

Backward Propagation



The gradient is

$$\frac{\partial L_t}{\partial W} = \sum_{k=1}^t \frac{\partial L_t}{\partial h_t} \cdot \underbrace{\frac{\partial h_t}{\partial h_k}}_{\frac{\partial h_t}{\partial h_{t-1}} \cdot \frac{h_{t-1}}{\partial h_{t-2}} \cdot \dots \cdot \frac{\partial h_{k+1}}{\partial h_k}} \cdot \frac{\partial h_k}{\partial W}$$

Inside $\frac{\partial h_t}{\partial h_k}$ we have the chain rule

$$\frac{\partial L_t}{\partial W} = \sum_{k=1}^t \frac{\partial L_t}{\partial h_t} \cdot \left(\prod_{l=k}^{t-1} \frac{\partial h_{l+1}}{\partial h_l} \right) \cdot \frac{\partial h_k}{\partial W}$$

So the gradient is a recursive product of hidden activation gradients (Jacobian). We want to bound the update to compensate for the vanishing/exploding gradients. We have

$$h_l = \tanh(W_h h_{l-1} + W_{in} x_l)$$

, so

$$\frac{\partial h_{l+1}}{\partial h_l} = D_{l+1} W_{hl}^T$$

where the activation Jacobian is

$$D_{l+1} = \text{diag}(1 - \tanh^2(W_h h_l + W_{in} x_{l+1}))$$

meaning a diagonal matrix with $\tanh^{-1}(\partial_i^t)$ on the diagonal.

$$\frac{\partial L_t}{\partial h_k} = \frac{\partial L_t}{\partial h_t} \left(\prod_{l=k}^{t-1} \frac{\partial h_{l+1}}{\partial h_l} \right) = \frac{\partial L_t}{\partial h_t} \prod_{l=k}^{t-1} D_{l+1} W_{hl}^T$$

We're interested in the magnitude

$$\left\| \frac{\partial L_t}{\partial h_k} \right\| = \left\| \frac{\partial L_t}{\partial h_t} \right\| \prod_{l=k}^{t-1} \sigma(D_{l+1}) \sigma(W_{hl}^T)$$

We bound the spectral radius σ . Can shrink to zero or increase exponentially:

$\sigma < 1 \Rightarrow$ vanishing

$\sigma > 1 \Rightarrow$ exploding

Gradient Clipping for Exploding Gradients Easiest to solve, by just limiting the gradient. With $g = \frac{\partial L_t}{\partial W}$ our gradient and Θ_0 our max value for the gradient

$$\|g\| > \Theta_0 \Rightarrow g = \frac{\Theta_0}{\|g\|} g$$

Rescaling doesn't work for gradient vanish as total gradient is a sum of time dependent gradients (preserving relative contribution from each time makes it exponentially decay).

Constant Error Propagation Solution seem to be having the Jacobian with spectral radius $\sigma = 1$.

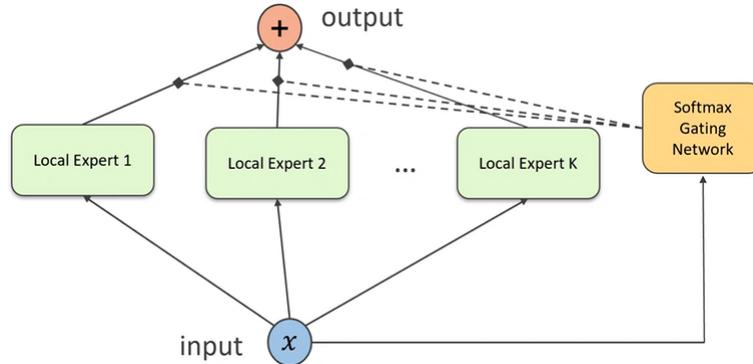
Activation functions The popular choices (sigmoids, tanhs) are always contractive ($\sigma < 1$). Alternatives: modReLU (ReLU generalized to the C domain) or identity.

Recurrent Weights Possible to achieve $\sigma = 1$

Orthogonal matrices $W^T W = 1$

Unitary matrices $W^H W = 1$

Gating Units Mixture of experts, origin of gating.



The softmax gating network tries to recognize the best expert to apply.

Forget Gate Constant Error Carousel (CEC)

Identify activation function

Identity weight matrix $h_t = h_{t-1} + \hat{c}(x_t)$

No forgetting

Hidden state saturation

CEC plus forget gate

CEC

Forget gate to "soft reset" units

$$f_t = \sigma(W_{fh}h_{t-1} + W_{fx}x_t + b_f)$$

$$h_t = f_t \cdot h_{t-1} + \hat{c}(x_t)$$

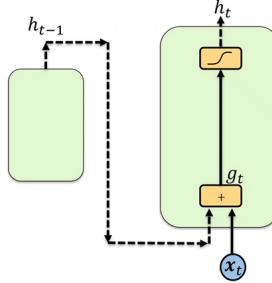
Adaptively forgets the past

Avoid saturation

No guarantees about constant propagation.

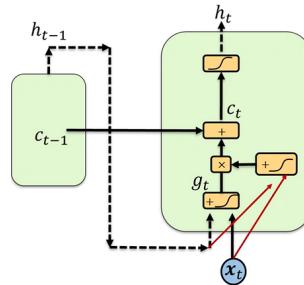
0.7.1 LSTM Cell

Long-Short Term Memory Cell Let's start with the vanilla RNN Unit



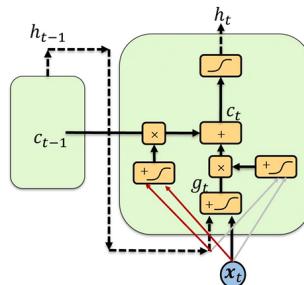
We introduce a linear recurrent update, with an additional hidden state which is the cell state c_t with current input x_t .

The **Input Gate** now controls how the input contributes to the internal state $I_t(x_t, h_{t-1})$ (logistic sigmoid)



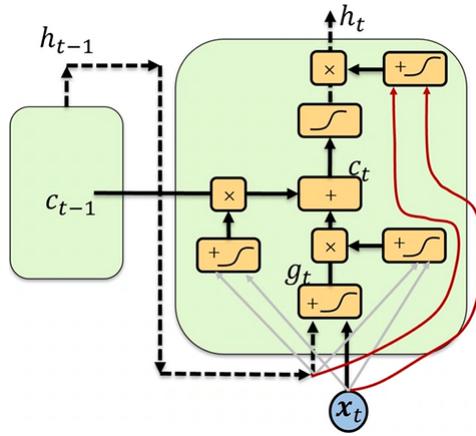
The **Forget Gate** controls how past internal state c_{t-1} contributes to c_t

$$F_t(x_t, h_{t-1})$$



The **Output Gate** controls what part of the internal state is propagated out of the cell

$$O_t(x_t, h_{t-1})$$



LSTM in Equations

Compute activation of input and forget gates

$$I_t = \sigma(W_{Ih}h_{t-1} + W_{In}x_t + b_I)$$

$$F_T = \sigma(W_{Fh}h_{t-1} + W_{Fin}x_t + b_F)$$

Compute input potential and internal state

$$g_t = \tanh(W_hh_{t-1} + W_{in}x_t + b_H)$$

$$c_t = f_T \odot c_{t-1} + I_t \odot g_t$$

(\odot element-wise multiplication)

Compute output gate and output state

$$O_t = \sigma(W_{Oh}h_{t-1} + W_{Oin}x_t + b_O)$$

$$h_t = O_t \odot \tanh(c_t)$$

Deep LSTM LSTM Layers extract information at increasing levels of abstraction (enlarging context)

Training LSTM Original LSTM training algorithm was a mixture of RTRL and Backpropagation Through Time:

BPTT on internal state gradient

RTRL-like truncation on other recurrent connections

No exact gradient calculation

All current LSTM implementation use full BPTT training, typically with Adam or RMSProp optimizer.

Regularizing LSTM

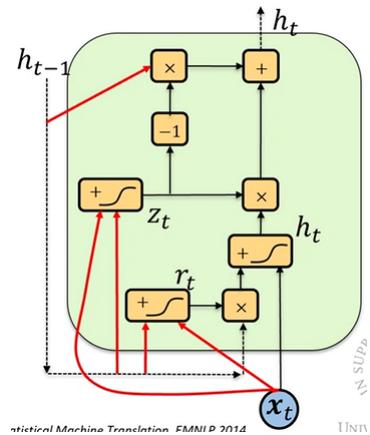
Dropout: randomly disconnect units from the network during training. Regulated by unit dropping hyperparameters, preventing **units coadaptation** (the relying of a unit on other units). The units are dropped for the whole sequence.

Need to adapt prediction phase.

Practicalities With a minibatch we have the problem of the length of the sequences.

0.7.2 GRU Cell

Gated Recurrent Unit



Reset and update gates are coupled, act as input and forget gates

$$z_t = \sigma(W_{zh}h_{t-1} + W_{zin}x_t + b_z)$$

$$r_t = \sigma(W_{rh}h_{t-1} + W_{rin}x_t + b_r)$$

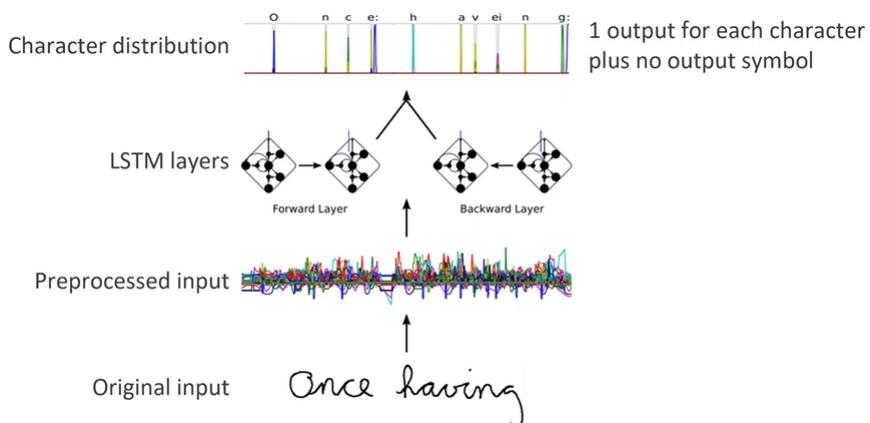
Reset acts directly on output state (no internal state and no output gate. \odot is the element-wise multiplication.

$$h_t = (1 - z_t) \odot h_{t-1} + z_t \odot h_t$$

$$h_t = \tanh(W_{hh}(r_t \odot h_{t-1})W_{hin}x_t + b_h)$$

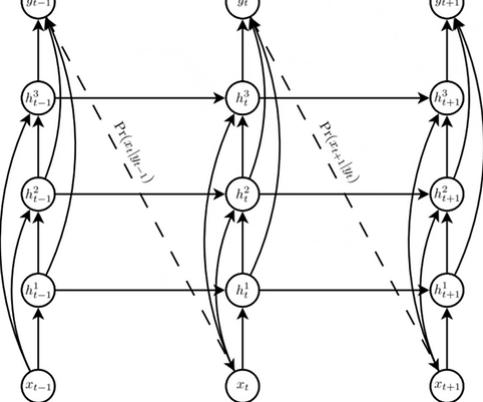
0.7.3 Applications

Bidirectional LSTM for Character Recognition

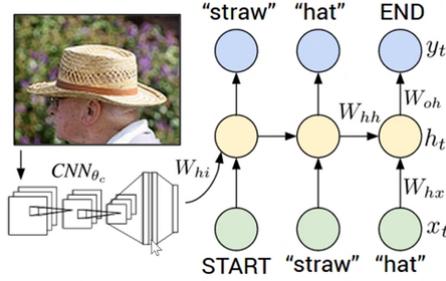


Language Modeling

Typical benchmark of RNNs.



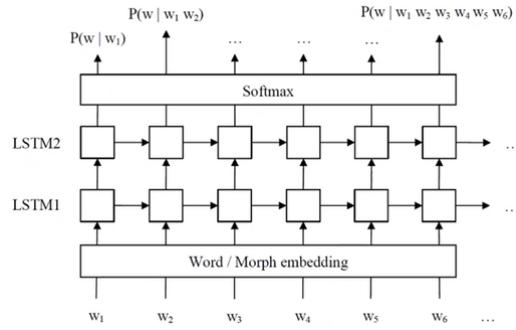
More Differentiable Compositions CNN-LSTM composition used for **image-to-sequence** (NeuralTalk)



No only for sequential/structured data: can also used to generate image of digits by learning to sequentially add color to a canvas.

0.7.4 Advanced Topics

Language Modeling - Dynamic Evaluation Language modeling is an unsupervised task, you can train the model at test time. Useful when you have domain shifts inside a large document.



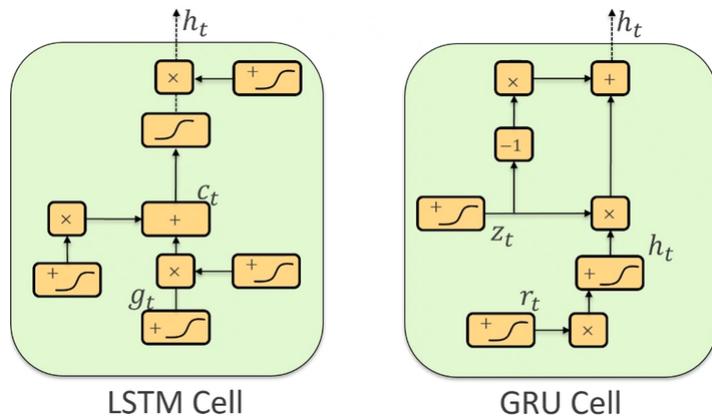
Meaning the backward pass and gradient descent update during test time, this because it's unsupervised.

Compression and Online Adaptation Take a large document and a randomly initialized RNN.

Compression: train and compress the file with arithmetic encoding.

Decompression: train the same random net and use the compressed file for lossless decompression.

Seq2Seq Models

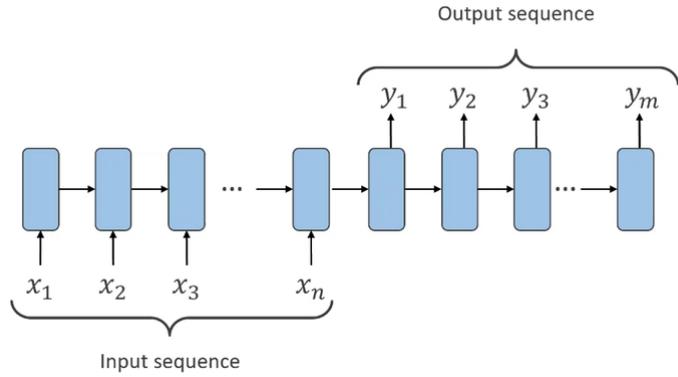


Gated RNN are excellent to handle size/topology varying data in input, but how can we handle size/topology varying outputs? **Sequence-to-sequence models**.

Structured data is compound information: efficient processing needs the ability to focus on certain parts of such information. **Attention mechanism**.

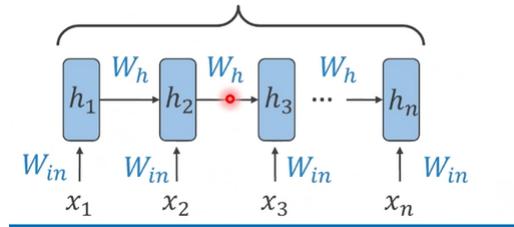
Sequence Transduction Input/output both sequences, maybe with different lengths. Example: machine translation. How to model the context?

We could take a single recurrent model, process the entire input sequence to encode it (and ignoring the outputs in this phase) and then process the output sequence (usually without knowing in advance how long it will be, and giving dummy inputs):

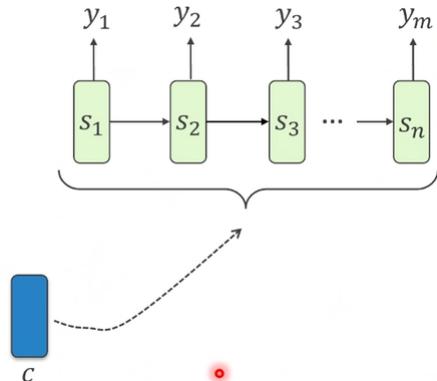


This idea doesn't really work well, because of forgetting. We need two separated models: **encoder-decoder schema**.

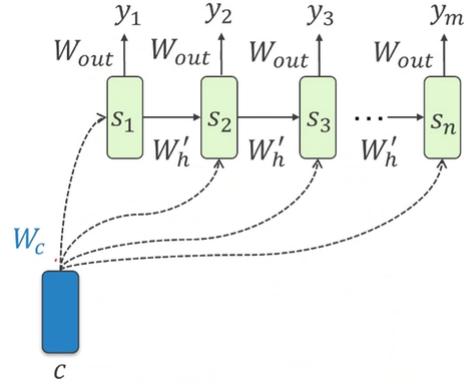
Encoder which builds a c , originally $c = h_n$



Decoder, LSTM/GRU layer of K cells seeded by the context vector c

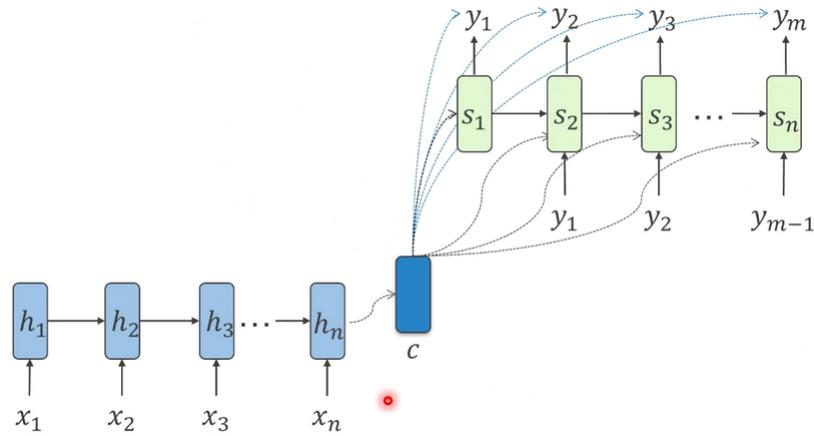


If we share the parameters between encoder and decoder, we can take $s_1 = c$ (or at least assume that c and s_1 have compatible sizes)



$$s_i = f(c, s_{i-1}, y_{i-1})$$

Different approaches to build this in practice. The problem is the risk of losing memory of c soon. It's better to work on a one-step-ahead scheme. Remember teacher forcing only at training time.



Encoder-decoder can share parameters, but it's uncommon

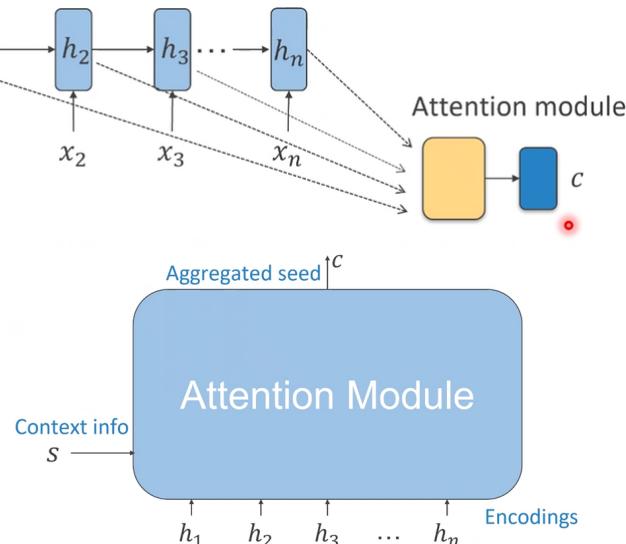
$$E_{en} = D_{en}^{-1}$$

But can be useful especially on lower resources.

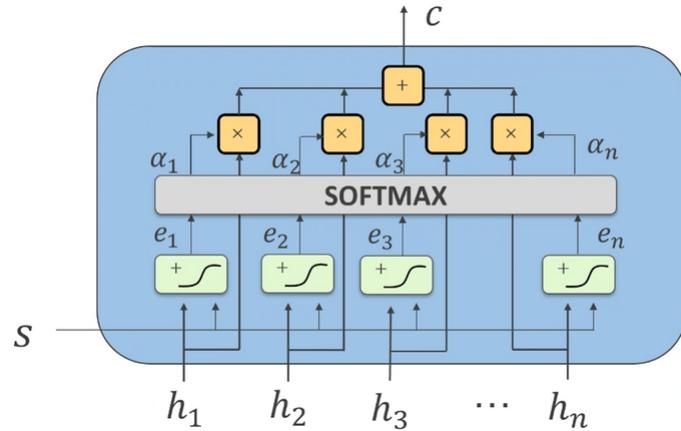
Encoder-decoder can be trained end-to-end but also independently.

Attention We would like to have alignment in the hidden states.

Encoder-decoder scheme assumes the hidden activation for the last input element summarizes sufficient information to the output: it's a bias toward the recent past, and it doesn't work well. Other parts of the input might be very informative for the task, possibly **elements appearing very far from sequence end**.



Context information S is used to remind what's important, used to look at different encodings. What's inside the black box? Gates!



The tanh layer fuse each encoding with the current context s

$$e_i = a(s, h_i)$$

e_i is the relevance between the context s and the hidden state h_i .
The we have the differentiable softmax max selecto operator

$$\alpha_i = \frac{e^{e_i}}{\sum_j e^{e_j}}$$

Then we aggregate with the seed by soft attention voting

$$c = \sum_i \alpha_i h_i$$

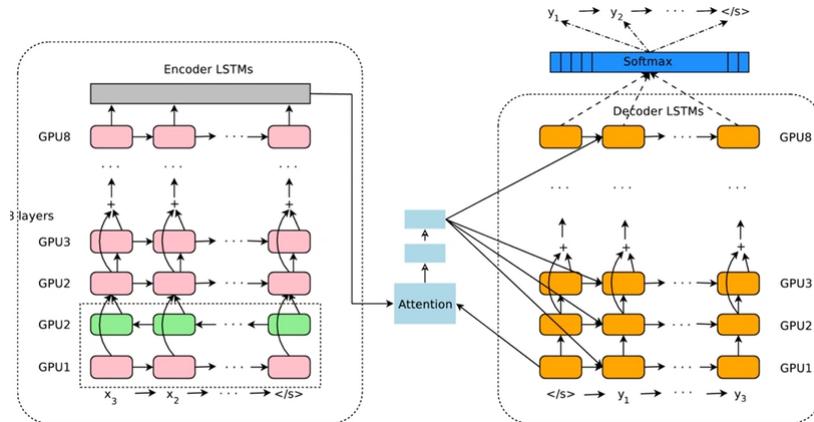
Basically

Relevance $e_i = a(s, h_i)$

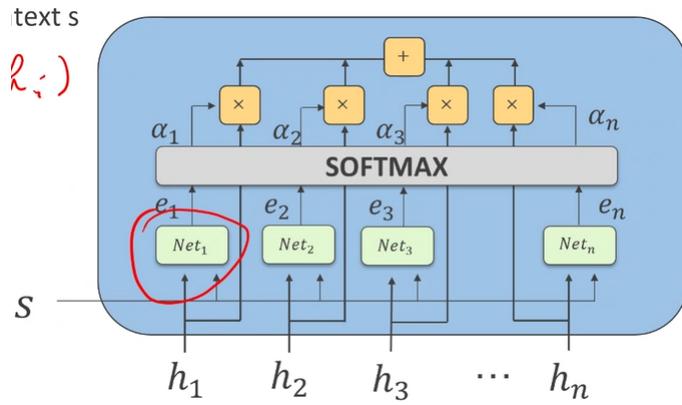
Normalization $\alpha_i = \frac{\exp(e_i)}{\sum_j \exp(e_j)}$

Aggregation $c = \sum_i \alpha_i h_i$

Context s is the past output state, the seed takes into account a subset of the input states.
An advanced example: Google Neural Machine Translation (GNMT)

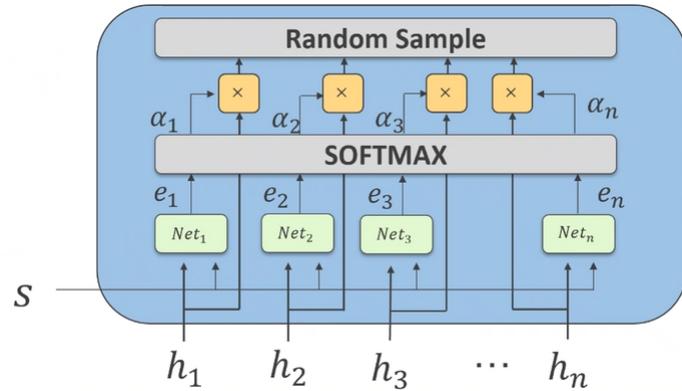


Generalized Relevance This component determines how much each h is correlated/associated with current context s :

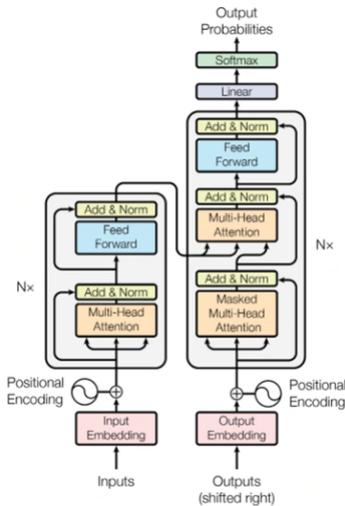


Net could be anything: a feedforward $FF(s, h_i)$, linear $s^T h_i \dots$

Hard Attention Sample a single encoding using probability α_i

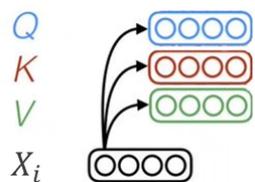


Transformers Pure attention model (no recurrence), self-attention. Evolution of GNMT.



Encoder-decoder architecture.

Self Attention For each element of an input sequence X_i project into three vectors: query, key and value. A kind of associative database.



For each vector, we compute attention over all other vectors.

$$SA(Q_i, K, V) = \sum_j \text{Softmax}_j \left(\frac{Q_i \cdot K^T}{\sqrt{d_k}} \right) V_j$$

Can be done multi-headed as well

$$\text{Attention}(Q, K, V) = \text{Softmax} \left(\frac{QK^T}{\sqrt{d_k}} \right) V$$

$$\text{MultiHead}(Q, K, V) = \text{Concat}(\text{head}_1, \dots, \text{head}_h)W^O$$

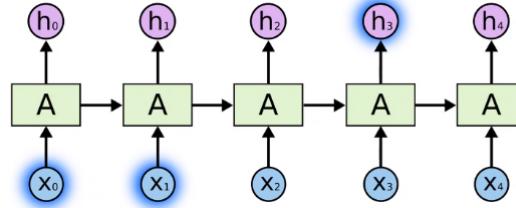
$$\text{head}_i = \text{Attention}(QW_i^Q, KW_i^K, VW_i^V)$$

Is self-attention a good mechanism to model temporal dependencies? What happens if I randomly shuffle some tokens? The result is going to be shuffled. We have a key for each timestep, but there's nothing that keeps track of time. So attention is order-independent, but we need temporal information.

So we sum word embed plus position embed.

0.7.5 Skipping State Updates

This mitigates vanishing gradients, because it reduces the distance between dependencies.



Convolutional Seq2Seq Use convolution instead of recurrence. Better parallelization on GPUs and smaller distance between long-range dependencies.

Vanishing gradients depend on the depth of the network.

Layer Type	Complexity per Layer	Sequential Operations	Maximum Path Length
Self-Attention	$O(n^2 \cdot d)$	$O(1)$	$O(1)$
Recurrent	$O(n \cdot d^2)$	$O(n)$	$O(n)$
Convolutional	$O(k \cdot n \cdot d^2)$	$O(1)$	$O(\log_k(n))$
Self-Attention (restricted)	$O(r \cdot n \cdot d)$	$O(1)$	$O(n/r)$

Recurrent are on the opposite side of the spectrum respect to self-attention.

Hierarchical RNNs Different approach: adds skip connections to the model (**static skip**) or learns when to skip updates (**adaptive skips**). Can skip units, block of units or entire layers.

Zoneout Regularization At each timestep, force some units to keep the same value by random sampling. Hard gate, avoiding the update improves the gradient propagation.

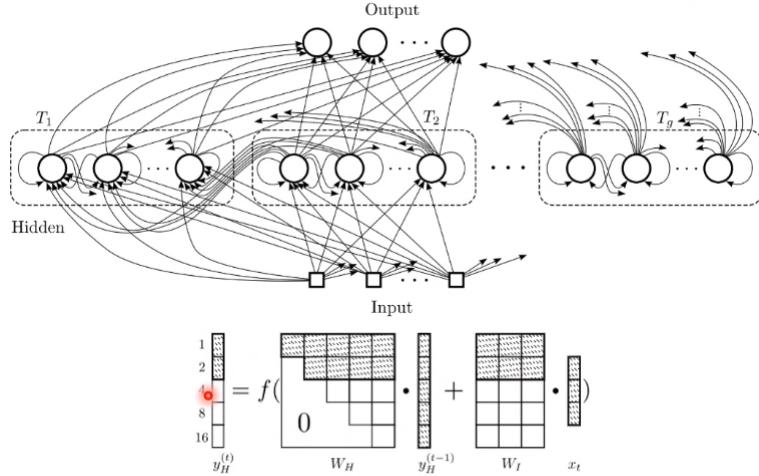
$$T = d_t \odot \tilde{T} + (1 - d_t) \odot 1$$

For comparison, the dropout is given by the formula

$$T = d_t \odot \tilde{T} + (1 - d_t) \odot 0$$

which zeroes the value of some units.

Clockwork RNN Modular recurrent layer where each module is update at different clock time. Modules interconnected only when destination clock time is larger. It works on blocks of units instead of single units.

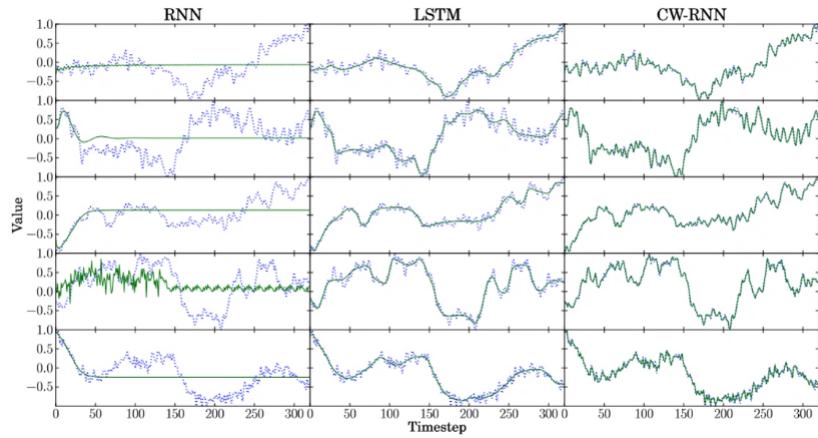


RNN update

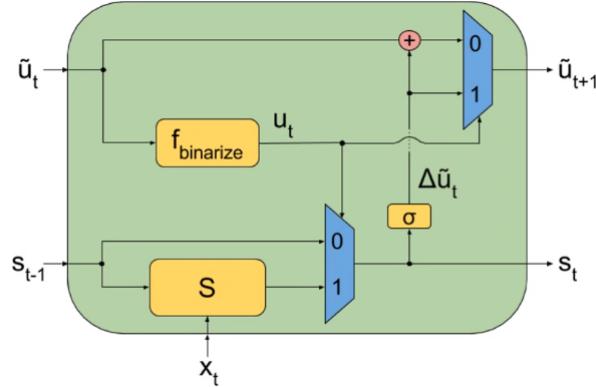
Block structure

Conditional update

Example of sequence generation:



Skip RNN



u_t is the binary state update gate, that determine if RNN state is going to be updated or copied (skip). Replacing gated update by copying increases the network memory (LSTM has an exponential fading effect due to the multiplicative gate).

0.7.6 Hierarchical Networks

Many sequences have latent hierarchical structures that we want to model explicitly. E.g.: Wikipedia represented as a sequence of characters. The hierarchy would be characters, words, sentences, paragraphs, documents.

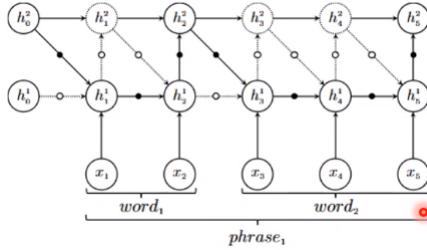
Explicit Boundaries If explicit boundaries are available, we can have different layers for each level of abstraction. Combining representations from the lower layer to obtain representation for the higher layer. The problem is that typically there are no explicit boundaries available.

Operations

Update: state update (LSTM cells) according to boundary detector

Copy: copies cell and hidden states from previous timestep to the current

Flush: sends summary to next layer and re-initialize the current layer's state



Recap

Recurrence: update at each timestep, linear scan of the sequence, path length = n

Convolution: update at each timestep but stop at the last k timesteps, path length = $\log_k(n)$

Attention: update the entire sequence in parallel, path length = 1

Zoneout: randomly disable unit update

CW-RNN: blocks of units with different update frequencies, static

Skip-RNN: adaptive gates learn to skip entire update, save computation

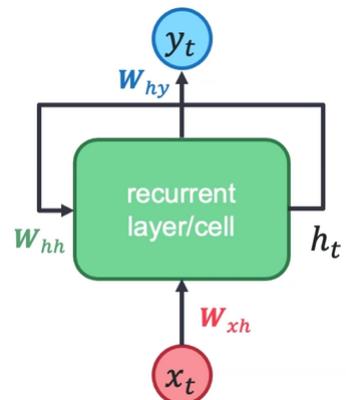
HM-RNN: each layer models more abstract features by learning the boundaries, adaptive

0.8 Reservoir Computing

It's an extremely efficient way of designing and training RNNs.

In a RNN we have a state update

$$h_t = \tanh(x_t W_{xh} + h_{t-1} W_{hh})$$



and an output function

$$y_t = h_t W_{hy}$$

Fading/Exploding memory is the situation where the influence of inputs far in the past vanishes/explodes in the current state, due to many non-linear transformation.

Gradients, too, might vanish/explode during propagation through many non-linear transformations, making it difficult to train on long-term dependencies.

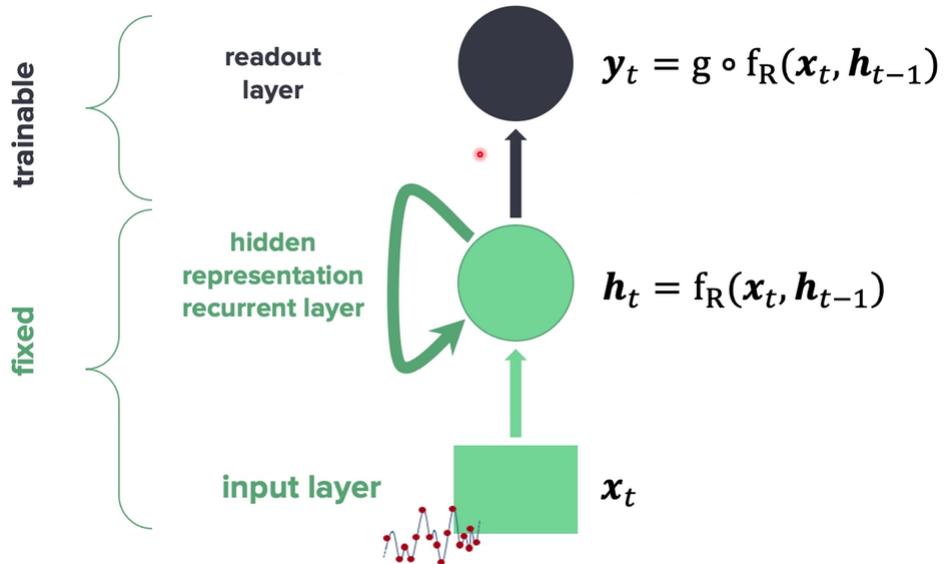
To overcome these instabilities, there are lots of approaches:

Gated architectures: pathways for uninterrupted gradient propagation. Like LSTMs and GRUs. But this makes the training slow

Smart initialization: reservoir computing. Training is limited.

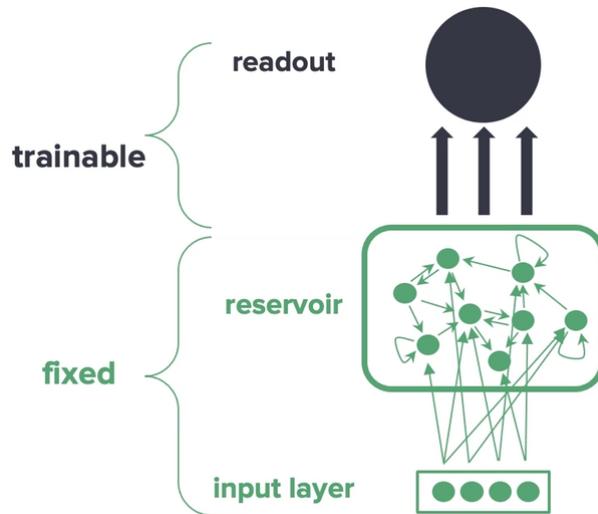
The essential part is introducing randomization in Deep Neural Networks. Trading a little bit of accuracy in favor for efficiency, mainly for energy consumption: ML has a lot of impact on the environment. Also we want to be able to use these systems on embedded applications.

Randomization is computationally cheaper than optimization.



Randomization means efficiency: training algorithms become cheaper and simpler. Model transfer: don't need to transmit all the weights. Amenable to neuromorphic implementations (ad hoc hardware implementations).

Reservoir Computing



I apply learning only on the output layer (**readout**), not in a recurrent way.
The state function is the classical of the RNN.

$$h_t = \tanh(x_t W_{xh} + h_{t-1} W_{hh})$$

But the weight matrices W_{xh} and W_{hh} are randomly initialized under stability conditions (**Echo State Property**) on the dynamical system and left fixed.

The **reservoir** is a large layer of recurrent units, sparsely connected, randomly initialized under the ESP and left **untrained**.

The **readout** is a linear combination of the reservoir state variables and can be trained in closed form.

$$\begin{aligned} y_y &= h_t W_{hy} \\ W_{hy} &= (H^T H)^{-1} H^T D \end{aligned}$$

Architecture Composed of a reservoir and readout

Setup Initialize W_{xh} and W_{hh} randomly and scale W_{hh} to meet the contractive/stability property.

Training Drive the network with the input signal. Discard an initial transient and train the readout.

Reservoir Non-linearly embeds the input into a higher dimensional feature space where the original problem is more likely to be solved linearly (Cover's Theorem). We use this randomized basis expansion computed by a pool of randomized filters. Provides a "rich" set of input-driven dynamics. Dynamics are driven by the state transition function.

$$\begin{aligned} F : R^{N_x} \times R^{N_h} &\rightarrow R^{N_h} \\ h_t &= F(x_t, h_{t-1}) = \tanh(x_t W_{xh} + h_{t-1} W_{hh}) \end{aligned}$$

The iterated version is the function applied to an arbitrarily long sequence $s = [x_1, \dots, x_t]$ returning the final state h_t

$$\hat{F} : (R^{N_x})^* \times R^{N_h} \rightarrow R^{N_h}$$

For example

$$\hat{F}(s, h_0) = \begin{cases} h_0 & \text{if } s = [] \\ F(x_t, \hat{F}([x_1, \dots, x_{t-1}], h_0)) & \text{if } s = [x_1, \dots, x_t] \end{cases}$$

Echo State Property A valid ESN should satisfy the ESP.

An ESN satisfies the ESP whenever

$$\forall s \in (R^{N_x})^N \wedge \forall h_0, z_0 \in R^{N_h} \text{ we have } \|\hat{F}(s, h_0) - \hat{F}(s, z_0)\| \rightarrow 0 \text{ as } N \rightarrow \infty$$

Sufficient condition involving the control of the maximum singular value of W_{hh} , this **theorem**: if the maximum singular value of W_{hh} is < 1 then the ESN satisfies the ESP for any possible input.

Necessary condition, involving the control of the maximum eigenvalue in modulus of W_{hh} , this **theorem**: if the spectral radius of W_{hh} is not < 1 then the ESN does not satisfy the ESP.

We know that

$$\rho(W_{hh}) \leq \|W_{hh}\|_n$$

Initialization Generate a random matrix W (e.g. from a uniform distribution $[-1, 1]$ and then scale by the desired spectral radius (< 1))

$$W_{hh} = \rho_{desired} \frac{W}{\rho(W)}$$

Now $\rho(W_{hh}) = \rho_{desired}$: the **spectral radius is a key hyperparameter of the reservoir**.

Dynamical Transient If the system is globally asymptotically stable, then all the possible trajectories will synchronize **after a transient**, not instantaneously.

Washout is initial part of the time-series in which the state could be still affected by initialization condition (where the ESP could still not hold). The washout states of the reservoir should be discarded.

ESN Training

Given a training set $\{(x_t, d_t)\}_{t=1}^N$

Run the reservoir on the input sequence and collect the states $H = [h_1, \dots, h_N]$

Remove the washout $H = H(N_W : N, :)$

Collect the target data similarly into a matrix $D = [d_{N_W}, \dots, d_N]$

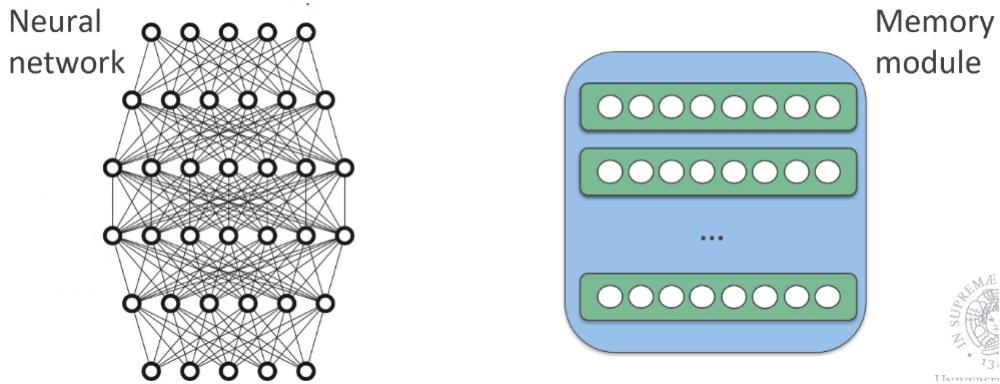
Solve the linear regression problem for the readout $\min_{W_{hy}} \|HW_{hy} - D\|_2^2$

To solve, typically the training is performed offline in closed form.

0.9 Neural Reasoning

Not using classical algorithms because we may not have a proper input but only sensor information (e.g. robot navigation). The model need to learn to encode the structure from the raw data and then solve the problem. Another example is question answering, where we need to memorize facts, the question and the answer: a bit too much for the dynamical RNN memories, so we need an external memory.

Memory Networks General Idea



The memory module typically is a matrix of N slots, sometimes with fixed size other times with dynamic memory. The neural network reads/writes from/to the memory module. Where to read and write can be done with attention.

0.9.1 Memory Network

Components

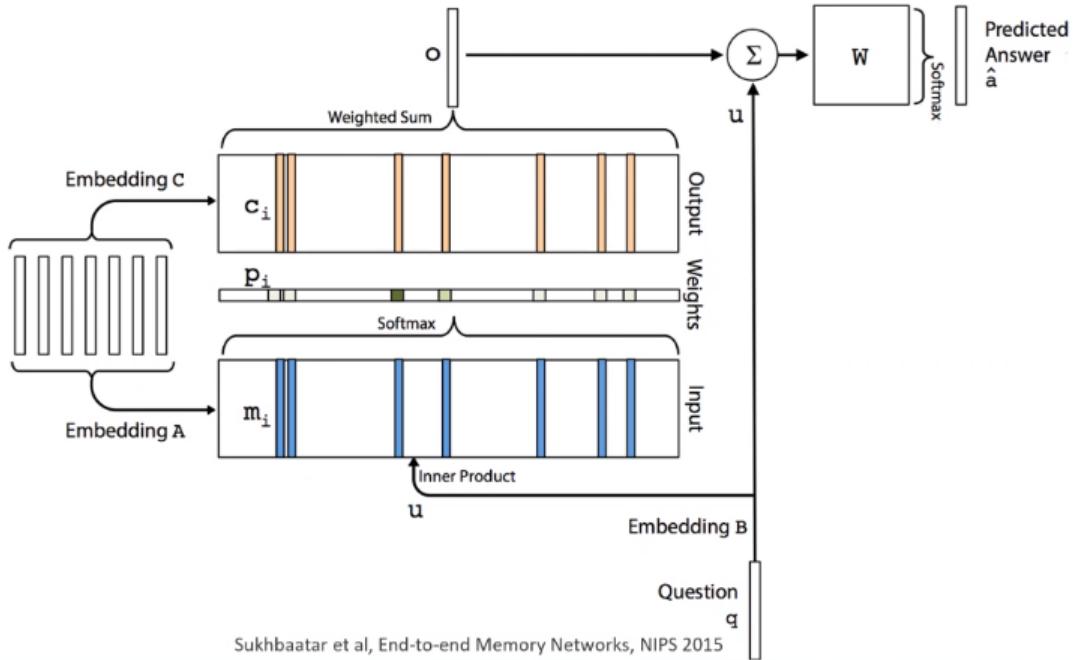
Input Feature Map I : encodes the input in a feature vector

Generalization G : decide what input (or function of it) to write to memory

Output Feature Map O : reads the relevant memory slots

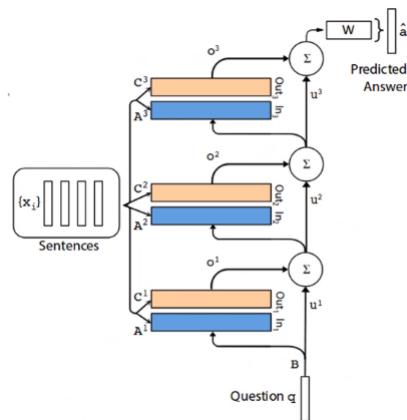
Response R : returns the prediction given the retrieved memories

End-to-End Memory Networks



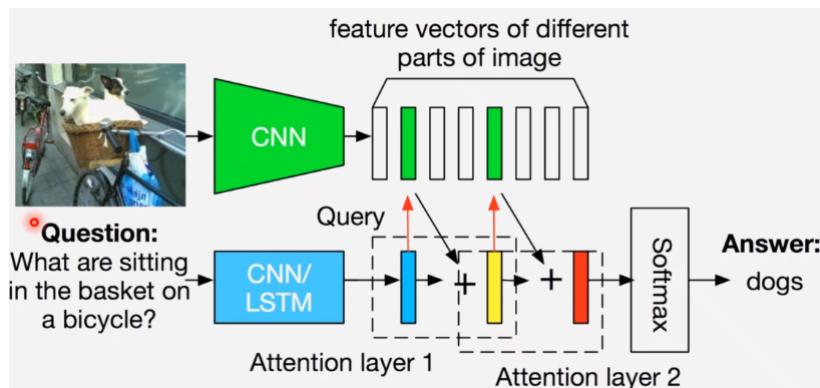
The memory contains the facts x_i . We search for memories matching the query, then we have a query driven soft-attention with the softmax. We combine the output memories.

We can have extensions, for example we may need more than a single step for more complex models (stacking multiple memory network layers to get several iterations of reasoning):



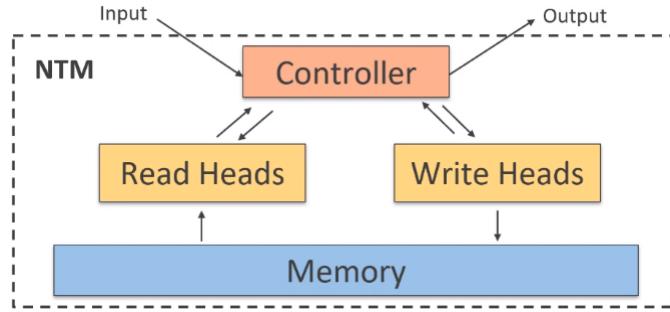
Often with tied weights.

Memory Nets for Visual Question Answering with Attention

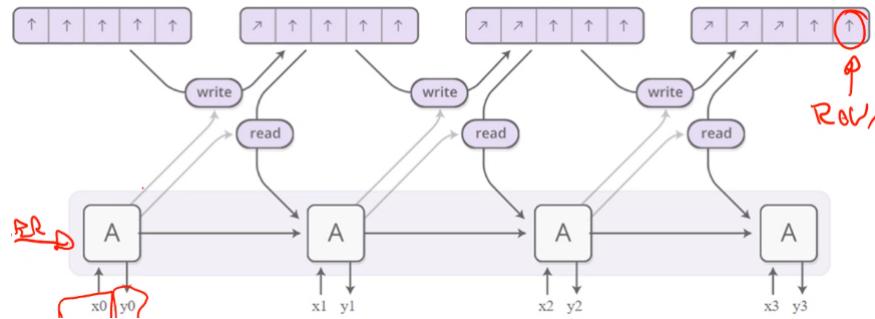


0.9.2 Neural Turing Machines

Memory networks that can read and write memories at both training and test. End-to-end differentiable.

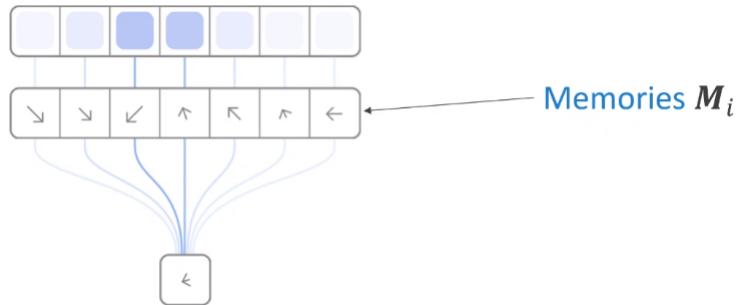


Controller Typically a RNN emitting vectors to control read and write from the memory.



The key to differentiability is to always read and write the whole memory.

Memory Read



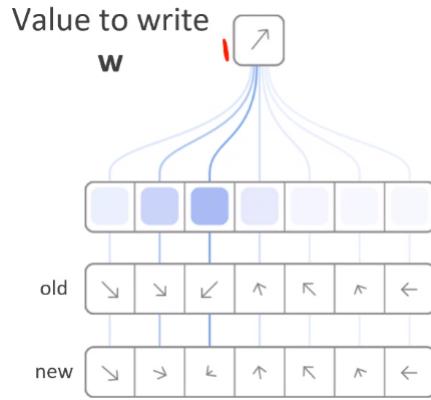
We have the attention coefficients (attention distribution vector a from the RNN, the darker the color the stronger the coefficient), which weights the sum.

$$r = \sum_i \alpha_i M_i$$

Memory Write

Location-based Addressing

Associative Memory: key-value pairs, and given an approximated version of a key we want to be able to recover the correspondent key.



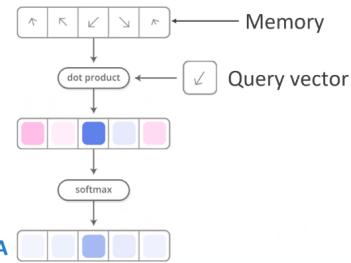
Given the value to write, the model builds the attention distribution vector a describing how much we change each memory.

$$M_i = a_i w + (1 - a_i) M_i$$

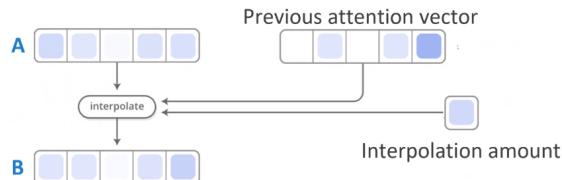
Write operation is actually performed by composing adding and erasing operations.

NTM Attention Focusing

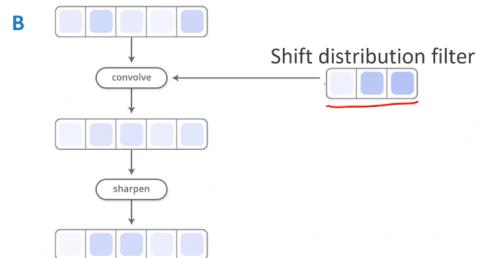
1. Generate content-based memory indexing



2. Interpolate with attention from previous time



3. Generate location-based indexing

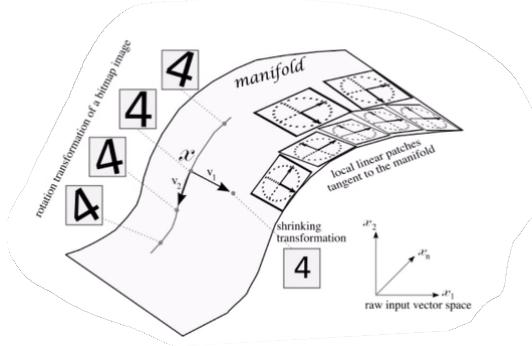


The convolution determines how we move between the locations in memory.
Then we sharpen the distribution for the final memory access.

Not yet of practical use, not straightforward to train.
Has advantages over GRNN when it comes to learn to program.

0.10 Unsupervised Learning

The Problem Characterize the data: data distribution and variances. Connects to autoencoders and manifold learning.



This to allow: understanding of data, generating new observation... and ultimately reasoning.

Another reason is that **labeled data is costly and difficult to obtain**. A sustainable future for deep learning: learning the latent structure of data, discover important features, learn task independent representations and introduce (if any) supervision only on few examples.

Why Generative? Focusing too much on discrimination rather than on characterizing data can cause issues, e.g.: reduced interpretability, adversarial examples.

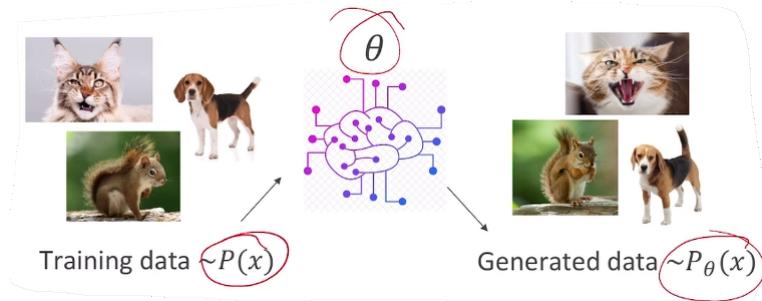
Generative model try to characterize data distribution.

Understand data to understand the world

Understand data variances to learn to steer them

Understand normality to detect anomalies

Approaching the Problem from a DL Perspective Given training data, learn a (deep) NN that can generate new samples form (an approximation of) the data distribution.



Two approaches:

Explicit (learn a model density $P_\theta(x)$), which can be visible (work only on visible data, tractable densities, sampling RNNs) and latent (assume there's something else, intractable densities).

Latent can also be variational (Variational Autoencoders) and stochastic (Boltzmann Machines).

Implicit (learn a process that samples data from $P_\theta(x) \simeq P(x)$). Can be divided into direct models (generative adversarial networks) and stochastic models (generative stochastic networks)

Learning With Fully Visible Information If all information is fully visible, the joint distribution can be computed from the chain rule factorization (Bayesian Networks).

$$P(x) = \prod_i^N P(x_i | x_1, \dots, x_{i-1})$$

Can be seen as conditioning the color of the pixel i based on previous pixels. Need to be able to define a sensible ordering for the chain rule, and conditional distribution difficult to compute.

We can use RNNs: use their ability to handle previous data. Scan the image according to a schedule and encode the dependency from previous pixels in the states of an RNN.

With only visible information we try to learn the θ parametrized model distribution

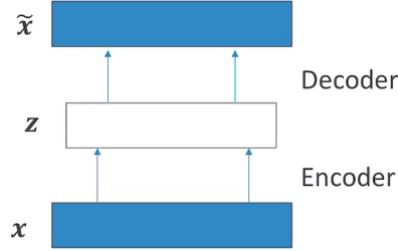
$$P_\theta(x) = \prod_i^N P_\theta(x_i | x_1, \dots, x_{i-1})$$

Then we introduce a latent process regulated by unobservable variables z

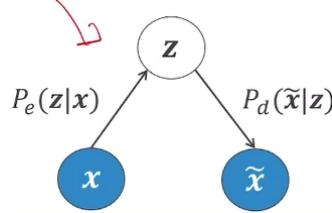
$$P_\theta(x) = \int P_\theta(x | z) P_\theta(z) dz$$

But $P_\theta(x | z)$ it's typically intractable.

NN With Latent Variables? An Autoencoder.



We introduced a probabilistic twist on AE



There's sampling on z , so I can't backpropagate from \tilde{x} .

We need a **deeper probabilistic push**: as an additional push in the probabilistic interpretation, we assume to be able to generate the reconstruction from a sampled latent representation.

Transform $z \mapsto \tilde{x}$ with $P(x | z)$ transformed into something simpler: a stochastic part and a deterministic part. The stochastic part must be as simple as possible, and the deterministic part as powerful as possible. For example, a probability distribution from which we can sample is not multivariate nor discrete, is a Gaussian, so the stochastic part is a Gaussian. Sample z from the Gaussian, into the powerful deterministic part and transform to represent some other distribution.

So we sample latent variables z from the true prior $P(z)$ and sample from the true conditional $P(\tilde{x} | z)$

Of course we can't access the true distribution, we approximate them with a simpler distribution (e.g. Gaussian), decode z into \tilde{x} with a decoder g . At training time sample z conditioned on data x and train the decoder g to reconstruct x itself from z .

$$z \simeq N(\mu(x), \sigma(x))$$

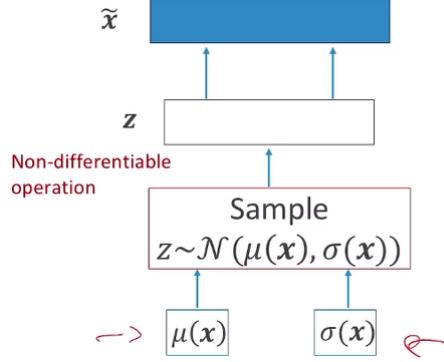
But it's not so easy, we would like to train maximizing something non differentiable ($P(x_i | z)P(z)$)

$$L(D) = \prod_{i=1}^N P(x_i) = \prod_{i=1}^N \int P(x_i | z) P(z) dz$$

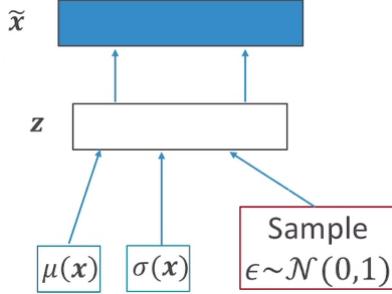
$P(x_i)$ is intractable so we lower bound it with variational approximation.

The non-differentiable part we handle with reparametrization.

Reparametrization



We can sample white noise and rescale that noise with the μ and the σ



The non-differentiable part is in $N(0, 1)$ so the gradient can flow to z because sampling is limited to non differentiated variable ϵ .

μ and σ are generated with an encoder.

Variational Approximation ELBO (Evidence Lower Bound)

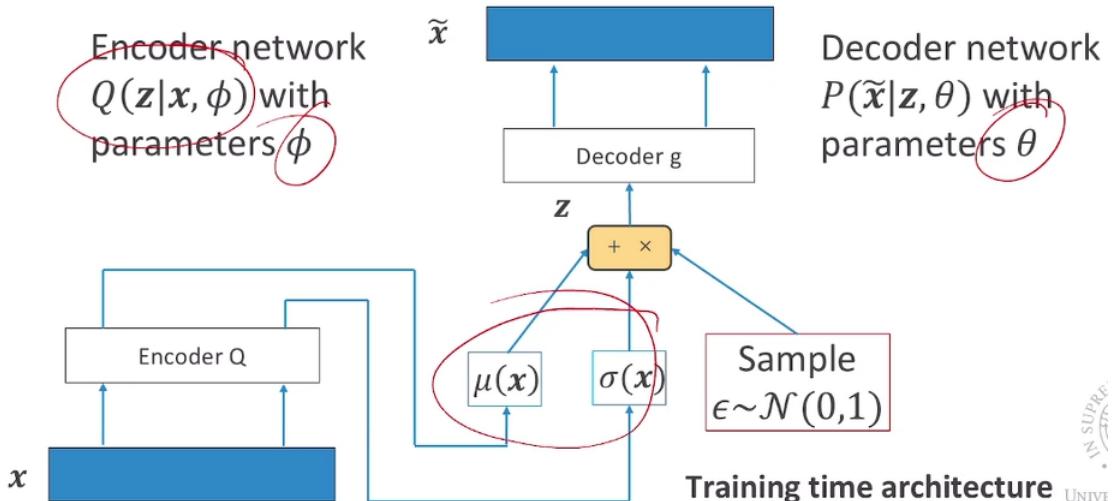
$$\log P(x | \theta) \geq E_Q[\log P(x, z)] - E_Q[\log Q(z)] = L(x, \theta, \phi)$$

The Q function is anything that can approximate the distribution with its own parameters. Maximizing the ELBO allows approximating from below the intractable log-likelihood $\log P(x)$

$$L(x, \theta, \phi) = E_Q[\log P(x | z)] + E_Q[\log P(z)] - E_Q[\log Q(z)]$$

$E_Q[\log P(z)] - E_Q[\log Q(z)] = KL(Q(z|\phi) \| P(z))$ is the KL, and $P(x | z)$ is the decoder estimate of the conditional (made possible through reparametrization). $P(z)$ is a probability from which is simple to sample form. $Q(z)$ has parameters, any function that can be matched against the distribution, so the Q can be played by a neural network (the encoder). This KL will enforce Q to behave like the prior.

0.10.1 Variational Autoencoder



Training Performed by backpropagation on θ, ϕ to optimize the ELBO

$$L(x, \theta, \phi) = E_Q[\log P(x|z = \mu(x) + \sigma^{1/2}(x) \cdot \epsilon, \theta)] - KL(Q(z|x, \phi) \| P(z|\theta))$$

To make it an appropriate loss function, the first term (E_Q) is the reconstruction and the second term is a regularization (KL). The Q function is a Gaussian, $P(z|\theta)$ is also a Gaussian, so KL is between two Gaussians and can be computed in closed form.

VAE Final Loss In principle we would like to optimize the following loss by SGD

$$E_{X \sim D}[E_{Z \sim Q}[\log P(x|z)] - KL(Q(z|x, \phi) \| P(z))]$$

Which with reparametrization becomes

$$E_{X \sim D}[E_{\epsilon \sim N(0,1)}[\log P(P(x|z = \mu(x) + \sigma^{1/2}(x) \cdot \epsilon, \theta))] - KL(Q(z|x, \phi) \| P(z))]$$

No expectation is w.r.t. distributions that depend on model parameters: we can move gradients into them.

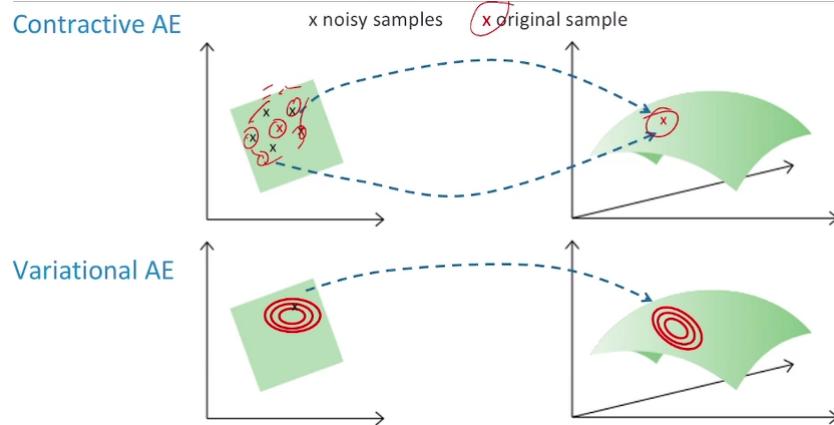
Information Theoretic Interpretation $\log P(x|z)$ tells the number of bits required to reconstruct x from z under the ideal encoding (i.e. $Q(z|x)$ is generally suboptimal).

The second part tells how much information of x we're giving away by picking it up from Q instead of the non-informative prior P . Meaning that $Q(z|x, \phi)$ is the number of bits required to convert an uninformative sample from $P(z)$ into a sample from $Q(z|x)$.

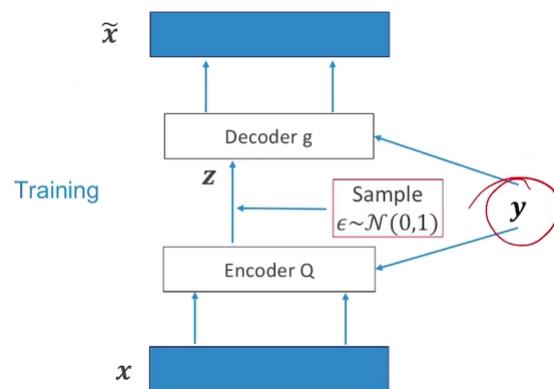
The information gain is the amount of extra information we get about X when z comes from $Q(z|x)$ instead of from $P(z)$.

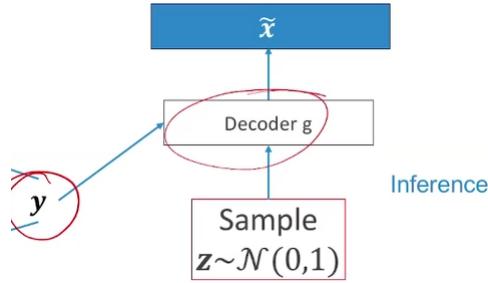
Testing Aka sampling the VAE. At test time detach the decoder, sample a random encoding and generate the sample as the corresponding reconstruction.

VAE vs Denoising/Contractive AE



Conditional Generation (CVAE) Learn conditional distributions





Learns the conditional distribution $P(x | y)$.
This is the simplest possible form of CVAE.

0.10.2 Implicit Models

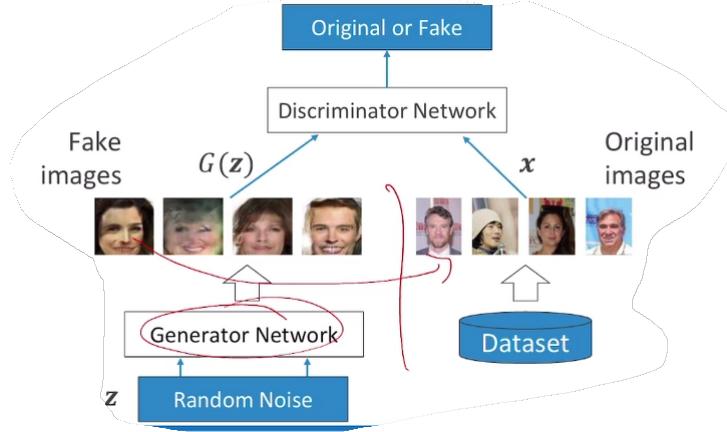
Distribution Learning vs Learning to Sample Variational AEs learn to approximate an intractable distribution

$$P_\theta(x) = \int P_\theta(x | z) P_\theta(z) dz$$

then sample it to generate the output. What if we learn to generate samples rather than learning the distribution? Generative Adversarial Networks (GAN), game theoretic approach.

The GAN Catch We need to learn to sample from a complex, high-dimensional training distribution: no straightforward way to do this. The catch: sample from a simple distribution (random noise, Gaussian) and train a differentiable distribution (Neural Network) to transform random noise to the training distribution.

Again a two step architecture.



$$C = \min_{\theta_G} \max_{\theta_D} (E_x[\log D_{\theta_D}(x)] - E_z[\log(1 - D_{\theta_D}(G_{\theta_G}(z)))])$$

With D being the output of the discriminator, and G the output of the GAN. So the term $D_{\theta_D}(x)$ is the discriminator output for real data x and the term $D_{\theta_D}(G_{\theta_G}(z))$ is the discriminator output for fake data $G(z)$.

An alternate optimization:

1. Discriminator gradient ascent

$$C_D = \max_{\theta_D} (E_x[\log D_{\theta_D}(x)] - E_z[\log(1 - D_{\theta_D}(G_{\theta_G}(z)))])$$

2. Generator gradient descent

$$C_G = \min_{\theta_G} (E_z[\log(1 - D_{\theta_D}(G_{\theta_G}(z)))])$$

The last doesn't really work well. The cost that the generator receives in response to generating $G(z)$ depends only on the discriminator response.

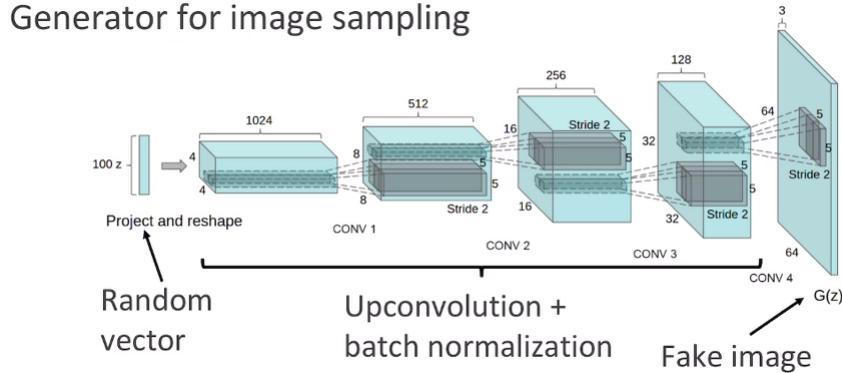
Hard Two-Player Game The optimal solution of a min-max problem is a saddle point. Little stability: initially lot of heuristic work, now converged to more principled solutions.

Wasserstein Distance Models Attempts to solve the hardness of training generators by optimizing the Wasserstein distance (EMD) between the generator and empirical distribution filtered through the discriminator function D

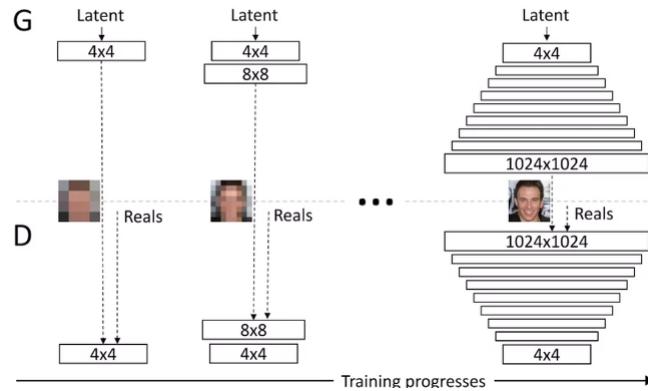
$$G^* = \arg \min_G W(\mu, \mu_G) = \arg \min_G \sup_{\|D\|_L \leq 1} (E_{x \sim \mu}[D(x)] - E_{x \sim \mu_G}[D(x)])$$

This is a constraint satisfaction problem due to $\sup_{\|D\|_L \leq 1}$, which makes the discriminator Lipschitz bounded by 1. In practice it's not easy, we put a constraint on the magnitude of the constraints.

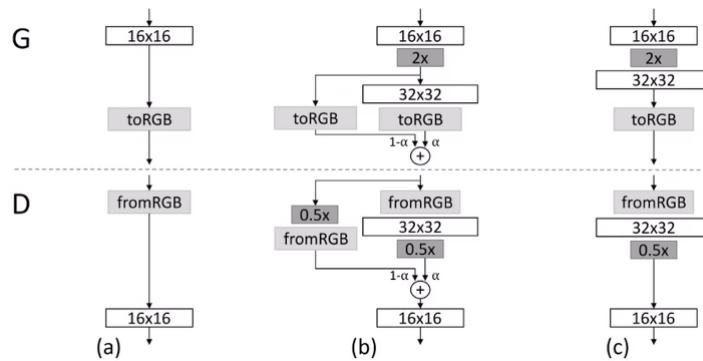
DCGAN Architecture



Progressive GAN Train with very small images, and learn to generate them. Then you move to slightly larger step by step.



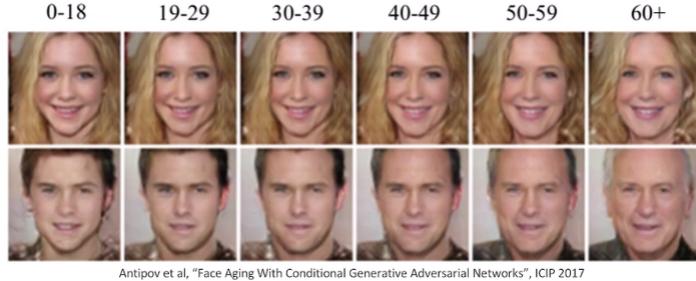
But the jump destroys what we've learned in previous steps. We need smooth transitions:



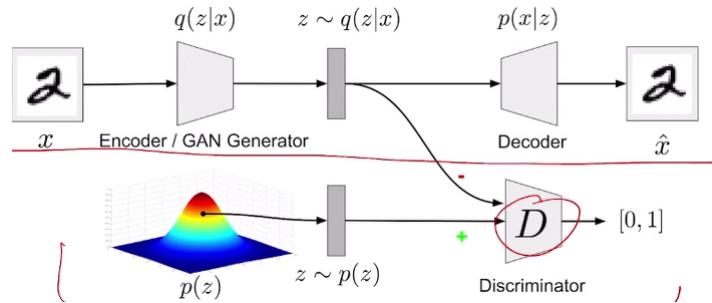
We use residual connections and α to trade between the smaller residual image slowly smoothly giving more weight to the larger image.

Conditional Generation Learn a mapping from an observed side information x and a random noise vector z to the fooling samples y

$$G : \{x, z\} \rightarrow y$$



Best of two worlds



The discriminator discriminates the zs from the encoder from the z from the distribution.

Training AAE We replace the second term with an adversarial loss

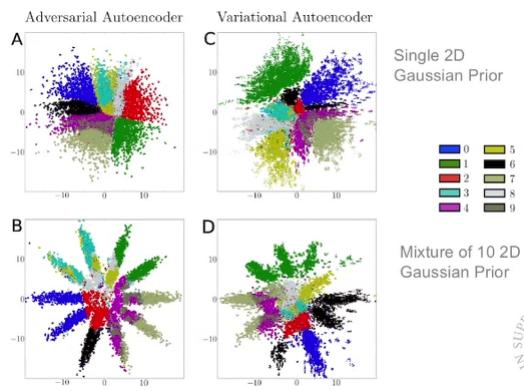
$$L(x) = E_Q[\log P(x | z)] - KL(Q(z | x) \| P(z))$$

Reconstruction phase: update encoder and decoder to minimize reconstruction error

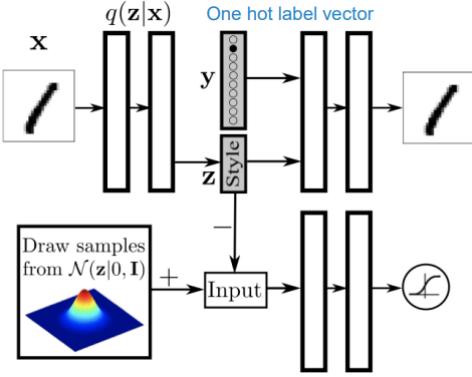
Regularization phase: update discriminator to distinguish true prior samples from generated samples. Update generator to fool the discriminator.

$KL(Q(z | x) \| P(z))$ is replaced by an adversarial loss. Adversarial regularization allows to impose priors for which we cannot compute the KL divergence.

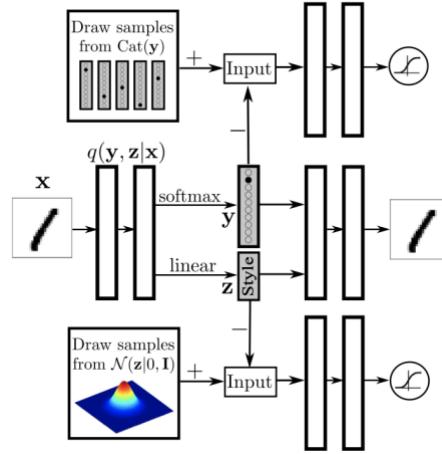
AAE yields a smoother coverage of the latent space.



AAE Style Transfer Supervised. Incorporate label information explicitly to force z to capture class-independent information (e.g. style).



AAE Semi-Supervised Learning Factorize latent code in one-hot encoding vector y and continuous code z . Distribution of y is made little distinguishable from a multinomial (induced from data).



Overview Generative Adversarial Networks **learn to sample** rather than learning the distribution. They obtain state-of-the-art generated sample quality, but are unstable/difficult to train, cannot perform inference (no distribution learning) and need differentiable generator.

Adversarial Autoencoders leverage adversarial penalties in place of KL regularization. useful to impose "complex" or empirical priors.

0.11 Continual Learning

Deep learning holds state-of-the-art performances in many tasks, mainly supervised training with **huge** and **fixed** datasets.

The Curse of Dimensionality is a very important topic: there are $3.9 \cdot 10^{372282}$ possible 277×277 RGB images. That's huge and you cannot consider them all.

We want to make AI sustainable in the long term, and we can achieve this by never stop learning.

Continual Learning

At t_1 , $X_1 \Rightarrow f_\theta : X_1 \rightarrow Y_1 \Rightarrow Y_1$

At t_2 , $X_2 \Rightarrow f_\theta : X_1 \cup X_2 \rightarrow Y_1 \cup Y_2 \Rightarrow Y_2$

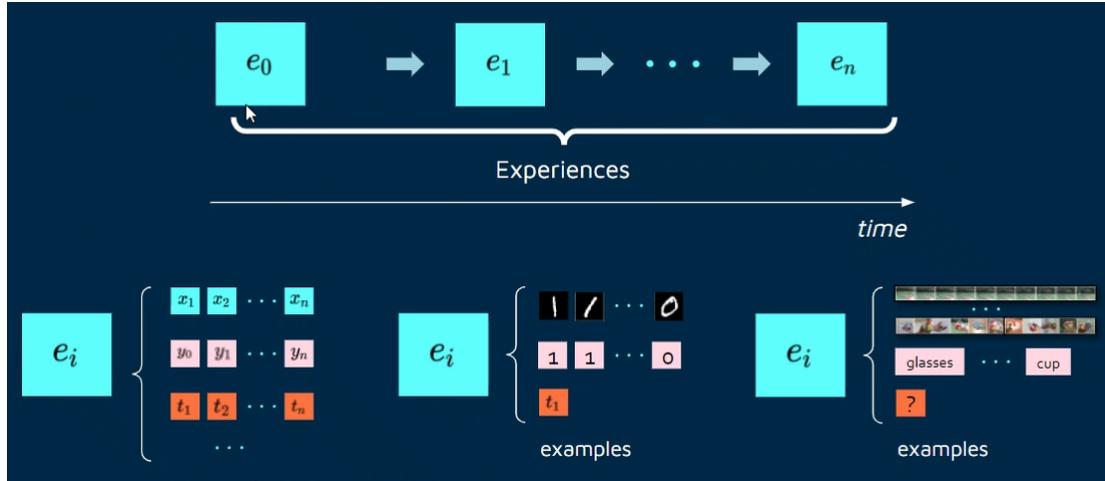
Higher and realistic time-scale where data (and tasks) become available only during time. **No access to previously encountered data**.

Constant computational and memory resources (efficiency concern), incremental development of ever more complex knowledge and skills (scalability concern): **efficiency + scalability = sustainability**.

Stability-plasticity dilemma Remembering past concepts (**stability**) while learning new concepts (**plasticity**) and being able to generalize. This is the cause of the first problem in deep learning: catastrophic forgetting.

This is not something to add to the existing machine learning systems, but a paradigm-changing approach to machine learning altogether, that will enable systems to continuously improve based on experience.

Formally



Assuming that the knowledge is non-conflicting.

Benchmarks The current focus is multi-task, few big tasks, toy datasets, mostly supervised and focused on accuracy. A better focus would be single-incremental-task, high-dimensional data streams, natural/unrealistic datasets, mostly unsupervised with focus on scalability and efficiency.

Scenarios Three main scenarios:

Task-Incremental: every experience is a different task

Class-Incremental: every experience contains example of different classes

Domain-Incremental: every experience contains examples of the same classes

But there are many others! Not only for data streams but also for sequences.

Common Baselines

Naive/Finetuning: just continuing the backprop

JointTraining/Offline: the best you can do with all the data starting from scratch

Ensemble: one model from scratch for each experience

Cumulative: for each experience, accumulate all data and re-train from scratch

Random Replay Basic approach:

Sample randomly from the current experience data

Fill your fixed **Random Memory** (RM)

Replace examples randomly to maintain an approximate equal number of examples for experience

0.12 Reinforcement Learning

Reinforcement learning is characterized, with respect to other ML tasks, by:

no supervisor, only a *reward* signal

delayed **asynchronous feedback**

time matters (sequential data, continual learning)

agent's actions affect the subsequent data it receives (inherently non-stationarity)

Reinforcement learning is used in:

learning to maneuver vehicles and control robots (walking, navigation, manipulation...)

playing games

discovering new molecules

end-to-end learning with discrete structures

0.12.1 Fundamentals

Rewards A reward R_t is a scalar feedback signal. It indicates how well an agent is doing at step t , and the agent's job is to maximize cumulative reward. Reinforcement learning is based on the **reward hypothesis**: all goals can be described by the maximization of expected cumulative reward.

Most of the engineering work in reinforcement learning is about the realization of the reward function.

Sequential Decision Making Goal: select actions that maximize total future reward. Actions may have long-term consequences, and the reward may be delayed. It may be better to sacrifice immediate reward to gain more long-term reward.

Examples:

financial investment, may take months to mature

refueling a helicopter, may prevent a crash in several hours

blocking opponent moves, might help winning chances many moves from now

Agent and Environment At each step t the agent:

executes action A_t

receives observation O_t

receives scalar reward R_t

The environment:

receives action A_t

emits observation O_{t+1}

emits scalar reward R_{t+1}

History and State The **history** is the sequence of observations, actions and rewards

$$H_t = O_1; R_1; A_1; \dots; A_{t-1}; O_t; R_t$$

I.e. all the observable variables up to time t . What happens next depends on the history: the agent selects an action, the environment selects observations/rewards.

State S_t is the information used to determine what happens next and is a **function of history**

$$S_t = f(H_t)$$

Environment State The environment state S_t^e is the environment e private representation at time t . Whatever information the environment uses to generate the next observation/reward.

The environment state is not usually visible to the agent (unobservable environment).

Even if S_t^e is visible, it may contain irrelevant information.

Agent State The agent state S_t^a is the internal representation owned by agent a : it's whatever information the agent uses to select its next action. Generally speaking is a function of history, $S_t^a = f(H_t)$, and its the information used by reinforcement learning algorithms.

Information (Markov) State An information state (Markov state) contains all useful information from the history. A state S_t is Markov $\Leftrightarrow P(S_{t+1} | S_1, \dots, S_t) = P(S_{t+1} | S_t)$.

The future is independent of the past given the present (**d-separation**), meaning $H_{1:t} \rightarrow S_t \rightarrow GH_{t+1:\infty}$. The state is a **sufficient statistic for the future**. The environment state S_t^e and the history H_t are Markov.

Observability

Fully Observable Environment: full observability means that the agent **directly** observes the environment state.

$$O_t = S_t^a = S_t^e$$

Formally, it's a **Markov Decision Process (MDP)**.

Partially Observable Environment: partial observability means that the agent **indirectly** observes the environment. E.g.: a robot with a camera may not know its absolute location, a trading agent only observes current prices, a poker player only observes public cards...

Formally, $S_t^a \neq S_t^e$ and the problem is a **Partially Observable Markov Decision Process (POMDP)**. The agent need to build its own state representation S_t^a :

History $S_t^a = H_t$

Beliefs on environment state $S_t^a = [P(S_t^e = s^1), \dots, P(S_t^e = s^N)]$

Dynamic memory (RNN) $S_t^a = \sigma(W_s S_{t-1}^a + W_O O_t)$

0.12.2 Components

A RL agent may include one or more of these:

Policy: agent's behavior function (may be implicit in the value function)

Value function: how good is each state and/or action

Model: agent's representation of the environment

Policy A policy π is the agent's behavior. It's a map from state s to action a :

Deterministic policy $a = \pi(s)$

Stochastic policy $\pi(a | s) = P(A_t = a | S_t = s)$

Value Function How "good" is a specific state/action for an agent? The value function v is a predictor of future reward. Used to evaluate goodness/badness of a states, therefore used to select between actions

$$v_\pi(s) = E_\pi[R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \dots | S_t = s]$$

Expected (discounted) future reward following policy π from state s .

Model A model predicts what the environment will do next.
Predict next state s' following an actions a

$$P_{ss'}^a = P(S_{t+1} = s' | S_t = s, A_t = a)$$

Predict next reward

$$R_s^a = E[R_{t+1} | S_t = s, A_t = a]$$

Characterizing RL Agents

Value based: implicit policy, value function given

Policy based

Model free: might have policy and/or value function, but no model

Model based

0.12.3 Problems

Learning vs Planning Two fundamental problems in sequential decision making:

Reinforcement Learning: environment initially unknown, agent interacts with it and improves its policy

Planning (reasoning, introspection, search,...): a model of the environment is known, the agent performs computations with its model (no external interaction) and it improves its policy

Exploration vs Exploitation Reinforcement learning is based on trial-and-error. The agent should discover a good policy from its experiences of the environment, without losing too much reward along the way. Exploration finds more information about the environment, while exploitation exploits known information to maximize reward. Effective reinforcement learning requires a trade between exploration and exploitation.

Prediction vs Control Prediction is evaluating the future, given a policy, while control is optimizing the future (finding the best policy).

Overview Reinforcement Learning is a general-purpose framework for decision making. It's for an agent with the capacity to act and observe. The state is the sufficient statistics to characterize the future, depends on the history of actions and observations (environment state vs agent state).

Success is measured by a scalar reward signal: the goal is to select actions to maximize future reward (exploit) and in order to be effective we should not forget to explore.

MDP Markov decision processes formally describe an environment for reinforcement learning: environment is **fully observable**, i.e. the current state completely characterizes the process. Almost all Reinforcement Learning problems can be formalized as MDPs, for example:

Partially observable problems can be converted into MDPs

Bandits are MDPs with one state

Optimal control primarily deals with continuous MDPs

A Markov decision process is a Markov chain with rewards and actions. It is an environment in which all states are Markov: it's a tuple (S, A, P, R, γ) where

S finite set of states

A finite set of actions a

P state **transition matrix** such that $P_{ss'}^a = P(S_{t+1} = s' | S_t = s, A_t = a)$

R is a **reward function** such that $R_s^a = E[R_{t+1} | S_t = s, A_t = a]$

$\gamma \in [0, 1]$ **discount factor**

The **return** G_t is the total discounted reward from time-step t

$$G_t = R_{t+1} + \gamma R_{t+2} + \dots = \sum_{k=0}^{\infty} \gamma^k R_{t+k+1}$$

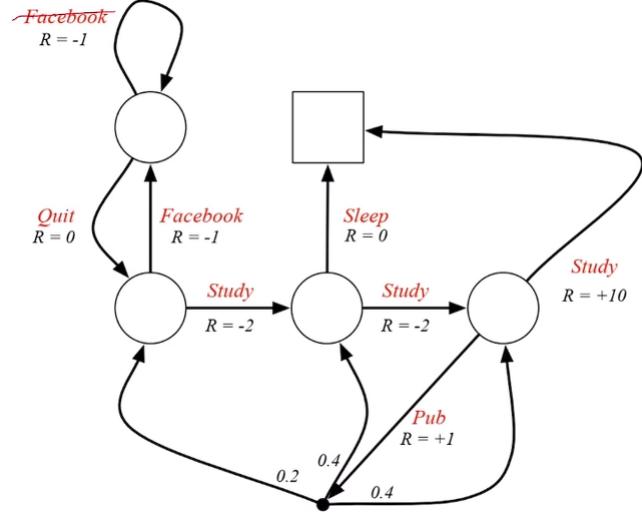
The values of receiving reward R after $k + 1$ time steps is $\gamma^k R$. γ values immediate reward vs delayed reward:

$\gamma \approx 0$ leads to "myopic" evaluation

$\gamma \approx 1$ leads to "fat-sighted" evaluation

It's mathematically convenient to discount rewards as it avoids infinite returns in cyclic Markov processes. Uncertainty about the future may not be fully represented. Also it's application independent and it's still possible to use undiscounted Markov reward processes (i.e. $\gamma = 1$) if all sequences terminate.

An example: student Markov decision process



Value Function Measures the long-term value of being in a certain state s . The state-value function $v(s)$ of a Markov decision process is the **expected return starting from state s**

$$v(s) = E[G_t | S_t = s]$$

Bellman Equation for MDPs The value function $v(S_t)$ can be decomposed into two parts

Immediate reward R_{t+1}

Discounted value of successor state $\gamma v(S_{t+1})$

$$\begin{aligned} v(s) &= E[G_t | S_t = s] = E \left[\sum_{k=0}^{\infty} \gamma^k R_{t+k+1} | S_t \right] = \\ &= E \left[R_{t+1} + \sum_{k=1}^{\infty} \gamma^k R_{t+k+1} | S_t \right] = E \left[R_{t+1} + \gamma \left(R_{t+2} + \sum_{k=2}^{\infty} \gamma^k R_{t+k+1} \right) | S_t \right] = \\ &= E[R_{t+1} + \gamma G_{t+1} | S_t] = E[R_{t+1} + \gamma v(S_{t+1}) | S_t] = \\ &= E[R_{t+1} | S_t = s] + \gamma E[v(S_{t+1}) | S_t = s] = \\ &= R_s + \gamma \sum_{s'} P_{ss'} v(s') \end{aligned}$$

With $E[v(S_{t+1}) | S_t = s]$ being the expected state-value of being in any state reachable from s , and $s \sim P(s' | s)$

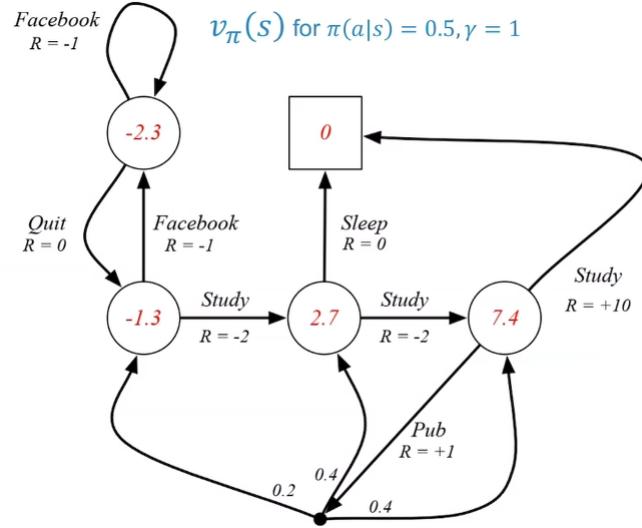
Policy A policy π is a distribution over actions a given states s

$$\pi(a | s) = P(A_t = a | S_t = s)$$

Defines the behavior of an agent, MDP policies depends only on the current state (**Markovian**) and are **stationary** (time-independent): $\forall t > 0 A_t \sim \pi(\cdot | s)$

Value Function with Policy The action-value function $q_\pi(s, a)$ is the expected return starting from state s taking action a and then following the policy π

$$q_\pi(s, a) = E_\pi[G_t | S_t = s, A_t = a]$$



Bellman Expectation Equation with Value and Action-Value function The state-value function can be decomposed into immediate reward plus discounted value of successor state

$$v_\pi(s) = E_\pi[R_{t+1} + \gamma v_\pi(S_{t+1}) | S_t = s] = \sum_{a \in A} \pi(a | s) q_\pi(s, a)$$

Similarly we can decompose the action-value function

$$q_\pi(s, a) = E_\pi[R_{t+1} + \gamma q_\pi(S_{t+1}, A_{t+1}) | S_t = s, A_t = a] = R_s^a + \gamma \sum_{s' \in S} P_{ss'}^a v_\pi(s')$$

One more step of nesting: the expected return of being in a state reachable from s through action a and then continue following the policy π

$$v_\pi(s) = \sum_{a \in A} \pi(a | s) \left(R_s^a + \gamma \sum_{s' \in S} P_{ss'}^a v_\pi(s') \right)$$

The expected return of any action a' taken from states reachable from s through action a (and then following policy π)

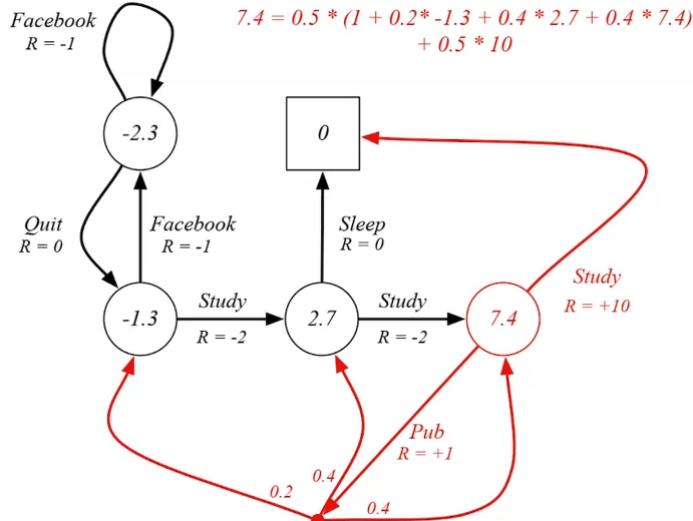
$$q_\pi(s, a) = R_s^a + \gamma \sum_{s' \in S} P_{ss'}^a \sum_{a' \in A} \pi(a' | s') q_\pi(s', a')$$

Matrix form: linear system

$$v_\pi = R^\pi + \gamma P^\pi v_\pi$$

With direct solution

$$v_\pi = (I - \gamma P^\pi)^{-1} R^\pi$$



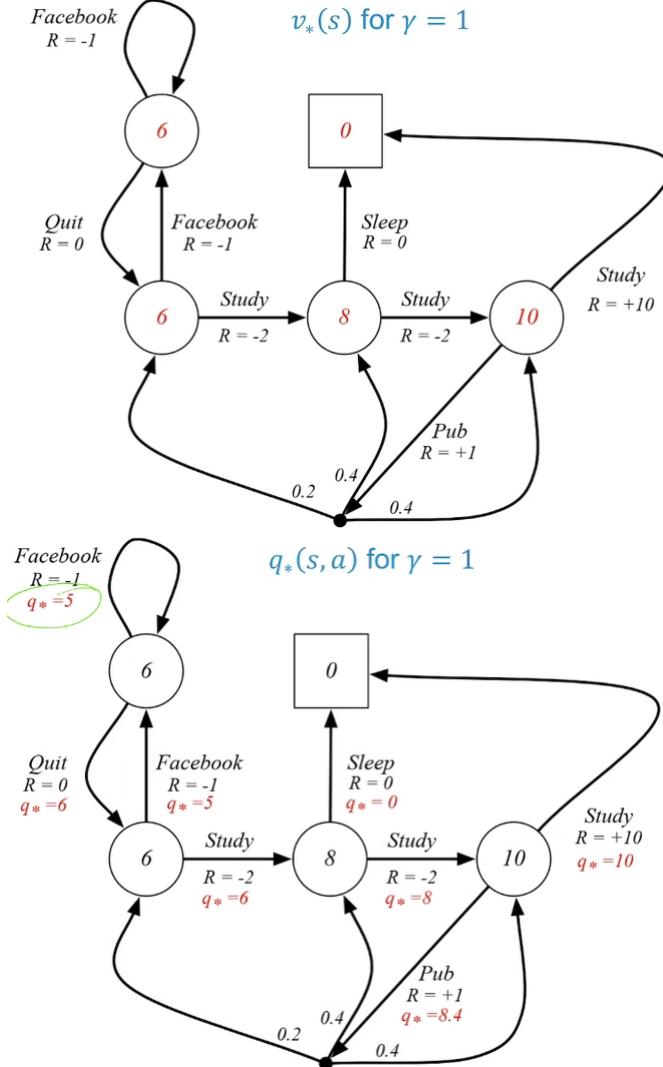
Optimal Value Function The optimal state-value function $v_*(s)$ is the maximum value function over all policies

$$v_*(s) = \max_{\pi} v_{\pi}(s)$$

The optimal action-value function $q_*(s, a)$ is the maximum action-value function over all policies

$$q_*(s, a) = \max_{\pi} q_{\pi}(s, a)$$

The optimal value function determines the best possible performance in the MDP. And MDP is solved when we know the optimal value function.



Optimal Policy Define a partial ordering over policies

$$\pi \geq \pi' \text{ if } \forall s \ v_{\pi}(s) \geq v_{\pi'}(s)$$

Theorem For any Markov Decision process

There exist an optimal policy π_* that is better or equal than all the other: $\forall \pi \ \pi_* \geq \pi$

All optimal policies achieve the optimal value function: $v_{\pi_*}(s) = v_*(s)$

All optimal policies achieve the optimal action-value function: $q_{\pi_*}(s, a) = q_*(s, a)$

An optimal policy can be found by maximizing over $q_*(s, a)$

$$\pi_*(a | s) = \begin{cases} 1 & \text{if } a = \arg \max_{a \in A} q_*(s, a) \\ 0 & \text{otherwise} \end{cases}$$

There is always a deterministic optimal policy for any MDP. If we know $q_*(s, a)$ we straightforwardly find the optimal policy.

Bellman Optimality Equations Optimal value functions are recursively related Bellman-style

$$v_*(s) = \max_{a \in A} q_*(s, a) = \max_{a \in A} R_s^a + \gamma \sum_{s' \in S} P_{ss'}^a v_*(s')$$

$$q_*(s, a) = R_s^a + \gamma \sum_{s' \in S} P_{ss'}^a v_*(s') = R_s^a + \gamma \sum_{s' \in S} P_{ss'}^a \left(\max_{a' \in A} q_*(s', a') \right)$$

0.12.4 Model-Based Planning

Dynamic Programming Dynamic: problem with sequential or temporal component.

Programming: optimizing a program, i.e. a policy.

It's a method for solving complex problems by breaking them down into subproblems: solve the subproblems and combine the partial solutions.

Something already seen in the Viterbi Algorithm.

Planning by Dynamic Programming Dynamic programming assumes full knowledge of the MDP.

Prediction:

input: MDP (S,A,P,R, γ) and policy π or MKRP (S,P,R, γ)

output: value function v_π

Control:

Input: MDP (S,A,P,R, γ)

output: optimal value function v_{π_*} and an optimal policy π_*

Iterative Policy Evaluation Problem: evaluating a given policy π .

Solution: iterative application of Bellman expectation backup

$$v_1 \rightarrow \dots \rightarrow v_\pi$$

Using **synchronous backup**:

At each iteration $k + 1$

for each state $s \in S$

Update $v_{k+1}(s)$ from $v_k(s')$ where s' is a successor state of s

Formally

$$\begin{aligned} v_{k+1}(s) &= \sum_{a \in A} \pi(a | s) \left(R_s^a + \gamma \sum_{s' \in S} P_{ss'}^a v_k(s') \right) \\ v_{k+1} &= R^\pi + \gamma P^\pi v_k \end{aligned}$$

Improving a Policy Given a policy π

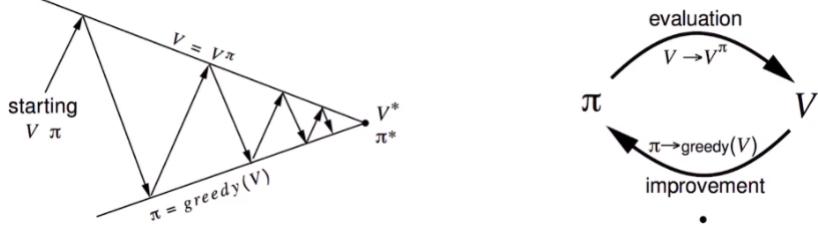
Evaluate the policy: $v_\pi(s) = E[R_{t+1} + \gamma R_{t+2} + \dots | S_t = s]$

Improve the policy by acting greedily with respect to v_π : $\pi' = \text{Greedy}(\pi)$

Repeat

This process always converges to π_* , under the assumption that the improvement is greedy.

Formal Improvement



Policy Evaluation: estimate v_π , iterative policy evaluation

Policy Improvement: generate $\pi' \geq \pi$, greedy policy improvement

Consider a deterministic policy $a = \pi(s)$, we can improve it by acting greedily

$$\pi'(s) = \arg \max_{a \in A} q_\pi(s, a)$$

This improves the value from any state s over one step

$$q_\pi(s, \pi'(s)) = \max_{a \in A} q_\pi(s, a) \geq q_\pi(s, \pi(s)) = v_\pi(s)$$

Therefore improving the value function

$$v_{\pi'}(s) \geq v_\pi(s)$$

If improvement stops

$$q_\pi(s, \pi'(s)) = \max_{a \in A} q_\pi(s, a) = q_\pi(s, \pi(s)) = v_\pi(s)$$

We satisfy Bellman optimality

$$v_\pi(s) = \max_{a \in A} q_\pi(s, a)$$

Therefore, $\forall s \in S$ $v_\pi(s) = v_*(s)$ and π is an optimal policy.

Modified Policy Improvement Does policy evaluation need to converge to v_{π_*} ? Introduce a stopping condition, e.g. ϵ -convergence of value function or stopping after k iterations (e.g. $k = 3$ sufficient in smallgridworld).

Why update the policy every iteration? Stop after $k = 1$, equivalent to value iteration.

Generalized policy iteration

Policy Evaluation: estimate v_π , **any** policy evaluation

Policy Improvement: generate $\pi' \geq \pi$, **any** policy improvement algorithm

Optimality Principle Any optimal policy can be subdivided into two components:

An optimal first action a^*

Followed by an optimal policy from successor state s'

Theorem Principle of Optimality

A policy $\pi(a | s)$ achieves the optimal value from state s (i.e. $v_\pi(s) = v_*(s)$) \Leftrightarrow for any state s' reachable from s , π achieves the optimal value from state s' , $v_\pi(s') = v_*(s')$

Deterministic Value Iteration If we know the solution to subproblems $v_*(s')$, then solution $v_*(s)$ can be found by one-step lookahead

$$v_*(s) = \max_{a \in A} R_s^a + \gamma \sum_{s' \in S} P_{ss'}^a v_*(s')$$

Value iteration applies these updates iteratively. Intuition: start with final rewards and work backwards (still works with loopy, stochastic MDPs).

Value Iteration Problem: find optimal policy π

Solution: iterative application of Bellman optimality backup

$$v_1 \rightarrow \dots \rightarrow v_\pi$$

Using synchronous backups: at each iteration $k + 1$ and for all states $s \in S$, update $v_{k+1}(s)$ from $v_k(s')$.

Unlike policy iteration, there is no explicit policy. Intermediate value function may not correspond to any policy. Formally

$$\begin{aligned} v_{k+1}(s) &= \max_{a \in A} \left(R_s^a + \gamma \sum_{s' \in S} P_{ss'}^a v_k(s') \right) \\ v_{k+1} &= \max_{a \in A} (R^a + \gamma P^a v_k) \end{aligned}$$

Extensions

Asynchronous Backups Dynamic programming methods described so far used synchronous backups, where all states are backed up in parallel.

Asynchronous dynamic programming backs up states individually, in any order: for each selected state, apply the appropriate backup. This can significantly reduce computation, and it's guaranteed to converge if all states continue to be selected. There are three simple approaches for asynchronous dynamic programming:

In-Place Dynamic Programming

Synchronous value iteration stores two copies of value function: for all $s \in S$

$$\begin{aligned} v_{new}(s) &= \max_{a \in A} R_s^a + \gamma \sum_{s' \in S} P_{ss'}^a v_{old}(s') \\ v_{old}(s) &= v_{new}(s) \end{aligned}$$

In-place value iteration only stores one copy of value function: for all $s \in S$

$$v(s) = \max_{a \in A} R_s^a + \gamma \sum_{s' \in S} P_{ss'}^a v(s')$$

Prioritized Sweeping

Use the magnitude of Bellman error to guide state selection

$$\left| \max_{a \in A} \left(R_s^a + \gamma \sum_{s' \in S} P_{ss'}^a v(s') \right) - v(s) \right|$$

Backup the state with the largest remaining Bellman error.

Update Bellman error of affected states after each backup.

It requires knowledge of reverse dynamics (predecessor states), and can be implemented efficiently by maintaining a priority queue.

Real-Time Dynamic Programming

Intuition: only states that are relevant to agent. Use the agent's experience to guide the selection of states.

After each time step S_t, A_t, R_{t+1} , backup the state S_t

$$v(S_t) = \max_{a \in A} \left(R_{S_t}^a + \gamma \sum_{s' \in S} P_{S_t s'}^a v(s') \right)$$

Full-Width Backup Dynamic Programming uses full-width backup. For each backup (sync or async): every successor state and action is considered, using knowledge of the MDP transitions and reward function.

Dynamic programming is effective for medium-sized problems (millions of states), but for large problems it suffers Bellman's curse of dimensionality: the number of states $n = |S|$ grows exponentially with the number of state variables. Even one backup can become too expensive.

Sample Backup From now on we will consider sample backups. Using sample rewards and sample transitions (S, A, R, S') , instead of reward function R and transition function P .

Pros: model-free (no advance knowledge of MDP required), breaks the curse of dimensionality through sampling, the cost of backup is constant, independent of $n = |S|$.

0.12.5 Model-Free Reinforcement Learning

So far: solve a known MDP (states, transition, rewards, actions). In a model-free context: no environment model, neither knowledge of MDP transition/rewards.

Model-Free Prediction: estimate the value function of an unknown MDP

Model-Free Control: optimize the value function of an unknown MDP

Monte-Carlo Reinforcement Learning

MC MV methods learn directly from episodes of experience. An episode terminates, so we learn from a sequence of state-action-reward. We assume to accumulate a certain number of these episodes, and MC works based on statistics. MC is model-free: no knowledge of DMP transition/rewards. Also **complete episodes** (they terminate, no bootstrapping).

Uses the simplest idea: the value is the mean return across episodes.

But can only apply Monte-Carlo to episodic MDPs, where **all episodes must terminate**.

MC Policy Evaluation Goal: learn v_π from episodes of experience under policy π

$$S_1, A_1, R_2, \dots, R_k \sim \pi$$

Recall that return is the total discounted reward

$$G_t = R_{t+1} + \gamma R_{t+2} + \dots + \gamma^{T-1} R_T$$

At time t I am in state S_t . Recall the value function is the expected return

$$v_\pi(s) = E[G_t | S_t = s]$$

MC policy evaluation uses empirical mean return instead of expected return.

Incremental Mean MC Update Update $V(s)$ incrementally after episode $S_1, A_1, R_2, \dots, R_T$. For each state S_t with return G_t

Increment counter $N(s) = N(s) + 1$ (number of time I've seen state s)

Update value function with incremental mean

$$V(S_t) = V(S_t) + \frac{G_t - V(S_t)}{N(S_t)}$$

In non-stationary problems track a running mean (forget old episodes)

$$V(S_t) = V(S_t) + \alpha(G_t - V(S_t))$$

Temporal-Difference Learning

TD methods learn directly from episodes of experience. Model-free (no knowledge of MDP transition/rewards), **also learns from incomplete episodes** by bootstrapping, and updates a guess towards a guess.

MC vs TD Goal: learn v_π from episodes of experience under policy π

Incremental every-visit MC

Update value $V(S_t)$ toward actual return G_t

$$V(S_t) = V(S_t) + \alpha(G_t - V(S_t))$$

Simplest temporal-difference learning algorithm (TD(0))

Update value $V(S_t)$ toward estimated return $R_t + \gamma V(S_{t+1})$

$$V(S_t) = V(S_t) + \alpha(R_t + \gamma V(S_{t+1}) - V(S_t))$$

$R_t + \gamma V(S_{t+1})$ is TD target and $R_t + \gamma V(S_{t+1}) - V(S_t)$ is TD error δ_t

Bias-Variance Tradeoff Return $G_t = R_{t+1} + \gamma R_{t+2} + \dots$ is an unbiased estimate of $v_\pi(S_t)$. (MC zero bias)
 True TD target $R_{t+1} + \gamma v_\pi(S_{t+1})$ is an unbiased estimate of $v_\pi(S_t)$.
 TD target $R_{t+1} + \gamma V(S_{t+1})$ is a biased estimate of $v_\pi(S_t)$
 TD target is much lower variance than the return: return depends on many random actions-transition-rewards, TD target depends on one random action-transition-reward.

MC has high variance, zero bias

Good convergence properties, even with function approximation. Not very sensitive to initial value and very simple to understand and use.

TD has low variance, some bias

Usually more efficient than MC. TD(0) converges to $v_\pi(s)$ but not always with function approximation. Is also more sensitive to initial value.

Batch MC and TD MC and TD converge $V(s) \rightarrow v_\pi(s)$ as experience $\rightarrow \infty$. But what about batch solution for finite experience?

$$s_1^1, a_1^1 r_2^1, \dots, s_{T_1}^1$$

⋮

$$s_1^K, a_1^K r_2^K, \dots, s_{T_1}^K$$

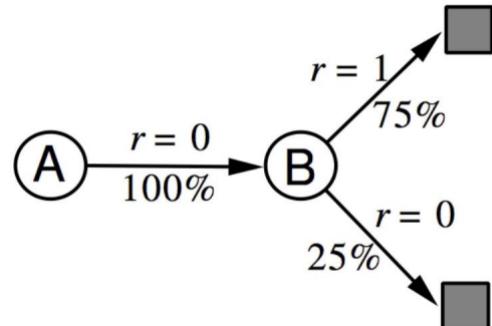
E.g. repeatedly sample episode $k \in [1, K]$, apply MC or TD(0) to episode k .

Simple example Two states, A, B, no discounting,
 8 episodes of experience:

1. A, 0, B, 0
2. B, 1
3. B, 1
4. B, 1
5. B, 1
6. B, 1
7. B, 1
8. B, 0

$V(B) = \frac{6}{8} = 75\%$ on MC, sum over number of visits, and 0.75 for TD.

$V(A) = 0$ on MC but 0.75 = 75% on TD.



Certainty Equivalence

MC converges to solution with minimum mean-squared error, best fit to the observed returns

$$\sum_{k=1}^K \sum_{t=1}^{T_k} (G_t^k - V(s_t^k))^2$$

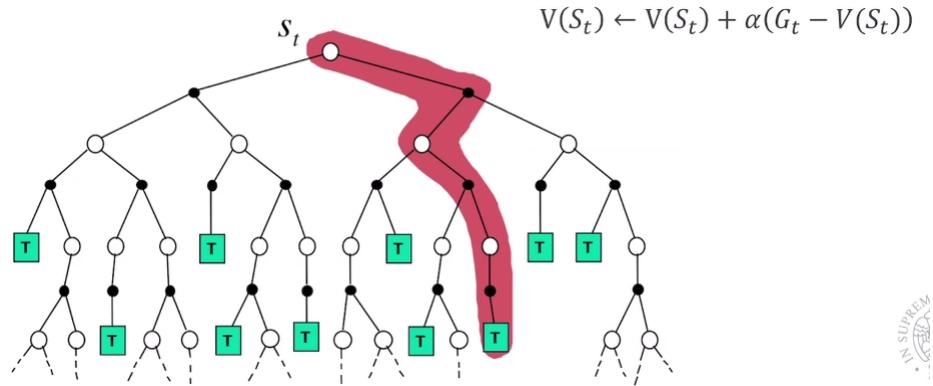
TD(0) converges to solution of maximum likelihood Markov model, solution to the MDP (S, A, P, R, γ) that best fits the data

$$\begin{aligned} \hat{P}_{ss'}^a &= \frac{1}{N(s, a)} \sum_{k=1}^K \sum_{t=1}^{T_k} 1(s_t^k, a_t^k, s_{t+1}^k; s, a, s') \\ \hat{R}_s^a &= \frac{1}{N(s, a)} \sum_{k=1}^K \sum_{t=1}^{T_k} 1(s_t^k, a_t^k; s, a) r_t^k \end{aligned}$$

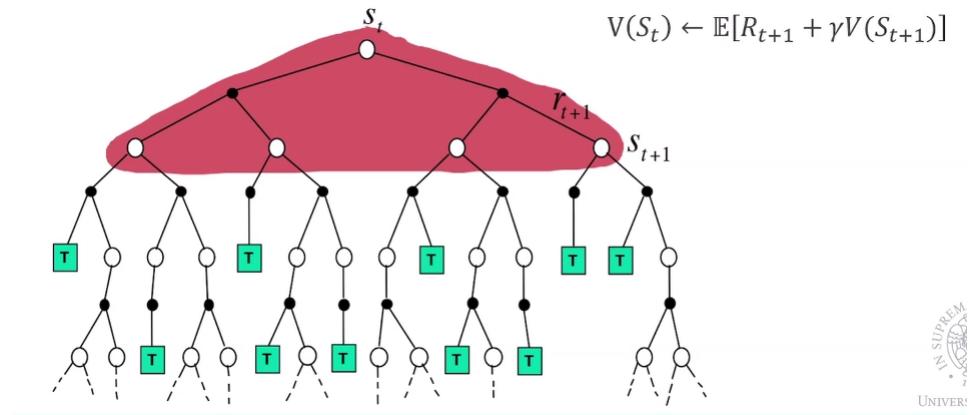
TD exploits Markov property (usually more efficient in Markov environments), while MC doesn't (usually more effective in non-Markov environments).

Unifying and Generalizing

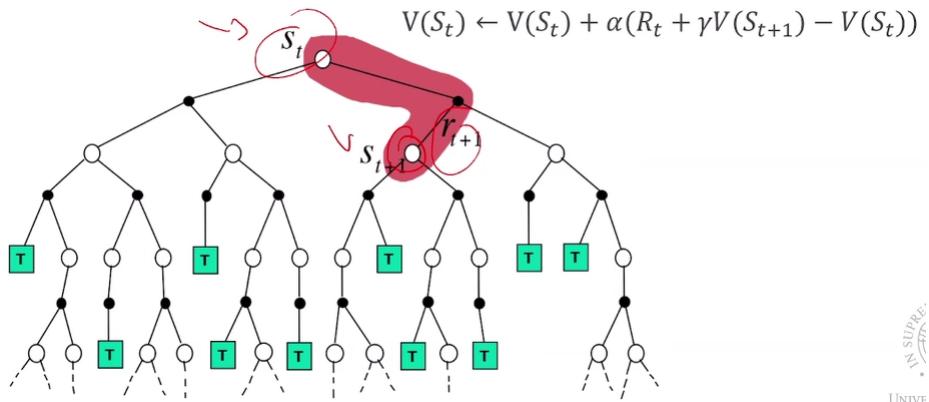
MC Update



Dynamic Programming

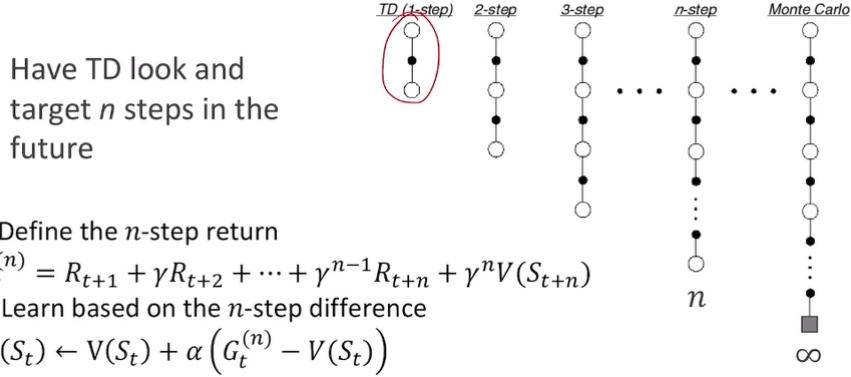


TD Update



Are there something in between MC and TD? Yes, TD(0) because looks just one step ahead.

n-step Prediction TD looks and targets n steps in the future



Define the n -step return

$$G_t^{(n)} = R_{t+1} + \gamma R_{t+2} + \dots + \gamma^{n-1} R_{t+n} + \gamma^n V(S_{t+n})$$

Learn based on the n -step inference

$$V(S_t) = V(S_t) + \alpha (G_t^{(n)} - V(S_t))$$

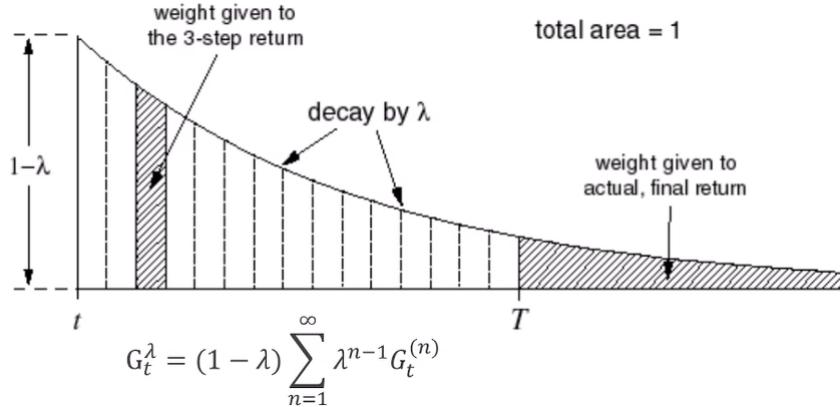
λ -returns (Forward View) The λ -return G_t^λ combines all n -step returns $G_t^{(n)}$. Using weight $(1 - \lambda)\lambda^{n-1}$

$$G_t^\lambda = (1 - \lambda) \sum_{n=1}^{\infty} \lambda^{n-1} G_t^{(n)}$$

Update as appropriate (TD(λ))

$$V(S_t) = V(S_t) + \alpha (G_t^\lambda - V(S_t))$$

But this Forward View need until infinity, so I need to do the computation in a smart way.
The TD(λ) weight function is:



Eligibility Traces Tradeoff between

Frequency heuristic

Recency heuristic

Eligibility trace combines both

$$E_0(s) = 0$$

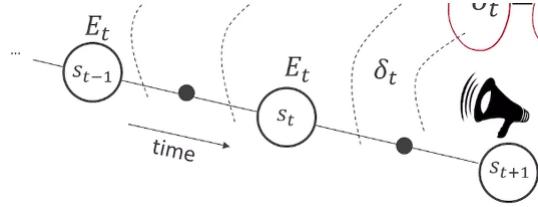
$$E_t(s) = \lambda \gamma E_{t-1}(s) + 1(S_t; s)$$

With fading memory of the past (frequency) and adds 1 whenever $S_t = s$ (recency)

Backward View TD(λ) Keep an eligibility trace for every state s . Update value $V(s)$ for every state s , in proportion to TD-error δ_t and eligibility trace $E_t(s)$

$$\delta_t = R_{t+1} + \gamma V(S_{t+1}) - V(S_t)$$

$$V(s) = V(s) + \alpha \delta_t E_t(s)$$



On-Policy Learning

Model-Free Policy Iteration Using Action-Value Function

Greedy policy improvement over $V(s)$ requires model of MDP

$$\pi'(s) = \arg \max_{a \in A} R_s^a + P_{ss'}^a V(s')$$

The problem is $P_{ss'}^a$, which is the model

Greedy policy improvement over $Q(s, a)$ is model-free

$$\pi'(s) = \arg \max_{a \in A} Q(s, a)$$

ϵ -greedy Exploration Simplest idea for ensuring continual exploration.
All m actions are tried with non-zero probability:

With probability $1 - \epsilon$ choose the greedy action

With probability ϵ choose an action at random

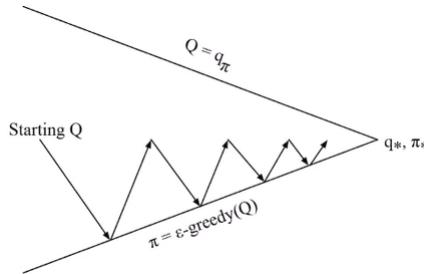
$$\pi(a | s) = \begin{cases} \frac{\epsilon}{m} + (1 - \epsilon) & \text{if } a^* = \arg \max_{a \in A} Q(s, a) \\ \frac{\epsilon}{m} & \text{otherwise} \end{cases}$$

Monte-Carlo Control

Every episode

Policy Evaluation: Monte-Carlo policy evaluation $Q \simeq q_\pi$

Policy Improvement: ϵ -greedy policy improvement



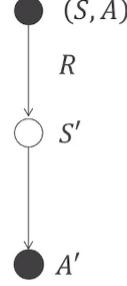
On-Policy Control with SARSA

With TD-learning, every time-step

Policy Evaluation: SARSA $Q \simeq q_\pi$

Policy Improvement: ϵ -greedy policy improvement

Updating Action-Value Functions with SARSA



$$Q(S, A) \leftarrow Q(S, A) + \alpha(R + \gamma Q(S', A') - Q(S, A))$$

Expected SARSA

$$q_*(s, a) = R_s^a + \gamma \sum_{a' \in A} \pi(a' | s') q_*(s', a')$$

We sample also future action $A' \simeq \pi$ (instead of leveraging policy to compute expectation).

Off-Policy Learning

Evaluate target policy $\pi(a | s)$ to compute $v_\pi(s)$ or $q_\pi(s, a)$, while following behavior policy $\mu(a | s)$

$$\{S_1, A_1, R_2, \dots, S_T\} \sim \mu$$

Q-Learning Off-policy learning of action-values $Q(s, a)$

Next action is chosen using behavior policy $A_{t+1} \sim \mu(\cdot | S_t)$

But we consider alternative successor action $A' \sim \pi(\cdot | S_t)$

And update $Q(S_t, A_t)$ toward value of alternative action

$$Q(S_t, A_t) = Q(S_t, A_t) + \alpha(R_{t+1} + \gamma Q(S_{t+1}, A') - Q(S_t, A_t))$$

Note that $R_{t+1} + \gamma Q(S_{t+1}, A')$ is TD learning

Off-Policy Control by Q-Learning Allow both behavior and target policies to improve. The target policy π is greedy with respect to $Q(S_t, A_t)$

$$\pi(S_{t+1}) = \arg \max_{a'} Q(S_{t+1}, a')$$

The behavior policy μ is ϵ -greedy with respect to $Q(s, a)$.

The Q-learning target then simplifies to

$$R_{t+1} + \gamma Q(S_{t+1}, A') = R_{t+1} + \gamma Q(S_{t+1}, \arg \max_{a'} Q(S_{t+1}, a')) = R_{t+1} + \max_{a'} \gamma Q(S_{t+1}, a')$$

Theorem Q-learning control converges to the optimal action-value function, $Q(s, a) \rightarrow q_*(s, a)$

$$Q(S, A) = Q(S, A) + \alpha(R + \max_{a'} \gamma Q(S'_{t+1}, a') - Q(S, A))$$

Wrap-Up

	<i>Full Backup (DP)</i>	<i>Sample Backup (TD)</i>
Bellman Expectation Equation for $v_\pi(s)$	$v_\pi(s) \leftarrow s$ Iterative Policy Evaluation	$v_\pi(s') \leftarrow s'$ TD Learning
Bellman Expectation Equation for $q_\pi(s, a)$	$q_\pi(s, a) \leftarrow s, a$ Q-Policy Iteration	$q_\pi(s', a') \leftarrow a'$ Sarsa
Bellman Optimality Equation for $q_*(s, a)$	$q_*(s, a) \leftarrow s, a$ Q-Value Iteration	$q_*(s', a') \leftarrow a'$ Q-Learning

Model-free prediction is value function estimation of an unknown MDP, based on sample updates, Monte-Carlo methods (estimating value function by averaging sample returns on episodic tasks) or TD learning (learn from existing estimates of future return, generalization to n th step).

Model-free control leverages action-value function: need to maintain sufficient exploration (ϵ -greedy), off-policy controls learns a value function of a target policy from data generated by a different behavior policy. SARSA(λ) on-policy and Q-learning off-policy.