V. Joint Probability Distributions And Markov Chains

Q 1] The joint distribution of two random variables X and Y is as follows:

X \ Y	-4	2	7
1	1/8	1/4	1/8
5	1/4	1/8	1/8

Compute the following:

(a) $E(X)$ and $E(Y)$ b) $E(XY)$ c) σ_X and σ_Y

d) $cov(X, Y)$ e) $\rho(X, Y)$

Sol^n Distribution of X:

x_i :	1	5
$f(x_i)$:	1/2	1/2

Distribution of Y:

y_j :	-4	2	7
$g(y_j)$:	3/8	3/8	1/4

a) $E(X) = \sum x_i f(x_i) = 1\left(\frac{1}{2}\right) + 5\left(\frac{1}{2}\right) = 3$

$E(Y) = \sum y_j\, g(y_j) = (-4)\left(\frac{3}{8}\right) + 2\left(\frac{3}{8}\right) + 7\left(\frac{1}{4}\right) = 1$

$\mu_x = E(X) = 3$ & $\mu_Y = E(Y) = 1$

b) $E(XY) = \sum x_i y_j J_{ij}$

$= 1(-4)\left(\frac{1}{8}\right) + 1(2)\left(\frac{1}{4}\right) + 1(7)\left(\frac{1}{8}\right) + 5(-4)\left(\frac{1}{4}\right) + (5)(2)\left(\frac{1}{8}\right) + 5(7)\left(\frac{1}{8}\right)$

$= \frac{3}{2}$

$\boxed{E(XY) = \frac{3}{2}}$

c) $\sigma_x^2 = E(X^2) - \mu_x^2$; $\sigma_y^2 = E(Y^2) - \mu_y^2$

$E(X^2) = \sum_i x_i^2 f(x_i)$

$= (1)(\frac{1}{2}) + (25)(\frac{1}{2}) = \underline{13}$

$E(Y^2) = \sum_j y_j^2 g(y_j)$

$= (16)(\frac{3}{8}) + (4)(\frac{3}{8}) + (49)(\frac{1}{4})$

$= \underline{79/4}$

$\sigma_x^2 = 13 - 3^2 = \underline{4}$; $\sigma_y^2 = \frac{79}{4} - 1 = \underline{\frac{75}{4}}$

d) $cov(X, Y) = E(XY) - \mu_x \mu_y$

$= \frac{3}{2} - (3)(1) = \underline{-\frac{3}{2}}$

$cov(X, Y) = -\frac{3}{2}$

e) $\rho(X, Y) = \frac{cov(X, Y)}{\sigma_x \sigma_y} = \frac{-3/2}{(2)\sqrt{\frac{75}{4}}} = \underline{-0.1732}$

$\therefore \boxed{\rho(X, Y) = -0.1732}$

Q 2] The joint probability distribution of two discrete random variables X & Y is given by $f(x, y) = k(2x + y)$ where x & y are integers such that $0 \le x \le 2$, $0 \le y \le 3$.

(a) Find the value of the constant k

(b) Find the marginal probability distribution of X & Y

(c) Show that the random variables X & Y are dependent.

Sol)

$X = \{x_i\} = \{0, 1, 2\}$

$Y = \{y_j\} = \{0, 1, 2, 3\}$

Given: $f(x, y) = k(2x+y)$

X\Y	0	1	2	3	Sum
0	0	k	2k	3k	6k
1	2k	3k	4k	5k	14k
2	4k	5k	6k	7k	22k
	6k	9k	12k	15k	42k

a) $42k = 1$

$\Rightarrow k = \frac{1}{42}$

b) Marginal distribution

x_i	0	1	2
$f(x_i)$	$1/7$ $= \frac{6}{42}$	$1/3$ $= \frac{14}{42}$	$11/21$ $= \frac{22}{42}$

y_j	0	1	2	3
$g(y_j)$	$\frac{6}{42}$ $= \frac{1}{7}$	$9/42$ $= \frac{3}{14}$	$12/42$ $= \frac{2}{7}$	$15/42$ $= \frac{5}{14}$

c) $f(x_i)\, g(y_j) \neq J_{ij}$

$\therefore$ Random variables are dependent.

3) Given the following joint distribution of the random variables X & Y, find the corresponding marginal distribution. Also compute co-variance and the correlation of the random variables X & Y (2) (6)

Sol

X\Y	1	3	9
2	1/8	1/24	1/12
4	1/4	1/4	0
6	1/8	1/24	1/12

Sol Marginal distributions of X & Y are

x_i	2	4	6	y_j	1	3	9
$f(x_i)$	1/4	1/2	1/4	$g(y_j)$	1/2	1/3	1/6

$$Cov(XY) = E(XY) - \mu_X \mu_Y$$

$$E(X) = \sum x_i f(x_i)$$

$$= 2\left(\frac{1}{4}\right) + 4\left(\frac{1}{2}\right) + 6\left(\frac{1}{4}\right) = \underline{\underline{4}}$$

$$E(Y) = \sum y_j g(y_j)$$

$$= 1\left(\frac{1}{2}\right) + 3\left(\frac{1}{3}\right) + 9\left(\frac{1}{6}\right) = \underline{\underline{3}}$$

$$E(XY) = \sum x_i y_j J_{ij}$$

$$= 2(1)\left(\frac{1}{8}\right) + 2(3)\left(\frac{1}{24}\right) + 2(9)\left(\frac{1}{12}\right) +$$

$$4(1)\left(\frac{1}{4}\right) + 4(3)\left(\frac{1}{4}\right) + 4(9)(0) +$$

$$6(1)\left(\frac{1}{8}\right) + 6(3)\left(\frac{1}{24}\right) + 6(9)\left(\frac{1}{12}\right)$$

$$= \underline{\underline{12}}$$

3 a

(i) $\text{cov}(X\ Y) = 12 - (4)(3) = 0$

$\therefore \text{cov}(X, Y) = 0$

Correlation of X & Y - $\rho(X, Y) = \dfrac{\text{cov}(X, Y)}{\sigma_X \sigma_Y}$

$\therefore \boxed{\rho(X, Y) = 0}$

4] Find the unique fixed probability vector of the regular stochastic matrix $A = \begin{bmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix}$

Sol^n We have to find $V = (x\ y)$ where $x + y = 1$ & $VA = V$

$\Rightarrow [x, y] \begin{bmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix} = [x\ y]$

$\left[\frac{3}{4}x + \frac{1}{2}y \quad \frac{1}{4}x + \frac{1}{2}y\right] = [x\ y]$

$\frac{3}{4}x + \frac{1}{2}y = x \quad (1)$; $\frac{1}{4}x + \frac{1}{2}y = y \quad (2)$

And solving using $x + y = 1$;

i.e. $y = 1 - x$ in (1)

$\frac{3}{4}x + \frac{1}{2}(1 - x) = x$

$3x + 2 - 2x = 4x$

$\therefore x = 2/3$

& $y = 1 - x = 1 - \frac{2}{3} = \frac{1}{3}$

$V = \left(\frac{2}{3}, \frac{1}{3}\right)$

$\therefore \left(\frac{2}{3}, \frac{1}{3}\right)$ is the unique fixed probability vector

③ ⑤

5) Find the unique fixed probability vector for the regular stochastic matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{bmatrix}$

Sol) $V = (x \ y \ z)$ where $x+y+z=1$ such that $VA = V$

$$[x \ y \ z] = \begin{bmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{bmatrix} [x \ y \ z]$$

$$\left[\frac{y}{6}, \ x + \frac{y}{2} + \frac{2z}{3}, \ 0 + \frac{y}{3} + \frac{z}{3}\right] = (x \ y \ z]$$

$$\frac{y}{6} = x, \quad x + \frac{y}{2} + \frac{2z}{3} = y, \quad \frac{y}{3} + \frac{z}{3} = z$$

$$y = 6x, \quad 6x + 3y + 4z = 6y, \quad y + z = 3z$$

And $x + y + z = 1$

Solving $x = \frac{1}{10}$, $y = \frac{6}{10}$, $z = \frac{3}{10}$

6) Show that $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}$ is a regular stochastic matrix. Also find the associated unique fixed probability vector

Sol

$$P^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

find: P^3, P^4

$$P^5 = \begin{bmatrix} 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 1/8 & 3/8 & 1/2 \end{bmatrix}$$

$\therefore$ P^5 all the entries are positive

$\therefore$ P is a regular stochastic matrix

$V = (x \; y \; z)$ where $x+y+z=1$ such that $VP = V$ (4) (a)

$$[x \; y \; z] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} [x \; y \; z]$$

$$\left[\frac{z}{2}, \; x+\frac{z}{2}, \; y\right] = (x \; y \; z]$$

$$\frac{z}{2} = x, \quad x + \frac{z}{2} = y, \quad y = z$$

$\therefore$ using $z = 2x$, $z = y = \cancel{4}2x$ in $x+y+z=1$

$$\therefore \cancel{x} = \frac{1}{5}, \quad y = z = \frac{2}{5}$$

$\therefore \left(\frac{1}{5}, \frac{2}{5}, \frac{2}{5}\right)$ is the required unique fixed probability vector of P.

7) Prove that the Markov chain whose transition probability matrix is

$$P = \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$ is irreducible

Sol)

$$P^2 = \frac{1}{36}\begin{bmatrix} 18 & 6 & 12 \\ 9 & 21 & 6 \\ 9 & 12 & 15 \end{bmatrix}$$

Since all the entries in P^2 are positive we conclude that the t.p.m, P is regular.

8] A student's study habits are as follows. If he studies one night, he is 70% sure not to study the next night. On the other hand if he does not study one night, he is 60% sure not to study the next night.

In the long run, how often does he study.

(4)
(6)

Sol A: study
B: not study

$$P = \begin{array}{c} \\ A \\ B \end{array}\begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix} \end{array}$$

To find the happening in the long run, we have to find unique fixed probability vector v of P.

$v = (x \; y)$ such that $vP = v$ where $x + y = 1$

i.e. $[x \; y] \begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix} = [x \; y]$

$0.3x + 0.4y = x$
$0.7x + 0.6y = y$

And $x + y = 1$
Solving, we have $x = \frac{4}{11}$; $y = \frac{7}{11}$

$\therefore$ In the long run the student will study $\frac{4}{11}$ of a time or 36.36% of the time

9) Three boys A, B, C are throwing ball to each other. A always throws the ball to B & B always throws the ball to C. C is just as likely to throw the ball to B as to A. If C was the first person to throw the ball, find the probabilities that after three throws (i) A has ball (ii) B has the ball (iii) C has the ball

Sol

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \end{array}$$

Initially, if C has the ball, $p^{(0)} = (0, 0, 1)$

Since the probabilities are desired after three rows

$$p^{(3)} = p^{(0)} p^3$$

$$= (0 \quad 0 \quad 1) \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

$$= \left[\frac{1}{4}, \frac{1}{4}, \frac{1}{2} \right]$$

Thus, after three rows, probability that the ball is with $A = \frac{1}{4}$, $B = \frac{1}{4}$ & $C = \frac{1}{2}$

10) A gambler's luck follows a pattern. If he wins a game, the probability of winning the next game is 0.6. However if he loses a game, the probability of losing the next game is 0.7. There is an even chance of gambler winning the first game. If so

a) What is the probability of he winning the second game.

b) What is the probability of he winning the third game

c) In the long run, how often he will win.

Sol:- $P = \begin{bmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{bmatrix}$

$p^{(0)} = \begin{bmatrix} 1/2 & 1/2 \end{bmatrix}$ (Initial probability vector)

a) $p^{(1)} = p^{(0)} P = \begin{bmatrix} 9/20 & 11/20 \end{bmatrix}$

b) $p^{(2)} = p^{(1)} P = \begin{bmatrix} 87/200 & 113/200 \end{bmatrix}$

c) $V = (x, y)$ $\quad V = VP$ & $x + y = 1$

$$V = \left[\frac{3}{7}, \frac{4}{7} \right]$$

Some More Examples

1] A fair coin is tossed thrice. The random variables X & Y are defined as, X denotes 0 or 1 according as head or a tail occurs on first toss. Y denotes the number of heads.

Find a) The marginal distribution of X & Y
b) The joint distribution of X & Y
c) Expectations of X, Y, XY
d) S.D's of X & Y
e) Co-variance and co-relation of X & Y

Sol) $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

$X = \{x_i\} = \{0, 1\}$ $\quad Y = \{y_j\} = \{0, 1, 2, 3\}$

a) Marginal distribution of X & Y

x_i :	0	1
$f(x_i)$:	1/2	1/2

y_j :	0	1	2	3
$g(y_j)$:	1/8	3/8	3/8	1/8

b) Joint distribution of X & Y

X \ Y	0	1	2	3
0	0	1/8	1/4	1/8
1	1/8	1/4	1/8	0

i, $J_{11} = P(X=0, Y=0) = 0$
$J_{12} = P(X=0, Y=1) = 1/8$ (corresponding to HTT)

$J_{13} = P(X=0, Y=2) = \frac{2}{8} = \frac{1}{4}$ (corresponding to HHT, HTH) ⓒ ⓐ

and so on

b) $E(X) = \mu_X = \sum_i x_i f(x_i) = 0\left(\frac{1}{2}\right) + 1\left(\frac{1}{2}\right) = 1/2$

$E(Y) = \mu_Y = \sum_j y_j g(y_j) = 3/2$

$E(XY) = \sum_{i,j} x_i y_j J_{ij}$

$= 0 + \left(0 + \frac{1}{4} + \frac{2}{8} + 0\right) = \frac{1}{2}$

c) $\sigma_X^2 = E(X^2) - \mu_X^2 = \left(0 + \frac{1}{2}\right) - \frac{1}{4} = \frac{1}{4}$

$\sigma_Y^2 = E(Y^2) - \mu_Y^2 = 3/4$

$\therefore \sigma_X = \frac{1}{2}$ & $\sigma_Y = \frac{\sqrt{3}}{2}$

d) $\mathrm{cov}(X,Y) = E(XY) - \mu_X \mu_Y$

$= 1/2 - 3/4 = -\frac{1}{4}$

$\rho(X,Y) = \frac{\mathrm{cov}(X,Y)}{\sigma_X \sigma_Y}$

$= \frac{-1/4}{\sqrt{3}/4} = -\frac{1}{\sqrt{3}}$

2] From the above example, considering the joint distribution of X & Y compute

a) $P(X \le 1, Y \le 2)$ b) $P(X+Y \ge 2)$

(b) ③

(Sol)
a) $P(X \leq 1, Y \leq 2)$

(0, 0) (0, 1) (0, 2) (1, 0) (1, 1) (1, 2)

$P(X \leq 1, Y \leq 2) = 0 + \frac{1}{8} + \frac{1}{4} + \frac{1}{8} + \frac{1}{4} + \frac{1}{8} = \underline{\frac{7}{8}}$

b) $P(X + Y \geq 2)$

(0, 2) (0, 3) (1, 1) (1, 2) (1, 3)

$P(X + Y \geq 2) = \frac{1}{4} + \frac{1}{8} + \frac{1}{4} + \frac{1}{8} + 0 = \underline{\frac{3}{4}}$

3) Given the joint distribution.

X\Y	0	1
0	0.1	0.2
1	0.4	0.2
2	0.1	0

Determine the

a) Marginal distribution of X & Y

b) Find the conditional distribution $h(x/y=1)$

(Sol)
a)

x_i	0	1	2
$f(x_i)$	0.3	0.6	0.1

y_j	0	1
$g(y_j)$	0.6	0.4

b) $h(x/y=1)$

$h(0/1) = \frac{0.2}{0.4} = \underline{0.5}$

$h(1/1) = \frac{0.2}{0.4} = \underline{0.5}$

$h(2/1) = \underline{0}$

4) A salesman's territory consists of 3 cities A, B & C. He never sells in the same city on successive days. If he sells in city A, then the next day he sells in city B. However, if he sells in either B or C, then the next day he is twice as likely to sell in city A as in other city. In the long run, how often does he sell in each of the cities.

Sol

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 2/3 & 0 & 1/3 \\ 2/3 & 1/3 & 0 \end{bmatrix} \end{array}$$

& $VP = V$

$$(x\ y\ z)\begin{bmatrix} 0 & 1 & 0 \\ 2/3 & 0 & 1/3 \\ 2/3 & 1/3 & 0 \end{bmatrix} = (x\ y\ z) \quad \& \ x+y+z=1$$

$$\frac{2}{3}y + \frac{2}{3}z = x$$

$$x + \frac{1}{3}z = y$$

$$\frac{1}{3}y = z$$

Solving these, $x = \frac{2}{5}$ $y = \frac{9}{20}$ $z = \frac{3}{20}$

$\therefore$ In long run he sells 40% of time in city A, 45% in B & 15% of time in city C.

5) A man's smoking habits are as follows. If he smokes filter cigarettes one week, he switches to non filter cigarettes the next week with probability 0.2. On other hand, if he smokes non filter cigarettes one week, there is a probability of 0.7 that he will smoke non filter cigarettes the next week

as well. In long run how of Co does he smoke filter cigarettes.

6f

$$P = \begin{array}{c|cc} & F & NF \\ \hline F & 0.8 & 0.2 \\ NF & 0.3 & 0.7 \end{array}$$

$VP = V$

$$(x \;\; y)\begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix} = (x \;\; y) \quad \& \quad x + y = 1$$

Solving $x = \frac{3}{5}$, $y = \frac{2}{5}$

$\therefore$ Man smokes filter cigarettes 60% time in long run.