UNIT-I

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Numerical solution of a algebric & trans-cendental equation.

* Property :- let $f(x)=0$ be the equation possesses real root in interval (a,b) only when
$$f(a).f(b) < 0.$$
i.e $f(a), f(b)$ will be of opposite sign.

* Method of false position (Regular falsi method)
let $f(x)=0$ be the equation possesing real root in (a,b) then,
$$x = \frac{a.f(b) - b.f(a)}{f(b) - f(a)}.$$

Problems.

01. Find real root of equation $x^3-2x-5=0$ by the method of false position to three decimal places

→ $x^3-2x-5=0$ but $f(x) \neq 0$.

$f(x) = x^3-2x-5$

$f(0) = -5$, $f(2) = -1$

$f(1) = -6$, $f(3) = +16$. } (2,3). [∵ when there comes (+) & (−) sign's after substitution then roots lies in them]

∴ roots lie in $(\overset{a}{2}, \overset{b}{3})$.

∴

x	$f(x)$
2	−1
2·1	+0·061
2·2	1·248
2·3	2·567
⋮	⋮
2·9	13·589
3	16.

} (2, 2·1).

NOTE:- Calc.
mode (7) → f(x) →
Start (a) → End (b) →
step (0·1).

∴ Roots lie in (2, 2·1).

where, $a = 2$, $b = 2.1$.

$\therefore f(a) = f(2) = (2)^3 - 2(2) - 5 = -1$. $\quad \therefore f(b) = f(2.1) = (2.1)^3 - 2(2.1) - 5 = +0.061$.

$$\therefore x_0 = \frac{a.f(b) - b.f(a)}{f(b) - f(a)} = \frac{(2)(0.061) - (2.1)(-1)}{(0.061 - (-1))}.$$

$\therefore x_0 = 2.0942$

$\therefore f(x_0) = (2.0942)^3 - 2(2.0942) - 5$

∴ ~~Roots lie in (2.09~~ $\therefore f(x_0) = -0.00392$.

$$\begin{array}{ccc} - & - & + \\ f(a) & f(x_0) & f(b) \end{array}$$

$\therefore$ Roots lie in $(\underset{a}{2.0942}, \underset{b}{2.1})$.

$$\therefore x_1 = \frac{a.f(b) - b.f(a)}{f(b) - f(a)} = \frac{(2.0942)(0.061) - 2.1(0.00392)}{(0.061 + 0.00392)}.$$

$\therefore x_1 =$ ~~1.8409~~ 2.0945

consider upto 3 decimal.

$\therefore$ Root of equation is, $x_1 = 2.0945$.

Q2. Solve $x^4 = 2$ with $a = 1$, $b = 2$ upto 4 decimals.

$\rightarrow f(x) = x^4 - 2$.

$\left.\begin{array}{l} f(a) = f(1) = -1. \\ f(b) = f(2) = 14 \end{array}\right\} (1, 2)$.

$\therefore$ The roots lie in $(1, 2)$.

x	$f(x)$
1	-1
1.1	-0.535
1.2	0.0736
⋮	

$\} (1.1, 1.2)$.

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$\therefore$ Roots lie in $(\underset{a}{1.1}, \underset{b}{1.2})$.

$\therefore f(a) = f(1.1) = (1.1)^4 - 2.$

$\therefore f(a) = -0.535$

$\therefore f(b) = f(1.2) = (1.2)^4 - 2$

$\therefore f(b) = +0.0736.$

$$\therefore x_0 = \frac{a \cdot f(b) - b \cdot f(a)}{f(b) - f(a)} = \frac{(1.1)(0.0736) + (1.2)(0.535)}{0.0736 + 0.535}.$$

$\therefore x_0 = 1.1879.$

$\therefore f(x_0) = \cancel{+0.0148}\ -0.0087$

$\therefore$

$$\begin{array}{ccc} - & - & + \\ f(a) & f(x_0) & f(b) \end{array}$$

$\therefore$ Root lie in $(\underset{a}{1.187}, \underset{b}{1.2})$.

$f(a) = -0.0087 \quad f(b) = 0.0736.$

$$\therefore x_1 = \frac{a \cdot f(b) - b \cdot f(a)}{f(b) - f(a)} = \frac{(1.187)(+0.0736) - (1.2)(-0.0087)}{(0.0736 + 0.0087)}.$$

$\therefore x_1 = 1.188$

$\therefore$ Root of equation is $x_1 = 1.188$ up to three decimals.

Q3. Find real root of equation $x \cdot \log_{10} x = 1.2$ by regular false method correct to four decimal places.

$\rightarrow f(x) = x \cdot \log_{10} x - 1.2.$

$f(0) = \infty$ (not defined).

$f(1) = -1.2$

$f(2) = -0.5979.$

$f(3) = 0.2313.$

$\} (2,3).$

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$\therefore$ The roots lie in $(2, 3)$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad a \; b$

x	$f(x)$
2	-0.597
2.1	-0.523
2.2	-0.446
⋮	⋮
2.7	-0.035
2.8	0.052

$\}\ (2.7, 2.8)$

$\therefore$ The roots lie in $(2.7, 2.8)$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad a \; b$

$\therefore f(a) = f(2.7) = -0.035, \quad f(b) = f(2.8) = 0.052$

$$x_0 = \frac{a.f(b) - b.f(a)}{f(b) - f(a)} = \frac{(2.7)(0.052) + (2.8)(0.035)}{0.052 + 0.035}$$

$\therefore x_0 = 2.7402$

$\therefore f(x_0) = -5.634 \times 10^{-4}$

$\therefore f(x_0) = -0.000563$

$-$ $\quad$ $-$ $\quad$ $+$

$f(a) \quad f(x_0) \quad f(b)$

The roots lie in $(2.740, 2.8)$.
$\quad\quad\quad\quad\quad\quad\quad\quad\quad a \quad b$

$$\therefore x_1 = \frac{a.f(b) - b.f(a)}{f(b) - f(a)} = \frac{(2.740)(0.052) + (2.8)(0.000563)}{(0.052) + (0.000563)}$$

$\therefore x_1 = 2.7406$

$f(x_1) = -4.0202 \times 10^{-5}$

$f(x_1) = -0.0000402$.

$-$ $\quad$ $-$ $\quad$ $+$

$f(a) \quad f(x_1) \quad f(b)$

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$\therefore$ The roots lie in $(\underset{a}{2.7406}, \underset{b}{2.8})$.

$$\therefore x_2 = \frac{a.f(b) - b.f(a)}{f(b)-f(a)} = \frac{(2.7406)(0.052)+(2.8)(0.0000402)}{(0.052+0.0000402)}.$$

$$\therefore x_2 = 2.7406.$$

$\therefore$ The root lie in $(2.7406, 2.8)$.

Q4. Find fourth root of 12, correct three decimal places by method of false position.

$\rightarrow$ $x = \sqrt[4]{12}$.

$x^4 = 12$

$x^4 - 12 = 0$

$\therefore f(x) = x^4 - 12$.

$\therefore f(0) = -12$

$\therefore f(1) = -11$

$f(2) = 4$ } $(1,2)$

$\therefore$ The root lie in $(\underset{a}{1}, \underset{b}{2})$.

x	$f(x)$
1	-11
1.1	-10.53
1.2	-9.926
⋮	
1.8	-1.502
1.9	1.0321

} $(1.8, 1.9)$.

$\therefore$ The root lie in $(\underset{a}{1.8}, \underset{b}{1.9})$.

$\therefore f(a) = f(1.8) = -1.502$, $f(b) = f(1.9) = 1.0321$.

$$x_0 = \frac{a.f(b) - b.f(a)}{f(b)-f(a)} = \frac{(1.8)(1.0321)+(1.9)(1.502)}{(1.0321+1.502)}.$$

H.W. 5) $\cos x = \sqrt{x}$, 6) $x + \ln x = 2$ in the interval $(1, 2)$.

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$\therefore x_0 = 1.859.$

$\therefore f(x_0) = -0.0499.$

$$\begin{array}{ccc} - & - & + \\ f(a) & f(x_0) & f(b) \end{array}$$

$\therefore$ The root lies in $(\underset{a}{1.859}, \underset{b}{1.9})$

$$x_1 = \frac{a.f(b) - b.f(a)}{f(b) - f(a)} = \frac{(1.859)(1.0321) + (1.9)(0.0499)}{(1.0321 + 0.0499)}.$$

$\therefore x_1 = 1.860.$

$\therefore f(x_1) = -0.0311;$

$\therefore x_2 / x_1$ $\therefore$ The root lie in $(\underset{a}{1.860}, \underset{b}{1.9})$.

$$\therefore x_2 = \frac{a.f(b) - b.f(a)}{f(b) - f(a)} = \frac{(1.860)(1.0321) + (1.9)(0.0311)}{(1.0321 + 0.0311)}.$$

$\therefore x_2 = 1.861.$

$\therefore$ Root of the equation, $x = 1.861$.

05. Find the third root of $x + \ln x = 2$ correct three decimals places by method of false position.

$\rightarrow$ $f(x) = x + \ln x - 2.$

$f(0) = \infty.$

$$\left.\begin{array}{l} f(1) = -1. \\ f(2) = 0.6931 \end{array}\right\} (1, 2).$$

$\therefore$ The root lie in $(1, 2)$.

x	$f(x)$
1.5	-0.094
1.6	0.07

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$\therefore$ The root lie in $(\underset{a}{1.5}, \underset{b}{1.6})$.

$a = 1.5, \quad b = 1.6.$

$$x_0 = \frac{a.f(b) - b.f(a)}{f(b) - f(a)} = \frac{(1.5)(0.07) + (1.6)(0.094)}{(0.07) + (0.094)}.$$

$$x_0 = 1.5573.$$

$$f(x_0) = (1.557) + \ln(1.557) - 2$$
$$= -0.000253.$$

$\therefore$ The root lie in $(\underset{a}{1.5573}, \underset{b}{1.6})$.

$-$ $\quad$ $-$ $\quad$ $+$

$f(a)$ $\quad$ $f(x_0)$ $\quad$ $f(b)$

$$x_1 = \frac{a.f(b) - b.f(a)}{f(b) - f(a)}.$$

$$\therefore x_1 = \frac{(1.5573)(0.07) + (1.6)(0.000253)}{(0.07 + 0.000253)}. = 1.5574$$

$\therefore$ The root lie in, $x = 1.557$

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Imp

Newton-Repulsion method :-

let $f(x) = 0$ be the equation, let x_0 be approximate root of equation, x_1 be exact root equation.

$$x_1 = x_0 + h.$$

i.e. $f(x_1) = 0.$

$$f(x_0+h) = 0.$$

Now by Taylor's expansion,

$$f(x_0+h) = f(x_0) + h.f'(x_0) + \frac{h^2}{2!}.f''(x_0) + \frac{h^3}{3!}f'''(x_0) + \dots \quad -(1).$$

In this case, 'h' is very small hence neglecting h^2, h^3, ...

So, (1) becomes,

$$0 = f(x_0) + h.f'(x_0) + 0\dots.$$

$$h = -\frac{f(x_0)}{f'(x_0)}.$$

$$\therefore x_1 = x_0 + h.$$

$$\boxed{\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}}$$

$$\therefore x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

$$\therefore x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}.$$

$$\vdots$$

So on, $\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$ for $n = 0, 1, 2, 3 \dots$

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Problems

Q1) Find root of equation $x^3-3x+1=0$ in $(1.5, 1.6)$. Correct to 3 decimal places by N R method.

→ $f(x) = x^3-3x+1$

$f'(x) = 3x^2-3.$

$f(1.5) = -0.125.$

$f(1.6) = 0.296.$

$\therefore |f(1.5)| = 0.125$ and $|f(1.6)| = 0.296.$

$\therefore$ Since, $|f(1.5)|$ is near to zero by NR method.

Take, $x_0 = 1.5.$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \qquad [\because f'(x_0) = 3.75]$$

$$\therefore x_1 = 1.5 - \frac{(-0.125)}{3.75} = 1.5 + \frac{(0.125)}{3.75}$$

$\therefore x_1 = 1.5333$

$$\therefore x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

$$\therefore x_2 = 1.533 - \frac{(+3.698 \times 10^{-3})}{4.0502} \qquad [\because f'(x_1) = 4.0502]$$

$\therefore x_2 = 1.5320$

Q2) Find root of equation, $3x = \cos x + 1$, correct to 3 decimal places, by NR method.

→ $3x = \cos x + 1$

$\cos x - 3x + 1 = 0.$

$\therefore f(x) = \cos x - 3x + 1$ Cal.c :- mode(7) → f(x) →

$\therefore f'(x) = -\sin x - 3$ Start (0) → End (1) →

$\therefore f(0) = 2. > 0$ Step (0.1).

$\therefore f(1) = -1.45 < 0.$

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Root lie in (0,1).

x	$f(x)$
0.6	0.025
0.7	-0.335

$\}$ (0.6, 0.7).

$\therefore$ Root lie in (0.6, 0.7).

$|f(0.6)| = 0.025$, $|f(0.7)| = 0.335$

$\therefore |f(0.6)|$ is nearer to zero.

Take $x_0 = 0.6$.

By NR method,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

where, $f(x_0) = \cos(0.6) - 3(0.6) + 1 = 0.025$.

$$x_1 = 0.6 - \frac{0.025}{(-3.564)}$$

$f'(x_0) = f'(0.6) = -3.564$.

$\therefore x_1 = 0.6070$

$$\therefore x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

where, $f(x_1) = \cos(0.6070) - 3(0.6070) + 1 = 0.000362$

$$= 0.6070 - \left(\frac{0.000362}{-3.5704}\right).$$

$f'(x_1) = -\sin(0.6070) - 3 = -3.5704$.

$\therefore x_2 = 0.6071$

$\therefore$ Root of equation is, $x = 0.6071$.

Q3) Design Newton iteration formula compute $\sqrt[3]{7}$; $x_0 = 2$.

$\rightarrow$ $x = \sqrt[3]{7}$

$x^3 = 7$

$x^3 - 7 = 0$.

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$\therefore f(x) = x^3 - 7$.
$\therefore f'(x) = 3x^2$

By NR method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

$$x_{n+1} = x_n - \left[\frac{x_n^3 - 7}{3x_n^2}\right].$$

$$x_{n+1} = \frac{3x_n^3 - x_n^3 + 7}{3x_n^2}.$$

$$\therefore \boxed{x_{n+1} = \frac{2x_n^3 + 7}{3x_n^2}}$$

for $x_0 = 2$

$$x_1 = \frac{2x_0^3 + 7}{3(x_0)^2}$$

$$x_1 = \frac{2(2)^3 + 7}{3(2)^2}$$

$\therefore x_1 = 1.9166$ //

$$x_2 = \frac{2(1.9166)^3 + 7}{3(1.9166)^2}$$

$\therefore x_2 = 1.9129$.

$$x_3 = \frac{2x_2^3 + 7}{3x_2^2}$$

$$x_3 = \frac{2(1.9129)^3 + 7}{3(1.9129)^2}$$

$x_3 = 1.9129$

$\therefore$ Root of equation is, $x = 1.9129$ //

Q4) Design Newton iteration formula compute $\sqrt[3]{24}$, correct to 4 decimal places.

→ $x = \sqrt[3]{24}$.
$x^3 = 24$
$x^3 - 24 = 0$.
f(x)

$$f(x) = x^3 - 24.$$
$$f'(x) = 3x^2.$$

By NR method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

$$x_{n+1} = x_n - \left[\frac{x_n^3 - 24}{3x_n^2}\right].$$

$$x_{n+1} = \frac{3x_n^3 - x_n^3 + 24}{3x_n^2}$$

$$\therefore \boxed{x_{n+1} = \frac{2x_n^3 + 24}{3x_n^2}}$$

Now,

$$f(x) = x^3 - 24$$

$\therefore f(1) = -23.$

$\therefore f(2) = -16.$

$\therefore f(3) = 3$

$\left.\begin{matrix} f(2) = -16 \\ f(3) = 3 \end{matrix}\right\} (2,3).$

$\therefore$ Root lie in $(2,3)$

x.	$f(x)$
2.8	-2.048
2.9	0.389.

$|f(2.9)|$. is nearer to zero.

$$x_0 = \cancel{0.382}\ 2.9.$$

$$\therefore x_1 = \frac{2x_0^3 + 24}{3x_0^2}$$

$\therefore x_1$ 17

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$\therefore x_1 = \frac{2(2.9)^3 + 24}{3(2.9)^2} = 2.88458$.

$\therefore x_2 = \frac{2x_1^3 + 24}{3x_1^2} = \frac{2(2.8845)^3 + 24}{3(2.8845)^2}$

$= 2.88449$.

$\therefore x_3 = \frac{2x_2^3 + 24}{3x_2^2} = \frac{2(2.8844)^3 + 24}{3(2.8844)^2}$

$\therefore x_3 = 2.8845$.

Fixed point Iteration,

Let $f(x) = 0$, be equation having real root in (a,b).

We write, $x = \phi(x)$.

$|\phi'(x)| < 1 \quad \forall x \in (a,b)$.

then formula to find root of equ^n,

$x_{n+1} = \phi(x_n)$.

Q1) Find root of equation, $f(x) = x^3 + x^2 - 1 = 0$ by fixed point iteration correct to 4 decimal places.

→ $f(x) = x^3 + x^2 - 1$.

$f(0) = -1$

$f(1) = 1$.

$\}$ $(0,1)$.

$\therefore$ Root lie in $(0,1)$.

x	$f(x)$
0.7	−0.167
0.8	0.152.

Root lie in $(0.7, 0.8)$.

Since, $|f(0.8)|$ is nearer to zero,

$\therefore x_0 = 0.8$.

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$$f(x) = x^3 + x^2 - 1 = 0.$$

$$x^3 + x^2 - 1 = 0$$

$$x^2(x+1) = 1.$$

$$x^2 = \frac{1}{(x+1)}.$$

$$\therefore x = \frac{1}{\sqrt{(x+1)}}. = \phi(x).$$

$$\therefore \phi(x) = \frac{1}{\sqrt{(x+1)}} = (x+1)^{-1/2}$$

$$\therefore \phi'(x) = -\frac{1}{2}(1+x)^{-3/2}\cdot(1).$$

$$\therefore |\phi'(x)| = \frac{1}{2}\left(\frac{1}{(1+x)^{3/2}}\right).$$

$$\therefore x = x_0 = 0.8.$$

$$\therefore |\phi'(x)| = 0.20 < 1 \text{ for } x = x_0.$$

$$\therefore \phi(x) = \frac{1}{\sqrt{x+1}}$$

$\therefore$ By fixed point method.

$$x_1 = \phi(x_0)$$

$$x_1 = \frac{1}{\sqrt{x_0+1}}$$

$$\therefore x_1 = 0.74535.$$

$$x_2 = \phi(x_1). = \frac{1}{\sqrt{x_1+1}} = 0.75693.$$

$x_3 = \phi(x_2) = \frac{1}{\sqrt{x_2+1}} = 0.75443.$

$x_4 = \phi(x_3) = 0.75487.$

$x_5 = \phi(x_4) = 0.75485.$

$x_6 = \phi(x_5) = 0.75488.$

$x_7 = \phi(x_6) = 0.75487.$

$\therefore$ Root of equation, $x = 0.7549.$

Q2) $f(x) = x - 0.5\cos x = 0$; $x_0 = 0.$

$x - 0.5\cos x = 0.$

$x = 0.5\cos x = \phi(x).$

$\phi(x) = 0.5\cos x$

$\phi'(x) = -0.5\sin x.$

$|\phi'(x)| = 0 < 1$ for $x = x_0 = 0.$

$\therefore \phi(x) = 0.5\cos x.$

By fixed point method,

$x_1 = \phi(x_0) = 0.5\cos x_0.$

$\therefore x_1 = 0.5$

$x_2 = \phi(x_1) = 0.5\cos x_1$

$\therefore x_2 = 0.4387$

$x_3 = \phi(x_2) = 0.5\cos x_2$

$\therefore x_3 = 0.4526.$

$x_4 = 0.5\cos x_3 = \phi(x_3).$

$\therefore x_4 = 0.4496.$

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$\therefore x_5 = \phi(x_4) = 0.5\cos(x_3)$
$= 0.4503$.

$\therefore x_6 = \phi(x_5) = 0.5\cos(x_4)$.
$\therefore x_6 = 0.4501$

$\therefore$ Root of equation is, $x = 0.450$.

Q3) Find root of equation, $x \cdot \cosh x = 1$.

→ $x \cdot \cosh x = 1$.

$\therefore x \cdot \left[\frac{e^x - e^{-x}}{2}\right] = 1$.

$\therefore x = \left(\frac{2}{e^x - e^{-x}}\right) = f(x)$.

$\therefore f(x) = \frac{2}{(e^x - e^{-x})}$.

$\therefore f(x) = x\cosh x - 1$.
$\therefore f(0) = -1$
$\therefore f(1) = 0.54$.

$\therefore$ Root lie in $(0, 1)$

x	$f(x)$
0.7	-0.121
0.8	0.069

$|f(0.8)|$ is nearer to zero
$x_0 = 0.8$.

$\therefore x \cdot \cosh x = 1$.
$\therefore x = \frac{1}{\cosh x} = \phi(x)$. $= \frac{u}{v}$

H.W.

(ii) Find smallest +ve solution of $\sin x = e^{-x}$.

$\rightarrow x = \sin^{-1}(e^{-x}) = \phi(x)$ — (I)

$$\phi'(x) = \frac{0 - \sinh x(1)}{(\cosh x)^2}$$

$x \cdot \sin x = e^{-x} \cdot x$

$x = \frac{x \cdot e^{-x}}{\sin x} = \phi(x)$.

$$\phi'(x) = \frac{-\sinh x}{(\cosh x)^2}$$

Ans :- 0.5885.

$\therefore |\phi'(x)| = 0.49 < 1$ for $x = x_0 = 0.8$.

9th iteration.

$$\phi(x) = \frac{1}{\cosh x}$$

By fixed point method.

$\therefore x_1 = \phi(x_0) = \phi(0.8) = \frac{1}{\cosh(0.8)}$

$= 0.7476$.

$\therefore x_2 = \phi(x_1) = \phi(0.7476) = \frac{1}{\cosh(0.7476)}$.

$= 0.7735$

$\therefore x_3 = \phi(x_2) = \phi(0.7735) = \frac{1}{\cosh(0.7735)}$.

$= 0.7608$.

$\therefore x_4 = \phi(x_3) = 0.7670$.

$\therefore x_5 = \phi(x_4) = 0.7640$.

$\therefore x_6 = \phi(x_5) = 0.7655$.

$\therefore x_7 = \phi(x_6) = 0.7647$.

$\therefore x_8 = \phi(x_7) = 0.7651$

$\therefore x_9 = \phi(x_8) = 0.7649$.

$\therefore x_{10} = \phi(x_9) = 0.7650$

$\therefore x_{11} = \phi(x_{10}) = 0.76500$.

$\therefore x_{12} = \phi(x_{11})$ 12.

$\therefore$ Root of equation,

$x = 0.7650$

Q4) Find the root of equation upto 4 decimals. OR
Find the smallest (+)ve solution of $\sin x = e^{-x}$

→ $\sin x = e^{-x}$

let, $f(x) = \sin x - e^{-x} = 0$.

$\therefore f(0) = \sin(0) - e^{-(0)} = -1$
$f(1) = \sin(1) - e^{-(1)} = 0.4735$ } $(0,1)$.

$\therefore$ The root lie's in $(0,1)$

x	$f(x)$
0.5	−0.127
0.6	0.0158

$\therefore$ The root lie's in $(0.5, 0.6)$.

$\therefore f(0.5) = -0.127$, $f(0.6) = 0.0158$

Now, $|f(0.5)| = 0.127$. & $|f(0.6)| = 0.0158$.

where $|f(0.6)| = 0.0158$ is nearer to zero
So, $x_0 = 0.6$.

Now, $\sin x = e^{-x}$
$x = \sin^{-1}(e^{-x})$.
$\phi(x) = \sin^{-1}(e^{-x})$.

$\phi'(x) = \frac{-1 \cdot e^{-x}}{\sqrt{1-e^{-2x}}} < 0$. for $x = x_0 = 0.6$.

Hence, $\phi(x) = \sin^{-1}(e^{-x})$.
By Fixed point method,
$x_1 = \sin^{-1}(e^{-x_0}) = \sin^{-1}(e^{-0.6}) = 0.5809$.
$x_2 = \sin^{-1}(e^{-x_1}) = \sin^{-1}(e^{-0.5809}) = 0.5936$.

$x_3 = \sin^{-1}(e^{-x_2}) = 0.5851$

$x_4 = \sin^{-1}(e^{-x_3}) = 0.8907$

$x_5 = \sin^{-1}(e^{-x_4}) = 0.5870.$

$x_6 = \sin^{-1}(e^{-x_5}) = 0.5895.$

$x_7 = \sin^{-1}(e^{-x_6}) = 0.5878.$

$x_8 = \sin^{-1}(e^{-x_7}) = 0.5889.$

$x_9 = \sin^{-1}(e^{-x_8}) = 0.5882$

$x_{10} = \sin^{-1}(e^{-x_9}) = 0.5886.$

$x_{11} = \sin^{-1}(e^{-x_{10}}) = 0.5884.$

$x_{12} = \sin^{-1}(e^{-x_{11}}) = 0.5886$

Hence, the root of equation is $x = 0.5886$.

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Numerical Solutions of ordinary Differential equations :-

Q1 Taylor's Series method.

consider first order equation,

$$\frac{dy}{dx} = f(x,y); \quad y(x_0) = y_0 \quad \longrightarrow \text{Initial condition}$$

(Initial value problem)

then Taylor's Series, is given by,

$$y(x) = y_0 + (x-x_0)(y'_0)_0 + \frac{(x-x_0)^2}{2!}(y''_0)_0 + \frac{(x-x_0)^3}{3!}(y''')_0 + \ldots$$

$$y' = f(x,y')$$

* Find approximate value of "y" using Taylor Series method.

01) $\frac{dy}{dx} = x^2y - 1$, $y(0) = 1$ find $y(0.1)$, $y(0.2)$.

$\longrightarrow$ $\frac{dy}{dx} = y' = x^2y - 1$.

$y(0) = 1$.

$x_0 = 0$, $y_0 = 1$

$\therefore y' = x^2y - 1$

Now replacing 'x' with 'x_0' & 'y' with 'y_0'

$\therefore (y')_0 = x_0^2 y_0 - 1$.

$= (0)^2(1) - 1$

$\therefore (y')_0 = -1$.

$y'' = x^2y' + 2xy$

$(y'')_0 = x_0^2 (y')_0 + 2x_0y_0$

$= (0)^2(-1) + 2(0)(1)$

$= 0$.

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$(y''') = x^2y'' + 2xy' + 2(1)y + 2xy'$

$(y''')_0 = x_0^2(y'')_0 + 2x_0(y')_0 + 2y_0 + 2x_0(y')_0$

$= 2(1) = 2$

$(y^{iv}) = x^2y''' + 2xy'' + 2xy'' + 2y' + 2y' + 2xy'' + 2xy''$

$(y^{iv}) = x^2y''' + 6xy'' + 6y'$

$(y^{iv})_0 = x_0^2(y''')_0 + 6x_0(y'')_0 + 6(y')_0$

$(y^{iv})_0 = 6(-1) = -6$

By Taylor's Series,

$$y(x) = y_0 + (x-x_0)(y')_0 + \frac{(x-x_0)^2}{2!}(y'')_0 + \frac{(x-x_0)^3}{3!}(y''')_0 + \frac{(x-x_0)^4}{4!}(y^{iv})_0 + \ldots$$

$$\therefore y(x) = 1 + (x-0)(-1) + \frac{(x-0)^2}{2!}(0) + \frac{(x-0)^3}{6}(2) + \frac{(x-0)^4}{24}(-6) + \ldots$$

$$\therefore y(x) = 1 - x + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \quad \text{—— (1)}$$

To find $y(0.1)$ put $x = 0.1$ in (1)

$$\therefore y(0.1) = 1 - (0.1) + \frac{(0.1)^3}{3} - \frac{(0.1)^4}{4} + \ldots = 0.9003$$

$$\therefore y(0.2) = 1 - (0.2) + \frac{(0.2)^3}{3} - \frac{(0.2)^4}{4}$$

$$\therefore y(0.2) = 0.802$$

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Q2) $\frac{dy}{dx} = 3x + y^2$, $y(0) = 1$ find $y(0.1)$.

$\longrightarrow$ $\frac{dy}{dx} = 3x + y^2 = y'$

$y(0) = 1$

where, $x_0 = 0$, $y_0 = 1$.

$\therefore y' = 3x + y^2$. $\quad (y')_0 = 3x_0 + y_0^2$
$= 3(0) + (1)^2$
$= 1$.

$\therefore y'' = 3 + 2y \cdot y'$, $\quad (y'')_0 = 3 + 2y_0 (y')_0$
$= 3 + 2(1)(1)$
$= 5$.

$\therefore y''' = \cancel{3}\ 2y \cdot y'' + 2y' \cdot y'$, $\quad (y''')_0 = 2y_0 (y'')_0 + 2(y')_0^2$
$= 2y \cdot y'' + 2(y')^2$ $\quad = 2(1)(5) + 2(1)^2$
$= 12$.

$\therefore y^{iv} = 2y \cdot y''' y' + 2y'' \cdot y'y' + 4(y')(y')$.
$= 2y \cdot y''' \cdot y' + 2y''(y')^2 + 4(y')^2$.

$\therefore (y^{iv})_0 = 2y_0 (y''')_0 (y')_0 + 2(y'')_0 (y')_0^2 + 4(y')_0^2$
$= 2(1)(12)(1) + 2(5)(1)^2 + 4(1)^2$.
$= 24 + 10 + 4$

$\therefore (y^{iv})_0 = 38$.

By Taylor's Series,

$y(x) = y_0 + (x - x_0)(y')_0 + \frac{(x-x_0)^2}{2!}(y'')_0 + \frac{(x-x_0)^3}{3!}(y''')_0 + \dots$

$y(x) = 1 + (x-0)(1) + \frac{(x-0)^2}{2}(5) + \frac{(x-0)^3}{6}(12) + \dots$

$y(x) = 1 + x + \frac{5}{2}x^2 + 2x^3 + \dots$

$\boxed{\log x = \ln}$ $\boxed{\log_{10} x = \log}$ in calc.

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put $x = 0.1$ in above,

$$y(0.1) = 1 + (0.1) + \frac{5}{2}(0.1)^2 + 2(0.1)^3 + \cdots$$

$$\therefore y(0.1) = 1.127$$

Q3) $\frac{dy}{dx} = \log(xy)$; $y(1) = 2$. find $y(1.1)$; $y(1.2)$
$= 2.075$ $= 2.164$

$\rightarrow$ $\because y' = \log(xy)$.

$y(1) = 2$

$x_0 = 1$, $y_0 = 2$

$y' = \log(xy)$, $(y')_0 = \log(1 \times 2) = \ln(2)$
$= 0.693$.

$$y'' = \frac{1}{xy}[xy' + y]$$

$$y'' = \frac{xy' + y}{xy}$$

$$(y'')_0 = \frac{x_0 (y')_0 + y_0}{x_0 y_0}$$

$$(y'')_0 = \frac{(1)(0.693) + 2}{(1)(2)}$$

$$(y'')_0 = 1.346.$$

$$y''' = \frac{xy[xy'' + y' + y'] - [xy' + y][xy' + y]}{(xy)^2}$$

$$y''' = \frac{xy[xy'' + 2y'] - [xy' + y]^2}{(xy)^2}$$

$$(y''')_0 = \frac{x_0 y_0 [x_0 (y'')_0 + 2(y')_0] - [x_0 (y')_0 + y_0]^2}{(x_0 y_0)^2}$$

$$= \frac{(1)(2)[1(1.346) + 2(0.693)] - [1(0.693) + 2]^2}{(1 \times 2)^2}$$

$$= -0.447.$$

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By Taylor's method,

$$y(x) = y_0 + (x-x_0)(y')_0 + \frac{(x-x_0)^2}{2!}(y'')_0 + \frac{(x-x_0)^3}{3!}(y''')_0 + \dots$$

$$y(x) = 2 + (x-1)(0.6931) + \frac{(x-1)^2}{2}(1.3465) - \frac{(x-1)^3}{6}(0.447) + \dots$$

For, $x = 1.1$.

$$y(1.1) = 2 + (1.1-1)(0.6931) + \frac{(1.1-1)^2}{2}(1.3466) - \frac{(1.1-1)^3}{6}(0.447) + \dots$$

$\therefore y(1.1) = 2.075$,

|||ly $y(1.2) = 2.164$

Q4) $\frac{dy}{dx} = e^x - y^2$; $y(0) = 1$: find $y(0.1)$, $y(0.2)$.

→ $y' = e^x - y^2$. $y(0) = 1$ where $x_0 = 0$, $y_0 = 1$.

$y' = e^x - y^2$, $(y')_0 = e^0 - 1 = 0$.

$y'' = e^x - 2y.y'$, $(y'')_0 = e^{x_0} - 2y_0.y'_0$
$= 1 - 0 = 1$

$y''' = e^x - 2[y.y'' + y'.y']$, $(y''')_0 = e^x - 2[y_0.(y'')_0 + (y'_0)^2]$.
$= 1 - 2[1(1)]$
$= -1$.

By Taylor's Series,

$$y(x) = y_0 + (x-x_0)(y')_0 + \frac{(x-x_0)^2}{2!}(y'')_0 + \frac{(x-x_0)^3}{3!}(y''')_0 + \dots$$

$$y(x) = 1 + (x-0)(0) + \frac{(x-0)^2}{2!}(1) + \frac{(x-0)^3}{3!}(-1) + \dots$$

$$y(x) = 1 + \frac{x^2}{2} - \frac{x^3}{6} + \dots$$

To get $y(0.1)$ we put $x = 0.1$ in above,

$$y(0.1) = 1 + \frac{(0.1)^2}{2} - \frac{(0.1)^3}{6} + \dots$$

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$\therefore y(0.1) = 1.0048$;

$$y(0.2) = 1 + \frac{(0.2)^2}{2} + \frac{(0.2)^3}{6} + \dots$$

$y(0.2) = 1 + \dots = 1.018$.

Euler's Method :-

Consider ; $\frac{dy}{dx} = f(x,y)$; $y(x_0) = y_0$.

then, $x_1 = x_0 + h$, $x_2 = x_0 + 2h$ ...... $x_n = x_0 + nh$.

$\therefore y_1 = y(x_1) = y_0 + h.f(x_0, y_0)$.

$y_2 = y_1 + h.f(x_1, y_1)$.

$y_3 = y_2 + h.f(x_2, y_2)$.

$\vdots$

$y_n = y_{n-1} + h.f(x_{n-1}, y_{n-1})$.

Find approximate value of y for following problems using Euler's method.

Q1) $\frac{dy}{dx} = x+y$ & $y=0$ when $x=0$, $h=0.2$ find. $y(1)$.

$\rightarrow$ $x_0 = 0$ & $h = 0.2$, $x_1 = x_0 + h = 0.2$, $x_2 = x_0 + 2h = 0.4$.

$x_3 = x_0 + 3h = 0.6$, ...

x	y	$\frac{dy}{dx} = f(x,y) = x+y$	old (y) + h.f(x,y) = new (y).
$x_0 = 0$	0	$0+0=0$	$0 + (0.2)(0) = 0$.
$x_1 = 0.2$	0	$0+0.2 = 0.2$	$0 + 0.2(0.2) = 0.04$.
$x_2 = 0.4$	0.04	$0.4 + 0.04 = 0.44$	$0.04 + 0.2(0.44) = 0.128$.

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$x_3 = 0.6$	0.128	$0.6 + 0.128 = 0.728$	$0.128 + 0.2[0.728] = 0.2736$
$x_4 = 0.8$	0.2736	$0.8 + 0.2736 = 1.0736$	$0.2736 + 0.2[1.0736] = 0.4883$
$x_5 = 1$	0.4883		

$\therefore y(1) = 0.4883$.

Q2) $\frac{dy}{dx} = \frac{y-x}{y+x}$ & $y(0) = 1$. find $y(0.1)$. by taking $h = 0.02$.

$\rightarrow$ $\frac{dy}{dx} = \frac{y-x}{y+x} = f(x,y)$. $y(0) = 1$

$y_0 = 1$, $x_0 = 0$.

x	y	$\frac{dy}{dx} = f(x,y) = \frac{y-x}{y+x}$	$old(y) + h f(x,y) = new(y)$
$x_0 = 0$	1	$\frac{1-0}{1+0} = 1$	$1 + 0.02[1] = 1.02$.
$x_1 = 0.02$	1.02	$\frac{1.02-0.02}{1.02+0.02} = 0.96$	$1.02 + 0.02[0.96] = 1.039$.
$x_3 = 0.04$	1.039	$\frac{1.039-0.04}{1.039+0.04} = 0.925$	$1.039 + 0.02[0.925] = 1.057$
$x_4 = 0.06$	1.057	$\frac{1.057-0.06}{1.057+0.06} = 0.892$	$1.057 + 0.02[0.892] = 1.074$.
$x_5 = 0.08$	1.074	$\frac{1.074-0.08}{1.074+0.08} = 0.861$	$1.07 + 0.02[0.861] = 1.091$.
$x_6 = 0.1$	1.091		

$y(0.1) = 1.091$

H.W.

Q1) $\frac{dy}{dx} = x.y^2$, $y(0) = 1$. $h = 0.1$, $y(1)$.

Q3) $\frac{dy}{dx} = x.y^2$, $y(0) = 1$, $h = 0.1$ find $y(1)$

→ $\frac{dy}{dx} = x.y^2 = f(x,y)$. $y(0) = 1$;
$x_0 = 0$, $y_0 = 1$.

x	y	$\frac{dy}{dx} = f(x,y) = x.y^2$	old(y) + h.f(x,y) = new(y).
$x_0 = 0$	1	0	$1 + 0.1(0) = 1.$
$x_1 = 0.2$	1	0.2	$1 + 0.1(0.2) = 1.02$
$x_2 = 0.4$	1.02	$(0.4)(1.02)^2 = 0.41$	$1.02 + 0.1(0.41) = 1.06$
$x_3 = 0.6$	1.06	$(0.6)(1.06)^2 = 0.67$	$1.06 + 0.1(0.67) = 1.12$
$x_4 = 0.8$	1.12	$(0.8)(1.12)^2 = 1.01$	$1.12 + 0.1(1.01) = 1.22.$
$x_5 = 1$	1.22 .		

$\therefore y(1) = 1.22$

Modified Euler Method :-

Accuracy is poor in Euler method hence we will correct the solution using modified Euler method.

Consider, $\frac{dy}{dx} = f(x,y)$,

$y(x_0) = y_0$,

$y_1^{(0)} = y_0 + hf(x_0;y_0)$ [Euler Formula] .

to correct y_1 to desired accuracy,

$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1,y_1^{(0)})\right]$.

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right].$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right].$$

$\vdots$

Continue process till we get root of desired accuracy.

NOTE:- Euler's & modified Euler's methods are called as "Euler's predictor corrector form"

Q1) Find approximate value of y using modified Euler method. $\frac{dy}{dx} = x^2+y$, $y(0)=1$ find $y(0.1)$ by taking $h=0.05$.

→ $\frac{dy}{dx} = f(x,y) = x^2+y$, when $x_0=0$

$x_1 = x_0+h$. $x_2=x_0+2h$. $y_0=1$.

$x_0 = 0$, $x_1=0.05$, $x_2=0.1$.

$$y_1^{(0)} = y_0 + h\,f(x_0,y_0) = 1 + (0.05)\left[x_0^2+y_0\right] = 1.05.$$

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right].$$

$$= 1 + \frac{0.05}{2}\left[0^2+(1) + (0.05)^2 + (1.05)\right] = 1.0513$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right].$$

$$= 1 + \frac{0.05}{2}\left[0^2+(1) + (0.05)^2 + 1.0513\right]$$

$$= 1.0513.$$

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$y_1 = 1.0513.$

$x_1 = 0.05, \quad y_1 = 1.0513.$

$y_2^{(0)} = y_1 + h.f(x_1, y_1). = 1.0513 + 0.05[(0.05)^2 + 1.0513]$
$= 1.1039.$

$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(0)})\right].$

$= 1.0513 + \frac{0.05}{2}\left[(0.05)^2 + 1.0513 + (0.1)^2 + (1.1039)\right].$

$= 1.10549.$

$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)})\right] = 1.10553.$

$\therefore y_2 = y(0.1) = 1.1055.$

Q2) $\frac{dy}{dx} = 1-y, \quad y(0) = 0$ find $y(0.1), y(0.2), y(0.3).$ (x_1, x_2, x_3)

[NOTE:- solve until value of "x" (ie x_1, x_2, x_3) get finished)

$\frac{dy}{dx} = f(x, y) = 1-y.$

$x_0 = 0, \quad y_0 = 0, \quad x_1 = 0.1, \quad x_2 = 0.2, \quad x_3 = 0.3.$

$\therefore h = x_1 - x_0 = 0.1.$

$y_1^{(0)} = y_0 + h.f(x_0, y_0). \quad = 0.1.$

$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right].$

$= 0 + \frac{0.1}{2}\left[(1) + (1 - 0.1)\right].$

$= 0.095$

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$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)}) \right] = 0.095.$$

$\therefore y_1 = 0.095$.

$x_2 = 0.2$

$x_1 = 0.1, \quad y_1 = 0.095.$

$$y_2^{(0)} = y_1 + h \cdot f(x_1, y_1) = 0.095 + 0.1[1 - 0.095]$$
$$= 0.1855.$$

$$y_2^{(1)} = y_1 + \frac{h}{2} \cdot \left[f(x_1, y_1) + f(x_2, y_2^{(0)}) \right] = 0.1809$$

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)}) \right] = 0.1812.$$

$y_2 = y(0.2) = \cancel{0.1088}\ 0.181$

$x_3 = 0.3$

$$y_3^{(0)} = y_2 + h \cdot f(x_2, y_2) = 0.181 + 0.1(1 - 0.181)$$
$$= 0.2629.$$

$$y_3^{(1)} = y_2 + \frac{h}{2}\left[f(x_2, y_2) + f(x_3, y_3^{(0)}) \right] = 0.2588.$$

$$y_3^{(2)} = y_2 + \frac{h}{2}\left[f(x_2, y_2) + f(x_3, y_3^{(1)}) \right] = 0.2590.$$

$\therefore y_3 = 0.259.$ End [$\because$ No further values of x] like x_4, x_5 etc.

Q3) $\frac{dy}{dx} = x + |\sqrt{y}|$, $y(0) = 1$. for range $0 \le x \le 0.6$, in steps 0.2 h

$\rightarrow \frac{dy}{dx} = x + |\sqrt{y}| = f(x, y).$

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H.W. ① $\frac{dy}{dx} = x + |\sqrt{y}|$; $y(0) = 1$. for range $0 \le x \le 0.6$, in steps 0.2 = h.

② $\frac{dy}{dx} = \log(x+y)$, $y(0) = 2$, $y(0.2)$, $y(0.4)$.

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$y(0) = 1$ $x_1 = 0.2$; $x_2 = 0.4$; $x_3 = 0.6$.

$\therefore$ $x_0 = 0$, $y_0 = 1$. $h = 0.2$

$y_1^{(0)} = y_0 + h \cdot f(x_0, y_0) = 1 + 0.2[0+1] = 1.2$

$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(0)})]$

$= 1 + 0.2/2\,[1 + (0.2 + \sqrt{1.2})] = 1.2295.$

$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})]$.

$= 1 + 0.2/2\,[1 + (0.2 + \sqrt{1.2295})] = 1.2308.$

$y_1^{(3)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(2)})]$

$= 1 + 0.2/2\,[1 + (0.2 + \sqrt{1.2308})] = 1.2309.$

$\therefore \boxed{y_1 = 1.230}$

$y_1 = 1.230$, $x_1 = 0.2$, $x_2 = 0.4$.

$y_2^{(0)} = y_1 + h\,f(x_1, y_1) = 1.230 + (0.2)[0.2 + \sqrt{1.230}] = 1.4918$

$y_2^{(1)} = y_1 + h/2\,[f(x_1, y_1) + f(x_2, y_2^{(0)})]$

$= 1.230 + 0.2/2\,[1.309 + (0.4 + \sqrt{1.4918})] = 1.5230.$

$y_2^{(2)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(1)})]$

$= 1.230 + 0.2/2\,(1.309 + 0.4 + \sqrt{1.5230})] = 1.5243.$

$y_2^{(3)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(2)})]$

$= 1.230 + 0.2/2\,[1.309 + 0.4 + \sqrt{1.5243}] = 1.5243.$

$\therefore \boxed{y_2 = 1.5243}$

$x_2 = 0.4$; $x_3 = 0.6$.

$y_3^{(0)} = y_2 + h \cdot f(x_2, y_2) = 1.524 + (0.2)[1.6345] = 1.8509.$

$y_3^{(1)} = y_2 + \frac{h}{2} \cdot [f(x_2, y_2) + f(x_3, y_3^{(0)})]$

$= 1.524 + (0.2/2)[1.6345 + 0.6 + \sqrt{1.8509}] = 1.8834.$

$y_3^{(2)} = y_2 + h/2\,[f(x_2, y_2) + f(x_3, y_3^{(1)})] = 1.8846.$

$y_3^{(3)} = y_2 + h/2\,[f(x_2, y_2) + f(x_3, y_3^{(2)})] = 1.8847.$

$\therefore \boxed{y_3 = 1.884.}$

Q4) $\frac{dy}{dx} = \log(x+y)$, $y(0) = 2$, $y(0.2)$, $y(0.4)$

$\rightarrow$ $\frac{dy}{dx} = \log(x+y) = f(x, y)$.

Now, $y(0) = 2$. where, $x_0 = 0$, $y_0 = 2$, $x_1 = 0.2$, $x_2 = 0.4$

$\therefore h = x_1 - x_0 = 0.2 - 0 = 0.2$

$\therefore y_1^{(0)} = y_0 + h.f(x_0, y_0) = 2 + (0.2)[\log(0+2)]$
$= 2.13$.

$\therefore y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$

$= 2 + \frac{0.2}{2}\left[0.6931 + 0.7601\right] = 2.1453$

$\therefore y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$

$= 2 + \frac{0.2}{2}\left[0.6931 + 0.7632\right] = 2.1456$

$\therefore y_1 = 2.145$, $x_1 = 0.2$, $x_2 = 0.4$

$y_2^{(0)} = y_1 + h.f(x_1, y_1) = 2.145 + (0.2)[0.8522] = 2.3154$.

$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(0)})\right]$

$= 2.145 + \left(\frac{0.2}{2}\right)\left[0.8522 + 0.9989\right] = 2.3301$.

$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)})\right]$.

$= 2.145 + \left(\frac{0.2}{2}\right)\left[0.8522 + 0.9725\right] = 2.327$

$y_2^{(3)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(2)})\right] = 2.330.$

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$\therefore y_2 = 2.330$

Runge-Kutta Method :-

Consider $\frac{dy}{dx} = f(x,y)$; $y(x_0) = y_0$, the constants are given as,

$$k_1 = h.f(x_0, y_0).$$

$$k_2 = h.f\left(x_0 + \left(\frac{h}{2}\right); y_0 + \left(\frac{k_1}{2}\right)\right).$$

$$k_3 = h.f\left(x_0 + \left(\frac{h}{2}\right), y_0 + \left(\frac{k_2}{2}\right)\right).$$

$$k_4 = h.f(x_0 + h, y_0 + k_3)$$

$$\therefore k = \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right].$$

$$y_1 = y_0 + k$$

Using RK method, find approximate value of 'y' for following ODE's.

Q1) $\frac{dy}{dx} = x + y$, $y(0) = 1$, find $y(0.2)$.

$\rightarrow$ $\frac{dy}{dx} = f(x,y) = x + y$

$y(0) = 1$

$x_0 = 0$, $y_0 = 1$, $x_1 = 0.2$, $h = x_1 - x_0$
$= 0.2 - 0$
$= 0.2$

By RK method,

$k_1 = h.f(x_0, y_0). = (0.2)(0+1) = 0.2.$

$k_2 = h.f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.2)\left(0 + \frac{0.2}{2}, 1 + \frac{0.2}{2}\right)$
$= 0.24.$

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$k_3 = h.f(x_0 + h/2, y_0 + k_2/2) = 0.2.f(0.1, 1.12)$
$= 0.244.$

$k_4 = h.f(x_0 + h, y_0 + k_3) = 0.2.f(0.2, 1.244).$
$= 0.288$

$\therefore k = \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$

$= \frac{1}{6}\left[0.2 + 2(0.24) + 2(0.244) + 0.288\right].$

$\therefore k = 0.2428$

$\therefore y_1 = y_0 + k = 1 + 0.2428$
$\therefore y_1 = 1.2428$

Q27 $\frac{dy}{dx} = 3.e^x + 2y$; $y(0) = 0$, $h = 0.1$ find $y(0.1)$.

$\longrightarrow$ $\frac{dy}{dx} = f(x_0, y_0) = 3e^x + 2y.$

$y(0) = 0$
$x_0 = 0$, $y_0 = 0$. $x_1 = 0.1$. $h = 0.1$.

By RK method,

$k_1 = h.f(x_0, y_0) = (0.1).(3.e^0 + 2(0)) = 0.3.$

$k_2 = h.f(x_0 + h/2, y_0 + k_1/2) = (0.1)f\left[0 + \frac{0.1}{2}, 0 + \frac{0.3}{2}\right]$
$= (0.1)f(0.05, 0.15).$
$= 0.345.$

$k_3 = h.f(x_0 + h/2, y_0 + k_2/2) = (0.1)f\left[0 + \frac{0.1}{2}, 0 + \frac{0.345}{2}\right]$
$= (0.1).f(0.05, 0.1726)$
$= 0.3499.$

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$k_4 = h \cdot f(x_0+h ; y_0+k_3) = (0.1) f(0+0.1, 0+0.3499)$.

$= (0.1) \cdot f(0.1, 0.3499)$.

$= (0.1)(3 \times e^{(0.1)} + 2(0.3499))$.

$= 0.4015$.

$\therefore k = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$= \frac{1}{6}[0.3 + 2(0.345) + 2(0.3499) + 0.4015]$

$\therefore k = 0.34855$

$\therefore y_1 = y_0 + k = 0 + 0.34855$

$= 0.34855$.

03) $\frac{dy}{dx} = x+y^2$, $y(0) = 1$, find $y(0.2)$. , $h = 0.1$

$\rightarrow \frac{dy}{dx} = f(x,y) = x+y^2$

$y(0) = 1$ where, $x_0 = 0$, $y_0 = 1$, $h = 0.1$. $x_1 = 0.1$.

$x_2 = 0.2$..

Now, $k_1 = h \cdot f(x_0, y_0) = h.(0,1) = (0.1)[0+1^2] = 0.1$.

$k_2 = h \cdot f(x_0 + \frac{h}{2}, y_0 + k_1/2) = (0.1) \cdot f(0 + \frac{0.1}{2}, 1 + \frac{0.1}{2})$.

$= (0.1) \cdot f(0.05, 1.05)$.

$= (0.1) \cdot (0.05 + 1.1025)$

$\therefore k_2 = 0.1075$.

$k_3 = h \cdot f(x_0 + h/2, y_0 + k_2/2) = (0.1) \cdot f(0.05, 1 + \frac{0.1075}{2})$.

$= (0.1) f(0.05, 1.05375)$.

$\therefore k_3 = 0.1160$

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$k_4 = h \cdot f(x_0 + h, y_0 + k_3) = (0.1) \cdot f(0 + 0.1, 1 + 0.1160)$.
$= (0.1) \cdot f(0.1, 1.1160)$.
$k_4 = 0.1345$.

$\therefore k = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$= \frac{1}{6}[0.1 + 2(0.1075) + 2(0.1160) + 0.1345]$

$\therefore k = 0.1135$.

$\therefore y_1 = y_0 + k \Rightarrow y_1 = 1 + 0.1135$

$\therefore y_1 = 1.1135$.

$\therefore$ Here, $x_1 = 0.1$, $y_1 = 1.1135$

$k_1 = h.f(x_1, y_1) = 0.1.f(0.1, 1.1135)$.
$= 0.1.(0.1 + (1.1135)^2)$
$k_1 = 0.1339$.

$k_2 = h.f(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}) = (0.1).f(0.1 + \frac{0.1}{2}, 1.1135 + \frac{0.1339}{2})$

$= (0.1).f(0.15, 1.1804)$.
$= (0.1)(0.15 + (1.1804)^2)$
$= 0.1543$.

$k_3 = h.f(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}) = (0.1).f(0.1 + \frac{0.1}{2}, 1.1135 + \frac{0.1543}{2})$

$= (0.1).f(0.15, 1.1906)$.
$= (0.1)(0.15 + (1.1906)^2)$
$= 0.1567$

$k_4 = h.f(x_1 + h, y_1 + k_3) = (0.1).f(0.1 + 0.1, 1.1135 + 0.1567)$
$= (0.1).f(0.2, 1.2702)$.
$= 0.1813$.

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Now, $K = \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]$

$= \frac{1}{6}\left[0.1339 + 2(0.1549) + 2(0.1567) + 0.1813\right]$

$K = 0.1564$.

$\therefore \quad y_2 = y_1 + K = y(0.2) = 1.1135 + 0.1564$
$= 1.2699$

---

(1) Method of false :- $x = \frac{a.f(b) - b.f(a)}{f(b) - f(a)}$

(2) Newton Repulsion method :- $x_1 = x_0 + h$, $x_n = x_0 + nh$.

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ ...... $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.

(3) Fixed point method :- $f(x) = 0$ $\Rightarrow$ convert given equation in terms of 'x'

ie $\phi(x) = x = 0$.

$\phi'(x) < 0 \;\forall\; x \in (a,b)$ $\Rightarrow$ Formula $\Rightarrow x_1 = \phi(x_0) =$

$x_2 = \phi(x_1) = \cdots$

(4) Taylor's Series method :- $y(x) = y_0 + (x - x_0).(y')_0 + \frac{(x-x_0)^2}{2!}(y'')_0 + \frac{(x-x_0)^3}{3!}(y''')_0 + \cdots$

(5) Euler's method :- $x_1 = x_0 + h$, $x_2 = x_0 + 2h$, ... $x_{n+1} = x_0 + nh$.

$y(x_0) = y_0$

x	y	$\frac{dy}{dx} = f(x,y)$	$old(y) + h.f(x,y) = new(y)$

(6) Modified Euler's method :- $y_1^{(0)} = y_0 + h.f(x_0, y_0)$, $y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0,y_0) + f(x_1, y_1^{(0)})]$

$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})]$ ......

(7) Runge Kutta method :- $K_1 = h.f(x_0, y_0)$, $K_2 = h.f(x_0 + h/2, y_0 + K_1/2)$.

$K_3 = h.f(x_0 + h/2, y_0 + K_2/2)$, $K_4 = h.f(x_0 + h/2, y_0 + K_3/2)$. $K = \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$

$\boxed{y_1 = y_0 + K}$

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Application Problems :-

01. Heating - cooling.

At what value of x, the process governed by $f_1(x)$ $f_1(x) = 100(1-e^{-0.2x})$ & $f_2(x) = 40.e^{-0.01x}$ reach same temperature. Also find the latter by Newton method.

→ For some x, $f_1(x) = f_2(x)$.

$$100(1-e^{-0.2x}) = 40.e^{-0.01x}$$

$$5(1-e^{-0.2x}) = 2.e^{-0.01x}.$$

$$\Rightarrow 5-5.e^{-0.2x} - 2.e^{-0.01x} = 0.$$

$$f(x) = 0.$$

$$\therefore f(x) = 5-5.e^{-0.2x} - 2.e^{-0.01x}.$$

$$\therefore f'(x) = -5(-0.2).e^{-0.2x} - 2(-0.01).e^{-0.01x}$$

$$\therefore f'(x) = e^{-0.2x} + 0.02\,e^{-0.01x}.$$

$\therefore f(2) = -0.311$, $f(3) = 0.315$. $\therefore$ Root lie in $(2,3)$.

$$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

x	$f(x)$
2.4	−0.046
2.5	0.0167

$\therefore |f(2.5)|$ is nearer to zero.

$$x_0 = 2.5$$

$$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

$$= x - \left[\frac{5-5e^{-0.2x} - 2.e^{-0.01x}}{e^{-0.2x} + 0.02.e^{-0.01x}}\right].$$

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$\therefore x_1 = 2.4732$

$\therefore x_2 = 2.473$.

$\therefore x = 2.473$.

$$\left[f_2(x)\right]_{x=2.473} = 40 \cdot e^{-0.01x} = 39.02^{\circ} F.$$

Q2) Vibrating-beam.

Find solution of $\cos x \cdot \cosh x = 1$ near $x = 3\pi/2$ by NR method.

$\longrightarrow$ $\cos x . \cosh x = 1$

$f(x) = \cos x . \cosh x - 1$

$f'(x) = \cos x[\sinh x] + (-\sin x)[\cosh x]$.

$f'(x) = \cos x . \sinh x - \sin x . \cosh x$

given, $x_0 = \frac{3\pi}{2}$.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= x - \left[\frac{\cos x . \cosh x - 1}{\cos x . \sinh x - \sin x . \cosh x}\right]$$

$x_1 = 4.7303$.

$\therefore x_2 = 4.7300$.

$\therefore x = 4.73$.

Q3) Legendre polynomials.

Find largest root of legendre polynomial $P_5(x)$ given by, $P_5(x) = \frac{1}{8}\left[63x^5 - 70x^3 + 15x\right]$ by newton's method.

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$\rightarrow f(x) = \frac{1}{8}[63x^5 - 70x^3 + 15x]$.

$f'(x) = \frac{1}{8}[315x^4 - 210x^2 + 15]$.

nearer value to zero

$f(0.9) = -0.04$, $f(1) = 1$.

$\therefore$ Root lie in $(0.9, 1)$.

$\therefore |f(0.9)|$ is nearer to zero.

$\therefore x_0 = 0.9$

$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.9063$.

$\therefore x_2 = 0.9061$.

$\therefore x = 0.906$

04) Logistic population model

$\frac{dy}{dx} = y - y^2$, $y(0) = 0.2$, $h = 0.1$. find $y(0.1)$ &

solve exactly. compute the error by modified Euler method.

$\rightarrow \frac{dy}{dx} = f(x,y) = y - y^2$.

$x_0 = 0$, $y_0 = 0.2$, $h = 0.1$

$x_1 = 0.1$, $y_1 = ?$

$y_1^{(0)} = y_0 + h.f(x_0, y_0)$.

Euler method,

$y_1^{(0)} = y_0 + h.f(x_0, y_0) = 0.216$.

Modified Euler method,

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right].$$

$$y_1^{(1)} = 0.2 + \frac{0.1}{2}\left[y_0 - y_0^2 + y_1^{(0)} - (y_1^{(0)})^2\right].$$

$$y_1^{(1)} = 0.2164.$$

$$\therefore y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right].$$

$$\therefore y_1^{(2)} = 0.2164.$$

$$\therefore y_1 = 0.2164.$$

Exact Solution,

$$\frac{dy}{dx} = y - y^2$$

$$\frac{dy}{y - y^2} = dx$$

$$\frac{dy}{y(1-y)} = dx \Rightarrow dy\left[\frac{1}{y(1-y)}\right] = dx.$$

$$\therefore dy\left[\frac{A}{y} + \frac{B}{(1-y)}\right] = dx \qquad [\because \text{partial fractions}]$$

$$\therefore dy\left[\frac{1}{y} + \frac{1}{1-y}\right] = dx.$$

Integrating on both sides:

$$\int \frac{1}{y}\,dy + \int \frac{1}{(1-y)}\,dy = dx.$$

$$\therefore \log(y) - \log(1-y) = x + c.$$

$$\therefore \log\left(\frac{y}{1-y}\right) = x + c$$

$$\frac{y}{1-y} = e^{x+c} \rightarrow \frac{y}{(1-y)} = e^{x} \cdot e^{c}$$

$$\therefore \frac{y}{(1-y)} = c_1 e^{x} \quad \text{—} \ (1) \qquad \text{where, } c_1 = e^{c}.$$

$\therefore$ I.C, $y(0) = 0.2$

then, (1) becomes, $\frac{0.2}{1-0.2} = c_1 . e^{0}$.

$$\therefore c_1 = \frac{0.2}{1-0.2} = 0.25.$$

$\therefore \frac{y}{1-y} = 0.25e^{x}$. is exact Solution.

put, $x = 0.1$

$$\therefore \frac{y}{1-y} = 0.25e^{(0.1)}.$$

$$\therefore \frac{y}{(1-y)} = 0.276.$$

$$\therefore y = 0.276(1-y)$$

$$\therefore y = 0.276 - 0.276y.$$

$$\therefore y + 0.276y = 0.276.$$

$$\therefore y = \frac{0.276}{1.276}$$

$$\therefore y = 0.2163.$$

$\therefore$ Error $= y_1 \left[\begin{matrix}\text{modified} \\ \text{Euler}\end{matrix}\right] - y_1 \left[\begin{matrix}\text{Exact} \\ \text{solution}\end{matrix}\right]$

$= 0.2164 - 0.2163.$

$\therefore$ Error $= 0.0001$

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Q5) Apply RK method, to find approximate value of 'y', $y' = 1+y^2$ given $y(0)=0$, $h=0.1$ find $y(0.1)$. Solve exactly.

$\rightarrow$ $\frac{dy}{dx} = 1+y^2 = f(x,y)$.

$x_0 = 0$, $y_0 = 0$, $h = 0.1$, $x_1 = 0.1$, $y_1 = ?$

By RK method,

$K_1 = h.f(x_0, y_0) = (0.1)(1+0^2) = 0.1$.

$K_2 = h.f[x_0 + h/2, y_0 + K_1/2]$.

$= (0.1).f(0 + 0.1/2, 0 + 0.1/2) = (0.1) f(0.05, 0.05)$

$\therefore K_2 = 0.10025$.

$\therefore K_3 = h.f(x_0 + h/2, y_0 + K_2/2)$

$= (0.1).f(0 + 0.05, 0 + \frac{0.101}{2}) = (0.1).f(0.05, 0.0505)$

$\therefore K_3 = (0.1)(1 + (0.0505)^2) = 0.10025$

$\therefore K_4 = h.f(x_0+h, y_0+K_3) = (0.1) f(0.1, 0.10025)$

$= 0.1010$.

$\therefore K = \frac{1}{6}[K_1 + K_2 + K_3 + K_4]$

$= 0.10033$.

$\therefore y_1 = y_0 + K = 0.10033$

Exact Solution, $\frac{dy}{dx} = 1+y^2$.

$\frac{dy}{(1+y^2)} = dx$.

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Integrating both sides,

$$\int \frac{dy}{(1+y^2)} = \int dx + c$$

$$\therefore \tan^{-1}(y) = x + c.$$

(Initial condition) $\therefore$ IC , $y(0) = 0$.

$$\therefore \tan^{-1}(0) = 0 + c$$

$$c = 0,$$

$$\therefore \tan^{-1} y = x,$$

$$\therefore y = \tan x \text{ is exact solution.}$$

$\therefore x = 0.1$, $y = \tan(0.1) = 0.10033$

$\therefore$ Error less.

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