Ex. Find the frequency of a vibrating beam which is given by $\cos x \cdot \cosh x = 1$ near the point $x_0 = 3\pi/2$.

→ Numerical methods for ordinary differential equations (ODE):

Consider the eq^n., $\frac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$ is called 1^{st} order DE with the condition defined at only one point at $x = x_0$ which is called initial value problem.

Following methods are used to solve the initial value problem.

1) Taylor method.
2) Euler's method.
3) Modified Euler's method.
4) Runge-Kutta method.

1) Taylor's method:

It is also called single step method which is used to solve the initial value problem of the form $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$

The taylor series formula is given by,

$$y(x) = y(x_0) + (x - x_0) y'(x_0) + \frac{(x - x_0)^2}{2!} y''(x_0)$$
$$+ \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots$$

where,

$y'(x_0)$, $y''(x_0)$, $y'''(x_0)$ ... represents the derivatives of y at $x = x_0$.

Procedure :

1) Rewrite the given DE. in the form

$\frac{dy}{dx} = f(x_0)$

2) Write down the formula.

3) Find $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots$ at $x = x_0$.

4) Substitute the value in the formula.

Ex. Find appr. value of y when $x = 0.1$, given that $\frac{dy}{dx} = x - y^2$; $y(0) = 1$ $\rightarrow y(x_0) = y_0$

using Taylor's method.

→ Given $x_0 = 0$ $y_0 = 1$

$$y(x) = y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2!} y''(x_0) + \ldots \longrightarrow (1)$$

Consider,

$\frac{dy}{dx} = x - y^2 = y'(x_0)$ at $x = x_0 = 0$

$$y'(x_0) = y'(0) = 0 - 1^2 = -1$$
$$y'(x_0) = -1$$

Diff. w.r.t. x,

$\frac{d^2y}{dx^2} = 1 - 2yy'$ at $x = x_0 = 0$

$$y''(x_0) = y''(0) = 1 - 2(1)(-1)$$
$$y''(x_0) = 3$$

III^{ly} $\frac{d^3y}{dx^3} = -2[yy'' + y'y']$

$= -2yy'' - 2y'^2$ at $x = x_0 = 0$

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$y'''(x_0) = -2(-1)^2 - 2(1)(3)$

$y'''(x_0) = -8.$

$y^{iv}(x) = -2[yy''' + y''y'] - 2[2y'y'']$ at $x = x_0$.

$y^{iv}(x_0) = -2(1\times -8 + 3\times -1) - 4(-1\times 3)$

$= 34$

Substituting all values in eqn (1),

$$y(x) = 1 + (x-0)(-1) + \frac{(x-0)^2}{2!}(3) + \frac{(x-0)^3}{3!}(-8)$$

$$+ \frac{(x-0)^4}{4!}(34)$$

$$y(x) = 1 - x + \frac{3x^2}{2!} - \frac{8x^3}{3!} + \frac{34x^4}{4!}$$

$y(0.1) = 0.91380$

Ex. Find the value of y at $x = 1.2$ using Taylor's method & compare with exact soln.

$y' = x+y$; $y(1) = 0$.

→ $x_0 = 1$ $y_0 = 0$

$$y(x) = y(x_0) + (x-x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \dots \rightarrow (1)$$

Consider,

$y' = x+y = y'(x)$ at $x = x_0 = 1$.

$y'(x_0) = 1 + 0$

$y'(x_0) = 1.$

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$y'' = 1 + y'$ at $x = x_0 = 1$

$y''(x_0) = 1 + 1 = 2.$

$y''' = y''$ at $x = x_0 = 1$

$y'''(x_0) = 2.$

$y^{iv}(x_0) = 2.$

~~y(x) =~~ Substituting values,

$$y(x) = 0 + (x-1)1 + \frac{(x-1)^2}{2!} \times 2 + \frac{(x-1)^3}{3!} \times 2 + \frac{(x-1)^4}{4!}$$

$$y(x) = (x-1) + (x-1)^2 + \frac{(x-1)^3}{3} + \frac{(x-1)^4}{12}$$

$y(1.2)_{app^x} = 0.2428$

Consider the given eqn.

$$\frac{dy}{dx} = x + y \qquad \frac{dy}{dx} - y = x.$$

$$\text{IF (integral factor)} = e^{\int P dx} = e^{\int -1 dx} = e^{-x}$$

$$e^{-x}\frac{dy}{dx} - ye^{-x} = xe^{-x}$$

$$\frac{d}{dx}\left[ye^{-x}\right] = xe^{-x}$$

Integrating the above eqn.

$$\int \frac{d}{dx}\left[ye^{-x}\right] dx = \int xe^{-x} dx + c$$

$$ye^{-x} = \left[x\left(\frac{e^{-x}}{-1}\right) - 1\left(\frac{e^{-x}}{(-1)^2}\right)\right] + c$$

$$y(x)e^{-x} = -xe^{-x} - e^{-x} + c$$

$$y(x) = -x - 1 + ce^{x}$$

Given that $y(1) = 0$

$$y(1) = -1 - 1 + ce^{1}$$

$$\therefore \quad c = \frac{2}{e}$$

Exact solution of given DE,

$$y(x) = -x - 1 + \frac{2}{e}e^{x}$$

$$y(1.2)_{exact} = 0.242805$$

The error is given by,

$$error = y_{exact} - y_{appr.}$$
$$= 0.000005$$

Ex. Evaluate $y(x)$ at $x = 0.1$ using $y' = xy + 1$ where, $y = 1$ at $x = 0$.

→ $x_0 = 0 \quad y_0 = 1$

$x = x_0 = 0$.

$$y' = xy + 1$$
$$y'(x_0) = 0 \times 1 + 1 = 1$$

$$y'' = xy' + y \quad \cancel{+1}$$
$$y''(x_0) = 0 \times 1 + 1 = 1$$

$$y''' = xy'' + y' + y'$$
$$y'''(x_0) = 0 + 1 + 1 = 2$$

$$y^{iv} = xy''' + y'' + 2y'' \quad \therefore y^{iv}(x_0) = 3$$

Substituting the values,

$$y(x) = 1 + (x-0)\,1 + \frac{(x-0)^2}{2!} \times 1 + \frac{(x-0)^3}{3!} \times 2 + \frac{(x-0)}{4!}$$

$$y(x) = 1 + x + \frac{x^2}{2!} + \frac{2x^3}{3!} + \frac{3x^4}{4!}$$

$$y(0.1) = 1.105345$$

HW:-

1) $y' = 3x + y^2$, $y(0) = 1$

2) $y' = e^x - y^2$, $y(0) = 1$ find $y(0.1)$

2) Euler's method :-

This method is used to solve the initial value problem $y' = f(x,y)$; $y(x_0) = y_0$

Let $y(x)$ be a function defined in the closed interval $[x_0, x_n]$. Divide the interval $[x_0, x_n]$ such that $x_0 < x_1 < x_2 < x_3 \cdots < x_n$

taking $x_1 - x_0 = h \Rightarrow x_1 = x_0 + h$

$x_2 - x_1 = h \Rightarrow x_2 = x_0 + 2h$

$\vdots \quad \vdots$

$x_n - x_{n-1} = h \Rightarrow x_n = x_0 + nh$

Consider $x \in [x_0, x_1]$

Integrating the given DE from x_0 to x_1 we have,

$$y(x_1) = y(x_0) + hf(x_0, y_0)$$

$$y_1 = y_0 + hf(x_0, y_0)$$

$|||^{ly}$

$$y_2 = y_1 + hf(x_1, y_1)$$

In general the formula is given by,

$$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1}).$$

1) Find $y(0.2)$ taking $h = 0.1$. $y' + 0.2y = 0$
$y(0) = 5$

→ $\frac{dy}{dx} = f(x, y)$ $\quad y(x_0) = y_0$

$\frac{dy}{dx} = -0.2y$ $\quad y(0) = 5$

where $f(x, y) = -0.2y$ $\quad x_0 = 0 \quad y_0 = 5.$

3) Modified Euler's method :

The modified Euler's formula is given by,

$$y_n^{(i+1)} = y_{n-1} + \frac{h}{2}\left[f(x_{n-1}\ y_{n-1}) + f(x_n\ y_n^{(i)})\right]$$

$i = 0, 1, 2 \ldots$

Procedure :-

1) Rewrite the given DE in the form,

$$\frac{dy}{dx} = f(x, y) \qquad y(x_0) = y_0.$$

2) Calculate x_n or h or n using the formula $x_n = x_0 + nh$.

3) At x_1 the modified Euler's formula is given by,

$$y_n^{(i+1)} = y_{n-1} + \frac{h}{2}\left[f(x_{n-1}\ y_{n-1}) + f(x_n\ y_n^{(i)})\right] \quad i = 0, 1, 2 \ldots$$

4) Put $i = 0$ in the above eq^n.

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1\ y_1^{(0)})\right]$$

5) Calculate initial approximation $y_1^{(0)}$ by using the formula,

$$y_1^{(0)} = y_0 + h f(x_0, y_0).$$

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6) Substituting $y_1^{(0)}$ in the above eq^n. and calculate $y_1^{(1)}$.

7) Calculate the sequence of values $y_1^{(2)}, y_1^{(3)} \ldots$ until the sequence converges to the approximate solution.

Ex. Solve : $y' = x^2 + y$, $y(0) = 1$ using MEM find $y(0.1)$. taking $h = 0.05$.

$\longrightarrow$ Given $f(x,y) = x^2 + y$

$x_0 = 0 \quad y_0 = 1 \quad h = 0.05 \quad x_n = 0.1$

$x_n = x_0 + nh$

$0.1 = 0 + n(0.05) \qquad \therefore n = 2.$

At $x = x_1 = 0.05$

$$y_1^{(i+1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(i)})\right] \qquad i = 0, 1, \ldots$$

Put $i = 0$,

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$y_1^{(1)} = 1 + \frac{h}{2}\left[f(0,1) + f(0.05, y_1^{(0)})\right]$$

$$y_1^{(0)} = y_0 + h f(x_0, y_0)$$

$$y_1^{(0)} = 1 + 0.05 f(0,1)$$

$$\therefore y_1^{(0)} = 1.05$$

$$y_1^{(1)} = 1 + \frac{0.05}{2}\left[f(0,1) + f(0.05, 1.05)\right]$$

$$= 1 + \frac{0.05}{2}\left[0^2 + 1 + (0.05)^2 + 1.05\right]$$

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$\therefore y_1^{(1)} = 1.05131$

$|||^{ly}$,
$$y_1^{(2)} = 1 + \frac{0.05}{2}\left[0^2 + 1 + (0.05)^2 + 1.05131\right]$$

$\therefore y_1^{(2)} = 1.051345$

At $x = 0.05$
$$y(x_1) = y(0.05) = 1.05134$$

At $x = x_2 = 0.1$
$$y_2^{(i+1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(i)})\right] \quad i = 0, 1, 2 \ldots$$

Put $i = 0$,
$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(0)})\right]$$

$$y_2^{(1)} = \overset{1.05134}{\cancel{1}} + \frac{0.05}{2}\left[f(0.05, 1.05134) + f(0.1, y_2^{(0)})\right]$$

$\cancel{y_2^{(1)} = 1.105834}$

$$y_2^{(1)} = 1.05134 + \frac{0.05}{2}\left[(0.05)^2 + 1.05134 + (0.1)^2 + 1.1040\right]$$

$y_2^{(1)} = 1.105536$.

$y_2^{(2)} = 1.10557$.

At $x = 0.1$,
$$y(x_2) = y(0.1) = 1.10557$$

Ex. Solve: $y' = 1 - y$; $y(0) = 0$ using MEM. find $y(0.3)$ ~~compute~~ carry out 3 steps.

OR.

taking $h = 0.1$ carry out 3 steps.

→ Given,

$f(x,y) = 1 - y$. $x_0 = 0$ $y_0 = 0$

$x_n = 0.3$ $n = 3$.

wkt,

$x_n = x_0 + nh$

$0.3 = 0 + 3h$ $\therefore h = 0.1$

At $x = x_1 = 0.1$,

$$y_1^{(i+1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(i)})\right] \quad i = 0, 1, \ldots$$

$$y_1^{(i+1)} = y_0 + \frac{h}{2}\left[1 - y_0 + 1 - y_1^{(i)}\right] \longrightarrow (1)$$

Put $i = 0$,

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[1 - y_0 + 1 - y_1^{(0)}\right]$$

$$y_1^{(1)} = 0 + \frac{0.1}{2}\left[1 - 0 + 1 - y_1^{(0)}\right]$$

$$y_1^{(0)} = y_0 + hf(x_0, y_0)$$

$$= 0 + 0.1(1 - 0).$$

~~$y_1^{(0)}$~~ $y_1^{(0)} = 0.1$

$y_1^{(1)} = 0.095$.

$y_1^{(2)} = 0.09525$.

At $x_1 = (0.1)$

$y(x_1) = y(0.1) = 0.09525$

At $x = x_2 = 0.2$.

$$y_2^{(i+1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(i)})\right]$$

$$y_2^{(i+1)} = y_1 + \frac{h}{2}\left[1 - y_1 + 1 - y_2^{(i)}\right]$$

At $i = 0$,

$$y_2^{(1)} = 0.09525 + \frac{0.1}{2}\left[1 - 0.09525 + 1 - y_2^{(0)}\right]$$

$$y_2^{(0)} = 0.09525 + 0.1(1 - 0.09525)$$

$$y_2^{(0)} = 0.18572.$$

$$y_2^{(1)} = 0.09525 + \frac{0.1}{2}\left[1 - 0.09525 + 1 - 0.18572\right]$$

$$y_2^{(1)} = 0.18120.$$

$$y_2^{(2)} = 0.18142.$$

$$y(x_2) = y(0.2) = 0.18142$$

At $x = x_3 = 0.3$.

$$y_3^{(i+1)} = y_2 + \frac{h}{2}\left[1 - y_3 + 1 - y_3^{(i)}\right]$$

At $i = 0$,

$$y_3^{(1)} = 0.18142 + \frac{0.1}{2}\left[1 - 0.18142 + 1 - y_3^{(0)}\right]$$

$$y_3^{(0)} = 0.18142 + 0.1(1 - 0.18142)$$

$$y_3^{(0)} = 0.263278$$

$$y_3^{(1)} = 0.18142 + \frac{0.1}{2}\left[1 - 0.18142 + 1 - 0.263278\right]$$

$y_3^{(1)} = 0.2591851$

$y_3^{(2)} = 0.25938$

$y(x_3) = y(0.3) = 0.25938.$

Homework :-

1) Solve $y' = 1-y^2$ $y(0)=0.2$ $h=0.1$
Carry out 3 steps.

2) $y' = y$; $y(0) = 1$ $h=0.1$

3) $y' = 2(1+y^2)$; $y(0) = 0$ $h = 0.05$

4) $xy^2 = y'$; $y(0) = 1$ $h = 0.1$.

4) Runge - Kutta method :-

This method is used to solve initial value problem, $\frac{dy}{dx} = f(x,y)$; $y(x_0) = y_0$.

The 4th order Runge - Kutta method is given by,

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$y(x_1) = y_1 = y_0 + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

$$y(x_2) = y_2 = y_1 + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right].$$

In general,

$$k_1 = h f(x_{n-1}, y_{n-1})$$

$$k_2 = h f\left(x_{n-1} + \frac{h}{2},\ y_{n-1} + \frac{k_1}{2}\right)$$

$$k_3 = h f\left(x_{n-1} + \frac{h}{2},\ y_{n-1} + \frac{k_2}{2}\right)$$

$$k_4 = h f(x_{n-1} + h,\ y_{n-1} + k_3).$$

$$y_n = y_{n-1} + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

Ex. Using 4^{th} order RK method solve the DE,
$\frac{dy}{dx} = \frac{4x}{y - xy}$, $y(0) = 3$. & find y at $x = 0.1$ & $x = 0.2$.

→

$x_0 = 0 \longleftrightarrow x_1 = 0.1 \longleftrightarrow x_2 = 0.2.$

Given that, the DE is of the form,

$\frac{dy}{dx} = \frac{4x}{y - xy}$ $\quad f(x, y)$ $\quad y(x_0) = y_0.$

$\frac{dy}{dx} = \frac{4x}{y - xy}$ $\quad x_0 = 0 \quad y_0 = 3. \quad h = 0.1$

At $x = x_1 = 0.1$, 4^{th} order RKM is given by,

$k_1 = h f(x_0, y_0).$

$= 0.1 f(0, 3) \quad = 0.1\left(\frac{4 \times 0}{3 - 0 \times 3}\right) \quad = 0.$

$\therefore k_1 = 0.$

$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$

$= 0.1 f\left(0 + \frac{0.1}{2}, 3 + \frac{0}{2}\right).$

$\therefore k_2 = 0.00701$

~~$\therefore k_2 = 0.0001$~~

$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$

$= 0.1 f\left(0 + \frac{0.1}{2}, 3 + \frac{0.00701}{2}\right).$

$\therefore k_3 = 0.007009$

$k_4 = h f(x_0 + h, y_0 + k_3).$

$= 0.1 f(0 + 0.1, 3 + 0.007009)$

$\therefore k_4 = 0.01478$

$y_1 = y_0 + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$

$= 3 + \frac{1}{6}\left(0 + 2 \times 0.00701 + 2 \times 0.007009 + 0.01478\right)$

$\therefore y_1 = 3.0071$

At $x = x_2 = 0.2.$

$k_1 = h f(x_1, y_1)$

$= 0.1 f(0.1, 3.007)$

$\therefore k_1 = 0.0147$

$k_2 = h f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$

$$= 0.1\, f\left(0.1 + \frac{0.1}{2},\ 3.007 + \frac{0.0147}{2}\right)$$

$\therefore\ k_2 = 0.0234.$

$$k_3 = \cancel{0.1}\, h f\left(x_1 + \frac{h}{2},\ y_1 + \frac{k_2}{2}\right)$$

$$= 0.1\, f\left(0.1 + \frac{0.1}{2},\ 3.007 + \frac{0.0234}{2}\right)$$

$\therefore\ k_3 = 0.0233$

$$k_4 = h f(x_1 + h,\ y_1 + k_3)$$
$$= 0.1\, f(0.1 + 0.1,\ 3.007 + 0.0233)$$

$\therefore\ k_4 = 0.0330.$

$$y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 3.007 + \frac{1}{6}\left(0.0147 + 2 \times 0.0234 + 2 \times 0.0233 + 0.0330\right)$$

$\therefore\ y_2 = 3.0305.$

Ex. Solve the DE, $10y' = x^2 + y^2$; $y(0) = 1$. find y at $x = 0.4$ taking $h = 0.2$.

$\rightarrow$

$x_0 = 0$ $\quad$ $x_1 = 0.2$ $\quad$ $x_2 = 0.4$

Given that,

$$f(x, y) = \frac{x^2 + y^2}{10} \qquad x_0 = 0 \quad y_0 = 1 \quad h = 0.2.$$

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At $x = x_1 = 0.2$

$k_1 = hf(x_0, y_0)$.

$= 0.2 f(0, 1)$

$\therefore k_1 = 0.02$.

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$.

$= 0.2 f\left(0 + \frac{0.2}{2}, 1 + \frac{0.02}{2}\right)$

$\therefore k_2 = 0.02060$

$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$

$= 0.2 f\left(0 + \frac{0.2}{2}, 1 + \frac{0.0206}{2}\right)$

$\therefore k_3 = 0.02061$

$k_4 = hf(x_0 + h, y_0 + k_3)$

$= 0.2 f(0 + 0.2, 1 + 0.02061)$

$\therefore k_3 = 0.02163$

$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= 1 + \frac{1}{6}(0.02 + 2 \times 0.02060 + 2 \times 0.02061 + 0.02163)$

$\therefore y_1 = 1.0206$.

At $x = x_2 = 0.4$

$k_1 = hf(x_0, y_1)$

$= 0.2 f(0.2, 1.0206)$

$\therefore k_1 = 0.02163$

$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$

$= 0.2\, f\left(0.1 + \frac{0.2}{2}, 1.0206 + \frac{0.02163}{2}\right)$

$\therefore k_2 = 0.02207$

$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$

$= 0.2\, f\left(0.1 + \frac{0.2}{2}, 1.0206 + \frac{0.02207}{2}\right)$

$\therefore k_3 = 0.02208$

$k_4 = hf(x_1 + h, y_1 + k_3)$

$= 0.2\,(0.1 + 0.2, 1.0206 + 0.02208)$

$\therefore k_4 = 0.02354$

$y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= 1.0206 + \frac{1}{6}(0.02163 + 2 \times 0.02207 + 2 \times 0.02208 + 0.02354)$

$\therefore y_2 = 1.0428$

Homework :-

1) Find $y(0.1)$ given $y' = 3e^x + 2y$; $y(0) = 0$ $h = 0.05$

2) $y' - xy^2 = 0$; $y(0) = 1$ $h = 0.1$

3) $y' = y - y^2$; $y(0) = 0.2$ $h = 0.1$

4) $y' = (1 - x^{-1})y$; $y(1) = 1$ $h = 0.1$

5) $y' = -y\tan x + \sin 2x$; $y(0) = 1$ $h = 0.1$