Maths Review for Computer Graphics

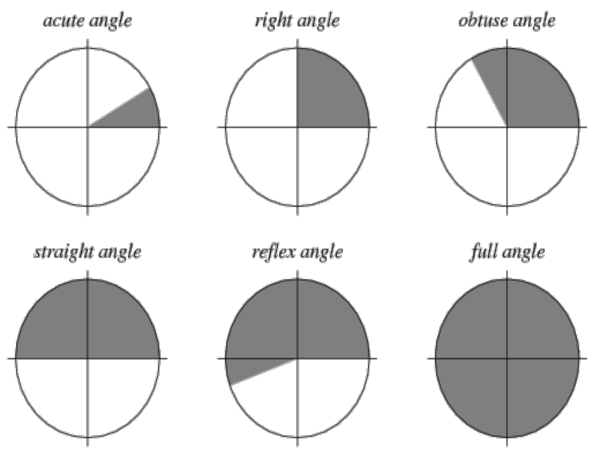
This is a completely optional review of some fundamental geometry.
Understanding it will help make sense of the more mathematical sections of the javascript code we are looking at.

I have assumed that you are familiar with the Cartesian coordinate system (x, y graphs), basic algebra and have a basic understanding of angles.

If you feel there are concepts I have missed out or sections that are unclear let me know.
This is a work in progress, so any suggestions for improvement are welcome.

The images are not my own, they have been collected through Google searches.
The comic is from here: http://xkcd.com/.

I. Angles & Degrees



Angles are usually measured in *degrees.*

A *right angle* is equal to 90 degrees.

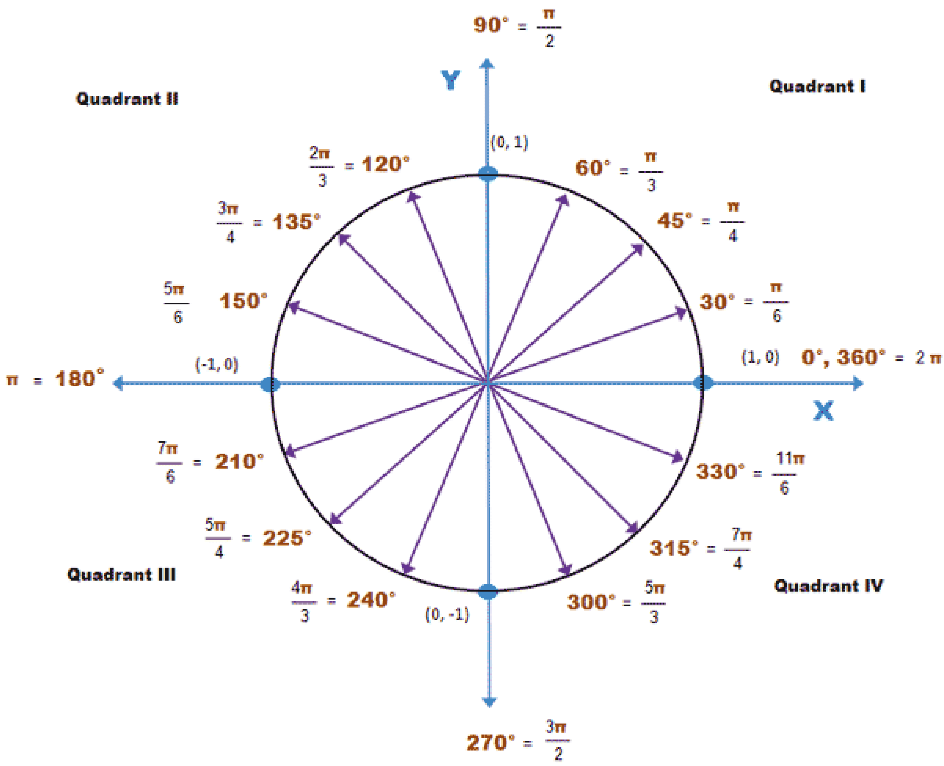
A circle is equivalent to a full angle of 360 degrees.

We can chop a circle into segments based on an angle.
For example:

- ¼ of a circle is equivalent to an angle of 90 degrees (360/4 = 90°).
- Similarly, 1/3 of a circle is equal to 120 degrees (360/3 = 120°).

(The names of the different types of angles on the right are not important to us, don't worry about memorizing them now.)

II. Radians



Radians are just another unit of measurement.
Just as we can measure distance in Kilometres or Miles, we can measure angles in *degrees* or *radians.*

There are **2π radians in a circle**.

(Remember, **π** is just a number equal to ~3.141592... Therefore **2π** is ~ 6.283...)

As you can see in the image on the right, we can give angles in radians as well as degrees.

- So, a right angle (¼ of a circle) is equal to π/2 radians.
- ¾ of a circle is equal to 3π/2 radians (270 degrees).

Notice how the angles increase counter-clockwise around the circle.

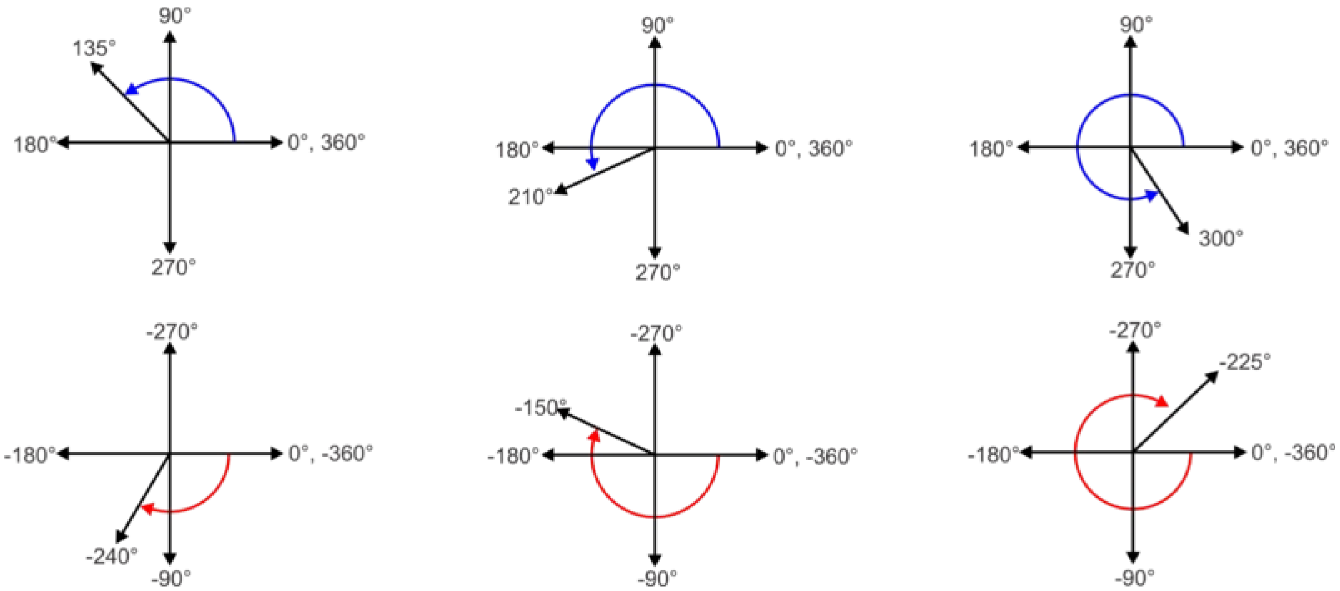
Here's the formula for going from degrees to radians, or vice versa, if you're interested.

$$\text{Radians} = \left(\frac{\pi}{180^{o}}\right) \times \text{degrees}$$

$$\text{Degrees} = \left(\frac{180^{o}}{\pi}\right) \times \text{radians}$$

III. Rotations

All this talk of radians and circles is leading up to our main focus – rotations. We need radians and circles so can describe the rotation of a shape. Look at the image below:



Look at the rotations described by the blue curved arrows.

Notice again, how we start at the positive x-axis (labelled 0 degrees) and we move in a counter-clockwise direction of rotation.
? Can you figure out how many radians the 1st rotation goes through (135 degrees)?

The rotations described by the red curved arrows are negative angles. You still start at the positive x-axis, but you travel in a clockwise direction.

IV. Unit Circle

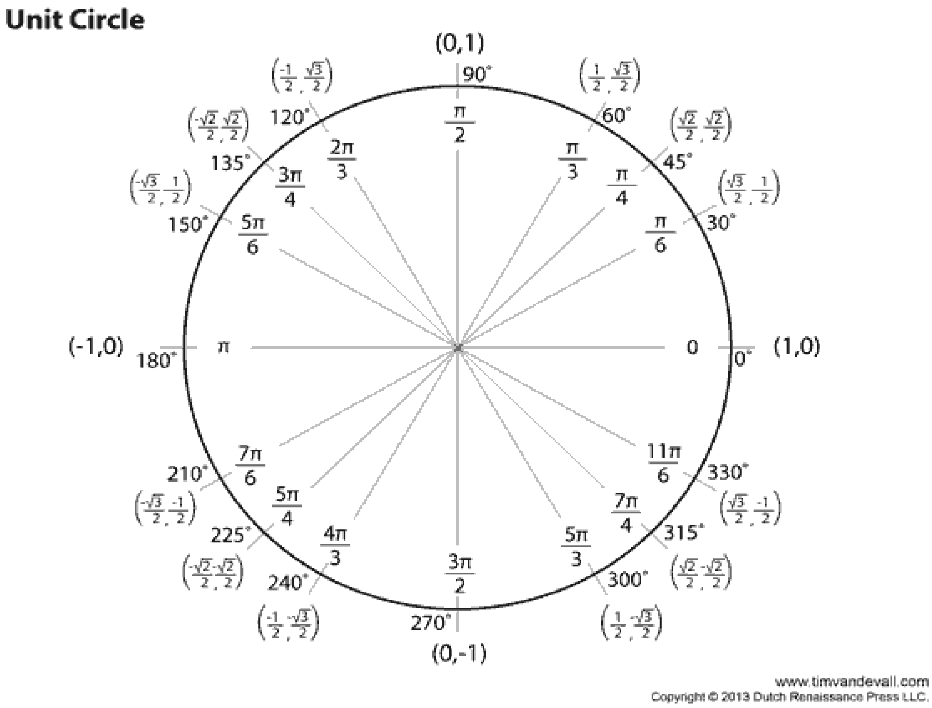
OK, let's put all this together and look at the *Unit Circle.*

This is just a circle with a radius of 1.

Look at the 0° position on the circle.
It has (x,y) coordinates of (1,0).

Similarly, 90° has coordinates of (0,1).

How are coordinates for 30° or 120° calculated?

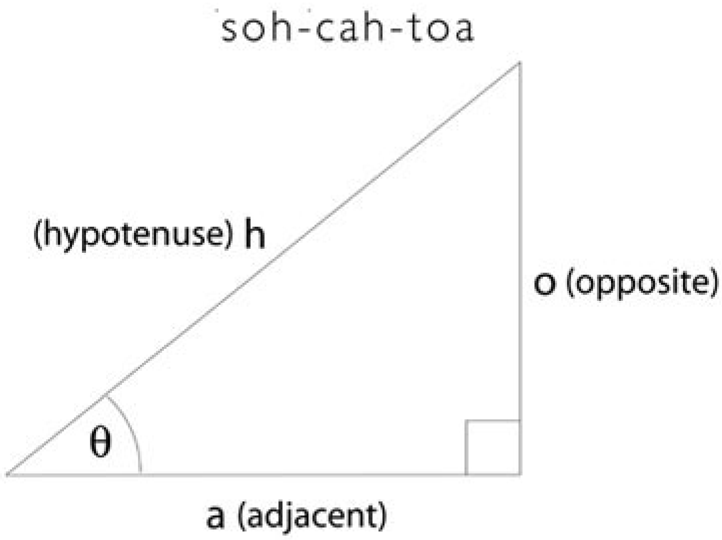


We can use some simple trigonometric ideas to work these out.

V. SOHCAHTOA

Let's review the trigonometry of right angle triangles (ie. Triangles with one 90° angle)

Take a triangle like the one here →



- We'll call one of the non-90° angles θ *(theta).*

- The side opposite θ is the opposite.
- The side opposite the right angle is the hypotenuse.
- The remaining side is the adjacent.

There are 3 basic trigonometric identities that allow us to relate the angles and sides of a triangle.

$$\text{sine}(\theta) = \frac{o}{h} \qquad \text{cosine}(\theta) = \frac{a}{h}$$

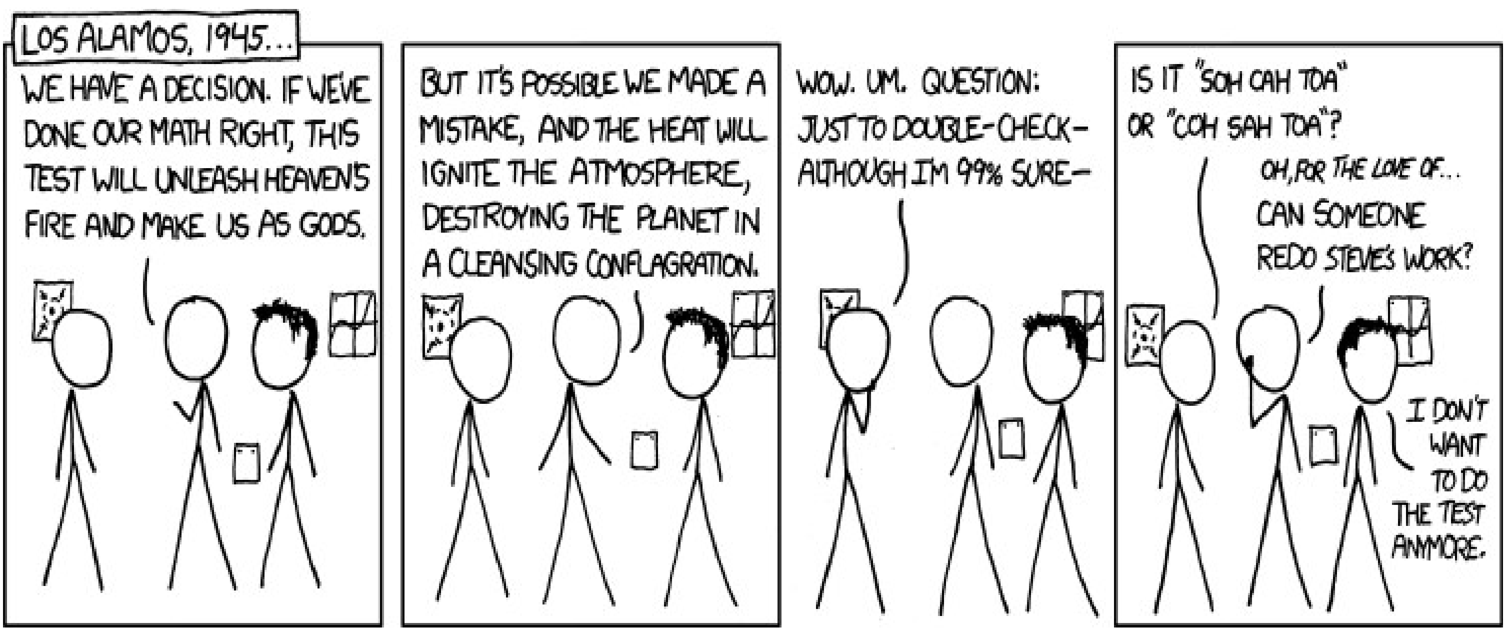
$$\text{tangent}(\theta) = \frac{o}{a} \qquad \text{tangent}(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

They are:

- sine
- cosine
- tangent

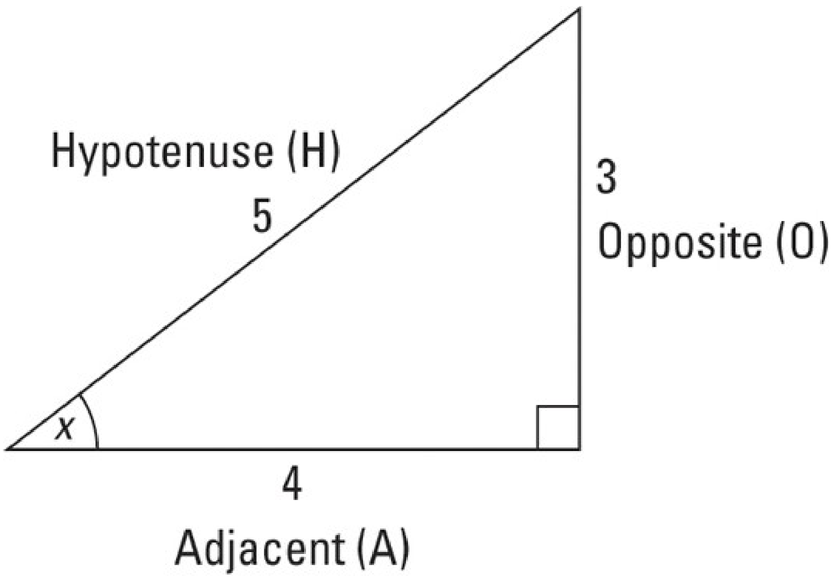
Looking at the formulas to the right, we can see that :

- the sine of θ is equal to opposite divided by the hypotenuse. (SOH)
- the cosine of θ is equal to adjacent divided by the hypotenuse (CAH).
- The tangent of θ is equal to opposite divided by the adjacent (TOA).



Let's do some calculations with SOHCAHTOA to get the hang of it.

Let's say we have this triangle, with has 3 known sides, a right angle and 2 unknown angles.



We're going to find angle x:

sin(x) = O/H

→ sin(x) = 3/5 = 0.6

To get from sin(x) to x, we take *inverse sine* of 0.6:

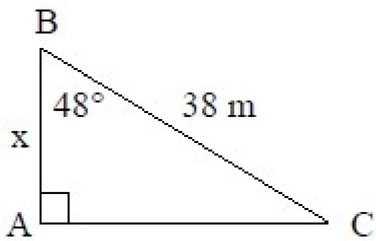
→ $\sin^{-1}$ (0.6) = x

(There is a $\sin^{-1}$ button on any scientific calculator).

→ x = 36.86° or 0.644 radians

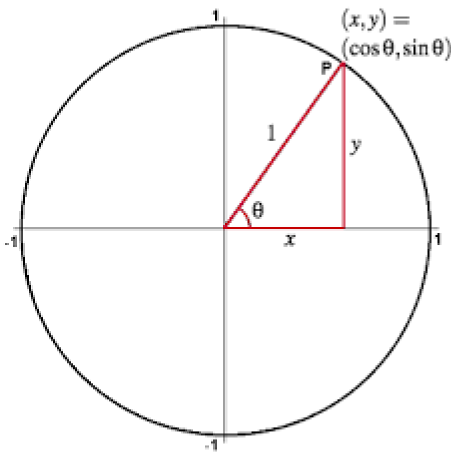
3. Calculate the value of z to the nearest hundredth: $\tan 24° = \dfrac{z}{34.627}$

4. Determine the length of side x to the nearest tenth.



Going back to the *Unit Circle:*

We can imagine a right angle triangle inscribed inside the unit circle.
The radius of the circle forms the hypotenuse.



Let's assume the angle θ = 60° = π/3
(From now on, we only deal with radians when referring to angles on the unit circle).

Can you calculate the coordinates point P on the circle?

With SOHCAHTOA:

- cos π/3 = x/1 = x = 0.500
- sin π/3 = y/1 = y = 0.866

If you look back at the Unit Circle diagram on Page 3, you'll see that the coordinates for a π/3 angle are (½, √3/2).

If you look at our calculations above (0.5, 0.866), you'll see that these are the decimal equivalents of the fractions.

You can use this table to check the coordinates for the main intervals on the unit circle.

degrees	0°	30°	45°	60°	90°
radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sin x	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos x	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tan x	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	—

The key thing to notice is that the x-coordinate of a point on the circle = cos x, and the y-coordinate = sin x.

VI. Back to Rotations

Let's use what we've learned to describe rotations of a point.

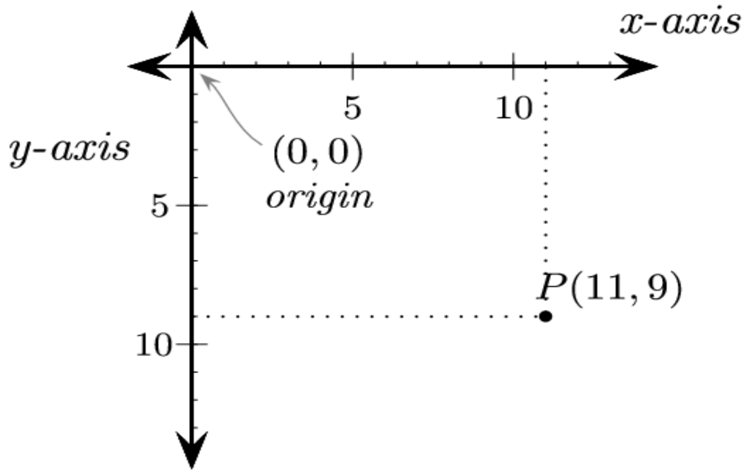
Imagine a point at (√2/2, √2/2) on the unit circle.

If we *rotate* the point by π/4, what will the coordinates of the point be now?

Well, the original point corresponded to angle of π/4 radians (look at the table above if you're unsure why).

Now we've added π/4 to it, so the point now corresponds to an angle of π/2 rads. The coordinates for such an angle are (0,1).

VII. Now forget everything you just learned...



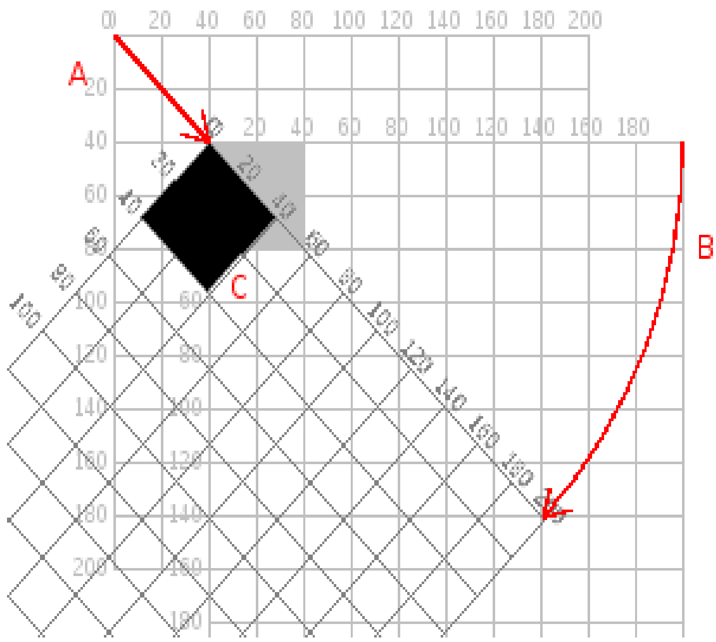
Due to reasons lost in the mists of time, computer graphics coordinates place the origin at the top-left corner of the screen.

When dealing with *x*, everything's the same. As *x* increases, you move from left to right.

y is reversed. As *y* increases, you move down instead of up.

Rotations are also reversed. If you rotate an object in a graphics program with a positive value, it will rotate **clockwise.**

Here's an example of a shape like the one we'll be rotating in our code.



Instead of a single point, we have the four corners of the shape to rotate.

In this example, we're rotating about the top-left corner, so it doesn't move.

The other 3 corners of the shape are being rotated by... (can you tell how many radians)?